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Strongly simply connected schurian algebras and multiplicative bases

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Abstract

In this paper, we define concepts of crowns and quasi-crowns, valid in an arbitrary schurian algebra, and which generalise the corresponding concepts in an incidence algebra. We show first that a triangular schurian algebra is strongly simply connected if and only if it is simply connected and contains no quasi-crown. We then prove that the absence of quasi-crowns in a triangular schurian algebra implies the existence of a multiplicative basis.

1. Introduction

The aim of this paper is to explore some of the relations between the existence of a multiplicative basis in a schurian algebra and its strong simple connectedness. Indeed, it is known since [4] (see also [20]) that a schurian strongly simply connected algebra

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admits a normed presentation, hence a multiplicative basis. Our starting point, however, is the criterion for the strong simple connectedness of an incidence algebra, that of the absence of crowns [20]. We then notice that Bongartz' well-known example of an algebra not admitting a multiplicative basis [14] contains a full convex subcategory isomorphic to a crown. Here, we define more general notions of crowns and quasi-crowns, valid in an arbitrary schurian algebra. We investigate how the absence of quasi-crowns implies the strong simple connectedness of the algebra, and show that this absence always implies the existence of a multiplicative basis.

Our motivation comes from the representation theory of finite dimensional algebras over an algebraically closed field k. For such an algebra A, there exists a (uniquely determined) quiver Q_A and (at least) a surjective algebra morphism from the path algebra kQ_A of Q_A onto A, whose kernel is denoted by I_ν, see, for instance, [13]. The algebra A is called *triangular* if Q_A has no oriented cycles. For each pair (Q_A, I_ν), called a *presentation* of A, one can define the *fundamental group* $\pi_1(Q_A, I_\nu)$, see [23,25]. A triangular algebra A is called *simply connected* if, for every presentation (Q_A, I_ν), the group $\pi_1(Q_A, I_\nu)$ is trivial [8], and *strongly simply connected* if every full convex subcategory of A is simply connected [33]. If A is a *schurian algebra* (that is, if, for each pair of primitive idempotents e, f of A, we have $\dim_k(eAf) \leqslant 1$), then all its presentations yield isomorphic fundamental groups [10], and A is simply connected if and only if so is the associated chain complex [16,17,29]. Simply connected algebras have played an important rôle in representation theory: indeed, covering techniques allow to reduce many problems to problems about simply connected algebras. While finding criteria for the simple connectedness of an algebra is an undecidable problem (because it can be reduced to a word problem), it is known (see [33]) that, if an algebra is separated in the sense of [11], then it is simply connected (but the converse is not true). On the other hand, the class of strongly simply connected algebras seems much easier to handle. Indeed, characterisations of strong simple connectedness were obtained, for instance, in [4], and the representation theory of the tame strongly simply connected algebras is largely known, see [28,31,32]. In particular, a question was asked by Skowroński in [33] whether it is true that a simply connected algebra is strongly simply connected if and only if it contains no full convex subcategory which is hereditary of type $\widetilde{\mathbb{A}}$. While the answer to this question is negative, even for incidence algebras (see example 1 in 3.1 below) there are many classes for which this statement holds true (see, for instance, [1]). In this paper, we return to the general case.

It was shown by Dräxler [20] that an incidence algebra is strongly simply connected if and only if its quiver contains no crowns. Crowns are well known in the combinatorics of posets, and are associated to their dismantlability (see, for instance, [19,21]). In this paper, we define a notion of dismantlability in an arbitrary schurian algebra.

On the other hand, we relate the strong simple connectedness to the vanishing of some of its (co)homology groups, namely, the Hochschild cohomology groups $HH^\bullet(A)$ of A with coefficients in the bimodule ${}_AA_A$ (see [18]) and the simplicial homology (and cohomology with coefficient in an abelian group G) groups $SH_\bullet(A)$ (and $SH^\bullet(A, G)$, respectively) of the simplicial complex associated with A.

We are now able to state our first main theorem.

Theorem A. *Let A be a schurian triangular algebra. The following conditions are equivalent*:

(a) *A is strongly simply connected.*
(b) *A is dismantlable.*
(c) *A is separated and contains no quasi-crowns.*
(d) *A is simply connected and contains no quasi-crowns.*
(e) $SH_1(A)=0$ *and A contains no quasi-crowns.*
(f) $SH^1(A,G)=0$ *for every abelian group G, and A contains no quasi-crowns.*
(g) *A is a quotient of an incidence algebra, $HH^1(A)=0$, and A contains no crowns.*

As a consequence of the equivalence of (a) and (b), we give an algorithm allowing to check whether a schurian triangular algebra is strongly simply connected or not (thus, in particular, verifying that the strong simple connectedness of a schurian triangular algebra is a decidable problem).

Predictably, we obtain much better results for quotients of incidence algebras, namely, in this case, we are able to replace "quasi-crowns" by "crowns" in the statement of the theorem above.

Also, we answer in the negative the conjecture saying that the presence of a bypass in the quiver of a schurian algebra prevents it from being simply connected. We show, on the other hand, that the presence of such a bypass in a simply connected schurian algebra implies the existence of a quasi-crown.

Our second main theorem is the following.

Theorem B. *Let A be a schurian triangular algebra containing no quasi-crowns. Then A admits a multiplicative basis.*

This clearly generalises the main result of [14], which states the existence of a multiplicative basis in a triangular representation-finite algebra. As an easy consequence of our result, only finitely many non-isomorphic schurian algebras of a given dimension do not contain quasi-crowns.

Our proofs rely heavily on the use of a Mayer–Vietoris sequence for a one-point extension, as in [15,26]. We also obtain as consequences some of the results of [20,22].

The paper is organised as follows. After a preliminary Section 2, we introduce our notions of crown and quasi-crown in Section 3, and our notion of dismantlability in Section 4. Section 5 is devoted to the proof of Theorem A and Section 6 to the proof of Theorem B.

2. Preliminaries

2.1. Notation

In this paper, by algebra, we always mean a basic and connected finite dimensional algebra over an algebraically closed field k. Given a quiver Q, we denote by Q_0 its set of points and by Q_1 its set of arrows. A *relation* in Q from a point x to a point y is a linear combination $\rho = \sum_{i=1}^{m} \lambda_i w_i$ where, for each i, $\lambda_i \in k$ is non-zero and w_i is a path of length at least two from x to y. A relation in Q is called a *monomial* if it equals a path, and a *commutativity relation* if it equals the difference of two paths. We denote by kQ the path algebra of Q and by $kQ(x, y)$ the k-vector space generated by all paths in Q from x to y. For an algebra A, we denote by Q_A its quiver. For every algebra A, there exists an ideal I in kQ_A, generated by a set of relations, such that $A \cong kQ_A/I$. The pair (Q_A, I) is called a *presentation* of A. An algebra $A = kQ/I$ can equivalently be considered as a k-category of which the object class A_0 is Q_0, and the set of morphisms $A(x, y)$ from x to y is the quotient of $kQ(x, y)$ by the subspace $I(x,\ y) = I \cap kQ(x, y)$, see [13]. A full subcategory B of A is called *convex* if any path in A with source and target in B lies entirely in B. An algebra A is called *triangular* if Q_A has no oriented cycles, and it is called *schurian* if, for all $x, y \in A_0$, we have $\dim_k A(x, y) \leqslant 1$. In this paper, we deal exclusively with schurian triangular algebras. For a point x in the quiver Q_A, we denote by e_x the corresponding primitive idempotent, and by P_x and I_x the corresponding indecomposable projective and injective A-module, respectively.

2.2. Simple connectedness

Let Q be a connected quiver without oriented cycles and I be an ideal of kQ generated by relations. A relation $\rho = \sum_{i=1}^{m} \lambda_i w_i \in I(x, y)$ is called *minimal* if $m \geqslant 2$ and, for every non-empty proper subset $J \subset \{1, 2, \ldots, m\}$, we have $\sum_{j \in J} \lambda_j w_j \notin I(x, y)$. For an arrow α, we denote by α^{-1} its *formal inverse*. A *walk* in Q from x to y is a formal composition $\alpha_1^{\varepsilon_1}\alpha_2^{\varepsilon_2}\cdots\alpha_t^{\varepsilon_t}$ (where $\alpha_i \in Q_1$ and $\varepsilon_i \in \{1, -1\}$ for all i) from x to y. The *homotopy relation* is the least equivalence on the set of all walks in Q such that:

(a) For each arrow $\alpha : x \to y$, we have $\alpha\alpha^{-1} \sim e_x$ and $\alpha^{-1}\alpha \sim e_y$.
(b) For each minimal relation $\sum \lambda_i w_i$, we have $w_i \sim w_j$ for all i, j.
(c) If $u \sim v$, then $wuw' \sim wvw'$, whenever these products are defined.

The set of all equivalence classes of walks starting and ending at a fixed base point x_0 is a group, called the *fundamental group* of (Q, I) and denoted by $\pi_1(Q, I)$. A triangular algebra A is called *simply connected* if, for any presentation (Q_A, I) of A, the group $\pi_1(Q_A, I)$ is trivial [8]. It is called *strongly simply connected* if every full convex subcategory of A is simply connected [33].

It is shown in [10] that, if an algebra $A \cong kQ_A/I$ is schurian and triangular, then the fundamental group $\pi_1(Q_A, I)$ does not depend on the presentation (Q_A, I) of A. We may thus use the unambiguous notation $\pi_1(A)$ to stand for $\pi_1(Q_A, I)$.

Moreover, it is known that, for every connected bound quiver (Q, I), there exists a CW-complex $\mathcal{B} = \mathcal{B}(Q, I)$, called its *classifying space*, such that $\pi_1(Q, I) = \pi_1(\mathcal{B})$, see [17]. If kQ/I is schurian and triangular, then the classifying space $\mathcal{B}(Q, I)$ is a simplicial complex, see [16,29], which coincides with the one considered in [15]. It is constructed as follows: an i-simplex is a set of $(i + 1)$-distinct objects $\{x_0, x_1, \ldots, x_i\}$ in A_0 such that, for any j with $1 \leqslant j \leqslant i$, there exists $a_j \in A(x_{j-1}, x_j)$ such that $a_i a_{i-1} \cdots a_1 \neq 0$. We denote by $C_\bullet(A)$ the corresponding chain complex.

For concepts and results from algebraic topology, we refer the reader to [30].

We need the following concept. Let B be a non-necessarily connected algebra, a B-module M is called *separated* if the supports of the distinct indecomposable summands of M lie in distinct connected components of B. For an algebra A, and for $x \in A_0$, let A^x denote the full subcategory of A generated by the non-predecessors of x in Q_A. Then x is called *separating* if the restriction to A^x of $\mathrm{rad}P_x$ is separated as an A^x-module. The algebra A is called *separated* if each $x \in A_0$ is separating. It is shown in [33] that any separated algebra is simply connected.

2.3. *Strong simple connectedness*

Let Q be a connected quiver without oriented cycles. A *contour* (p, q) in Q from x to y is a pair of parallel paths of positive length from x to y. A contour (p, q) is called *interlaced* if p and q have a common point besides x and y. It is called *irreducible* if there exists no sequence of paths $p = p_0, p_1, \ldots, p_m = q$ from x to y such that, for each i, the contour (p_i, p_{i+1}) is interlaced. A cycle C in Q is called *irreducible* if, either C is an irreducible contour, or C is not a contour, but satisfies the following condition and its dual: for each source x in C, no proper successor of x in Q is also a source of C, and exactly two proper successors of x in Q are sinks of C. This is equivalent to the definition of irreducibility given in [4, 1.5]. It is proven in [4, 2.4] that an algebra A is schurian and strongly simply connected if and only if

(a) all irreducible cycles are irreducible contours, and
(b) there exists a presentation $A \cong kQ_A/I$ such that for each irreducible contour (p, q), we have $p, q \notin I$ but $p - q \in I$.

Such a presentation is a normed presentation, in the sense of [12]. Its existence implies that such an algebra admits a multiplicative basis.

2.4. *Incidence algebras and their quotients*

Let $(\Sigma, \leqslant)$ be a finite poset (partially ordered set) with n elements. The *incidence algebra* $k\Sigma$ is the subalgebra of the algebra $M_n(k)$ of all $n \times n$ matrices over k consisting of the matrices $[a_{ij}]$ satisfying $a_{ij} = 0$ if $j \nleqslant i$. The quiver Q_Σ of $k\Sigma$ is the (oriented) Hasse diagram of Σ, and $k\Sigma \cong kQ_\Sigma/I_\Sigma$, where I_Σ is generated by all differences $p - q$, with (p, q) a contour in Q_Σ. The quiver Q_Σ has no *bypass*, that is, no subquiver of the form

$$\begin{tikzcd}
& \circ \arrow[r, "\beta_2"] & \circ \arrow[r] & \cdots \arrow[r] & \circ \arrow[r] & \circ \arrow[dr, "\beta_t"] & \\
\circ \arrow[ur, "\beta_1"] \arrow[rrrrrr, "\alpha"] & & & & & & \circ
\end{tikzcd}$$

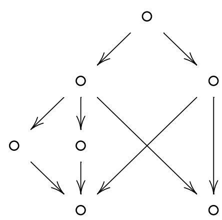
and, conversely, for any quiver Q having no bypass, there exists a poset Σ such that $Q = Q_\Sigma$.

In many places, we consider quotients of incidence algebras. For such a quotient $A \simeq kQ_A/I$, there exists a poset Σ with $Q_\Sigma = Q_A$ and, furthermore, $I = I_\Sigma + J$, where J is an ideal of kQ_Σ generated by monomials. It is well known that, if A is schurian strongly simply connected, then it is a quotient of an incidence algebra, see [20, 2.7], [4, 2.4].

3. Crowns

3.1. Before our main definitions, we give some motivating examples. We recall that in [33], Skowroński stated the following problem. Let A be a simply connected algebra. Is it true that A is strongly simply connected if and only if A contains no full convex subcategory which is hereditary of type $\widetilde{\mathbb{A}}$ (we then say that A is *strongly* $\widetilde{\mathbb{A}}$*-free*)? The answer to this question is negative in general, and even for incidence algebras.

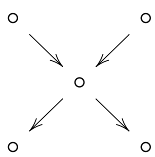
Example 1. Let indeed Σ be the poset with quiver



Clearly, the incidence algebra $k\Sigma$ is not strongly simply connected, but is simply connected and strongly $\widetilde{\mathbb{A}}$-free.

One could think of replacing the requirement that A be strongly $\widetilde{\mathbb{A}}$-free by the one that A contains no full subcategory which is hereditary of type $\widetilde{\mathbb{A}}$ (we then say that A is $\widetilde{\mathbb{A}}$*-free*). This, however, is not true, even if one assumes (as we do) that A is schurian, as is shown by the incidence algebra of the following poset (called a "cross").

Example 2.

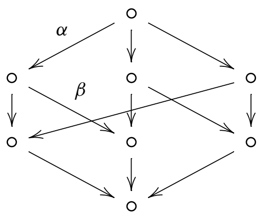


it is strongly simply connected, but not $\widetilde{\mathbb{A}}$-free.

However, it was shown in [20, 3.3] that an incidence algebra is strongly simply connected if and only if it contains no crown as full subcategory. We shall define here a concept of quasi-crown which makes sense for any schurian algebra, and reformulate Skowroński's question as follows: let A be a schurian and simply connected, is it true that A is strongly simply connected if and only if it contains no quasi-crown as a full subcategory? Our first main result answers this question positively. In case that A is a quotient of an incidence algebra we may replace "quasi-crown" by "crown" in this reformulation.

We notice that, in case A is not a quotient of an incidence algebra, the requirement that A is simply connected does not necessarily imply that A contains no crown, as is shown by the following example.

Example 3.



with all squares commutative except the upper left one, where we have the relation $\alpha\beta = 0$. Here A is simply connected even though it contains a crown. This example is called the "box," because its geometric realisation is a cubic box with all faces closed, except the upper one, which is half-open. Note that A is a tame quasi-tilted algebra. It is actually a semiregular branch enlargement of a tame concealed algebra (but is not iterated tilted of type $\widetilde{\mathbb{A}}$), see [3].

3.2. We now recall a few notions and results from the theory of split-by nilpotent extensions (see, for instance, [5,9]). Let A and B be two algebras, we say that B is a *split extension* of A by the two sided nilpotent ideal W if there exists a split surjective algebra morphism $\pi : B \to A$ whose kernel W is a nilpotent ideal of B. In this case, W is generated by arrows of the quiver of B. Indeed, let $B = kQ_B/I$, then a set S of generators of W is *special* if, for each $\rho + I \in S$, we have:

(a) If ρ is a path in Q_B then, for each proper subpath ρ' of ρ, we have $\rho' + I \notin W$.
(b) If $\rho = \sum_{i=1}^{m} \lambda_i w_i$ is a relation with $m \geqslant 2$, then for each non-empty proper subset $J \subset \{1, 2, \ldots, m\}$, we have $\sum_{j \in J} \lambda_j w_j + I \notin W$.

It is shown in [9, 1.3] that, if $B = kQ_B/I$ is a split extension of A by W, then W has a special set of generators and any such set consists of the classes modulo I of arrows in Q_B. We now give a criterion allowing to verify whether a bound quiver is a split extension or not.

Lemma. *Let $B = kQ_B/I$ be a schurian triangular algebra and W be an ideal in B generated by classes modulo I of a set of arrows. Then B is a split extension of B/W by W if and only if, for every pair of non-zero paths $\gamma = \gamma_1\gamma_2\cdots\gamma_r$, and $\gamma' = \gamma'_1\gamma'_2\cdots\gamma'_s$ bound by a minimal relation $\lambda\gamma + \mu\gamma'$ in B, if there exists an i (with $1 \leqslant i \leqslant r$) such that $\gamma_i \in S$, then there exists a j (with $1 \leqslant j \leqslant s$) such that $\gamma'_j \in S$.*

Proof. *Necessity*. Assume B is a split extension of B/W by W. Then the subalgebra B generated by the classes of arrows which are not in S is isomorphic to B/W, so we can assume that for any arrow $\beta \notin S$, the lifting of $\beta + W \in B/W$ to B is $\overline{\beta} = \beta + I$. Assume γ and γ' are as stated, and that $\gamma'_j \notin S$ for all j. Then $\gamma' + W = (\gamma'_1 + W)\cdots(\gamma'_s + W) \neq 0$ in B/W, hence $\overline{\gamma'} = \overline{\gamma'_1}\cdots\overline{\gamma'_s} \notin W$ in B. On the other hand, $\overline{\gamma} = \gamma + I \in W$ and $\lambda\gamma + \mu\gamma' \in I$ imply $\overline{\gamma'} \in W$ (because $\mu \neq 0$), a contradiction.

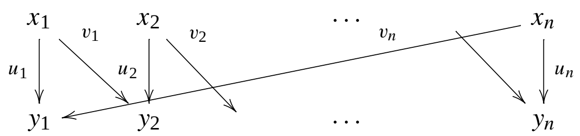
Sufficiency. We need only observe that our hypothesis implies that the subalgebra of B generated by all arrows not in S is isomorphic to B/W. Since it is obviously a subspace, it suffices to prove that, if γ, δ are paths in Q_B and $\overline{\gamma} = \gamma + I$, $\bar{\delta} = \delta + I$, then the product $(\overline{\gamma} + W)(\bar{\delta} + W) = \overline{\gamma\delta} + W$ yields the same value for all representatives of the classes $\overline{\gamma}$ and $\bar{\delta}$. However, if this is not the case, then there exist paths γ', δ' such that $\overline{\gamma} - \overline{\gamma'} \in W$, $\bar{\delta} - \overline{\delta'} \in W$ and $\overline{\gamma\delta} \in W$, while $\overline{\gamma'\delta'} \notin W$. Now, $\gamma\delta$ and $\gamma'\delta'$ being parallel paths are bound by a minimal relation, and we get a contradiction to our hypothesis. $\square$

3.3. In this section, all algebras are schurian triangular algebras. Let A be an algebra. We define the *interval* $[x, y]_A$, or more briefly $[x, y]$ between x and y (with $x, y \in A_0$) to be the full subcategory of A generated by all points $z \in A_0$ which lie on a non-zero path from x to y, that is, such that

$$A(x, z)A(z, y) \neq 0.$$

Clearly, if all paths from x to y in A are non-zero, then $[x, y]$ coincides with the full subcategory (x, y) of A generated by the convex hull of x and y. This is the case, for instance, whenever A is an incidence algebra.

3.4. The notion of crown is well known in the combinatorics of posets, see, for instance, [20,21]. We generalise it to schurian algebras as follows. Let C be a full subcategory of A consisting of $2n$ objects $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and $2n$ non-zero morphisms $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ with $n \geqslant 2$, and of the form



We say that C is a *crown* (of *width* n) in A if:

(a) $[x_i, y_j] \cap [x_h, y_l] \neq \emptyset$ if and only if $j = i$ and $(h, l) \in \{(i, i), (i-1, i), (i, i+1)\}$ or $j = i + 1$ and $(h, l) \in \{(i, i+1), (i, i), (i+1, i+1)\}$.
(b) The intersection of three distinct $[x_j, y_l]$ is empty.
(c) For each i, $[x_i, y_i] \cap [x_i, y_{i+1}] = \{x_i\}$ and $[x_i, y_i] \cap [x_{i-1}, y_i] = \{y_i\}$.

We agree to set $x_0 = x_n$, $x_{n+1} = x_1$, $y_0 = y_n$, $y_{n+1} = y_1$.

We now generalise this notion. Let C be a full subcategory of $A = kQ_A/I$. Then C is a *quasi-crown* if there exists a set of arrows $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ in Q_A such that, if R denotes the ideal of A generated by the arrows $\alpha_i + I$ (with $1 \leqslant i \leqslant r$), then

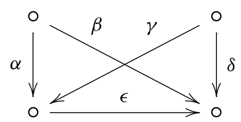
(a) C is a split extension of $C' = C/C \cap R$, and
(b) C' is a crown in A/R.

In this case, we say that the points of C induce a quasi-crown in A.

Intuitively, a quasi-crown C may be thought of as consisting of a crown C' together with some additional paths between the points of C', and these paths make C a split extension of C'.

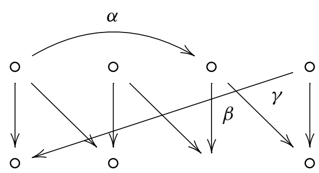
Quasi-crowns already appeared implicitly in Bongartz' proof [14] (see the proof of Lemma 3) and as we shall see in 3.8 below, also in [6, 2.4].

Example 4. The following is an example of a quasi-crown, taking $R = \langle \epsilon \rangle$. Let C be given by the quiver



bound by $\alpha\epsilon = 0$, $\gamma\epsilon = 0$.

Example 5. Clearly, in incidence algebras, quasi-crowns are crowns. But the two notions do not coincide even for quotients of incidence algebras, as is shown by the algebra given by the quiver



bound by $\alpha\beta = 0$ and $\alpha\gamma = 0$.

3.5. Recall that [20, 3.3] says that an incidence algebra is strongly simply connected if and only if contains no crowns. We have the following lemma.

Lemma. *Let A be a schurian strongly simply connected algebra, then A contains no quasi-crown.*

Proof. Assume that $A = kQ_A/I$ contains a quasi-crown C. Thus, there exists an ideal R of A generated by the classes modulo I of a set of arrows $\{\alpha_1, \alpha_2, \dots, \alpha_r\}$ of Q_A such that C is a split extension of $C' = C/C \cap R$, and C' is a crown in A/R. Therefore, there exist $2n$

objects $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ of C and $2n$ non-zero morphisms $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ of A/R with $n \geqslant 2$. Let Γ be the cycle given by

$$\begin{tikzcd}
x_1 \arrow[d, "u_1"'] \arrow[dr, "v_1"] & x_2 \arrow[d, "u_2"'] \arrow[dr, "v_2"] & & \cdots \arrow[dr] & x_n \arrow[d, "u_n"] \arrow[dllll, "v_n"'] \\
y_1 & y_2 & {} & \cdots & y_n
\end{tikzcd}$$

Clearly, Γ is not a contour. Since A is schurian strongly simply connected, we obtain by [4, 1.3 or 2.4], that Γ is reducible. Hence, there exists a path in Q_A from x_i to y_j (with $j \neq i, i+1$) or from x_i to x_j (with $j \neq i$) or dually from y_i to y_j (with $j \neq i$). We can assume that $j \geqslant i$ and that $j - i$ is minimal for this property. Let us first suppose that $j - i > 1$. If we have a path w from x_i to x_j, then

$$\begin{tikzcd}
x_i \arrow[rrrr, bend left, "w"] \arrow[dr, "v_i"'] & x_{i+1} \arrow[d, "u_{i+1}"'] \arrow[dr, "v_{i+1}"] & & \cdots \arrow[dr] & x_j \arrow[d, "u_j"] \\
& y_{i+1} & {} & \cdots & y_j
\end{tikzcd}$$

defines a cycle which must be irreducible by the minimality of $j - i$. This yields a contradiction to [4, 1.3] since A is schurian strongly simply connected. The other cases are similar.

Therefore, $j = i + 1$ and we can assume, up to duality, that there is a path from x_i to x_{i+1}, say w. We thus have a contour given by (v_i, wu_{i+1}). Since A is schurian strongly simply connected, there exists a binomial relation involving those paths. Now, A is a quotient of an incidence algebra and v_i is non-zero, hence we get that wu_{i+1} is also non-zero. Since $C' = C/C \cap R$ is a crown, then wu_{i+1} must be zero in A/R, otherwise $x_{i+1} \in [x_i, y_{i+1}] \cap [x_{i+1}, y_{i+1}]$. On the other hand, since wu_{i+1} is not zero in A, we can assume that there exists an arrow $\alpha \in R$ which is a subpath of wu_{i+1}. The binomial relation of A involving v_i and wu_{i+1} forces v_i to be zero in A/R (by 3.2), a contradiction. Hence, there is no quasi-crown contained in A. $\square$

Example 6. Assume $A = k\Sigma/J$ is a quotient of an incidence algebra. If Σ contains a (quasi-)crown this does not necessarily implies that A contains a (quasi-)crown. Let, for instance, A be the quotient of the incidence algebra of the poset with quiver

$$\begin{tikzcd}
\circ \arrow[d, "\alpha"'] \arrow[ddr] & \circ \arrow[ddl] \arrow[dd] \\
\circ \arrow[d, "\beta"'] & \\
\circ & \circ
\end{tikzcd}$$

by the ideal generated by $\alpha\beta$.

3.6. We recall that we always assume our algebras to be schurian and triangular. The following construction, due to Bretscher and Gabriel [15] is needed essentially in the sequel. Let s be a source in an algebra A. We define the following two sets of objects of A:

$$\Sigma_s = \bigl\{x \in A_0 \mid A(s,x) \neq 0\bigr\}$$

(that is, Σ_s consists of the objects in the support of the corresponding indecomposable projective A-module P_s), and

$$\Sigma_s' = \Sigma_s \setminus \{s\}$$

(that is, Σ_s' consists of the objects in the support of the radical of P_s). We partially order each of these sets by setting

$$x \leqslant y \quad \text{if and only if} \quad A(s,y)A(y,x) \neq 0$$

(that is, there exist non-zero paths from s to y and from y to x with non-zero composition). The incidence categories $k\Sigma_s$ and $k\Sigma_s'$ can be identified with subcategories of A, usually not full.

Example 7. Let A be given by the quiver

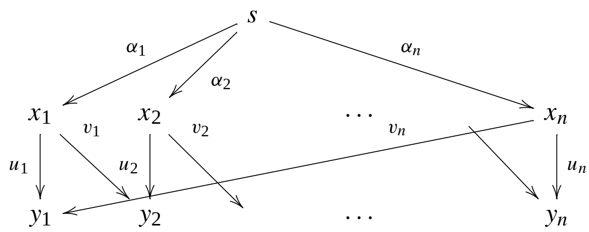
$$\begin{tikzcd}
 & s \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\
\circ \arrow[dr, "\beta"'] & & \circ \arrow[dl, "\delta"] \\
 & \circ &
\end{tikzcd}$$

bound by $\alpha\beta = 0$. Then Σ_s and Σ_s' are respectively given by the posets

$$\begin{tikzcd}
 & s \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\
\circ & & \circ \arrow[dl, "\delta"] \\
 & \circ &
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
\circ & & \circ \arrow[dl, "\delta"] \\
 & \circ &
\end{tikzcd}$$

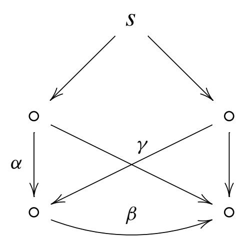
Lemma. *Let s be a source in A. If the points $\{x_1,\dots,x_n,y_1,\dots,y_n\}$ induce a crown Γ in $k\Sigma_s$, then the same points induce a quasi-crown in A.*

Proof. By hypothesis, there exists a full subcategory of $k\Sigma_s$ of the form



Let C' denote the full subcategory generated by the x_i and y_j. Also let R be the ideal of A generated by the classes of all the arrows which are not in $k\Sigma_s$. We show that the full subcategory C of A generated by C' induces a quasi-crown of A. Clearly, $C' = C/C \cap R$. Thus, it suffices to verify that C is a split extension of C'. Let x and y be points of C and thus of C' such that there exists a path γ from x to y in $k\Sigma_s$. We have to show that the class of no path from x to y in A lies in R. Suppose thus that there exists a non-zero path w from x to y in R. Since x is in C', it is also in $k\Sigma_s$ and there exists a path v from s to x in A such that $v\gamma$ is a non-zero path. Since w belongs to R, this means that $vw = 0$. On the other hand, A is a schurian algebra, thus there exists a scalar λ, such that $\gamma = \lambda w$ in A. Therefore, $v\gamma = \lambda(vw) = 0$ a contradiction which proves that no such path w exists. This shows that C is a split extension of C'. $\square$

Example 8. In general, the quasi-crown induced as in the lemma is not a crown in A, as is shown by the algebra given by the quiver



bound by all possible commutativity relations and $\alpha\beta = 0$, $\gamma\beta = 0$.

3.7. We have better results for quotients of incidence algebras.

Lemma. *Let A be a quotient of an incidence algebra and assume that $k\Sigma_s$ contains a crown. Then A contains a crown.*

Proof. By 3.6, A contains a quasi-crown Γ, induced by one of $k\Sigma_s$. We first claim that the interval from x to y in $k\Sigma_s$ coincides with the one of A. Let $z \in [x, y]_A$, that is, there exist paths $p : x \rightsquigarrow z$ and $q : z \rightsquigarrow y$ such that pq is not zero in A. Since x and y belong to $k\Sigma_s$, there exist non-zero paths $u : s \rightsquigarrow x$ and $v : s \rightsquigarrow y$. Since A is a quotient of an incidence algebra and (v, upq) is a contour, then upq is non-zero in A. Thus pq corresponds also to a non-zero path in $k\Sigma_s$. This proves that $z \in [x, y]_{k\Sigma_s}$. The other inclusion being obvious, this establishes our claim. Now, in order to show that Γ must be a crown in A, assume that this is not the case. In the notation of 3.4, this means that there exists a path from x_i to x_j (with $j \neq i$), from y_i to y_j (with $j \neq i$), or from x_i to y_j (with $j \neq i, i+1$). In each of these cases, we find a zero path parallel to a non-zero one, a contradiction to the fact that A is a quotient of an incidence algebra. □

3.8. As a consequence of 3.6, we connect the notion of quasi-crown with the results of [6, 2.4]. Let $A = B[M]$ be a one-point extension algebra, and s denote the extension point. Since all presentations of A give rise to isomorphic fundamental groups, we fix a presentation of A, and consider the induced presentation of B. Let $\cong$ be the least equivalence relation on the set of arrows of source s such that $\alpha_1 \cong \alpha_2$ whenever there exists a minimal relation of the form $\lambda_1(\alpha_1 v_1) + \lambda_2(\alpha_2 v_2)$. Let t be the number of equivalence classes $[\beta_1], \ldots, [\beta_t]$ of arrows with source s. For each i, with $1 \leqslant i \leqslant t$, let $l(i)$ be the number of tuples of paths $(u_1, v_1, \ldots, u_n, v_n)$ such that there are relations $\lambda'_{1,2}(\alpha_1 v_1) + \lambda''_{1,2}(\alpha_2 u_2), \ldots, \lambda'_{n-1,n}(\alpha_{n-1} v_{n-1}) + \lambda''_{n-1,n}(\alpha_n u_n), \lambda'_{n,1}(\alpha_n v_n) + \lambda''_{n,1}(\alpha_1 u_1)$ with $\alpha_1, \alpha_2, \ldots, \alpha_n$ distinct arrows in $[\beta_i]$.

Let further, $B = B_1 \times \cdots \times B_c$, where $B_1, \ldots, B_c$ are connected, then for each j, the embedding of B_j inside A induces a canonical group morphism $\phi_j : \pi_1(B_j) \to \pi_1(A)$, hence a morphism $\phi : \prod_{j=1}^{c} \pi_1(B_j) \to \pi_1(A)$.

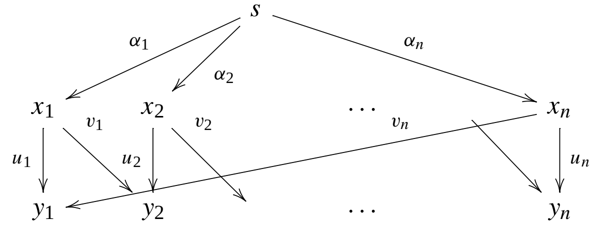
Corollary. *With the above notation*:

(a) *Assume that for some i with $1 \leqslant i \leqslant t$, we have $l(i) \neq 0$, then A contains a quasi-crown. If, in particular, A is a quotient of an incidence algebra, then A contains a crown.*
(b) *Assume A contains no quasi-crown, then, for each abelian group G, we have a short exact sequence of abelian groups*

$$0 \longrightarrow G^{t-c} \longrightarrow \mathrm{Hom}\big(\pi_1(A), G\big) \xrightarrow{\mathrm{Hom}\,(\phi, G)} \prod_{j=1}^{c} \mathrm{Hom}\big(\pi_1(B_j), G\big) \longrightarrow 0.$$

(c) *If A is simply connected, but one of the B_j is not, then A contains a quasi-crown.*

Proof. (a) Assume $l(i) \neq 0$ for some i, and $(u_1, v_1, \ldots, u_n, v_n)$ be a tuple as above, then A contains a subcategory of the form



It suffices, in view of 3.6, to show that $k\Sigma_s$ contains a crown. We may clearly, without loss of generality, assume that n is minimal. This implies immediately that the points x_i and y_j satisfy conditions (a) and (b) of the definition of crown, see 3.4. In the terminology of [7], these points induce a weak crown in $k\Sigma_s$. By [7, 3.2], the convex hull of these points (in $k\Sigma_s$) contains a crown. By 3.6, A contains a quasi-crown. This shows our first statement. For the second, assume that A is a quotient of an incidence algebra, then, as just seen, $k\Sigma_s$ contains a crown. By 3.7, A itself contains a crown.

(b, c) It is shown in [6, 2.4] that, for each abelian group G, there is an exact sequence of abelian groups

$$0 \longrightarrow G^{t-c} \longrightarrow \operatorname{Hom}\bigl(\pi_1(A), G\bigr) \xrightarrow{\operatorname{Hom}(\phi,G)} \prod_{j=1}^{c} \operatorname{Hom}\bigl(\pi_1(B_j), G\bigr) \longrightarrow \prod_{i=1}^{t} G^{l(i)}.$$

Both (b) and (c) then follow immediately from (a). □

4. Dismantlability

4.1. Let A be a schurian algebra. A point $x \in A_0$ is called a *doubly irreducible* (see [21]) if there is at most one arrow of target x, and at most one arrow of source x.

Given a doubly irreducible x in A, we define a new category $B = A(x)$ such that $B_0 = A_0 \backslash \{x\}$ as follows.

Assume first that $y \xrightarrow{\alpha} x \xrightarrow{\beta} z$. If $\alpha\beta \neq 0$, we let B be the full subcategory of A consisting of all its objects except x. If, on the other hand $\alpha\beta = 0$, we let B be the category whose object class is $B_0 = A_0 \backslash \{x\}$ and whose arrows are the same as those of A, except for the arrows α and β which are replaced by a new arrow $\alpha' : y \to z$. Finally, the relations of B are exactly those of A, except the relation $\alpha\beta = 0$ which disappears.

We define similarly B if $x \xrightarrow{\beta} z$ or if $y \xrightarrow{\alpha} x$.

Note that B is generally not schurian: if A is given by the quiver

$$\begin{tikzcd} & x \arrow[dr, "\beta"] & \\ y \arrow[ur, "\alpha"] \arrow[rr, "\gamma"] & & z \end{tikzcd}$$

bound by $\alpha\beta = 0$, then B is given by $y \overset{\alpha'}{\underset{\gamma}{\rightrightarrows}} z$.

As will be seen, the statement and proof of the following lemma hold true even if $A(x)$ is not schurian, by taking the fundamental group of the induced presentation.

Lemma. *Let A be a schurian algebra, and x be a doubly irreducible in A such that $A(x)$ is schurian. Then $\pi_1(A) \cong \pi_1(A(x))$.*

Proof. Since we are only interested in the homotopy relations in A and $A(x)$, we may, and shall, assume without loss of generality that all binomial relations are commutativity relations.

Assume first that $y \xrightarrow{\alpha} x \xrightarrow{\beta} z$. We have two cases: $\alpha\beta \neq 0$ and $\alpha\beta = 0$. We show that the second case may be reduced to the first. Indeed, suppose $\alpha\beta = 0$. It is known that the homotopy ignores the monomial relations. We let A' be the algebra (not necessarily schurian) given by the same quiver as A, and the same relations except $\alpha\beta = 0$, which disappears. Then the identity morphisms clearly induce an isomorphism between the fundamental groups of A and A'.

Replacing A by A' if necessary, we may thus assume from the start that A is a not necessarily schurian algebra, and that x is a doubly irreducible such that $y \xrightarrow{\alpha} x \xrightarrow{\beta} z$ and $\alpha\beta \neq 0$. Moreover, A is only bound by monomial relations or commutativity relations. Then the full subcategory $B = A(x)$ of A such that $B_0 = A_0 \backslash \{x\}$ is also bound by monomial relations and commutativity relations. We assume first that there is no relation of the form $\alpha\beta = \gamma_1 \dots \gamma_t$. Thus, the arrows α, β are replaced in B by a new arrow $\alpha' : y \to z$. We define a map $\bar{\varphi}$ from the set $\mathcal{W}_B$ of all walks in B to the set $\mathcal{W}_A$ of all walks in A by setting

$\bar{\varphi}(x') = x'$ for all $x' \in B_0$,
$\bar{\varphi}(\gamma) = \gamma$ for any arrow $\gamma \neq \alpha'$ in B, and
$\bar{\varphi}(\alpha') = \alpha\beta$.

We extend $\bar{\varphi}$ to any walk in $\mathcal{W}_B$ by the formula

$$\bar{\varphi}\left(\xi_1^{\epsilon_1} \dots \xi_r^{\epsilon_r}\right) = \bar{\varphi}(\xi_1)^{\epsilon_1} \dots \bar{\varphi}(\xi_r)^{\epsilon_r}$$

(here, ξ_i is an arrow in B, and $\epsilon_i \in \{1, -1\}$ for each i). This map is surjective: indeed, any irreducible closed walk in A involving α^ϵ, or β^ϵ (with $\epsilon \in \{1, -1\}$) involves $(\alpha\beta)^\epsilon$ because the point x is doubly irreducible. Since $\bar{\varphi}$ clearly respects the minimal relations, it induces a group epimorphism $\varphi : \pi_1(B) \to \pi_1(A)$.

We now define $\bar{\psi} : \mathcal{W}_A \to \mathcal{W}_B$ as follows:

$\bar{\psi}(x) = z$,
$\bar{\psi}(x') = x'$ for all $x' \neq x$ in A_0,
$\bar{\psi}(\alpha) = \alpha'$,
$\bar{\psi}(\beta) = z$, and
$\bar{\psi}(\gamma) = \gamma$ for any arrow $\gamma \neq \alpha, \beta$ in A.

We extend $\bar{\psi}$ to any walk as above. Since $\bar{\psi}$ respects the minimal relations, it induces a morphism $\psi : \pi_1(A) \to \pi_1(B)$. On the other hand, we have $\bar{\psi}\bar{\varphi} = 1_{\mathcal{W}_B}$ so that $\psi\varphi = 1_{\pi_1(B)}$ and so φ is a group isomorphism.

Assume now that there exists a minimal relation of the form $\alpha\beta = \gamma_1 \dots \gamma_t$. In this case, the arrows α and β are simply deleted in B.

We define $\bar{\varphi} : \mathcal{W}_B \to \mathcal{W}_A$ to be the inclusion. Clearly, it induces a group morphism $\varphi : \pi_1(A) \to \pi_1(B)$. We now define $\bar{\psi} : \mathcal{W}_A \to \mathcal{W}_B$ as follows:

$\bar{\psi}(x) = z$,
$\bar{\psi}(x') = x'$ for all $x' \neq x$ in A_0,
$\bar{\psi}(\alpha) = \gamma_1 \dots \gamma_t$,
$\bar{\psi}(\beta) = z$, and
$\bar{\psi}(\gamma) = \gamma$ for any arrow $\gamma \neq \alpha, \beta$ in A.

We extend $\bar{\psi}$ to walks in the usual way. Clearly, $\bar{\psi}$ is surjective. Also, it respects the minimal relations, hence it induces a group epimorphism $\psi : \pi_1(A) \to \pi_1(B)$. To finish the proof, it suffices to show that $\varphi\psi = 1_{\pi_1(A)}$. In order to do it, we prove that for every closed walk w in A, we have $\bar{\varphi}\bar{\psi}(w) \sim w$ (where $\sim$ denotes the homotopy relation). Clearly, we may consider only the case where $w = w_1\alpha\beta w_2$ (or, dually, $w = w_1\beta^{-1}\alpha^{-1}w_2$), and then we have $\bar{\varphi}\bar{\psi}(w) = \bar{\varphi}(w_1\gamma_1 \dots \gamma_t w_2) = w_1\gamma_1 \dots \gamma_t w_2 \sim w$ (or $\bar{\varphi}\bar{\psi}(w) = w_1\gamma_t^{-1} \dots \gamma_1^{-1} w_2 \sim w$, respectively).

Finally, the cases where $x \xrightarrow{\beta} z$ and $y \xrightarrow{\alpha} x$ are similar. □

4.2. We deduce that this construction preserves the strong simple connectedness of the algebra.

Corollary. *Assume A to be schurian strongly simply connected and that $x \in A_0$ is doubly irreducible. Then $B = A(x)$ is strongly simply connected.*

Proof. Let C be a full convex subcategory of B. Assume again that $y \xrightarrow{\alpha} x \xrightarrow{\beta} z$ (the other cases being similar). If y and z do not both lie in C, then C is (isomorphic to) a full convex subcategory of A, and hence is simply connected. Otherwise, there exists a full convex subcategory C' of A such that $C = C'(x)$. Since C' is simply connected, and $\pi_1(C') \cong \pi_1(C)$, then C is simply connected. □

4.3. A schurian algebra A is said to be *dismantlable* (by doubly irreducibles) if there exists an ordering $\{x_1, x_2, \dots, x_n\}$ of all objects of A such that x_1 is doubly irreducible in A and, for each $i \geqslant 1$, $A(x_1, \dots, x_i) = A(x_1, \dots, x_{i-1})(x_i)$ is schurian and the object x_{i+1} is doubly irreducible in $A(x_1, \dots, x_i)$.

Remark. Let A be a schurian algebra whose quiver contains no bypass. There exists a unique poset Σ such that $Q_\Sigma = Q_A$. We show in 4.6 below that, if A is dismantlable then so is $k\Sigma$. The converse, however, is not true.

Example 9. Let Σ be the poset given by the quiver

$$\begin{tikzcd}
 & \circ \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] & \\
\circ \arrow[dr, "\beta"'] & & \circ \arrow[dl, "\delta"] \\
 & \circ &
\end{tikzcd}$$

and J be the ideal of $k\Sigma$ generated by $\alpha\beta$ $(=\gamma\delta)$, then $A = k\Sigma/J$ is not dismantlable, even though $k\Sigma$ clearly is.

Proposition. *Let A be dismantlable. Then A is simply connected.*

Proof. By induction on $|A_0|$. For $|A_0| = 1$, there is nothing to show. Assume the statement holds for all dismantlable algebras A' such that $|A'_0| < |A_0|$, and let A be dismantlable. Let $\{x_1, \ldots, x_n\}$ be an ordering of the objects of A as in the above definition. By 4.1, $\pi_1(A(x_1)) \cong \pi_1(A)$. By the induction hypothesis, $A(x_1)$ is simply connected. Hence so is A. $\Box$

4.4. We now show that dismantlability implies strong simple connectedness.

Lemma. *Let A be a schurian dismantlable algebra, and let C be a full convex subcategory of A. Then C is dismantlable.*

Proof. By induction on $|A_0|$.The statement being clear for $|A_0| = 1$, assume that $|A_0| > 1$ and that A contains a full convex subcategory C which is not dismantlable. In particular, $C \neq A$. Since C is convex in A, there exists a source or a sink $a \in A_0 \backslash C_0$. We may then, up to duality, write $A = B[M]$, where B is the full convex subcategory of A with $B_0 = A_0 \backslash \{a\}$. We have $C \subseteq B$, and C is convex in B. Since $|B_0| = |A_0| - 1$, the induction hypothesis implies that B is not dismantlable. Since, however, A itself is dismantlable, there exists an ordering $\{x_1, \ldots, x_n\}$ of the objects of A as in the definition 4.3. In particular, $x_1 \neq a$ because otherwise B would be dismantlable. If $x_1 \notin C_0$, then C is (isomorphic to) a full convex subcategory of $A(x_1)$, and $A(x_1)$ is dismantlable with one object less that A, then the induction hypothesis yields a contradiction to the non-dismantlability of C. Therefore $x_1 \in C_0$. This implies that $C(x_1)$ is a full convex subcategory of $A(x_1)$. But then the induction hypothesis yields that $C(x_1)$ is dismantlable. Therefore C itself is dismantlable, another contradiction. $\Box$

4.5. This lemma and 4.3 imply immediately the following.

Corollary. *Let A be a schurian dismantlable algebra, then A is strongly simply connected.*

4.6. As promised, we prove that dismantlability of a schurian algebra implies that of a corresponding incidence algebra.

Proposition. *Let A be a schurian dismantlable algebra, then there exist a unique poset Σ such that $Q_\Sigma = Q_A$ and $k\Sigma$ is also dismantlable.*

Proof. By the above corollary and [2, 4.4] there exist a unique poset Σ such that $Q_\Sigma = Q_A$ and $k\Sigma$ is strongly simply connected. Thus, by [20, 3.3], $k\Sigma$ contains no crown. By [21, 2.3], $k\Sigma$ is dismantlable. $\Box$

4.7. We end this section by proving the converse of 4.5.

Proposition. *Let A be a schurian strongly simply connected algebra. Then A is dismantlable.*

Proof. By [2, 4.4], since A is strongly simply connected, there exists a strongly simply connected incidence algebra $k\Sigma$ such that A is a quotient of $k\Sigma$. By [20, 3.3], $k\Sigma$ contains no crown. By [21, 2.3], $k\Sigma$ is dismantlable. In particular, $k\Sigma$ contains a doubly irreducible x which is also doubly irreducible in A. Now, notice that $A(x)$ is schurian. This is clear if $A(x)$ is a full subcategory of A. Otherwise, there exist two arrows $\alpha : y \to x$ and $\beta : x \to z$ such that $\alpha\beta = 0$ in A. If x does not belong to a cycle, then the statement is clear. However, if it does, then we can clearly assume that there exists an irreducible cycle containing α and β, and this contradicts [4, 2.4]. By 4.2 above, $B = A(x)$ is strongly simply connected. Since $|B_0| < |A_0|$, induction says that B is dismantlable. Hence so is A. $\square$

5. The proof of Theorem A

5.1. This section is devoted to the proof of our first main theorem. Let s be a source in a schurian triangular algebra A. Then we can write $A = B[M]$ where B is the full convex subcategory of A such that $B_0 = A_0 \backslash \{s\}$. We define Σ_s and Σ'_s as in 3.6. By [15, 2.6], we have a short exact sequence of complexes

$$0 \longrightarrow C_\bullet\big(k\Sigma'_s\big) \xrightarrow{\left[\begin{smallmatrix} u \\ -v \end{smallmatrix}\right]} C_\bullet(k\Sigma_s) \oplus C_\bullet(B) \xrightarrow{[i\,j]} C_\bullet(A) \longrightarrow 0,$$

where u, v, i, j are induced by the inclusions. Passing to (simplicial) homology yields the Mayer–Vietoris sequence

$$\begin{aligned} \cdots \longrightarrow SH_2(A) &\xrightarrow{\delta} SH_1\big(k\Sigma'_s\big) \longrightarrow SH_1(k\Sigma_s) \oplus SH_1(B) \longrightarrow SH_1(A) \\ &\xrightarrow{\delta} SH_0\big(k\Sigma'_s\big) \xrightarrow{\iota} SH_0(k\Sigma_s) \oplus SH_0(B) \longrightarrow SH_0(A) \end{aligned}$$

and it is shown in [15, p. 34] that the morphism ι is injective if and only if the point s is separating. On the other hand, the poset Σ_s admits a maximal element s, hence the corresponding chain complex is contractible (because it is homeomorphic to a cone, see [16, 3.4]), therefore $SH_n(k\Sigma_s) = 0$ for all $n \geqslant 1$.

Lemma. *Let A be a schurian algebra, and s be a source in A.*

(a) *If $k\Sigma'_s$ contains no crowns, then there exists a monomorphism $SH_1(B) \to SH_1(A)$. Thus, $SH_1(A) = 0$ implies $SH_1(B) = 0$.*
(b) *If s is separating, then there exists an epimorphism $SH_1(B) \to SH_1(A)$. Thus, $SH_1(B) = 0$ implies $SH_1(A) = 0$.*
(c) *If $k\Sigma'_s$ contains no crowns and s is separating, then we have an isomorphism $SH_1(B) \cong SH_1(A)$.*

Proof. Since $SH_1(k\Sigma_s) = 0$, we have an exact sequence

$$\cdots \xrightarrow{\delta} SH_1\big(k\Sigma_s'\big) \longrightarrow SH_1(B) \longrightarrow SH_1(A) \xrightarrow{\delta} SH_0\big(k\Sigma_s'\big) \xrightarrow{\iota} \cdots.$$

If the incidence algebra $k\Sigma_s'$ contains no crown, then it is strongly simply connected. In particular, $\pi_1(k\Sigma_s')$ is trivial, and hence $SH_1(k\Sigma_s') = 0$ (because, by the Hurewicz–Poincaré theorem, $SH_1(k\Sigma_s')$ is the abelianisation of $\pi_1(k\Sigma_s')$). Hence (a) follows. If s is separating, then ι is injective, thus giving (b). Finally, (c) follows trivially. □

5.2. It is well known that, if A is a simply connected algebra (or, else, if $HH^1(A) = 0$), then every source in A is separating, see [6] (or [33], respectively). In the schurian case, we can say more.

Corollary. *Let A be a schurian algebra. If $SH_1(A) = 0$, then every source in A is separating.*

Proof. Indeed, it follows from the Mayer–Vietoris sequence that the morphism ι is injective. □

5.3. The following lemma is part of the proof of our Theorem A.

Lemma. *Let A be a schurian algebra, not containing quasi-crowns and such that $SH_1(A) = 0$. Then A is strongly simply connected.*

Proof. By induction on $|A_0|$. Since the statement is clear for $|A_0| = 1$, assume that it holds for all schurian algebras B without quasi-crowns such that $|B_0| < |A_0|$ and $SH_1(B) = 0$.

Let s be a source in A, and B be the full convex subcategory of A defined by $B_0 = A_0 \backslash \{s\}$. We claim that $k\Sigma_s'$ contains no crowns. If this is not the case, and Γ is a crown in $k\Sigma_s'$, then Γ is a crown in $k\Sigma_s$, hence by 3.6 there exists a quasi-crown in A which must lie in B (because $s \notin \Gamma_0$) and this yields a contradiction which establishes our claim. Therefore $SH_1(k\Sigma_s') = 0$.

Since, as pointed out above, $SH_1(k\Sigma_s)$ is zero, the Mayer–Vietoris sequence gives

$$0 = SH_1\big(k\Sigma_s'\big) \longrightarrow SH_1(B) \longrightarrow SH_1(A) = 0.$$

Hence, $SH_1(B) = 0$. Furthermore, B contains no quasi-crown. Therefore, B is strongly simply connected, by the induction hypothesis. Since, on the other hand, s is separating (by the above corollary), A is simply connected.

In order to show that A is strongly simply connected, we need to show that every proper full convex subcategory C of A is simply connected. But, since C is proper, there exists a source s (up to duality) of A such that $s \notin C_0$. Letting, as above, $B_0 = A_0 \backslash \{s\}$, and B be the full subcategory it generates, we get that B is strongly simply connected, and $C \subseteq B$. Therefore, C is simply connected. □

5.4. We may replace “quasi-crowns” by “crowns” in the case where A is a quotient of an incidence algebra.

Corollary. *Let A be a quotient of an incidence algebra, not containing crowns and such that $SH_1(A) = 0$. Then A is strongly simply connected.*

Proof. This follows from 3.7 and the proof of 5.3 above. □

5.5. We also deduce from 5.3 an alternative proof of 3.8(c).

Corollary. *Let $A = B[M]$ be a schurian simply connected algebra such that B is not simply connected. Then B contains a quasi-crown.*

Proof. Let s denote the extension point. We suppose that B contains no quasi-crowns and reach a contradiction. By the proof of 5.3, $k\Sigma'_s$ contains no crowns and so $SH_1(k\Sigma'_s) = 0$. On the other hand, the simple connectedness of A yields $SH_1(A) = 0$. By 5.1, $SH_1(B) = 0$. Since B has no quasi-crown, and satisfies $SH_1(B) = 0$, it follows from 5.3 that B is strongly simply connected, a contradiction to our hypothesis. □

5.6. It is a general problem to determine for which classes of algebras is simple connectedness equivalent to the vanishing of the first Hochschild cohomology group (see, for instance, [3,6,24,33]).

Proposition. *Let A be a connected quotient of an incidence algebra, containing no crowns. The following conditions are equivalent*:

(a) *A is separated,*
(b) *A is simply connected,*
(c) $HH^1(A) = 0$.

Proof. That (a) implies (b) follows from [33, 2.2]. Assume now A to be simply connected. Since A is schurian, it follows from [27] that $HH^1(A) \cong \mathrm{Hom}(\pi_1(A), k^+) = 0$. Thus, (b) implies (c).

We prove that (c) implies (a) by induction on $|A_0|$. Since the statement is clear for $|A_0| = 1$, assume that $|A_0| > 1$ and that the statement holds for any algebra B such that $|B_0| < |A_0|$ and $HH^1(B) = 0$.

Suppose that $HH^1(A) = 0$. We must prove that each object x in A_0 is separating. Let s be a source in A and let B be the full convex subcategory of A with object class $B_0 = A_0 \backslash \{s\}$. Then $A = B[M]$ and $B = \prod_{j=1}^{c} B_j$, where $B_1, \dots, B_c$ are connected. Moreover, since A is a quotient of an incidence algebra, then so are the B_j. By [33, 3.2], s is separating, so we may assume x to be different from s. Since A contains no crown, then, by 3.8(a, b), there exists, for any abelian group G, a short exact sequence

$$0 \longrightarrow G^{t-c} \longrightarrow \mathrm{Hom}\big(\pi_1(A), G\big) \longrightarrow \prod_{j=1}^{c} \mathrm{Hom}\big(\pi_1(B_j), G\big) \longrightarrow 0.$$

Taking $G = k^+$, we have $\mathrm{Hom}(\pi_1(A), k^+) \cong HH^1(A) = 0$. Hence, for any j, $HH^1(B_j) \cong \mathrm{Hom}(\pi_1(B_j), k^+) = 0$, so, by the induction hypothesis, each B_j is separated. Since x is different from s, it belongs to some B_j. Since B_j is separated, x is separating in B_j and therefore in A (because $A^x = B_j^x$). □

As will be seen shortly (and as is evident from 5.4) the equivalent conditions of the proposition are equivalent to strong simple connectedness.

5.7. We are now able to prove our first main theorem.

Theorem. *Let A be a schurian triangular algebra. The following conditions are equivalent*:

(a) *A is strongly simply connected.*
(b) *A is dismantlable.*
(c) *A is separated and contains no quasi-crowns.*
(d) *A is simply connected and contains no quasi-crowns.*
(e) $SH_1(A) = 0$ *and A contains no quasi-crowns.*
(f) $SH^1(A, G) = 0$ *for every abelian group G, and A contains no quasi-crowns.*
(g) *A is a quotient of an incidence algebra, $HH^1(A) = 0$ and A contains no crowns.*

Proof. (a) implies (b). By 4.7.

(b) implies (a). By 4.5.

(a) implies (c). By [33, 4.1], every strongly simply connected algebra is separated. We also apply 3.5.

(c) implies (d). By [33, 2.2], every separated algebra is simply connected.

(d) implies (e). Follows from the Hurewicz–Poincaré theorem.

(e) implies (a). By 5.3.

(e) is equivalent to (f). By the Dual Universal Coefficients Theorem, we have, for any abelian group G:

$$SH^1(A, G) \cong \mathrm{Hom}_{\mathbb{Z}}\big(SH_1(A), G\big) \oplus \mathrm{Ext}^1_{\mathbb{Z}}\big(SH_0(A), G\big).$$

Since A is connected, $SH_0(A) \cong \mathbb{Z}$ so that $SH^1(A, G) \cong \mathrm{Hom}_{\mathbb{Z}}(SH_1(A), G)$. Thus (e) implies (f). The converse follows upon taking $G = SH_1(A)$.

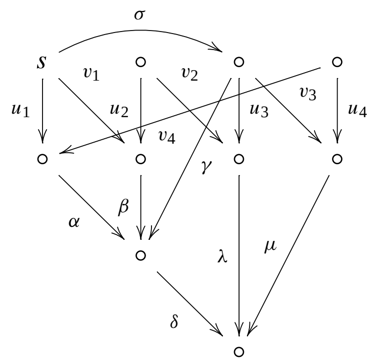
(a) implies (g). By [2, 4.5] (see also [4,20]), A is a quotient of an incidence algebra. Moreover, $HH^1(A) = 0$ and by 3.5, A contains no quasi-crown, then A contains no crowns.

(g) implies (c). By 5.6. □

As a direct consequence of the equivalence between the strong simple connectedness and the dismantlability of a schurian algebra, we have the following algorithm which allows us to verify the strong simple connectedness of a schurian algebra:

INPUT: A (which is a schurian algebra).
Check if there exists an $x \in A_0$ which is a doubly irreducible.
If there exists no doubly irreducible
OUTPUT: A is not strongly simply connected.
If there exists a doubly irreducible x
Check if $A(x)$ is schurian.
If $A(x)$ is not schurian
OUTPUT: A is not strongly simply connected.
If $A(x)$ is schurian, then set $A := A(x)$.
If A_0 is a singleton
OUTPUT: A is strongly simply connected.
If A_0 is not a singleton, return to input.

Example 10. The following is an example of a simply connected algebra containing a quasi-crown, and which is evidently not strongly simply connected. Let A be given by the quiver



bound by $u_1\alpha\delta = 0$, $\sigma u_3 = 0$, $\sigma v_3 = 0$, $u_1\alpha = v_1\beta = \sigma\gamma$, $u_2\beta\delta = v_2\lambda$, $\gamma\delta = u_3\lambda = v_3\mu$, $u_4\mu = v_4\alpha\delta$. Indeed, let B be a full convex subcategory of A with object class $B_0 = A_0 \setminus \{s\}$, then B is obviously a simply connected incidence algebra (because it has a minimal point) and the extension module $M = \operatorname{rad} P_s$ is indecomposable. Hence, by [6, 2.5], $A = B[M]$ is simply connected. Observe also that A is a quotient of an incidence algebra and contains a crown.

5.8. We may replace "quasi-crowns" by "crowns" in conditions (c)–(f) of Theorem A in the case of quotients of incidence algebras.

Corollary. *Let A be a quotient of an incidence algebra. The following conditions are equivalent*:

(a) *A is strongly simply connected.*
(b) *A is dismantlable.*
(c) *A is separated and contains no crowns.*

(d) *A is simply connected and contains no crowns.*
(e) $SH_1(A) = 0$ *and A contains no crowns.*
(f) $SH^1(A, G) = 0$ *for every abelian group G, and A contains no crowns.*
(g) $HH^1(A) = 0$ *and A contains no crowns.*

Proof. This follows from 5.7, 5.6, and 5.4. □

5.9. Remarks

We recall that, if A is an incidence algebra, then the following conditions are equivalent:

(a) A is strongly simply connected;
(b) A has no crown;
(c) A is dismantlable

(by [20, 3.3] and [21, 2.3]). These conditions imply each of the following:

(d) $HH^1(A) = 0$;
(e) A is simply connected;
(f) A is separated

(by [33]). However, the latter conditions are not equivalent and, while (f) implies (e), which implies (d), the other implications are not true. Let A be the incidence algebra of the poset with quiver

$$\begin{tikzcd}
& \circ \arrow[dl] \arrow[dr] & \\
\circ \arrow[d] \arrow[drr] & & \circ \arrow[d] \arrow[dll] \\
\circ & & \circ
\end{tikzcd}$$

Then A is simply connected but not separated, thus (e) does not imply (f). Finally, it is well known that (d) does not imply (e) (see, for instance, [16, 3.10]).

5.10. We also get the following obvious corollary.

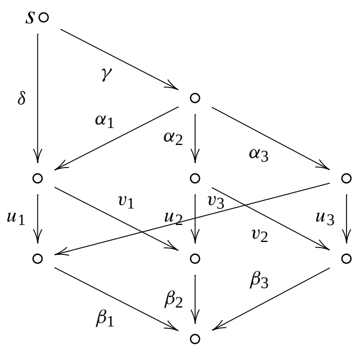
Corollary. *Let A be a schurian triangular algebra containing no quasi-crowns. The following conditions are equivalent*:

(a) *A is strongly simply connected.*
(b) *A is dismantlable.*
(c) *A is separated.*
(d) *A is simply connected.*
(e) $SH_1(A) = 0$.

(f) $SH^1(A, G) = 0$ *, for each abelian group* G.
(g) *A is a quotient of an incidence algebra, and* $HH^1(A) = 0$.

5.11. The question has arisen whether the presence of a bypass in the bound quiver of a schurian algebra may prevent this algebra from being simply connected. We now answer this question in the negative: the following is an example of schurian simply connected algebra containing a bypass.

Example 11. Let A be given by the quiver



bound by $\alpha_1 v_1 = 0$, $\gamma\alpha_1 = 0$, $\delta v_1 = \gamma\alpha_2 u_2$, $\gamma\alpha_3 = 0$, $\delta u_1 = 0$ and all other squares are commutative. Then the full convex subcategory B of A with objects class $B_0 = A_0 \setminus \{s\}$ is the "box" of Example 3 of 3.1 above and, in particular, is simply connected. The extension module $M = \operatorname{rad} P_s$ is indecomposable, so that $A = B[M]$ is simply connected. However, we notice that A contains a (quasi-)crown: this is a general fact.

Corollary. *Let A be a simply connected schurian algebra containing a bypass. Then A contains a quasi-crown.*

Proof. Assume that A is a simply connected algebra containing a bypass but no quasi-crown. By Theorem A, it is strongly simply connected. Hence there exists an incidence algebra $k\Sigma$ such that $Q_A = Q_{k\Sigma}$ (see, for instance, [2]). But then A contains no bypass, a contradiction. $\square$

5.12. The following is an easy consequence of the previous corollary.

Corollary. *Let A be a simply connected representation-finite algebra. Then A contains no bypass.*

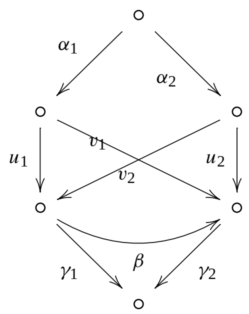
6. Schurian algebras not containing quasi-crowns

6.1. We now turn to the proof of our second main theorem. Let A be a schurian algebra. Following [12, 1.2], we call $\mathcal{B}$ a *multiplicative basis* of A if:

(a) $e_x \in \mathcal{B}$ for each $x \in A_0$,
(b) $\mathcal{B} \cap e_x(\operatorname{rad}^n A)e_y$ is a basis of $e_x(\operatorname{rad}^n A)e_y$ for all $x, y \in A_0$ and all $n \in \mathbb{N}$,
(c) $b \in \mathcal{B} \cap e_x A e_y$ and $c \in \mathcal{B} \cap e_y A e_z$ imply $bc \in \mathcal{B}$ or $bc = 0$.

The following is an example of an algebra having no multiplicative basis.

Example 12. Let A be given by a quiver



bound by $\alpha_1 u_1 = \alpha_2 v_2$, $\alpha_1 v_1 = \alpha_2 u_2$, $u_1\gamma_1 = v_1\gamma_2$, $v_2\gamma_1 = cu_2\gamma_2$, $u_1\beta = 0$, $v_1\beta = 0$, $\beta\gamma_2 = 0$ and $\operatorname{rad}^3 A = 0$ (where c is a constant different from 0 and 1). We notice that A contains a quasi-crown. Also, A is a split extension of the algebra in Bongartz' example [14], the latter being obtained by deleting the arrow β.

6.2. In the following lemma, we show that a schurian algebra A, not containing quasi-crowns, has only low-dimensional simplicial homology and cohomology groups. For our purpose, the key statement is that $SH_2(A) = 0$, all other statements follow easily from [26, 3.1]. We give however an independent proof for the convenience of the reader.

Lemma. *Let A be a schurian algebra not containing quasi-crowns. Then*

(a) $SH_n(A) = 0$ *for all* $n \geqslant 2$.
(b) $SH^n(A, G) = 0$ *for all* $n \geqslant 3$ *and all abelian groups* G.

Proof. (a) We use induction on $|A_0|$. Let s be a source in A, and B be the full subcategory of A with object class $B_0 = A_0 \setminus \{s\}$ (thus $A = B[M]$). For $n \geqslant 2$, we have an exact sequence

$$\longrightarrow SH_n(B) \longrightarrow SH_n(A) \longrightarrow SH_{n-1}\big(k\Sigma'_s\big) \longrightarrow .$$

By 3.7, $k\Sigma'_s$ has no crowns. Since it is an incidence algebra, it is (strongly) simply connected. In particular, $SH_1(k\Sigma'_s) = 0$. On the other hand, since B has no quasi-crowns either, the induction hypothesis implies that $SH_n(B) = 0 = SH_n(k\Sigma'_s)$ for all $n \geqslant 2$. Therefore $SH_n(A) = 0$ for $n \geqslant 2$.

(b) This follows from (a) and from

$$SH^n(A,G) \cong \mathrm{Hom}_{\mathbb{Z}}\big(SH_n(A),G\big) \oplus \mathrm{Ext}^1_{\mathbb{Z}}\big(SH_{n-1}(A),G\big). \qquad \square$$

6.3. We are now able to prove our second main Theorem B.

Theorem (Multiplicative basis). *Let A be a schurian algebra not containing quasi-crowns. Then A admits a multiplicative basis.*

Proof. Let $k^\times$ denote the multiplicative group of the non-zero scalars. Applying 6.2, the Dual Universal Coefficients Theorem yields

$$\begin{aligned} SH^2\big(A,k^\times\big) &\cong \mathrm{Hom}_{\mathbb{Z}}\big(SH_2(A),k^\times\big) \oplus \mathrm{Ext}^1_{\mathbb{Z}}\big(SH_1(A),k^\times\big) \\ &\cong \mathrm{Ext}^1_{\mathbb{Z}}\big(SH_1(A),k^\times\big) = 0, \end{aligned}$$

since $k^\times$ is a divisible abelian group (because k is an algebraically closed field). By [15, 2.2], this implies that A has a multiplicative basis. $\square$

Remark. In [26, 3.2], Martins and de la Peña prove the existence of a multiplicative basis in an algebra A such that $\mathrm{gl.dim}\, A \leqslant 2$ and $HH^2(A) = 0$. We replace both of these hypotheses by the one of the absence of quasi-crowns. Our result may thus be applied, for instance, to algebras of an arbitrarily large global dimension.

6.4. The next corollary follows immediately from our Theorem B.

Corollary. *For each natural number d, there exist only finitely many isomorphism classes of schurian algebras, not containing quasi-crowns, of dimension d.*

Proof. Indeed, this follows, from the facts that, for such an algebra, the number of points, the number of arrows and hence the number of paths are bounded, and a basis consists of classes of paths modulo the ideal. $\square$

6.5. A second immediate corollary is the following well-known result of Bongartz.

Corollary [14]. *Let A be a representation-finite triangular algebra, then A admits a multiplicative basis.*

6.6. As another corollary of our two main Theorems A and B, we obtain a new proof of [4, 2.4].

Corollary. *Let A be a schurian strongly simply connected algebra, then A admits a multiplicative basis.*

6.7. The following corollary is a direct consequence of [15, 2.2] and the fact that, if A is a schurian triangular algebra having no quasi-crowns, then $SH^2(A, k^\times) = 0$. We recall that $\mathcal{B}(A)$ denotes the classifying space of A, see [17].

Corollary. *Let A, A' be schurian triangular algebras such that A has no quasi-crowns and $\mathcal{B}(A) = \mathcal{B}(A')$, then there exists an isomorphism of k-algebras $A \cong A'$.*

6.8. To end this paper, we illustrate our methods by obtaining short proofs of some well-known results about strongly simply connected algebras.

Corollary [20, 2.4]. *A schurian algebra A is strongly simply connected if and only if, for every full convex subcategory C of A, we have $SH_1(C) = 0$.*

Proof. *Necessity*. Assume that A is strongly simply connected. Then any full convex subcategory C of A is also strongly simply connected. By Theorem A, $SH_1(C) = 0$.

Sufficiency. Let C be a full convex subcategory of A. By hypothesis, $SH_1(C) = 0$. Since A is schurian, it follows from [27] that

$$HH^1(C) \cong \mathrm{Hom}\big(\pi_1(C), k^+\big) \cong \mathrm{Hom}\big(SH_1(C), k^+\big) = 0.$$

By [33, 4.1], A is strongly simply connected. $\square$

6.9. The next corollary is expressed by saying that a schurian strongly simply connected algebra (or, more precisely, its classifying space) is acyclic.

Corollary [20, 2.6]. *Let A be a schurian strongly simply connected algebra, then*

(a) $SH_n(A) = 0$ *for all* $n \geqslant 1$.
(b) $SH^n(A, G) = 0$ *for all* $n \geqslant 1$ *and all abelian groups* G.

Proof. (a) By 6.2, this is clear if $n \geqslant 2$. For $n = 1$, this is granted by the simple connectedness of A.

(b) We recall that the strong simple connectedness of A implies that $SH^1(A, G) = 0$ for all abelian groups G, see 5.7. Moreover,

$$SH^2(A, G) \cong \mathrm{Hom}_{\mathbb{Z}}\big(SH_2(A), G\big) \oplus \mathrm{Ext}^1_{\mathbb{Z}}\big(SH_1(A), G\big) = 0.$$

Finally, for $n \geqslant 3$, this follows from 6.2. $\square$

6.10. We end this paper with a short proof of the following result of [22].

Corollary [22]. *Let A be a schurian strongly simply connected algebra, then the Hochschild cohomology ring $HH^*(A)$ of A is k.*

Proof. It follows from [17, (6.5)] that, for all $n \geqslant 1$, we have $HH^n(A) \cong SH^n(A, k^+)$. By 6.9(b), the latter vanishes. □

Observe that, in this case, if $A = B[M]$ is written as one-point extension, then we clearly have $\mathrm{Ext}^i_B(M, M) = 0$, for all i.

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