Falsity Preservation

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Abstract

In the following paper we examine falsity preservation phenomena for natural deduction and axiomatic logical systems. We start with Syllogistics and end with a very interesting form o Peano Arithmetic for Refutability in which Gödel's Incompleteness Theorems hold. The paper presents valid syllogistic moods of falsity preservation based on valid moods for truth preservation and also valid rules of falsity preservation in natural deduction. That such systems can be defined is surprising, but that they resemble truth preservation system is remarkable. That put us in front of an important question: what means do we have to distinguish truth and falsity from a logical point of view?

Key-words

Falsity preservation, Syllogistic, Natural Deduction, De Morgan's Laws, Peano Arithmetic, incompleteness, negation.

I) Aristotelian Syllogistic - Syllogisms Preserving Falsity

The following is based on a very simple concept: the concept of contradictory opposition. In the square of oppositions, for each categorical sentence, there corresponds only one categorical sentence which is its contradictory. If we assume that from two contradictories one being true the other should be false, then for each valid syllogism, i.e., for each syllogistic form preserving truth, it corresponds a syllogistic form preserving falsity.

Let's consider BARBARA schema and its corresponding falsity preservation form. It is possible to state a system for falsity preservation having "valid moods". So, corresponding to BARBARA we have a falsity preservation form BORBORO:

Some M is not P (O)

Some S is not M (O)

Some S is not P (O)

BORBORO is such that, in case both premises are false, the conclusion must be false. Jus notice that having two false categorical sentences of form *Some M is not P* (O) and *Some S is not M* (O), then, by the square of oppositions, we'll have two true categorical sentences of form *All M is P* (A) and *All S is M* (A), respectively. Next, as BARBARA is a valid truth preservation form, then *All S is P* (A) must be true and, again, by the square of oppositions, *Some S is not P* (O) must be false. Therefore, BORBORO is a valid falsity preservation form.

The above schema also applies to the any other valid syllogistic form:

Figure 1	Barbara	Celarent	Darii	Ferio		
Figure 2	Cesare	Camestres	Festino	Baroco		
Figure 3	Darapti	Disamis	Datisi	Felapton	Bocardo	Ferison
Figure 4	Bramantip	Camenes	Dimaris	Fesapo	Fresison	

i.e.:

Figure 1	Borboro	Cilorint	Doree	Firea		
Figure 2	Cisori	Comistris	Fistena	Boraca		
Figure 3	Doropte	Desomes	Dotese	Filoptan	Bacorda	Firesan
Figure 4	Bromontep	Cominis	Demores	Fisopa	Frisesan	

are valid falsity preservation forms.

The basic idea is so clear and so easily presentable that its absence from most of logic manuals is surprising. It seems that Aristotle and the tradition never considered those falsity preservation forms. A good reason why those forms were not apparently considered is that the main stream of occidental thinking always had in mind another project: the project of Second Analytics. It is, in some sense, a narrow epistemological project: to establish what is a science, what is to know the truth in a organized fashion, axiomatic fashion. We say narrow, because it can be argued that to know falsities in a domain is an expressive part of an epistemological project. Of course, let's make ourselves clear, we are not blaming Aristotle for not having examined these new forms, if he really didn’t. He had reasons to be interested in truth preservation.

Incidentally, if Lukasiewicz in his *Aristotle’s Syllogistic* (1951) were correct about Syllogistic as a theory of valid conditional statements, then there would be conditionals which originate from the above new forms that would also be valid. In other words, for BORBORO case, the conditional of form *Some S is not P* $\supset$ *Some M is not P* $\vee$ *Some S is not M* is valid (always true).

Let’s use a notation to represent each categorical form, borrowing it from Corcoran (1974): for *All x is y*, ***Axy***; for *No x is y*, ***Nxy***; for *Some x is y*, ***Sxy***; and for *Some x is not y*, ***$xy***. Having in mind that Aristotle himself was not a formalist and that his Syllogistic is not a formal system, we can, nonetheless, observe certain interesting properties. Substitution, in each categorical form, of the logical expression *A* for *$*, *$* for *A*, *N* for *S* and *S* for *N*, we’ll give us two syntactical indiscernible systems of valid forms, one preserving truth the other preserving falsity. So, from a formalistic point of view, preservation of truth and preservation of falsity are structurally similar, in regard of Syllogistic moods. In other words, it is impossible to distinguish preservation phenomena (truth preservation and falsity preservation), not, at least, from a syntactical point of view. As we certainly can distinguish truth and falsity, this fact seems to imply that those concepts require another level of conceptualization to be distinguished, maybe a semantical level. But, we are not even sure of that. At this moment, we just can say that we know what our syntactical system means because we intend it to mean that and, somehow, we are able to communicate such intention.

I) Negation, First Order Logic and First Order Peano Arithmetic

Certainly the idea of opposite contradiction involves the concept of negation. Usually, by means of negation in logical formulas, i.e. $\neg A$, we intend to express opposite contradiction, i.e. the contradictory of *A*. Also, usually we assume an intimate connection between negation and falsity. Negation and falsity are most of the time treated as a couple, in mathematics and philosophy of mathematics. Heyting (1956, pp. 18 and 19), for example, says that:

> Strictly speaking, we must well distinguish the use of "not" in mathematics from that in explanations which are not mathematical, but which are expressed in ordinary language. In mathematical assertions no ambiguity can arise: "not" has always the strict meaning. "The proposition p is not true", or "the proposition p is false" means "If we suppose the truth of p, we are led to a contradiction".

Therefore, if logical systems are envisaged as elucidation of logical relations connecting mathematical propositions, negation and falsity will appear as intimately related. However, we asks ourselves, is it correct that negation inside logical syntactic systems express falsity of a proposition?

In what follows we state a formal system for falsity preservation. Traditionally, proofs in logical systems should preserve truth. Notwithstanding, there is no reason to reject preservation of falsity as an interesting criteria to be met when we want to elucidate logical relations. The following system is advanced having such criteria in mind. It is structurally identical with First Order Peano Arithmetic and its statement follows closely that of Kleene (1952, p. 82):

	Peano Arithmetic for **Provability**	Peano Arithmetic for **Refutability**
Propositional	1a. $A\supset(B\supset A)$	1a. $A\not\supset(B\not\supset A)$
	1b. $(A\supset B)\supset((A\supset(B\supset C))\supset(A\supset C))$	1b. $(A\not\supset B)\not\supset((A\not\supset(B\not\supset C))\not\supset(A\not\supset C))$
	2. $\dfrac{A,\ A\supset B}{B}$	2. $\dfrac{A,\ A\not\supset B}{B}$
	3. $A\supset(B\supset(A\& B))$	3. $A\not\supset(B\not\supset(A\vee B))$
	4a. $A\& B\supset A$	4a. $A\vee B\not\supset A$
	4b. $A\& B\supset B$	4b. $A\vee B\not\supset B$
	5a. $A\supset A\vee B$	5a. $A\not\supset A\& B$
	5b. $B\supset A\vee B$	5b. $B\not\supset A\& B$
	6. $(A\supset C)\supset((B\supset C)\supset(A\vee B\supset C))$	6. $(A\not\supset C)\not\supset((B\not\supset C)\not\supset(A\& B\not\supset C))$
	7. $(A\supset B)\supset((A\supset\neg B)\supset\neg A)$	7. $(A\not\supset B)\not\supset((A\not\supset\sim B)\not\supset\sim A)$
	8. $\neg\neg A\supset A$	8. $\sim\sim A\not\supset A$
Predicational	9. $\dfrac{C\supset A(x)}{C\supset\forall xA(x)}$	9. $\dfrac{C\not\supset A(x)}{C\not\supset\exists xA(x)}$
	10. $\forall xA(x)\supset A(t)$	10. $\exists xA(x)\not\supset A(t)$
	11. $A(t)\supset\exists xA(x)$	11. $A(t)\not\supset\forall xA(x)$
	12. $\dfrac{A(x)\supset C}{\exists xA(x)\supset C}$	12. $\dfrac{A(x)\not\supset C}{\forall xA(x)\not\supset C}$
Arithmetical	13. $A(0)\&\forall x(A(x)\supset A(x'))\supset A(x)$	13. $A(0)\vee\exists x(A(x)\not\supset A(x'))\not\supset A(x)$
	14. $a'=b'\supset a=b$	14. $a'\neq b'\not\supset a\neq b$
	15. $\neg a'=0$	15. $\sim a'\neq 0$
	16. $a=b\supset(a=c\supset b=c)$	16. $a\neq b\not\supset(a\neq c\not\supset b\neq c)$
	17. $a=b\supset a'=b'$	17. $a\neq b\not\supset a'\neq b'$
	18. $a+0=a$	18. $a+0\neq a$
	19. $a+b'=(a+b)'$	19. $a+b'\neq(a+b)'$
	20. $a.0=0$	20. $a.0\neq 0$
	21. $a.b'=(a.b)+a$	21. $a.b'\neq(a.b)+a$

Each formula and each rule in each line on in both columns are structurally similar. Their difference lies on the logical constants used. However, it follows a pattern. Each of them results from the other by substitution of logical constants inside pairs of duals: [$\wedge$, $\vee$]; [$\supset$,$\not\subset$]; [$\forall$,$\exists$]; [$\bot$,$\top$]; [$\neg$,$\sim$] and [=, $\neq$]. We used a second symbol for negation in the new system because we are not sure if we could ascribe to it the same meaning in both systems. The "desimplication" ($\not\subset$) can be defined: $A\not\subset B\equiv_{df}\neg(B\supset A)$. Its truth table is:

A	B	A$\not\subset$B
V	V	F
V	F	F
F	V	V
F	F	F

Also, we observe that "$\neq$" can be seen as a basic predicate, so basic as "=". Actually, it seems that we don't need any notion of negation in order to acquaint the falsity of $1\neq 1$ or the truth of $1\neq 2$.

All these "new axioms" are false (refutable) and that all rules preserve falsity. A way of realizing that is of looking into the corresponding First Order Natural Deduction System. Also, this kind of system shows more intuitively the falsity preservation phenomena.

So, making the same substitutions based on pairs of duals we made before, we obtain a system where all rules preserve falsity[1]. The system bellow is structurally similar to the Intuitionist Natural Deduction System[2]:

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[1] In case the system for truth preservation was formulated using *falsum*, $\bot$, the system for falsity will be formulated with *verum*, $\top$.

[2] We follow Gentzen's and Prawitz's convention of using two kinds of variables: x, y, z … which are used only bounded; a, b, c … which are used as individual parameters, not bounded by any quantifier. Top-formulas surrounded by braces are being discharged by the rule having the same index. Other braces of form $[\alpha/\beta]$ indicate a syntactical substitution of β in place of α. We use t to indicate a term.

Introductions	$\dfrac{A_1 \qquad A_2}{A_1 \vee A_2}\ \vee^f i$	$\dfrac{\begin{matrix}\Gamma, [A_1]^i \\ \vdots \\ A_2\end{matrix}}{A_1 \not\subset A_2}{}^{i}\ \not\subset^f i$	* $\dfrac{\begin{matrix}\Gamma \\ \vdots \\ A\end{matrix}}{\exists x A[a/x]}\ \exists^f i$
	$\dfrac{A_1}{A_1 \wedge A_2}\ l\wedge^f i \qquad \dfrac{A_2}{A_1 \wedge A_2}\ r\wedge^f i$		$\dfrac{A[a/t]}{\forall x A[a/x]}\ \forall^f i$
Eliminations	$\dfrac{A_1 \vee A_2}{A_1}\ l\vee^f e \qquad \dfrac{A_1 \vee A_2}{A_2}\ r\vee^f e$	$\dfrac{A_1 \quad A_1 \not\subset A_2}{A_2}\ \not\subset^f e$	$\dfrac{\exists x A[a/x]}{A[a/t]}\ \exists^f e$
	$\dfrac{A_1 \wedge A_2 \quad \begin{matrix}\Gamma_1, [A_1]^i \\ \vdots \\ C\end{matrix} \quad \begin{matrix}\Gamma_2, [A_2]^i \\ \vdots \\ C\end{matrix}}{C}{}^{i}\ \wedge^f e$	$\dfrac{T}{A}\ Te$	* $\dfrac{\forall x A[a/x] \quad \begin{matrix}\Gamma, [A]^i \\ \vdots \\ C\end{matrix}}{C}{}^{i}\ \forall^f e$

All rules in this system should be read in the same way we would use for truth preservation. We would say that there is falsity preservation when it is guaranteed that the conclusion is false if all subsidiary derivations preserve falsity[3], but only in case open top formulas are all false. In order to apply rule $\exists^f i$ we should verify a restriction before. Individual parameter a must not occur in Γ. Rule $\forall^f e$ also has a restriction, parameter a must not occur in Γ neither C. There are no introduction or elimination rules for implication, but introduction and elimination rules for desimplication. The introduction rule establishes that

[3] In this context, when we say that falsity is preserved from suppositions to immediate premises it means that if all suppositions were false, then it is guaranteed that the conclusion of subsidiary derivations will be false. Anyway, premises that are not conclusion of a subsidiary derivation can be regarded, conventionally, as subsidiary derivations of one formula. Finally, we notice, it is problematic to say that there is preservation of falsity for $\exists^f i$ as much as it is to say that there is preservation of truth for $\forall i$, because the immediate premise is a propositional function, in these cases.

desimplication $A_1 \not\subset A_2$ is false just in case falsity is preserved from Γ, A_1 to A_2 and all formulas in Γ are false.

Regarding syntactical structure, rules in the system for falsity are identical to the rules in the system for truth preservation. Conjunction and disjunction exchanged places, universal and existential too. Rules for implication ($\supset$) and for desimplication ($\not\subset$), as well as *falsum* ($\perp$) and *verum* (T), are structurally similar. In this way, any valid structural property for truth preservation also applies to falsity preservation[4], in particular "consistency". Just consider that, strictly speaking, we can't distinguish what is conjunction from what is disjunction looking only at the structure of the rules without knowing if the system preserves truth or falsity.

The above falsity system would not count with intuitionist agreement, excepting its propositional chunk, because $\forall^f e$ would not be regarded as valid[5]. However, such a rejection is surprising since the rule is structurally identical with $\exists e$, which is admissible by intuitionists standards of the concept of canonical proof generally defined by means of the inversion principle[6]. We observe that $\forall^f i$ seems to be a good description of the conditions under which a universal proposition is false. In other terms, while universal introduction seems to be non-objectionable, the elimination rule is objectionable from the intuitionistic point of view, even if both rules are in agreement, at least if we follow a certain reading of the inversion principle.

The classical system is obtained from the above system adding an indirect proof principle. It could be an axiom, corresponding to the translation of the excluded middle (in this case the translation is $A \wedge \sim A$)[7] or one of the following rules corresponding to the so called *Consequentia Mirabilis*[8]:

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[4] We are refereeing here, above all, to the normalization and confluence theorems and all its derived corollaries. See Prawitz (1965).

[5] The reason is that there is no guarantee that we can show for some term t that the *falsum* would follow from $A[a/t]$ when it follows from $\forall x A[a/x]$.

[6] See Prawitz (1965).

[7] Negation can be define as $\neg A \equiv_{df} A \supset \perp$ (truth preservation) / $\neg A \equiv_{df} A \not\subset T$ (falsity preservation) or stay as a primitive.

$$\frac{\begin{array}{c}\Gamma, [A\not\subset T]^i\\ \vdots\\ A\end{array}}{A}\,i \qquad \frac{\begin{array}{c}\Gamma, [\sim A]^i\\ \vdots\\ A\end{array}}{A}\,i$$

With this rule it is not difficult to derive the formula $A\wedge{\sim}A$, from zero hypothesis, aided by other rules in the falsity system. Also, as we have completeness and soundness for classical first order logic system for truth preservation, then they should also be valid for falsity preservation.

Well, all rules and formulas of predicate calculus for falsity preservation, in Kleene's style system given above, are derivable from natural deduction rules for falsity. Moreover, all Peano Arithmetic proper axioms for refutability can be easily perceived as false, with one exception. The exception, where intuition suffers more, is the induction axiom. The natural deduction rule corresponding to this axiom in the system for truth preservation is identical, structurally speaking, to a rule in the system for falsity preservation:

$$\frac{A[a/0] \qquad \begin{array}{c}[A[a/b]]^i\\ \vdots\\ A[a/b']\end{array}}{A}\,i$$

Basically, if *A(0)* were false and from *A(b)* to *A(b')* there were preservation of falsity, then *A(a)* would be false, where *a* is an individual parameter and could be substituted by any number, but it cannot occur in open hypotheses. So, by applying five natural deduction rules preserving falsity, the induction axiom for falsity can be derived, from zero hypotheses:

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[8] Depending on if negation is taken as defined or primitive.

$$
\dfrac{\dfrac{\dfrac{[A(0)\vee\exists x(A(x)\not\subset A(x'))]^2}{A(0)}\,e\vee^f \qquad \dfrac{[A(b)]^1 \quad \dfrac{\dfrac{[A(0)\vee\exists x(A(x)\not\subset A(x'))]^2}{\exists x(A(x)\not\subset A(x'))}\,e\vee^f}{A(b)\not\subset A(b')}\,e\exists^f}{A(b')}\,e\not\subset^f}{A(a)}\,{}^1\,ind^f}{A(0)\vee\exists x(A(x)\not\subset A(x'))\not\subset A(x)}\,{}^2\,i\not\subset
$$

In the end, to guarantee correctness for the new axiomatic system, it would suffice to certify that each natural deduction rule preserves falsity, task which we leave for the readers.

Therefore, axiomatic Peano Arithmetic for Refutability seems correct. For each derivable formula, there will be another formula, structurally similar, derivable in the axiomatic system for provability, and vice-versa.

In a precise sense, truth and falsity are not syntactically distinguishable, in regard of the systems examined. Actually, also, the use of conjunction or of disjunction by means of syntactical rules cannot establish the entire meaning of them or the meaning we intuitively attach to them. The same happens with other logical constants and its duals: [$\supset,\not\subset$]; [$\forall,\exists$]; [$\bot,T$]; [$\neg,\sim$] and [$=,\neq$]. Negation could be an exception, since the rules are structurally equal on both systems. However, we are not sure if negation means the same in both systems.

III) De Morgan's Laws and Incompleteness

Let's define in what follows a syntactic operation over formulas. Operation * consists in the substitution of logical constants and basic predicates for its duals.

Refutability and provability systems are structurally identical and, by means of operation *, as there is a Godelian sentence *G* in the system for provability, there will be a Godelian sentence *G** in the system for refutability such that neither *G** nor its negation will be derivable in the system for refutability, under hypothesis of "consistency". If the refutability system is "consistent", neither G^* nor its negation are derivable in the system. Otherwise, if there were a derivability sequence for G^* in the refutability system, then there would be a corresponding sequence mimicking it in the provability system, consisting of formulas obtained by means of operation *. We observe that for any formula F, $F^{**}=F$. Thus, in that case, there would be a derivability sequence of G in the provability system, because $G^{**}=G$. But that is impossible, if the provability system is consistent. However, supposing the refutability system "consistent", the provability system must also be consistent, because, otherwise, there would be derivability sequences for every formula in the refutability system, contrary to the assumption.

Usually, we interpret sentence G as saying: *I'm not derivable inside the system for provability*. A worthwhile investigation is that of what G^* would mean. Does it mean *I'm not derivable in the system for refutability*? Well, a priori it sounds …. but not! It means *I'm derivable in the system for refutability*. As G^* is not derivable in the refutability system, then what it says - *I'm derivable in the system for refutability* - is false and the system, thus, doesn't derive a falsity, which it should if it were complete, but it cannot.

What operation * does over formulas corresponds closely to the content of De Morgan's Laws. Actually, we can devise an extension of De Morgan's Laws for the complete set of first order logical constants. In order to formulate the following theorem, let's regard all logical constants as belonging to one and same language and use only one negation symbol ($\neg$).[9]

Theorem I: Let A be a formula in a language where each basic predicate P has a dual basic predicate Q such that for any n-tuple of terms $t_1,\ldots,t_n$, $P(t_1,\ldots,t_n)\Leftrightarrow\neg Q(t_1,\ldots,t_n)$, then $A^*\Leftrightarrow\neg A$.[10]

Proof: By induction in the logical degree of A. If A is a basic formula, easy. If A is a conjunction or a disjunction, we obtain the stated equivalence proving it by means of

[9] The object language equivalence constant can be defined from the others.

[10] We use $\Leftrightarrow$ to express metalingüistic logical equivalence. It can be regarded as semantic equivalence or deductive equivalence.

natural deduction rules, because by induction hypothesis $B^*\Leftrightarrow\neg B$ and $C^*\Leftrightarrow\neg C$, so $A^*\equiv(\neg B^*\vee\neg C^*)\Leftrightarrow\neg(B\wedge C)\equiv\neg A$.[11] If $A\equiv\bot$, then $A^*\equiv\neg T$, immediate. If $A\equiv T$, similar to the precedent case. If $A\equiv(B\supset C)$, then $A^*\equiv(B^*\not\subset C^*)$, but by induction hypothesis $B^*\Leftrightarrow\neg B$ and $C^*\Leftrightarrow\neg C$, so $A^*\equiv(\neg B\not\subset\neg C)\Leftrightarrow\neg(\neg C\supset\neg B)\Leftrightarrow\neg(C\vee\sim B)\Leftrightarrow\neg(B\supset C)\equiv\neg A$. If $A\equiv(B\not\subset C)$, similar. If $A\equiv\forall xB[b/x]$, then $A^*\equiv\exists x(B^*[b/x])$, but by induction $B^*\Leftrightarrow\neg B$, so $A^*\equiv\exists x\neg B[b/x]\Leftrightarrow\neg\forall xB[b/x]\equiv\neg A$. If $A\equiv\exists xB[b/x]$, similar. *QED*

The above theorem was heuristically suggested by the intuition that a formula A in one of the axiomatic systems, be it for truth or for falsity, can be put in correspondence with A^* in the other system. Therefore, if one of the formulas is true, the other should be false. Thus, one of them must be equivalent to the negation of the other.

We can extend the operation * in order to obtain a more general theorem. Let's define ° as the operation which is similar to *, but, in case of basic predicates or propositional variables, it adds a negation in front of the formula, instead of changing it by its dual as before.

Theorem II: $A^\circ\Leftrightarrow\neg A$.

Proof: Similar to the proof of theorem I.

Corollary: De Morgan's Laws are valid for the complete set of first order logical constants: $\{\neg, \wedge, \vee, \supset, \not\subset, \forall, \exists\}$.

Proof: Using theorem II and substitution of equivalences of form $\neg\neg A\Leftrightarrow A$ for atomic formulas.

It is amazing that, after all, we can relate incompleteness and De Morgan's Laws. Because of the above theorem $G^*\Leftrightarrow\neg G$, i.e., Gödel first incompleteness theorem can be restated by saying that there is a formula G such that neither G nor the equivalent of G^* are derivable in the system, be it a system for provability or for refutability.

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[11] We are using the symbol $\equiv$ to express syntactical equivalence.

Thus, if our interpretations are correct, there is a pair of dual formulas – G and G^*, in one hand, $\neg G$ and $\neg G^*$, by the other – such that in each pair one is true and the other false, but none of them is derivable in both systems, under hypothesis of consistency. For each pair of duals, derivability of one of them, in some system, implies derivability of the other. Any proof in one of those systems corresponds to another proof in the other by means of * operation.

If we agree that to be provable means to be true and that to be refutable means to be false, then, since we can't distinguish provability and refutability, from a classical point of view, it seems we can't distinguish truth and falsity. In the end the distinction between truth and falsity is not expressible syntactically and, thus, there must be some faculty in the interpreter of language that allows them to distinguish cases, unless the malign genius really make us to take the true by the false and vice-versa. However, it is clear that in some way we convey the intended meaning of our systems, only is not done syntactically. How is it done? This a good question.

Coming back to the issue about negation, it usually is understood as representing falsity of a proposition, in mathematical propositions. However, it should represent truth in the system for refutability, because any provable negated formula is false and the proposition being negated must be true. As the systems are syntactically indistinguishable – it is only a matter of convention to say that "$\wedge$" means "and" etc. –, we can't say without further ado that negation represents falsity more than truth, at least not without making explicit our intentions with the proposed system: preservation of truth or preservation of falsity. Nonetheless, someone could claim that when we examine mathematical assertions we are certainly dealing with truth. That is correct, but we could then ask: apart of using formal systems to capture the truth, if we observe that they could capture falsity, what else would allow us to distinguish truth and falsity?

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