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Derived-tame blowing-up of tree algebras

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Abstract

Let k be an algebraically closed field and A be a tree algebra. We consider here a class obtained by the blowing-up of a tree algebra A at a set of vertices D of A, such an algebra is denoted by $A\{D\}$. The objective of this paper is to prove the equivalence between the derived-tameness and the non-negativity of the Euler form for algebras of this form. We also show that, in this case, if D is a non-empty set then $A\{D\}$ must be derived equivalent to a special incidence algebra, called semichain algebra.

Keywords: Tame derived category; Euler form; Blowing-up of algebra; Tree algebra

1. Introduction

Let k be a fixed algebraically closed field. For a finite-dimensional k-algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules. We are interested in the description of the derived category $D^b(A)$ of bounded complexes over $\operatorname{mod} A$. Our interest in this problem is partially motivated by the fact that the derived categories of certain categories of coherent sheaves are related to the derived categories of finite-dimensional algebras. For instance, it was shown by D. Happel [13] that, if A is derived equivalent to the derived category of a hereditary category, then $D^b(A)$ is triangle equivalent to $D^b(H)$,

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where H is a hereditary algebra, or $D^b(A)$ is triangle equivalent to $D^b(\operatorname{coh}\mathbb{X})$ where $\operatorname{coh}\mathbb{X}$ is the category of coherent sheaves over the weighted projective line $\mathbb{X}$ in the sense of [7].

A precise description of the category $D^b(A)$ is known for relatively few classes of algebras. One of those is given by algebras whose derived category is cycle-finite [1]. In these cases, A is derived equivalent to a tame hereditary algebra or to a tubular algebra. The latest ones are in fact derived equivalent to some $\operatorname{coh}\mathbb{X}$. In general, very little is known about the derived category. However, quite a lot is known for (skewed) gentle algebras (which includes semichain) whose derived category was studied in [10].

In this paper, we are interested in the case where A is derived-tame. We recall that, by [11], if the global dimension of A is finite, then the derived category $D^b(A)$ is triangle equivalent to the stable module category of the repetitive category $\widehat{A}$. Thus, following [16], we say that an algebra A, of finite global dimension, is *derived-tame* provided the category $\widehat{A}$ is tame, that is, each finite full subcategory of $\widehat{A}$ is tame, see [6]. For further discussion on derived tameness, we refer the reader to [9]. It is an interesting question to seek a combinatorial criterion allowing to verify whether a given algebra is derived-tame or not. Thus, for instance, arose the problem to determine which algebras are derived-tame if and only if their Euler quadratic forms are non-negative. We recall that, if A is an algebra of finite global dimension, then its Euler quadratic form is defined on the Grothendieck group of A by $\chi_A(\operatorname{\mathbf{dim}} M) = \sum_{i=0}^{\infty}(-1)^i \dim_k \operatorname{Ext}_A^i(M, M)$ for any A-module M.

This problem was first solved for tree algebras [5,8]. It was shown that a tree algebra A is derived-tame if and only if χ_A is non-negative. Moreover, in this case, A is derived equivalent to one of the following: a hereditary algebra of type $\mathbb{E}$, $\widetilde{\mathbb{E}}$, a tubular algebra, or a special type of incidence algebras, called semichain algebra (see Section 2.2 for the definition). Semichain algebras play a prominent role in this paper.

We consider here a class of algebras obtained by the blowing-up of a tree algebra A at a set of vertices D, in the sense of [5], such an algebra is denoted by $A\{D\}$, see also Section 2.3. These blowing-up of tree algebras are natural in our context since they are generalizations of tree algebras which contains the class of semichain algebras. The objective of this paper is to give the equivalence between derived-tameness and non-negativity of the Euler form for algebras of this form. We prove the following theorem.

Theorem. *Let A be a tree algebra and D a non-empty set of vertices of A. The following conditions are equivalent*:

(a) *The blowing-up $A\{D\}$ is derived-tame.*
(b) *The Euler form $\chi_{A\{D\}}$ of $A\{D\}$ is non-negative.*
(c) *The blowing-up $A\{D\}$ is derived equivalent to a semichain algebra $S(c, m)$.*

In particular, in this case the derived class of the blowing-up algebra $A\{D\}$ is uniquely determined by the number of vertices, the corank and the Dynkin type of its Euler form.

It is easily seen in Section 2.4 that each of the equivalent conditions above implies that A is derived-tame. We will, as in the proof for tree algebras, consider two classes of derived-tame tree algebras. We say that a derived-tame tree algebra is *derived of type* $\mathbb{E}$ if it is derived equivalent to a hereditary algebra of type $\mathbb{E}_p$, $\widetilde{\mathbb{E}}_p$ ($p = 6, 7, 8$) or to a tubular algebra. Otherwise, we say that it is *derived* $\mathbb{E}$*-free*. Thus, the proof of the theorem is divided in two cases depending whether the tree algebra is derived of type $\mathbb{E}$ or not. If the tree algebra A is derived $\mathbb{E}$-free, the proof is given in two propositions shown in Sections 3 and 4, respectively. Section 5 presents the proof when A is derived of type $\mathbb{E}$. We recall some concepts and general facts in Section 2. We also introduce a subclass of blowing-up of tree algebras which are derived equivalent to semichain. This subclass will be used in the proof of the main theorem.

2. Derived-tame algebras and blowing-up

2.1. Notation

Let A be a basic algebra of the form $A = kQ/I$, where Q is a finite quiver and I is an admissible ideal of the path algebra kQ. We usually suppose that our algebra is connected. By a vertex of A, we mean a vertex of the quiver Q. Observe that each vertex x of A is associated to a primitive idempotent of A, denoted by e_x.

In this paper, we usually assume that A is a *tree algebra*, that means that the underlying graph of Q is a tree. In this case, there is a minimal set of paths generating the ideal I. We refer to these monomial generators $\rho = a_1 \to \cdots \to a_t$ in kQ as relations of A and indicate them by dotted lines.

We also consider A as a k-category whose objects are its vertices and in which the morphisms set from x to y is $e_y A e_x$.

Given two algebras A and B, we say that they are *derived equivalent* if their respective derived categories $D^b(A)$ and $D^b(B)$ are triangle equivalent.

We say that A is a *reflection* of B if there exist an algebra C, not necessarily connected, and a C-module M such that $A = C[M]$ and $B = [M]C$. In this case, we also say that B is a *reflection* of A. An algebra A is *reflection equivalent* to B if there exists a sequence of algebras $A = A_1, A_2, \dots, A_t = B$ where A_i is a reflection of A_{i+1} for each i. Two algebras which are reflection equivalent are also derived equivalent, see [19, (4.10)].

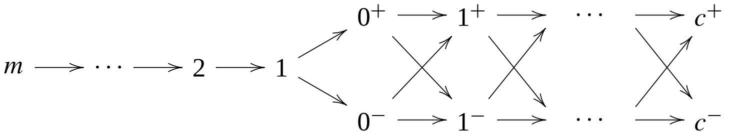
2.2. Derived-tame algebras

Let A and B be two derived equivalent algebras. One can show that the non-negativity, and in this case, the corank of the Euler form are preserved. Moreover, by [16, (1.3)], if A is derived-tame then so is B. Recall that the following algebras are derived-tame and have a non-negative Euler form.

Examples. (a) By [11], tame hereditary algebras $A = kQ$ are derived-tame. If A is representation-finite (hence Q is of type $\mathbb{A}_n$, $\mathbb{D}_n$ or $\mathbb{E}_p$ for $p = 6, 7, 8$) then χ_A is positive definite. If A is representation-infinite (hence Q is of type $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$ or $\widetilde{\mathbb{E}}_p$ for $p = 6, 7, 8$) then χ_A is non-negative with corank $\chi_A = 1$.

(b) By [14], tubular algebras in the sense of [17] are derived-tame with a non-negative Euler form of corank 2.

(c) Another example of derived-tame algebras, see [10,16], is given by the incidence algebra $S(c,m)$, called *semichain algebra*, of the following poset:



This algebra has a non-negative Euler form of corank c, see [16]. We agree that c can be taken as -1. In this case, $S(-1,m)$ is a hereditary algebra of type $\mathbb{A}_m$. Also observe that we have that $S(0,m)$ is a hereditary algebra of type $\mathbb{D}_{m+2}$, for $m \geqslant 2$, and that $S(1,m)$ is derived equivalent to a hereditary algebra of type $\widetilde{\mathbb{D}}_{m+3}$, for $m \geqslant 1$.

2.3. Blowing-up

Let $A = kQ/I$ be an algebra and let D be a set of vertices of Q. We define the *blowing-up of A at D* to be the algebra $A\{D\} = kQ\{D\}/I\{D\}$ given by the quiver $Q\{D\}$ and ideal $I\{D\}$ describe below.

The quiver $Q\{D\}$ is obtained from Q by replacing each vertex d of D by the vertices d^- and d^+ and each arrow $\alpha : x \to d$ with $d \in D$ by the arrows $\alpha^- : x \to d^-$ and $\alpha^+ : x \to d^+$ and dually for each arrow $\beta : d \to y$ with $d \in D$.

There is an obvious quiver epimorphism $Q\{D\} \to Q$ which extends uniquely to a surjective algebra morphism $\pi : kQ\{D\} \to kQ$. We define the ideal $I\{D\}$ of $kQ\{D\}$ to be the ideal generated by all linear combinations $\rho = \sum_{i=1}^{m} \lambda_i w_i$ of paths w_i (having the same starting and ending vertices) such that $\pi(\rho) \in I$. If $D = \{d\}$, we usually denote the blowing-up $A\{D\}$ by $A\{d\}$.

Remark that blowing-up can be defined in a more general way using a finite set F instead of the set $\{+,-\}$, see [5, (2.5)].

From now on, we suppose that $A = kQ/I$ is a tree algebra. In this case, the ideal I is generated by paths. Thus, the ideal $I\{D\}$ is generated by all paths w of $Q\{D\}$ such that $\pi(w) \in I$ and by all commutativity relations $\beta^+\alpha^+ = \beta^-\alpha^-$ whenever there are arrows

$$x \xrightarrow{\alpha^+} d^+ \xrightarrow{\beta^+} y \quad \text{and} \quad x \xrightarrow{\alpha^-} d^- \xrightarrow{\beta^-} y$$

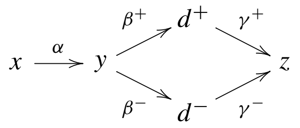
in $Q\{D\}$ such that $\pi(d^+) = \pi(d^-)$ and $\pi(\alpha^+) = \pi(\alpha^-)$ as well as $\pi(\beta^+) = \pi(\beta^-)$. Observe that, in this case, the blowing-up $A\{D\}$ is a quotient of an incidence algebra.

Examples. (a) If the algebra $A = kQ$ is a chain

$$s_m \to \cdots \to s_1 \to d_0 \to \cdots \to d_c$$

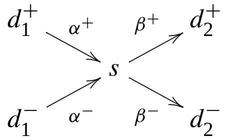
and $D = \{d_0, \ldots, d_c\}$, then $A\{D\}$ coincides with the semichain $S(c,m)$.

(b) Let $A = kQ/I$ be given by the quiver $Q = x \xrightarrow{\alpha} y \xrightarrow{\beta} d \xrightarrow{\gamma} z$ and I be the ideal generated by the path $\alpha\beta\gamma$. Then the blowing-up $A\{d\}$ is given by the following quiver:



bound by the relations $\beta^+\gamma^+ = \beta^-\gamma^-$ and $\alpha\beta^+\gamma^+ = 0$.

(c) Let Q be the quiver $d_1 \xrightarrow{\alpha} s \xrightarrow{\beta} d_2$ and I be the ideal generated by $\alpha\beta$. The blowing-up $A\{D\}$, with $A = kQ/I$ and $D = \{d_1, d_2\}$, is given by the quiver:



bound by the relations $\alpha^+\beta^+ = \alpha^+\beta^- = \alpha^-\beta^+ = \alpha^-\beta^- = 0$.

The next proposition, which follows directly from [5, (2.2)], tells us how reflection equivalent algebras yield reflection equivalent blowing-up of these algebras. This property is an essential tool in the sequel.

Proposition [5, (2.2)]. *Let A be an algebra and D be a set of vertices of A. If A is reflection equivalent to an algebra B, then there exists a set E of vertices of B corresponding to vertices of D under the sequence of reflections such that the blowing-up $A\{D\}$ is reflection equivalent to the blowing-up $B\{E\}$.*

2.4. Full subcategory

Observe that the algebra A is a full subcategory of every blowing-up $A\{D\}$. Therefore, it is natural to ask whether derived tameness and non-negativity of the Euler form are preserved under full subcategories.

Proposition. *Let A be a triangular algebra and B be a full subcategory of A. Then*

(a) *If A is derived-tame then so is B.*
(b) *If the Euler form of A is non-negative then so is the Euler form of B.*

Proof. (a) Recall that the repetitive category $\widehat{A}$ of A is the algebra (without unity) given by the doubly infinite matrix

$$\widehat{A} = \begin{bmatrix} \ddots & & & & 0 \\ \ddots & A & & & \\ & \mathrm{D}A & A & & \\ & & \mathrm{D}A & A & \\ 0 & & & \ddots & \ddots \end{bmatrix}$$

whose elements are matrices with a finite number of non-zero coefficients. The sum is the usual matrix one and the multiplication is induced from the canonical morphisms $A \otimes_k \mathrm{D}A \to A$, $\mathrm{D}A \otimes_k A \to A$ and the zero morphism $\mathrm{D}A \otimes_k \mathrm{D}A \to 0$ where $\mathrm{D}A = \mathrm{Hom}_k(A, k)$.

Any two objects of $\widehat{B}$ are of the form $x[i]$ and $y[j]$ where $x, y \in B_0$ and $i, j \in \mathbb{Z}$. The set of morphisms in $\widehat{A}$ from $x[i]$ to $y[j]$ is given by

$$\widehat{A}\big(x[i], y[j]\big) = \begin{cases} A(x, y) \times \{i\} & \text{if } i = j, \\ \mathrm{D}A(y, x) \times \{i\} & \text{if } j = i + 1, \\ 0 & \text{else.} \end{cases}$$

Since B is a full subcategory of A, one can easily see that $\mathrm{D}B(y, x) = \mathrm{D}A(y, x)$ whenever x, y are objects in B. Thus, the repetitive category $\widehat{B}$ of B is a full subcategory of the repetitive category $\widehat{A}$ of A. Therefore, if A is derived-tame then $\widehat{A}$ is tame and, so is $\widehat{B}$. Hence, B is derived-tame. This fact also easily follows from the equivalents definitions of derived tameness given in [9].

(b) This result is well known if B is convex. Let us then suppose that B is not convex. We prove this result by induction on the number of vertices of A. Let x be a vertex of A which is not in B. There exists an algebra A' which is reflection equivalent to A such that x corresponds to a source in A'. Moreover, there exists a full subcategory B' of A' which is reflection equivalent to B. In fact, B' is a full subcategory of $A'^{(x)} = A'/A'e_xA'$. But $A'^{(x)}$ is a convex subcategory of A'. Since A' is derived equivalent to A, it follows that $\chi_{A'}$ is non-negative. Therefore, $\chi_{A'^{(x)}}$ is also non-negative. By the induction hypothesis, $\chi_{B'}$ is non-negative. The result follows from the fact that B is derived equivalent to B'. □

The following remark on hereditary algebras will also be a useful tool.

Remark. Observe that for a hereditary algebra, the concepts of derived-tameness, tameness and non-negativity of the Euler form coincide.

2.5. B-tree algebras

To prove the main theorem, we introduce a subclass of the blowing-up algebras. Observe that a similar class of algebras, called semi-tree, was introduced in [5, (2.6)]. In fact, we made the condition (D5) stronger in such a way that this class corresponds to algebras which are derived equivalent to semichain algebras.

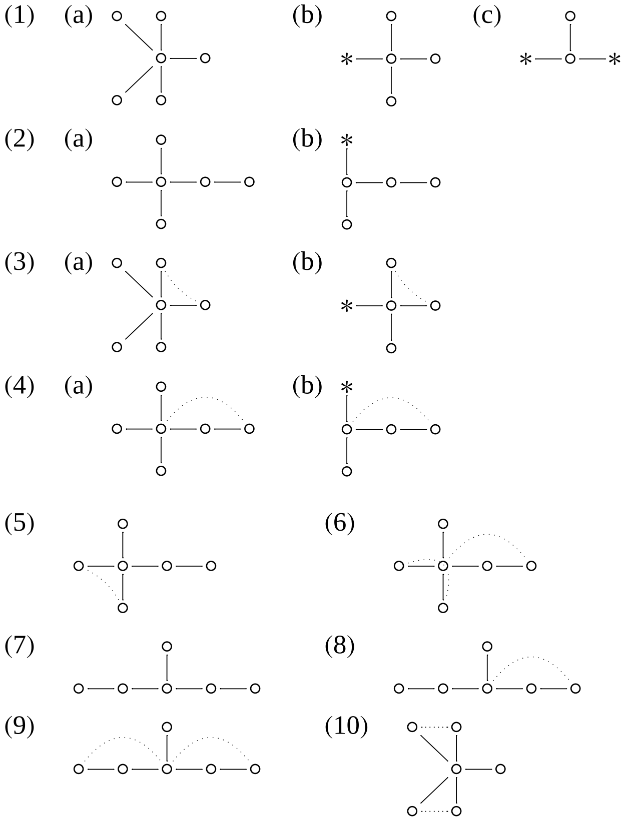
Definition. Let Λ be an algebra, $A = kQ/I$ be a tree algebra and D be a set of vertices of A. We say that Λ is a *B-tree for* (A, D) if $\Lambda = A\{D\}$ and the pair (A, D) satisfies the following conditions:

(D1) At each vertex of D starts at most one arrow and at each vertex of D ends at most one arrow.
(D2) The ideal I is generated by a set of paths of length two or tree.

(D3) If $\epsilon : a \xrightarrow{\alpha} b \xrightarrow{\beta} c$ is one of the generators of the ideal I, then the middle vertex b does not belong to D. Moreover, all other generators of I that contain the arrow α end in the vertex b, and all other generators of I that contain the arrow β start in b.

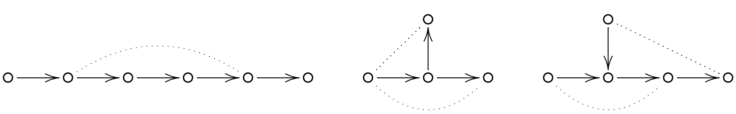
(D4) The generators of I of length three have the form $\epsilon : a \to a' \xrightarrow{\alpha} b \xrightarrow{\beta} c'$ or dually, $\epsilon' : a' \xrightarrow{\alpha} b \xrightarrow{\beta} c' \to c$ or they come as pairs $(\epsilon : a \to a' \xrightarrow{\alpha} b \xrightarrow{\beta} c', \epsilon' : a' \xrightarrow{\alpha} b \xrightarrow{\beta} c' \to c)$. In each case, the vertices a', b and c' do not belong to D and no other arrow starts nor ends in a' or in c'. Moreover, no other generator of I contains the arrows α or β.

(D5) There is no (full) convex subcategory of A of one of the following forms, where the vertex $*$ belongs to D:

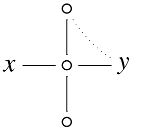


If there exists such a pair (A, D), then we simply say that Λ is a B-tree.

Examples. (a) The following algebras are not B-trees for $(A, \emptyset)$ where $A = A\{\emptyset\}$ is the corresponding algebra.

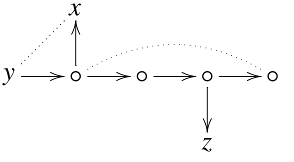


(b) The blowing-up of the tree algebra



at the set $\{x, y\}$ is not a B-tree.

(c) The blowing-up of the tree algebra



at the set $\{x, y, z\}$ is a B-tree.

Remark. Each blowing-up described in (D5), (1)–(4), is derived equivalent to a wild (that means not tame) hereditary algebra and thus is not derived-tame, nor has a non-negative Euler form. While, those described in (D5), (5)–(10), are derived equivalent to hereditary algebras of type $\mathbb{E}_6$.

2.6. Derived-tame tree algebras

We recall the result of T. Brüstle on tree algebras. C. Geiss obtained some of the implications for a larger class of algebras.

Theorem [5,8]. *Let A be a tree algebra. The following conditions are equivalent*:

(a) *A is derived-tame.*
(b) *χ_A is non-negative.*
(c) *A is derived equivalent to a hereditary algebra of type $\mathbb{E}_p$, $\widetilde{\mathbb{E}}_p$ $(p = 6, 7, 8)$ or to a tubular algebra or to precisely one of the semichain algebras $S(c, m)$.*

In particular, the derived class of the tree algebra A is uniquely determined by the number of vertices, the corank and the Dynkin type of its Euler form.

Observe that algebras which are derived equivalent to a hereditary algebra of type D_n or $\mathrm{D}t_n$ are also derived equivalent to $S(0, n - 2)$ or $S(1, n - 3)$, respectively.

The above theorem and proposition 2.4 imply that if $A\{D\}$ is derived-tame or if its Euler quadratic form is non-negative then A is derived-tame. We recall that a derived-tame tree algebra is said to be *derived of type* $\mathbb{E}$ if it is derived equivalent to a hereditary algebra of type $\mathbb{E}_p$, $\widetilde{\mathbb{E}}_p$ $(p = 6, 7, 8)$ or to a tubular algebra. Otherwise, it is said to be *derived* $\mathbb{E}$*-free*. We will consider those classes of algebras independently. We recall the connection between the algebras derived of type $\mathbb{E}$ and those which contain a full subcategory derived equivalent to a hereditary algebra of type $\mathbb{E}_6$. This proposition follows from [2, (2.1)] and [5, (1.3)].

Proposition. *Let A be a derived-tame tree algebra. The following conditions are equivalent*:

(a) *A is derived of type $\mathbb{E}$.*
(b) *A contains a convex subcategory which is derived equivalent to a hereditary algebra of type $\mathbb{E}_q$, $\widetilde{\mathbb{E}}_q$ $(q = 6, 7, 8)$ or to a tubular algebra.*
(c) *A contains a full subcategory which is derived equivalent to a hereditary algebra of type $\mathbb{E}_6$.*

We obtain the following lemma which connects B-tree algebras with derived $\mathbb{E}$-free algebras. This lemma will allow us to apply results of [5] to B-tree algebras.

Lemma. *Let A be a tree algebra and D be a set of vertices of A such that the blowing-up $A\{D\}$ is derived-tame or the Euler form of $A\{D\}$ is non-negative. Then $A\{D\}$ is a B-tree for (A, D) if and only if (A, D) satisfies the conditions* (D1)–(D4) *and A is derived $\mathbb{E}$-free.*

Proof. The sufficiency follows from remark 2.5. The hypothesis together with proposition 2.4 and theorem 2.6 imply that A is derived-tame. By the above proposition, it is sufficient to verify that A does not contain a full subcategory which is derived equivalent to a hereditary algebra of type $\mathbb{E}_6$. This is easily done by looking at the list of tree, algebras derived equivalent to $\mathbb{E}_6$, see [2, Section 3]. □

In the next two sections, we prove the main theorem when A is a derived $\mathbb{E}$-free tree algebra. Let D be a set of vertices of A such that the blowing-up $A\{D\}$ is derived-tame or the Euler form of $A\{D\}$ is non-negative. First, we prove that $A\{D\}$ is derived equivalent to a B-tree algebra. Then, we show that B-tree algebras are derived equivalent to semichain algebras.

3. Blowing-up derived equivalent to B-tree

This section is devoted to prove the following proposition.

Proposition 3.1. *Let A be a derived $\mathbb{E}$-free tree algebra and D be a set of vertices of A such that the blowing-up $A\{D\}$ is derived-tame or the Euler form of $A\{D\}$ is non-negative. Then $A\{D\}$ is derived equivalent to a B-tree algebra.*

Observe that the derived equivalence obtained here is, in fact, an equivalence given by a sequence of reflections.

As in [5], we remove, by reflections, some relations. Let $A = kQ/I$ be a tree algebra and $\rho = a_0 \to \cdots \to a_r$ be a generator of the ideal I. We say that ρ is *thin* if

(T1) both a_0 and a_r are end vertices of Q, that means that they only have one neighbor, and

(T2) the projective A-module P_{a_0} has support $\{a_0, \dots, a_{r-1}\}$ and the injective A-module I_{a_r} has support $\{a_1, \dots, a_r\}$.

The following lemma enables us to assume that we deal with an algebra without thin relations.

Lemma 3.2 [5, (4.1)]. *Let A be a tree algebra. Then there exists a tree algebra B without thin relations such that A is reflection equivalent to B.*

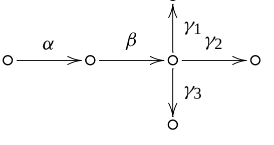
Moreover, if A is derived $\mathbb{E}$-free, then there exists a pair (B', E') satisfying the conditions (D1)–(D4) *such that $B'\{E'\} = B$. In addition, the vertices of E' are end vertices of a monomial relation.*

Remark. Since Q_B is a tree, the vertices of E' are always end vertices of Q_B.

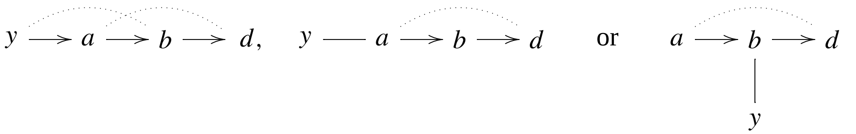
Lemma 3.3. *Let A be a derived $\mathbb{E}$-free algebra and D be a set of vertices of A such that the blowing-up $A\{D\}$ is derived-tame or the Euler form of $A\{D\}$ is non-negative. Then there exists a tree algebra B without thin relations which is a B-tree for $(B, \emptyset)$ and a set E of vertices of B such that $B\{E\}$ is reflection equivalent to $A\{D\}$.*

Proof. By the previous lemma, there exists a tree algebra B' without thin relations which is reflection equivalent to A. On the other hand, there exists a pair (B, E'') which satisfies the conditions (D1)–(D4) such that $B\{E''\} = B'$. By proposition 2.6, B' is derived $\mathbb{E}$-free and so is B which is also a derived-tame tree algebra. By lemma 2.6, $B\{E''\}$ is a B-tree for (B, E''). Obviously, B is also a B-tree for $(B, \emptyset)$. The above lemma guarantees us that each vertex of E'' is the beginning or the end of a monomial relation. In particular, these vertices are end vertices of Q_B.

On the other hand, by proposition 2.3, $A\{D\}$ is derived equivalent to $B'\{E'\}$ for some set E' of vertices of $B' = B\{E''\}$. We prove that the vertices of E' are, in fact, vertices of B. To do so, suppose that there exists a vertex $d \in E''$ such that $d^+ \in E'$ or $d^- \in E'$, respectively. We can assume without loss of generality that d is the end of a monomial relation. First, let us suppose that this relation is of length 3. In this case, $B'\{E'\} = (B\{E''\})\{E'\}$ contains a convex subcategory C of the form:



with $\alpha\beta\gamma_i = 0$. Since C is derived equivalent to a wild hereditary algebra of type 1.a, see the list in (D5), the remark 2.4 gives us the desired contradiction. Consequently, d must be the end of a monomial relation of length 2, say $\epsilon : a \to b \to d$. However, ϵ is a relation of $B' = B\{E''\}$ and thus is not thin. Therefore, there exists a neighbor y of a or of b and depending on the case, B contains one of the following convex subcategories:



By passing to the blowing-up $B'\{E'\}$, we obtain a full subcategory which is derived equivalent to a wild hereditary algebra of type 2.a, see the list in (D5). Once more, this yields a contradiction. In conclusion, the sets E' and E'' can be identified with sets of vertices of B and $B'\{E'\} = B\{E\}$ where $E = E' \cup E''$. $\square$

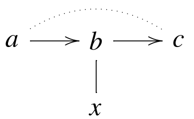
By the above lemma, we can assume that A is a tree algebra without thin relations which is a B-tree for $(A, \emptyset)$. Let D be a set of vertices of A such that $A\{D\}$ is derived-tame or the Euler form of $A\{D\}$ is non-negative. With this assumption, we prove that $A\{D\}$ is a B-tree (not necessarily for (A, D)). First, we show that any such pair (A, D) satisfies the conditions (D2)–(D4). Then, we consider a suitable pair of the above blowing-up and prove that this one also satisfies the condition (D1). Finally, we use lemma 2.6 to conclude that the algebra given by $A\{D\}$ is a B-tree. We introduce the next notation to alleviate the proofs.

Notation. Let A be an algebra and F be a set of vertices of A. Denote by $A(F)$ the convex hull of F, that is the smallest convex subcategory of A which contains the vertices of F.

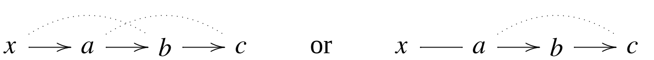
Lemma 3.4. *Let $A = kQ/I$ be a tree algebra without thin relations which is a B-tree for $(A, \emptyset)$ and let D be a set of vertices of A such that $A\{D\}$ is derived-tame or the Euler form of $A\{D\}$ is non-negative. Then (A, D) satisfies the conditions* (D2)–(D4).

Proof. Observe that since A is a B-tree for $(A, \emptyset)$, the condition (D2) is trivially satisfied. The same holds for the part of the conditions (D3) and (D4) that do not involve D. Let us start proving that (D3) holds.

Let $\epsilon : a \to b \to c$ be a generator of I of length two. To prove that (A, D) satisfies (D3), we only need to verify that b does not belong to D. Thus, let us suppose that $b \in D$. By hypothesis, ϵ is not a thin relation, which means that a, b or c have a different neighbor x. Observe that if x is a neighbor of b then there is no relation from a to x, nor from x to c and $A(\{a, b, c, x\})$ is of the following form:

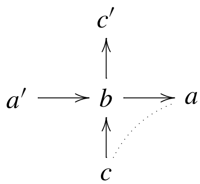


If x is a neighbor of a, dually for c, the convex subcategory $A(\{a, b, c, x\})$ is one of the following:

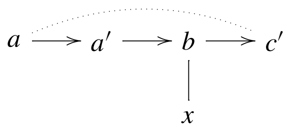


In each case, the blowing-up of $A(\{a, b, c, x\})$ at b is a full subcategory of $A\{D\}$ which is derived equivalent to a wild hereditary algebra. The remark 2.4 gives us a contradiction. That means that $b \notin D$ and that (A, D) satisfies (D3).

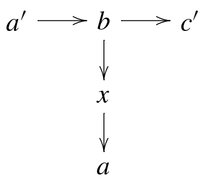
Now, let us show that (A, D) also satisfies (D4). Recall that $(A, \emptyset)$ satisfies this condition. We first consider when there is a pair of generators of I of length three. Let $(\epsilon : a \to a' \to b \to c', \epsilon' : a' \to b \to c' \to c)$ be one of these generators. We have to show that $\{a', b, c'\} \cap D = \emptyset$. Let $x \in \{a', b, c'\} \cap D$. The blowing-up of $A(\{a, a', b, c', c\})$ at x is a full subcategory of $A\{D\}$ which is derived equivalent to the blowing-up of



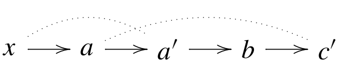
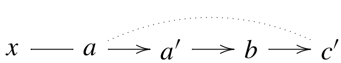
at the vertex x. By the above and the remark 2.5, this blowing-up is not derived-tame, neither does it have a non-negative Euler form. This yields the desired contradiction. Consequently, we can assume, up to duality, that there exists a generator of I of length three $\epsilon : a \to a' \to b \to c'$ with c' an end vertex and $y \in \{a', b, c'\} \cap D$. By hypothesis, ϵ is not a thin relation which means that there exists a neighbor x of a or b different from a' and c'. If x is a neighbor of b, then $A(\{a, a', b, c', x\})$ is of the following form:



In this case, the blowing-up of $A(\{a, a', b, c', x\})$ at y is derived equivalent to the blowing-up of the hereditary algebra



at the vertex y. This blowing-up is not derived-tame, neither does it have a non-negative Euler form. Thus, we obtain a contradiction. If x is a neighbor of a, then $A(\{a, a', b, c', x\})$ is of one of the following forms:

 or 

In both cases, the blowing-up of $A(\{a, a', b, c', x\})$ at the vertex y is derived equivalent to the blowing-up of

$$\begin{array}{ccccccc} & & & & & & b' \\ & & & & & \nearrow & \\ x & \text{---} & a & \longrightarrow & a' & & \\ & & & & & \nwarrow & \\ & & & & & & c' \end{array}$$

at the same vertex. Once more, this blowing-up is not derived-tame, neither does it have a non-negative Euler form. This last contradiction completes the proof that the pair (A, D) satisfies (D4). $\square$

To satisfy condition (D1), we need to define a concept of maximality for the pair (A, D).

Definition. Let A be a tree algebra and D be a set of vertices of A. We say that (A, D) is a *maximal pair* if there is no other pair (A', D') such that $A'\{D'\} = A\{D\}$ with A' a subtree of A and D' a set of vertices of A' containing D.

Remark. Let $A = kQ/I$ be a tree algebra without thin relations which is a B-tree for $(A, \emptyset)$ and let D be a set of vertices of A such that $A\{D\}$ is derived-tame or the Euler form of $A\{D\}$ is non-negative. Then there exists a maximal pair (B, E) such that $A\{D\} = B\{E\}$. Moreover, B is a tree algebra without thin relations which is a B-tree for $(B, \emptyset)$ since B is a subtree of A. Clearly, $B\{E\}$ is derived-tame or has a non-negative Euler form.

We thus have the following lemma.

Lemma 3.5. *Let $A = kQ/I$ be a tree algebra without thin relations which is a B-tree for $(A, \emptyset)$ and let D be a set of vertices of A such that $A\{D\}$ is derived-tame or the Euler form of $A\{D\}$ is non-negative. Moreover, suppose that (A, D) is maximal. Then (A, D) satisfies the conditions* (D1)–(D4).

Proof. By the Lemma 3.4, (A, D) satisfies the conditions (D2)–(D4). Thus, it remains to show that (D1) holds. By duality, we can suppose that there exist at least two arrows ending in a vertex b of D, say $a \to b \leftarrow c$. Since the blowing-up of $a \to b \leftarrow c$ at the vertices a and b is a wild hereditary algebra given by the following quiver:

$$\begin{array}{ccccc} a^+ & \longrightarrow & b^+ & & \\ & \searrow\!\!\!\!\nearrow & & \nwarrow & \\ & & & & c \\ & & & \swarrow & \\ a^- & \longrightarrow & b^- & & \end{array}$$

We can suppose that neither a, nor c belong to D. Moreover, the vertices a and c are end vertices of A, since the blowing-up of

$$x \text{ --- } a \longrightarrow b \longleftarrow c \quad \text{or} \quad x \longrightarrow a \longrightarrow b \longleftarrow c$$

at the vertex b is derived equivalent to a wild hereditary algebra.

By conditions (D3) and (D4), we have no relation beginning in a, nor in c. Therefore, we obtained the same algebra $A\{D\}$ if we consider the blowing-up of A' at D' where A' is the subtree of A obtained by deleting the vertex c and $D' = D \cup \{a\}$. This contradicts the maximality of (A, D). $\square$

Lemma 3.6. *Let $A = kQ/I$ be a tree algebra without thin relations which is a B-tree for $(A, \emptyset)$ and let D be a set of vertices of A such that $A\{D\}$ is derived-tame or the Euler form of $A\{D\}$ is non-negative. Then $A\{D\}$ is a B-tree.*

Proof. By the last remark, we can suppose that (A, D) is maximal. Thus, by the previous lemma, (A, D) satisfies the conditions (D1)–(D4). The conclusion follows from lemma 2.6. $\square$

Therefore, the Proposition 3.1 is a consequence of Lemmas 3.3 and 3.6.

4. Derived-tameness of B-tree algebras

This section is devoted to the proof of the following proposition.

Proposition 4.1. *Let B be a B-tree. Then B is derived equivalent to a semichain algebra $S(c, m)$.*

We remark that the derived equivalence of the former proposition is obtained by a sequence of tilts, which in fact correspond to APR-tilts and reflections.

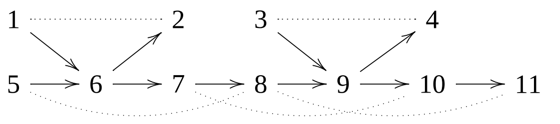
Using the above proposition, we get that all B-tree algebras are derived-tame and have non-negative Euler forms. Moreover, when the corank c is bigger than 2, they provide a class of derived-tame algebras which are neither derived equivalent to a hereditary algebra nor to a tubular one. Consequently, B-tree algebras are easy computable examples of algebras derived equivalent to semichain algebras.

In order to prove this result, we use essentially the results and ideas of Section 3 of the paper of Brüstle on derived-tame tree algebras [5]. In fact, the proof mainly consists in verifying that we can apply these results to B-tree algebras. This is done by induction on the number of monomial relations. Recall the following lemma.

Lemma 4.2 [5, (3.3)]. *Let A be a hereditary algebra and D be a set of vertices of A such that $A\{D\}$ is a B-tree. Then $A\{D\}$ is derived equivalent to a semichain algebra $S(c, m)$. If A is of type $\mathbb{A}_n$, then $c = |D| - 1$.*

Let $A = kQ/I$ be a tree algebra. Given two vertices x and y of Q, we denote by $\triangleright(x, y)$ the convex subcategory of A generated by all the vertices of Q which are nearer (in a walk) to x than to y together within the interval $[x, y]$, that is all vertices in a walk from x to y.

Example. Let A be the algebra given by the following bound quiver:



In this case, we have that $\triangleright(5,7)$ is the convex subcategory generated by 5, 6 and 7. On the other hand, $\triangleright(7,5)$ is the convex subcategory whose objects are 3, 4, 5, 6, 7, 8, 9 and 10.

For the rest of this section, let $A\{D\}$ be a B-tree for (A, D) with $A = kQ/I$. Recall that for each monomial relation of A, we can define a trisection of the quiver Q as fallow.

Now let $\epsilon : a \to b \to c$ be a monomial relation of length two as considered in (D3). For such a relation, we define three convex subcategories of A, $N_\epsilon^- = \triangleright(a,b)$, dually, $N_\epsilon^+ = \triangleright(c,b)$ and $N_\epsilon^0 = \bigcup_x \triangleright(x,b)$ where the union is taken over all neighbors x of b except a and c. Then, by construction, the union of these three subcategories covers the whole quiver Q. Moreover, by condition (D3), any relation of A different from ϵ is completely contained in one of the sets N_ϵ^-, N_ϵ^+, N_ϵ^0.

For a relation of length three, we also define a trisection with the same properties. By condition (D4), we either have one relation $\epsilon : a \to a' \xrightarrow{\alpha} b \xrightarrow{\beta} c'$ where c' is an end vertex of Q, or dually, a relation $\epsilon' : a' \xrightarrow{\alpha} b \xrightarrow{\beta} c' \to c$ where a' is an end vertex of Q, or there is a pair of relations $(\epsilon : a \to a' \xrightarrow{\alpha} b \xrightarrow{\beta} c', \epsilon' : a' \xrightarrow{\alpha} b \xrightarrow{\beta} c' \to c)$. In each case, we associate to ϵ or ϵ' or (ϵ, ϵ') a trisection of Q as follows. We define three convex subcategories of A, $N_\epsilon^- = \triangleright(a',b)$, dually, $N_\epsilon^+ = \triangleright(c',b)$ and finally, $N_\epsilon^0 = \bigcup_x \triangleright(x,b)$ where the union is taken over all neighbors x of b except a' and c'. Any other relation is thus completely contained in one of the sets N_ϵ^-, N_ϵ^+, N_ϵ^0.

Example. In the previous example, consider the pair of monomial relations $(\epsilon : 7 \to 8 \to 9 \to 10,\ \epsilon' : 8 \to 9 \to 10 \to 11)$. Then $N_\epsilon^- = \triangleright(8,9)$ is the convex subcategory whose objects are the vertices 1, 2, 5, 6, 7, 8 and 9. Moreover, the convex subcategory $N_\epsilon^+ = \triangleright(10,9)$ is supported by $\{9,10,11\}$ and $N_\epsilon^0 = \triangleright(3,9) \cup \triangleright(4,9)$ by $\{3,4,9\}$.

Remark that $A\{D\}$ is a B-tree for any $D \subseteq \{1,2,3,4,11\}$.

The following lemma describes which relations can be used in the inductive step.

Lemma 4.3 [5, (3.4)]. *Let $A\{D\}$ be a B-tree for (A, D) and let $A = kQ/I$. Then there exists a monomial relation ϵ or a pair of monomial relations (ϵ, ϵ') in I such that at most one of the convex subcategories N_ϵ^-, N_ϵ^+, N_ϵ^0, defined above contains some relation.*

We suppose from now on that $A\{D\}$ and the relation ϵ or the pair of relations (ϵ, ϵ') is the one described in the above lemma.

We need to consider a nicer form for this algebra. To do so, let us recall the following result.

Lemma 4.4 [5, (3.1)]. *Let A be an algebra of the following form*:

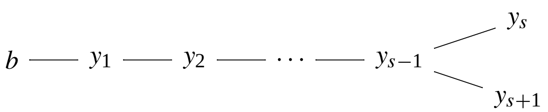
$$x_1 \text{ --- } x_2 \text{ --- } \cdots \text{ --- } x_r \text{ --- } s \, (A')$$

where A' denotes a convex subcategory of A and there is no relation starting or ending in one of the vertices x_i. Let D be a set of vertices of A such that $s \notin D$ and (A, D) satisfies the condition (D1). *Then $A\{D\}$ is derived equivalent to $B\{D\}$ where B is given by the following algebra or its dual*:

$$x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_r \longrightarrow s \, (A')$$

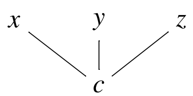
Lemma 4.5. *Up to derived equivalence, we can suppose that $A\{D\}$ and ϵ or (ϵ, ϵ'), respectively, are such that two of the convex categories N_ϵ^-, N_ϵ^+ and N_ϵ^0 are hereditary algebras of type $\mathbb{A}_m$ with all the arrows pointing in the same direction.*

Proof. We know that there exist at least two of the convex categories N_ϵ^-, N_ϵ^+ and N_ϵ^0 which are hereditary algebras. By (D5), they must be of type $\mathbb{A}_m$, $\mathbb{D}_m$ or $\widetilde{\mathbb{D}}_m$. Assume first that N_ϵ^0 is one of the hereditary categories. If N_ϵ^0 contains a convex subcategory of the form $x — b — y_1 — y_2$ (for an arbitrary orientation of the arrows) then A contains a convex subcategory of type 5 or 2.a of the list in (D5) whereas the relation is of length 2 or 3. On the other hand, if b has three neighbors $\{x, y, z\}$ in N_ϵ^0 then A contains a convex subcategory of type 3.a or 1.a of the list in (D5) whereas the relation is of length 2 or 3. Assume the quiver of N_ϵ^0 to be of the following form:



By (D5), y_s and y_{s+1} are not in D. Indeed, if y_s or y_{s+1} is in D then A admits a convex subcategory of type 2.b (if $s > 1$), 3.b or 1.b (if $s = 1$) whereas the relation ϵ is of length 2 or 3. Moreover, by applying an APR-tilt at y_s, if necessary, we may suppose that y_s and y_{s+1} are both sinks or sources. This process is not interfering with the blowing-up since y_{s-1} does not belong to D by (D1). Thus, we obtain the same algebra $A\{D\}$ if we replace the vertices y_s and y_{s+1} by a one vertex d_s which belongs to D. This new blowing-up is still a B-tree. Therefore, we can suppose that N_ϵ^0 is of the form $b - y_1 - \cdots - y_s$, whose of type $\mathbb{A}_m$. The result follows from Lemma 4.4. In the same way for N_ϵ^+ (dually N_ϵ^-), we need to verify that it contains none of the following subcategories where x, y and z are different from b (or from c', when ϵ is of length 3):

$$x \text{ --- } c \text{ --- } y \text{ --- } z,$$



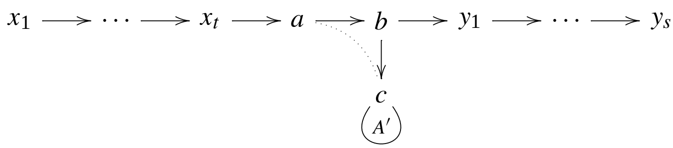
or $\quad d \text{ --- } c \text{ --- } y$

If this is the case then A contains a convex subcategory of the form 8 (or 7), 4.a (or 2.a) or 4.b (or 2.b) of the list in (D5) whereas the relation is of length 2 (or 3). As above, we obtain that N_ϵ^+ is of type $\mathbb{A}_m$. If $c \notin D$, the result follows from Lemma 4.4. Otherwise, if we do not have the desired orientation, there exists a source in N_ϵ^+. Take the nearest source from c, then since (A, D) satisfies condition (D1), it cannot belong to D. Thus, we can apply Lemma 4.4. □

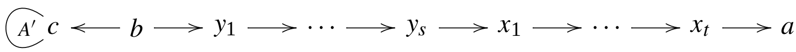
Remark. In the last lemma, N_ϵ^- (or N_ϵ^+) is a hereditary algebra of type $\mathbb{A}_m$ with all the arrows pointing in the direction of the monomial relation.

In order to show Proposition 4.1, we apply the proof of T. Brüstle. For this purpose, it is enough to verify that the algebras obtained at each induction step is a B-tree. We have four possibilities, up to duality. To know more on the derived equivalences used in the following, we refer the reader to [5].

First, consider the case where $\epsilon : a \xrightarrow{\alpha} b \xrightarrow{\beta} c$ is a relation of A of length 2 as considered in (D3) and N_ϵ^-, N_ϵ^0 contain no relation. By the above lemma, we can suppose that A is of the following form:

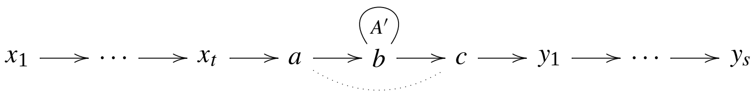


We know that A is derived equivalent to the algebra B given by

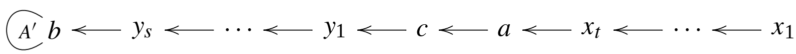


and $A\{D\}$ is derived equivalent to $B\{D\}$. Moreover, B has less monomial relations than A and (B, D) satisfies conditions (D1)–(D4). Thus, we just have to see that (B, D) satisfies condition (D5). If not, B contains a convex subcategory C of the list in (D5). Since (A, D) satisfies this condition, the subcategory $c \leftarrow b \rightarrow y_1 \rightarrow \cdots \rightarrow y_s \rightarrow x$ is contained in C, with $x = x_1$ if $t \neq 0$ and $x = a$ if $t = 0$. This implies that $s = 0$ and that C cannot be of type 1, 3, 4, 6, 9 or 10. On the other hand, if C is of type 2.a (or 2.b, 5, 7, 8), then A contains a convex subcategory of type 4.a (or 4.b, 6, 8 or 9, respectively). This yields the desired contradiction. Consequently, $B\{D\}$ is a B-tree.

Now, consider the case where $\epsilon : a \xrightarrow{\alpha} b \xrightarrow{\beta} c$ is a relation of length 2 as in (D3) and N_ϵ^-, N_ϵ^+ contain no relation. By Lemma 4.5, we can suppose that A is the following algebra:



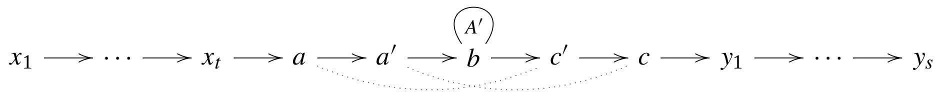
We have that A is derived equivalent to the following algebra B:



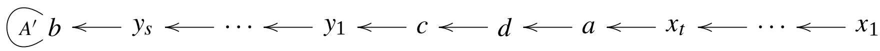
and $A\{D\}$ is derived equivalent to $B\{D\}$. Moreover, B has less relations than A and (B, D) satisfies the conditions (D1)–(D4). Suppose that (B, D) does not satisfies condition (D5), that means that B contains a convex subcategory C of the list in (D5). Since (A, D) satisfies this condition, we get that the subcategory $b \leftarrow y_s \leftarrow \cdots \leftarrow y_1 \leftarrow c \leftarrow a$ must be contained in C. Thus, $s = 0$ and C cannot be of type 1, 3, 4, 6, 9 or 10. On the other hand, if C is of type 2.a (or 2.b, 5, 7, 8), then A contains a convex subcategory of type 3.a (or 3.b, 10, 5, 6, respectively), a contradiction. Hence, $B\{D\}$ is a B-tree.

In the last two cases, we deal with a pair of monomial relations of length three, (ϵ, ϵ') as considered in condition (D3). We consider that one of the vertices a or c may not exist and therefore one of the relations ϵ or ϵ' does not appear.

We begin with the case when both convex categories N_ϵ^- and N_ϵ^+ are hereditary. Thus we may assume by Lemma 4.5 that A has the following form:

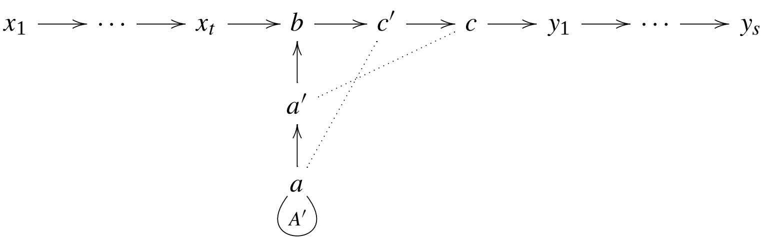


We have that A is derived equivalent to the algebra $B\{d\}$ where B is of the form

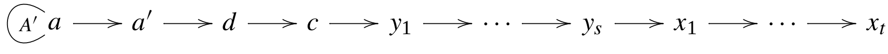


and $B\{D \cup \{d\}\}$ is derived equivalent to $A\{D\}$. Moreover, B has less relations than A and $(B, D \cup \{d\})$ satisfies the conditions (D1)–(D4). Suppose that $B\{D \cup \{d\}\}$ is not a B-tree, that means that B contains a convex subcategory C of the list in (D5). But (A, D) satisfies condition (D5). Thus, C must contain the subcategory $b \leftarrow y_s \leftarrow \cdots \leftarrow y_1 \leftarrow c \leftarrow d \leftarrow a$. Therefore, $s = 0$ and one of the vertices a or c does not exist. This implies that A must contain a convex subcategory of the same type as C, a contradiction.

We finish the proof with the case where we have a pair of monomial relations (ϵ, ϵ') as considered in (D4), and the convex subcategories N_ϵ^0 and N_ϵ^+ are hereditary. By Lemma 4.5, we can assume that A is the following algebra:



As in the previous case, one of the relations ϵ or ϵ' may not exist. We have that A is derived equivalent to the blowing-up $B\{d\}$ where B is of the following form:



and $B\{D \cup \{d\}\}$ is derived equivalent to $A\{D\}$. Moreover, B has less monomial relations than A and $(B, D \cup \{d\})$ satisfies the conditions (D1)–(D4). Suppose that B contains a convex subcategory C of the list in (D5). But, (A, D) satisfies condition (D5) and thus C

must contain the subcategory $a \to a' \to d \to c$. Therefore, one of the relations ϵ or ϵ' does not exist. In both cases, A contains a convex subcategory of the same type as C and we obtain a contradiction. Thus, $B\{D \cup \{d\}\}$ is a B-tree.

Consequently, we have proved that given a B-tree, there exists a derived equivalent B-tree such that the corresponding tree has strictly less monomial relations. Then Proposition 4.1 follows from induction and Lemma 4.2.

5. Blowing-up of derived type $\mathbb{E}$ tree algebra

Let A be a tree algebra and D be a non-empty set of vertices of A such that $A\{D\}$ is derived-tame or has a non-negative Euler form. To prove the main theorem, there remains to consider the case when A is derived of type $\mathbb{E}$. The purpose of this section is to prove the following lemma which yields a contradiction of the above hypothesis by using the proposition 2.4.

Lemma 5.1. *Let A be a tree algebra derived of type $\mathbb{E}$ and d be a vertex of A. Then $A\{d\}$ is not derived-tame, neither does it have a non-negative Euler form.*

Let A be a tree algebra derived of type $\mathbb{E}$ and d be a vertex of A. By definition, A is derived equivalent to a hereditary algebra of type $\mathbb{E}_p$, $\widetilde{\mathbb{E}}_p$ with $p = 6, 7, 8$ or to a tubular algebra. Since A is a triangular algebra, we know that A is reflection equivalent to an algebra B where d corresponds to a source of B which we also denote by d. Thus, by proposition 2.3, $A\{d\}$ is derived equivalent to $B\{d\}$. Therefore, it is sufficient to prove that $B\{d\}$ is not derived-tame, and that it does not have a non-negative Euler form. On the other hand, there exists an algebra C (not necessarily connected) and a C-module M such that $B = C[M]$ and d is the extension vertex, that is $M = \operatorname{rad} P_d$. We easily see that the blowing-up $B\{d\}$ corresponds to the one-point extension $B[M] = C[M][M]$. It is a well-known fact that Hochschild cohomology is preserved under derived equivalence. Since $H^1(A) = 0$, we have that $H^1(C[M]) = 0$. By [18], M is a *separated module*, that means that each indecomposable summand of M belongs to a different connected component of C.

First, let us suppose that C is connected and thus that M is indecomposable.

Lemma 5.2. *Let C be an algebra and M be an indecomposable C-module such that $C[M]$ is derived equivalent to a hereditary algebra of type $\mathbb{E}_p$, $\widetilde{\mathbb{E}}_p$ with $p = 6, 7, 8$ or to a tubular algebra. Then $C[M][M]$ is not derived-tame, neither does it have non-negative Euler form.*

Proof. If $B = C[M]$ is derived equivalent to a tubular algebra, there exists a B-module X such that $\dim \operatorname{Hom}_B(M, X) \geqslant 3$ or $\dim \operatorname{Hom}_B(X, M) \geqslant 3$. It follows from [17], that at least one of $B[M]$ or $[M]B$ is wild and thus, $B[M] = C[M][M]$ is not derived-tame. The result now follows from [4, (6.2)]. Therefore, we can suppose that B is derived equivalent to a hereditary algebra of type $\mathbb{E}_p$ or $\widetilde{\mathbb{E}}_p$ with $p = 6, 7, 8$.

Recall that d is the extension vertex of $C[M]$. Therefore, there exists a triangle equivalence $F : D^b(C[M]) \to D^b(H')$ where H' is a hereditary algebra of type $\mathbb{E}_p$ or $\widetilde{\mathbb{E}}_p$ with

$p = 6, 7, 8$ such that FP_d is an indecomposable H'-module. Moreover, we can suppose that $\mathrm{Hom}(FP_d, H') = 0$. We deduce easily that $\mathrm{Ext}^1(FP_d, FP_d) = 0$ and $\mathrm{End}\, FP_d \cong k$. Denote by $FP_d^{\perp}$ the full subcategory of $\mathrm{mod}\, H'$ whose objects are all the modules X such that $\mathrm{Hom}(FP_d, X) = 0 = \mathrm{Ext}^1(FP_d, X)$. By [11, (III.6.4)], $FP_d^{\perp} \cong \mathrm{mod}\, H$ where H is a hereditary algebra. Following [12, (3.3)], the derived category $D^b(H)$ can be identified with the full subcategory of $D^b(H')$ whose objects are all complexes N such that $\mathrm{Hom}_{D^b(H')}(FP_d, N[i]) = 0$ for all $i \in \mathbb{Z}$. We verify that for each indecomposable C-projective module P_x, we have $FP_x \in D^b(H)$. But, $\mathrm{Hom}_{D^b(H')}(FP_d, FP_x[i])$ is isomorphic to

$$\mathrm{Hom}_{D^b(C[M])}\big(P_d, P_x[i]\big) = \begin{cases} 0 & \text{if } i < 0, \\ \mathrm{Hom}(P_d, P_x) & \text{if } i = 0, \\ \mathrm{Ext}^i(P_d, P_x) & \text{if } i > 0. \end{cases}$$

Since d is a source, we get $FC \in D^b(H)$. It follows that C is derived equivalent to the hereditary algebra H. In fact, $C \cong \mathrm{End}_{D^b(H)} FC$. Since $C[M]$ is derived-tame, H must be tame.

Applying [3], there exists an indecomposable H-module N which is projective or regular such that $H[N]$ is derived equivalent to $C[M]$ and $H[N][N]$ is derived equivalent to $C[M][M]$.

If N is projective, then $H[N]$ is a hereditary algebra. By hypothesis, $H[N]$ is of type $\mathbb{E}_p$ or $\widetilde{\mathbb{E}}_p$ with $p = 6, 7, 8$. Since $H[N][N]$ is also a hereditary algebra, it is clear that it must be wild. By remark 2.4, $H[N][N]$ is not derived-tame, and does not have a non-negative Euler form, and the same is true of $C[M][M]$.

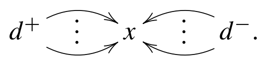
Therefore, we can suppose that N is regular and thus that $H[N]$ is derived equivalent to a hereditary algebra H' of type $\widetilde{\mathbb{E}}_p$ with $p = 6, 7, 8$. On the other hand, there exists an indecomposable H'-module N' such that $H'[N']$ is derived equivalent to $H[N][N]$. In fact the module N', which is the image of N under the derived equivalence, corresponds to a regular module of quasi-length 2 in a stable tube of H'. The result follows from [15]. □

Finally, we have to consider the case where C is not connected. Recall that $B = C[M]$ is connected. Of course, it is easy to see that C has at least five vertices since $B = C[M]$ is derived equivalent to a hereditary algebra of type $\mathbb{E}_p$ or $\widetilde{\mathbb{E}}_p$ where $p = 6, 7, 8$. Thus, it is sufficient to show the following lemma.

Lemma 5.3. *Let C be a non-connected algebra with at least three vertices and let M be a separated C-module such that $C[M]$ is connected. Then $C[M][M]$ is not derived-tame, neither does it have a non-negative Euler form.*

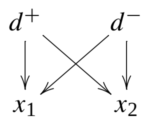
Proof. By hypothesis, there exist connected algebras $C_1, \ldots, C_r$ and for each i, an indecomposable C_i-module M_i such that $C = \prod_{i=1}^r C_i$ and $M = \bigoplus_{i=1}^r M_i$. Denote by d^+ and d^- the extension vertices of $C[M][M]$. Remark that $\mathrm{rad}\, P_{d^*} = M$ for $* \in \{+, -\}$.

Let x be a vertex of C. If $\dim M(x) \geqslant 2$, we obtain a full subcategory of $C[M][M]$ which is a wild hereditary algebra of the form

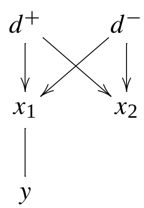


Thus, $\dim M(x) \leqslant 1$ for all x.

For each i, let x_i be a vertex of the support of $M_i / \operatorname{rad} M_i$. Since $r \geqslant 2$, $C[M][M]$ contains a convex subcategory of the form:



If $r \neq 2$, then C must contain a full subcategory which is a wild hereditary algebra. Thus, by remark 2.4, we can assume that $r = 2$. Since, C has more than three vertices, we can suppose that x_1 has a neighbor y. Suppose that $y \in \operatorname{supp}(M_1)$ or that $x_1 \leftarrow y$. Then, we obtain a convex subcategory of $C[M][M]$ which corresponds to the following wild hereditary algebra:



Thus, $y \notin \operatorname{supp}(M_1)$ and $x_1 \to y$. This implies that there is a monomial relation $d^+ \to x_1 \to y$. Since, $\operatorname{rad} P_{d^+} = M = \operatorname{rad} P_{d^-}$, we have also a monomial relation $d^- \to x_1 \to y$. Therefore, $C[M][M]$ contains a convex subcategory which is derived equivalent to the above wild hereditary algebra. The result follows from remark 2.4. □

This lemma finishes the proof of Lemma 5.1 and therefore of the theorem.

Remark. The proof of the theorem gives us a way of verifying whether a blowing-up of a tree algebra at some non-empty set is derived-tame (or equivalently have a non-negative Euler form) or not.

In fact, let $A\{D\}$ be a blowing-up of a tree algebra A. If A contains, as a convex subcategory, some algebras whose derived equivalent to $\mathbb{E}_6$, see [2, Section 3], then $A\{D\}$ is not derived-tame. If not, remove by reflection all thin relations, see Lemma 3.2. The algebra obtained in this way is a B-tree (for some maximal pair) if and only if $A\{D\}$ is derived-tame.

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