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RESEARCH

Conformally invariant skew curves for the total skew curvature on surfaces of $\mathbb{R}^3$

F. Narvaez[1,2*†] and R. Garcia[1,2†]

*Correspondence: lfnarvaez@unicauca.edu.co
†F. Narvaez and R. Garcia have contributed equally to this work.
[1]Departamento de Matemática, Universidad Del Cauca, Popayán, Cauca, Colombia
Full list of author information is available at the end of the article

Abstract

In differential geometry, curvature-based functionals, such as the total Gaussian curvature, the Willmore energy, and the total geodesic torsion, play a central role in both theoretical investigations and practical applications. In this paper, we study geometric properties of the extremal curves for the next functional

$$\mathfrak{F} := \int \sqrt{H^2 - K} ds = \int \frac{(k_1 - k_2)}{2} ds,$$

where ds is the arc element on S and k_1, k_2 are the principal curvatures. First, we establish that a necessary and sufficient condition for a surface to be a Dupin cyclide is that its lines of curvature and the extremal curves of functional $\mathfrak{F}$ intersect at a constant angle. Secondly, we demonstrate that the extremal curves of the functional $\mathfrak{F}$ are invariant under inversion. Finally, we show that the determination of functional extremal curves of $\mathfrak{F}$ for any cone, general cylinder, and surfaces of revolution can be reduced to quadratures.

Keywords: Skew curves, Loxodromic curves, Geodesic curves, Conformal transformations, Euler–Lagrange equation, Dupin cyclides

Mathematics Subject Classification: 53A05, 49K15, 34A05

1 Introduction

In differential geometry, there are several functionals that have been extensively studied. Some examples include:

- The length $\mathcal{L} = \int ds$ where ds is the element of arc length; see [3,24].
- The total geodesic torsion $\mathcal{T}_g = \int \tau_g ds$, where τ_g is the geodesic torsion; see [30].
- The total normal curvature $\mathcal{N}_r = \int k_n^r ds$, where k_n is the normal curvature; see [1].
- The total Gaussian curvature $\mathcal{G} = \int K dA$, where K is the Gaussian curvature, is a topological invariant; see [3].
- The total mean curvature $\mathcal{H} = \int H dA$, where H is the mean curvature, depends on the extrinsic geometry of the surface; see [24, Chapter 6].
- The Willmore energy $\mathcal{W} = \int (H^2 - K) dA$; see [41,42].

Curves or surfaces (depending on the functional being studied) that nullify the first variation of these functionals are of great importance, both in mathematics and in applied fields. It is worth noting that the last three functionals mentioned above depend solely on the principal curvatures, Curvature-based energies play an important role in the description of both physical and non-physical systems [37].

In [1] the authors investigate a variational problem closely related to the bending energy of curves on surfaces in three-dimensional real space forms. They derive the first variation formula and obtain the corresponding Euler–Lagrange equations for energies that depend on normal curvature. They look at cases when the Euler-Lagrange equations are satisfied by special curves, namely, geodesics, asymptotic lines and curvature lines. They also obtain rigidity results under certain geometric conditions.

Building on these variational approaches, further studies have considered broader classes of curvature-dependent energies. In [16], the authors examine the critical curves of energy functionals that depend on geodesic curvature, normal curvature, and geodesic torsion of curves on surfaces. They use variational methods to derive and analyze the corresponding Euler–Lagrange equations, explore several examples, and consider some possible applications.

A particularly notable example is the Willmore energy, which has been extensively studied due to its profound geometric and physical significance. This functional can also be written as

$$\mathcal{W} = \int \left(H^2 - K\right)\, dA = \int \frac{(k_1 - k_2)^2}{4}\, dA.$$

where H and K are the mean curvature and Gaussian curvature of the surface, respectively, and k_1 and k_2 denote the principal curvatures of the surface. As shown in [41], the Willmore energy is invariant under the conformal transformations of Euclidean space.

Motivated by the conformal invariance of the Willmore energy, by E. Kasner's work on natural families of curves [19,20], and by the contributions of A. Santaló to variational problems on surfaces [30–32], in the present work, we determine the geometric and dynamic properties of the curves that nullify the first variation of the functional:

$$\mathfrak{F} = \int \sqrt{H^2 - K} ds = \int \left(\frac{k_1 - k_2}{2}\right) ds. \tag{1}$$

The function in the integrand, which depends solely on the point on the surface, $\sqrt{H^2 - K}$, is known as the *skew curvature*; see [36,39]. Moreover, since at an umbilic point the principal curvatures are equals, the quantity $\frac{(k_1-k_2)}{2}$ precisely measures the local deviation from umbilicity. In this sense, the functional $\mathfrak{F}$ can be interpreted as the *total umbilic defect* along a curve.

This work is organized as follows. In Sect. 2 we recall the classical facts about inversion, loxodromic curves, and variational problems of curves on surfaces.

In Sect. 3, we determine the Euler–Lagrange equations for skew curves in the symmetric Weierstrass form, obtaining a second-order differential equation that characterizes these curves.

In Sect. 4 we show that the skew curves are invariant under inversions.

In Sect. 5 we present various examples of global behavior of the skew curves on surfaces of revolution.

In Sect. 6 a characterization of Dupin cyclides in terms of skew curves is obtained.

Section 7 characterizes Dupin cyclides using skew curves and D-curves (Darboux curves; [30]). For consistency, the term D-curve is used throughout this paper.

Finally, in Sect. 8 other examples of skew curves in cylinders and cones are analyzed.

2 Preliminaries

In this section, we provide some fundamental definitions related to differential geometry and the calculus of variations. For the differential geometry part, we follow the texts of [3,35]. For the calculus of variations, we refer to [11,28,29,40].

2.1 Inverse surface

Consider a surface $\widetilde{S}$ obtained under inversion of a given surface S of $\mathbb{R}^3$. Let the center of inversion be taken as the origin. Then, if c is the radius of inversion, the position vector $\widetilde{x}$ of a point on the inverted surface corresponding to the point x on S has a magnitude of $\frac{c^2}{\sqrt{\langle x,x\rangle}}$ and is therefore given by:

$$\widetilde{x} = \frac{c^2}{|x|^2}x, \tag{2}$$

since,

$$\begin{aligned}\widetilde{x}_u &= \frac{c^2}{\langle x, x\rangle}x_u - \frac{2c^2\langle x, x_u\rangle x}{\langle x, x\rangle^2},\\ \widetilde{x}_v &= \frac{c^2}{\langle x, x\rangle}x_v - \frac{2c^2\langle x, x_v\rangle x}{\langle x, x\rangle^2}.\end{aligned}$$

Therefore, the first fundamental form is given by the following:

$$\widetilde{E} = c^4\frac{E}{\langle x, x\rangle^2}, \qquad \widetilde{F} = c^4\frac{F}{\langle x, x\rangle^2}, \qquad \widetilde{G} = c^4\frac{G}{\langle x, x\rangle^2}. \tag{3}$$

Let N be the unit normal vector of S. Assuming $p = \langle x, N\rangle$, the coefficients of the second fundamental form are given by:

$$\begin{aligned}\widetilde{e} &= -c^2\frac{e}{\langle x, x\rangle} - \frac{2c^2pE}{\langle x, x\rangle^2},\\ \widetilde{f} &= -c^2\frac{f}{\langle x, x\rangle} - \frac{2c^2pF}{\langle x, x\rangle^2},\\ \widetilde{g} &= -c^2\frac{g}{\langle x, x\rangle} - \frac{2c^2pG}{\langle x, x\rangle^2}.\end{aligned} \tag{4}$$

Note that the next equalities are valid:

$$\begin{aligned}\widetilde{g}\widetilde{E} - \widetilde{e}\widetilde{G} &= -\frac{c^6}{\langle x, x\rangle^3}(gE - eG),\\ \widetilde{E}\widetilde{f} - \widetilde{e}\widetilde{F} &= -\frac{c^6}{\langle x, x\rangle^3}(Ef - Fe),\\ \widetilde{F}\widetilde{g} - \widetilde{G}\widetilde{f} &= -\frac{c^6}{\langle x, x\rangle^3}(Fg - Gf).\end{aligned} \tag{5}$$

It is worth recalling that the differential equation of the lines of curvature on a surface $\widetilde{S}$ is given by

$$(\widetilde{E}\widetilde{f} - \widetilde{F}\widetilde{e})u'^2 + (\widetilde{E}\widetilde{g} - \widetilde{G}\widetilde{e})u'v' + (\widetilde{F}\widetilde{g} - \widetilde{G}\widetilde{f})v'^2 = 0. \tag{6}$$

Substituting the expressions from equation (5) into the differential equation (6), and simplifying the common factor $-\frac{c^6}{\langle x,x\rangle^3}$, we observe that the differential equation of the curvature lines in $\widetilde{S}$ coincides with that of the curvature lines in S. Therefore, the curvature lines are preserved under inversion.

The principal curvatures of $\widetilde{S}$ in a principal chart are given by:

$$\widetilde{k}_1 = \frac{\widetilde{e}}{\widetilde{E}} = -\frac{\langle x,x\rangle k_1}{c^2} - \frac{2p}{c^2}, \quad \widetilde{k}_2 = \frac{\widetilde{g}}{\widetilde{G}} = -\frac{\langle x,x\rangle k_2}{c^2} - \frac{2p}{c^2}. \tag{7}$$

In this work, it is more advantageous to examine surfaces that are parametrized by principal curvature lines. Thus, we will consider S to represent this surface throughout the text. Next, we define the loxodromic curves and find the differential equation that characterizes this family.

Definition 1 Let (u, v) be an orthogonal system of curves on a surface S. We refer to the curves that intersect the coordinate curves $u = const$ and $v = const$ at a constant angle as loxodromic curves of the (u, v) system.

To find the differential equation of the loxodromic curves, consider (u, v) as an orthogonal system of curves on the surface S. Denoting θ as the angle that a curve $u = u(s)$, $v = v(s)$ makes with the curve $v = const$, we have:

$$tan^2(\theta) = \frac{Gv'^2}{Eu'^2}.$$

If θ is constant, then the derivative must be zero, that is,

$$(EG_u - E_uG)u'^2v' + (EG_v - E_vG)u'v'^2 - 2EG(u'v'' - u''v')u'v' = 0. \tag{8}$$

Note that this last equation is a second-order equation, like the differential equation of geodesic curves, D-curves, and other curves of geometric and physical significance; see [22].

Remark 1 When the coordinate curves (u, v) are the principal lines on the triaxial ellipsoid, the loxodromic curves were studied in [12]. In the torus of revolution the loxodromic curves are D-curves [30]. They can be dense in the ellipsoid and in the torus.

2.2 The Weierstrass equation

The calculus of variations has a well-defined origin. In June 1696, John Bernoulli issued a challenge to the world's greatest mathematicians to solve a new problem.

Given two points A and B in a vertical plane, determine the path AMB that a movable point M, subject to its weight, must take from A to B in the shortest possible time.

Solutions were submitted by Gottfried Wilhelm Leibniz, Isaac Newton, John Bernoulli, James Bernoulli, and Guillaume l'Hôpital. The curve obtained as a solution to the problem mentioned above is called the *brachistochrone*.

The brachistochrone is one of many problems in which we wish to determine a function $y(x)$, such that

$$y(a) = A, \qquad y(b) = B \tag{9}$$

and that minimizes or maximizes the integral

$$J[y(x)] = \int_a^b \Phi(x, y(x), y'(x))dx. \tag{10}$$

One of the pioneering contributions of Leonhard Euler was the development of a systematic method for solving problems of this type.

Theorem 1 *For the functional* (10)*, defined on the set of all functions* $y(x)$ *having a continuous first derivative and satisfying the boundary conditions* (9)*, to achieve an extremal on the given function* $y(x)$*, it is necessary that the function satisfies the Euler-Lagrange equation*

$$\Phi_y - \frac{d}{dx}\Phi_{y'} = 0.$$

The integral curves of Euler's equation are called extremals (Lagrangian curves).

Up to this point, our study has been limited to curves that can be expressed as $v = f(u)$, which is an overly restrictive assumption for most geometric problems. In several problems, it is more convenient and sometimes necessary to utilize a parametric representation of the curves.

$$u = \phi(t), \qquad v = \psi(t) \tag{11}$$

where the functions $\phi(t)$, $\psi(t)$ are continuous and have piecewise continuous derivatives, and $\phi'(t)^2 + \psi'(t)^2 \neq 0$. Suppose we have the functional

$$\int_a^b \Phi(u, v, u', v')dt \tag{12}$$

where $u'(t) = \frac{du}{dt}$, $v'(t) = \frac{dv}{dt}$. In order that the values of the functional (12) should depend on the curve and not on its parametrization, which can be accomplished in various ways, it is necessary and sufficient that the integrand not contain explicitly the parameter t and that it be positive homogeneous of first degree in the arguments u', v':

$$\Phi(u, v, ku', kv') = k\Phi(u, v, u', v'), \quad k > 0.$$

For example, in the functional

$$\int \sqrt{u'^2 + v'^2}dt$$

the integrand is positive homogeneous of first degree. In one sense, homogeneous problems are easy to solve. Any continuously differentiable solution must satisfy the two Euler-Lagrange equations:

$$\frac{\partial \Phi}{\partial u} - \frac{d}{dt}\left(\frac{\partial \Phi}{\partial u'}\right) = 0,$$

$$\frac{\partial \Phi}{\partial v} - \frac{d}{dt}\left(\frac{\partial \Phi}{\partial v'}\right) = 0.$$

However, these two equations cannot be entirely independent since the original non-parametric problem had only a single Euler-Lagrange equation. Weierstrass used the fact that the two Euler-Lagrange equations are not independent to argue that they can be replaced by a single equation that is symmetric in u and v.

Theorem 2 (Weierstrass's Symmetric Form) *For the functional*

$$\int \Phi(u, v, u', v')dt$$

attain an extremum on the given curve $(u(t), v(t))$ *it is necessary that the curve satisfy the Weierstrass's symmetric form of the Euler–Lagrange equation for the parametric problem*

$$\Phi_1(u'v'' - u''v') + \Phi_{uv'} - \Phi_{u'v} = 0, \tag{13}$$

where $\Phi_1 = -\frac{\Phi_{u'v'}}{u'v'} = \frac{\Phi_{u'u'}}{v'^2} = \frac{\Phi_{v'v'}}{u'^2}$.

(see, e.g., [11], pp. 258–259; [23], pp. 178–184).

3 Differential equation of skew curves

Before we explore skew curves, let's take a moment to recall the classical derivation of geodesics as extremals of the arc-length functional. On a surface, geodesic curves can be defined as curves whose normal vector is parallel to the surface normal at every point. Alternatively, we can derive them using the calculus of variations by examining the arc-length functional.

$$\mathcal{L} = \int ds = \int \sqrt{Eu'^2 + 2Fu'v' + Gv'^2}dt.$$

It is well known that every regular surface admits an isothermal parametrization. In such a parametrization, the geodesic functional takes the simplified form

$$\mathcal{L} = \int \Omega(u, v)\sqrt{u'^2 + v'^2}\,dt,$$

where $\Omega(u, v)$ is a conformal factor depending on the coordinates. In certain cases, employing this parametrization facilitates the derivation of a first integral of the geodesic equations, as occurs, for example, with Liouville surfaces, see [3, Page 315.].

In many geometric problems, choosing an appropriate parametrization of the surface facilitates the acquisition of the results. In this work, we will assume that the surface S is parametrized by curvature lines. Consequently, the coefficients of the first and second fundamental forms satisfy $f = F = 0$. In this parametrization, our functional is given by:

$$\begin{aligned}\mathfrak{F} &= \int \sqrt{H^2 - K}\,ds = \frac{1}{2}\int (k_1 - k_2)\,ds = \frac{1}{2}\int \left(\frac{e(u,v)}{E(u,v)} - \frac{g(u,v)}{G(u,v)}\right) ds\\ &= \frac{1}{2}\int \left(\frac{e(u,v)}{E(u,v)} - \frac{g(u,v)}{G(u,v)}\right)\sqrt{E(u,v)u'^2 + G(u,v)v'^2}\,dt.\end{aligned}$$

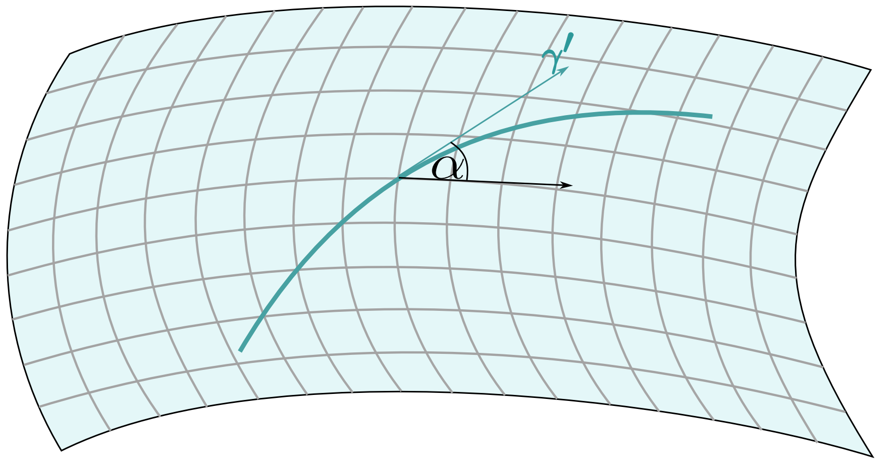

Fig. 1 The angle α between a skew curve γ and the principal direction $\frac{\partial}{\partial u}$

Throughout this work, we refer to the extremal curves of this functional as **skew curves**. Geometrically speaking, skew curves can be understood as geodesics with a conformal factor given by $(k_1 - k_2)$. In this case, using the Mainardi-Codazzi compatibility equations

$$e_v = \frac{E_v}{2}\left(\frac{e}{E} + \frac{g}{G}\right), \quad g_u = \frac{G_u}{2}\left(\frac{e}{E} + \frac{g}{G}\right),$$

the Weierstrass differential equation (13) for the extrema curves is given by:

$$L(v''u' - u''v') + Au'^3 + Bu'^2v' + Cv'^2u' + Zv'^3 = 0, \tag{14}$$

where,

$$\begin{aligned}
L &= -2G^2E^2(Eg - Ge),\\
A &= 2E^4(Gg_v - G_vg),\\
B &= -EG\left[-2G^2(Ee_u - E_ue) + (Eg - Ge)(EG_u - E_uG)\right],\\
C &= EG\left[2E^2(Gg_v - G_vg) - (Eg - Ge)(EG_v - E_vG)\right],\\
Z &= 2G^4(Ee_u - E_ue).
\end{aligned}$$

A classical technique in the calculus of variations involves introducing a new variable, in this case the angular variable, defined as the angle an extremal curve makes with the coordinate curve $u = const$, see Fig. 1. In the following result, we introduce this angular variable and derive a differential equation that can be interpreted as an analogue of the Liouville differential equation for geodesic curves on surfaces (see [3, Page 163.]).

Theorem 3 *Let (u, v) be a principal chart and γ be a skew curve parametrized by the arc length. Let α be the angle between γ and the principal direction $\frac{\partial}{\partial u}$. Then, the angle α satisfies the following differential equation:*

$$\alpha' = -\sin(\alpha)\frac{\frac{\partial k_1}{\partial u}}{\sqrt{E}(k_1 - k_2)} - \cos(\alpha)\frac{\frac{\partial k_2}{\partial v}}{\sqrt{G}(k_1 - k_2)}. \tag{15}$$

Proof Let (u, v) be a principal chart, then the principal curvatures are given by

$$k_1 = \frac{e}{E}, \qquad k_2 = \frac{g}{G}. \tag{16}$$

Replacing these values in the differential equation of skew curves (14) and carrying out simplification, we arrive at the next differential equation

$$L(u'v'' - u''v') + Au'^3 + Bu'^2 v' + Cu'v'^2 + Zv'^3 = 0. \tag{17}$$

The coefficients of this last differential equation can be expressed in terms of the coefficients of the first fundamental form, the principal curvatures, and their derivatives

$$\begin{aligned}
L &= 2EG(k_1 - k_2),\\
A &= 2E^2 \frac{\partial k_2}{\partial v},\\
B &= 2EG\frac{\partial k_1}{\partial u} + (k_1 - k_2)(EG_u - E_u G),\\
C &= 2EG\frac{\partial k_2}{\partial v} + (k_1 - k_2)(EG_v - E_v G),\\
Z &= 2G^2 \frac{\partial k_1}{\partial u}.
\end{aligned} \tag{18}$$

Let α be the angle between a skew curve and the principal direction $\frac{\partial}{\partial u}$. Then

$$u' = \frac{\cos(\alpha)}{\sqrt{E}}, \qquad v' = \frac{\sin(\alpha)}{\sqrt{G}},$$

calculating the second-order derivatives of the previous expressions

$$\begin{aligned}
u'' &= -\frac{\sin(\alpha)\alpha'}{\sqrt{E}} - \frac{1}{2}\frac{\cos(\alpha)\left[E_u\sqrt{G}\cos(\alpha) + E_v\sqrt{E}\sin(\alpha)\right]}{E^2\sqrt{G}},\\
v'' &= \frac{\cos(\alpha)\alpha'}{\sqrt{G}} - \frac{1}{2}\frac{\sin(\alpha)\left[G_u\sqrt{G}\cos(\alpha) + G_v\sqrt{E}\sin(\alpha)\right]}{G^2\sqrt{E}}.
\end{aligned} \tag{19}$$

Substituting these values into the differential equation for skew curves (17), and after simplification, we obtain

$$EG(k_1 - k_2)\alpha' + E\sqrt{G}\frac{\partial k_2}{\partial v}\cos(\alpha) + G\sqrt{E}\frac{\partial k_1}{\partial u}\sin(\alpha) = 0.$$

From the previous equation, we solve for the α'

$$\alpha' = -\sin(\alpha)\frac{\frac{\partial k_1}{\partial u}}{\sqrt{E}(k_1 - k_2)} - \cos(\alpha)\frac{\frac{\partial k_2}{\partial v}}{\sqrt{G}(k_1 - k_2)}.$$

This completes the proof of the theorem. □

Remark 2 The equation (15) can be rewritten as follows:

$$2\frac{d\alpha}{ds_c} = -\sin(\alpha)\theta_1 - \cos(\alpha)\theta_2, \tag{20}$$

where

$$ds_c = \frac{(k_1-k_2)}{2}ds, \tag{21}$$

is the differential of α with respect to the conformal arc length s_c, and θ_i are the conformal principal curvatures defined by

$$\theta_1 = \frac{4}{\sqrt{E}(k_1-k_2)^2}\frac{\partial k_1}{\partial u}, \quad \theta_2 = \frac{4}{\sqrt{G}(k_1-k_2)^2}\frac{\partial k_2}{\partial v}. \tag{22}$$

We emphasize equation (20) because all the quantities appearing in it are well known to be invariant under inversion (see [13,14,25]). In [14], the differential equation governing the D-curves is reformulated, producing an expression that involves the quantities (21) and (22), analogous to the equation (20) obtained here for the skew curves.

4 Skew curves are invariant under inversion

It is well known that the only conformal maps in a Euclidean space of dimension greater than 2 are those obtained through similarities and inversions (reflections) in spheres. Several examples of curves have been studied that preserve their geometric structure, or differential equations that remain invariant, under conformal transformations (see [2,18, 20]). According to Weatherburn (see [38, Page 163]), one of the most important properties of inverse surfaces is the preservation of curvature lines. In [34], the author shows that the D-curves are invariant under inversion. The following theorem demonstrates that the skew curves are also preserved under this transformation.

Theorem 4 *The skew curves are preserved under inversion.*

Proof Let $\widetilde{S}$ be a surface parametrized by principal curvature lines obtained through an inversion of the surface S. Taking into account that the differential equation of skew curves on $\widetilde{S}$ is given by

$$L(v''u' - u''v') + Au'^3 + Bu'^2v' + Cv'^2u' + Zv'^3 = 0. \tag{23}$$

The coefficients of the equation are written in terms of the first and second fundamental forms as follows

$$\begin{aligned}
L &= 2\widetilde{E}\widetilde{G}(\widetilde{k}_1 - \widetilde{k}_2),\\
A &= 2\widetilde{E}^2\frac{\partial \widetilde{k}_2}{\partial v},\\
B &= 2\widetilde{E}\widetilde{G}\frac{\partial \widetilde{k}_1}{\partial u} + (\widetilde{k}_1 - \widetilde{k}_2)(\widetilde{E}\widetilde{G}_u - \widetilde{E}_u\widetilde{G}),\\
C &= 2\widetilde{E}\widetilde{G}\frac{\partial \widetilde{k}_2}{\partial v} + (\widetilde{k}_1 - \widetilde{k}_2)(\widetilde{E}\widetilde{G}_v - \widetilde{E}_v\widetilde{G}),\\
Z &= 2\widetilde{G}^2\frac{\partial \widetilde{k}_1}{\partial u}.
\end{aligned} \tag{24}$$

Utilizing the values of $\widetilde{E}, \widetilde{G}, \widetilde{e}, \widetilde{g}, \widetilde{k}_2$ and $\widetilde{k}_1$ provided in (3), (4) and (7), we obtain

$$\widetilde{L} = 2\widetilde{E}\widetilde{G}(\widetilde{k}_1 - \widetilde{k}_2) = \left(-\frac{c^6}{\langle x, x\rangle^3}\right)[2EG(k_1 - k_2)]. \tag{25}$$

To determine the coefficient Z, let us observe that

$$\widetilde{k}_1 = -\frac{\langle x, x\rangle}{c^2}k_1 - 2\frac{\langle x, N\rangle}{c^2}.$$

Differentiating this expression with respect to the variable u it follows that

$$\frac{\partial \widetilde{k}_1}{\partial u} = -\frac{1}{c^2}\left(2\langle x, x_u\rangle k_1 + \frac{\partial k_1}{\partial u}\langle x, x\rangle\right) - 2\frac{\langle x, N_u\rangle}{c^2}. \tag{26}$$

Using the Olinde Rodrigues theorem (see [3, Page 147]) we have that

$$\langle x, N_u\rangle = -k_1\langle x, x_u\rangle.$$

Substituting this last equality into equation (26) we obtain

$$\frac{\partial \widetilde{k}_1}{\partial u} = -\frac{\partial k_1}{\partial u}\frac{\langle x, x\rangle}{c^2}.$$

Therefore, the coefficient Z is given by

$$Z = \left(-\frac{c^6}{\langle x, x\rangle^3}\right)(2G^2)\frac{\partial k_1}{\partial u}.$$

The remaining coefficients can be calculated analogously, finding for them the following expressions

$$A = 2E^2\frac{\partial k_2}{\partial v}\left(-\frac{c^6}{\langle x, x\rangle^3}\right),$$
$$B = \left[2EG\frac{\partial k_1}{\partial u} + (k_1 - k_2)(EG_u - E_uG)\right]\left(-\frac{c^6}{\langle x, x\rangle^3}\right),$$
$$C = \left[2EG\frac{\partial k_2}{\partial v} + (k_1 - k_2)(EG_v - E_vG)\right]\left(-\frac{c^6}{\langle x, x\rangle^3}\right).$$

Substituting these values into the differential equation of skew curves on the surface $\widetilde{S}$ and simplifying the factor $\left(-\frac{c^6}{\langle x,x\rangle^3}\right)$, we observe that the differential equation of skew curves in $\widetilde{S}$ is the same as in S, indicating that the skew curves invert into skew curves; see Fig. 2.

□

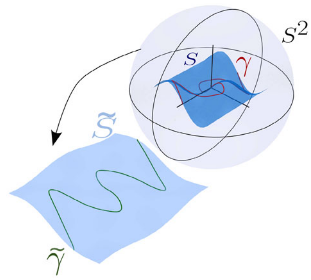

Fig. 2 The surface outside the sphere is obtained through an inversion of the blue surface. The curve on the surface inside the sphere, shown in red, is a skew curve on this surface and will be inverted into a skew curve on the exterior surface of the sphere

5 Skew curves on surfaces of revolution

The exploration of geodesics on surfaces of revolution is a traditional subject in differential geometry. In a surface of revolution, the second-order differential equation of geodesic curves contains a first integral called the Clairaut relation. The existence of this relation provides valuable insight into the geometric and dynamical properties of geodesics on surfaces of revolution (see Chapter 5, p. 132 of [10]). Let us consider a surface of revolution given by the following parametric equations.

$$X(u,v) = (u\cos(v), u\sin(v), \rho(u)), \tag{27}$$

where $\rho(u)$ is the function that gives the radius of the parallels according to the height u. Computing the coefficients of the first and second fundamental forms we get

$$E = 1 + \rho'^2, \qquad e = \frac{\rho''}{\sqrt{1+\rho'^2}}, \tag{28}$$

$$F = 0, \qquad f = 0, \tag{29}$$

$$G = u^2, \qquad g = \frac{u\rho'(u)}{\sqrt{1+\rho'^2}}. \tag{30}$$

Here, for the sake of convenience, we omit the argument of the function ρ and denote the derivative of the function ρ with respect to the variable u using an accent mark. Therefore,

$$\begin{aligned} E_u &= 2\rho'\rho'', & E_v &= 0,\\ G_u &= 2u, & G_v &= 0,\\ e_u &= \frac{\rho'''(1+\rho'^2) - \rho'\rho''^2}{(1+\rho'^2)^{\frac{3}{2}}}, & e_v &= 0,\\ g_u &= \frac{\rho'^3 + u\rho'' + \rho'}{(1+\rho'^2)^{\frac{3}{2}}}, & g_v &= 0. \end{aligned}$$

Theorem 5 *Let S be a surface of revolution parametrized by equation (27). Then the function*

$$J(u,v,\alpha) = u\,(k_2 - k_1)\sin(\alpha) \tag{31}$$

is a first integral for skew curves on S.

Proof The equation (15) for the skew curves on S satisfies the following system of differential equations

$$\begin{aligned} u' &= \frac{1}{\sqrt{E}}\cos(\alpha), \\ v' &= \frac{1}{\sqrt{G}}\sin(\alpha), \\ \alpha' &= -\sin(\alpha)\frac{\frac{\partial k_1}{\partial u}}{\sqrt{E}(k_1-k_2)}, \end{aligned} \tag{32}$$

hence

$$\frac{d\alpha}{du} = -\sin(\alpha)\frac{\frac{\partial k_1}{\partial u}}{(k_1-k_2)\cos(\alpha)}.$$

Since $\frac{\frac{\partial k_1}{\partial u}}{(k_1-k_2)}$ depends solely on u, we separate the variables and integrate, obtaining

$$\sin(\alpha)\left(\frac{\rho' + \rho'^3 - u\rho''}{(1+\rho'^2)^{\frac{3}{2}}}\right) = const,$$

which completes the proof. □

Corollary 1 *The only surfaces of revolution whose loxodromic curves are skew curves are those whose meridian section is either a circle or a line, that is, tori, cones, and cylinders of revolution. Among the tori of revolution, this includes surfaces generated by circles that intersect the axis of rotation; see Fig. 3.*

Proof By the previous theorem, along a skew curve,

$$\sin(\alpha)\left(\frac{\rho' + \rho'^3 - u\rho''}{(1+\rho'^2)^{\frac{3}{2}}}\right) = const. \tag{33}$$

Therefore, if a skew curve is a loxodromic curve, the function $\sin(\alpha)$ must be constant. Since the right-hand side of equation (33) is constant, it follows that

$$\left(\frac{\rho' + \rho'^3 - u\rho''}{(1+\rho'^2)^{\frac{3}{2}}}\right) = const_1.$$

By differentiating the previous equality with respect to the variable u, and after some simplifications, we obtain

$$(\rho'^2 + 1)\rho''' - 3\rho'\rho''^2 = 0. \tag{34}$$

This third-order differential equation is the differential equation for all circles in the plane, including lines as circles with infinite radius, which concludes the proof of the result. □

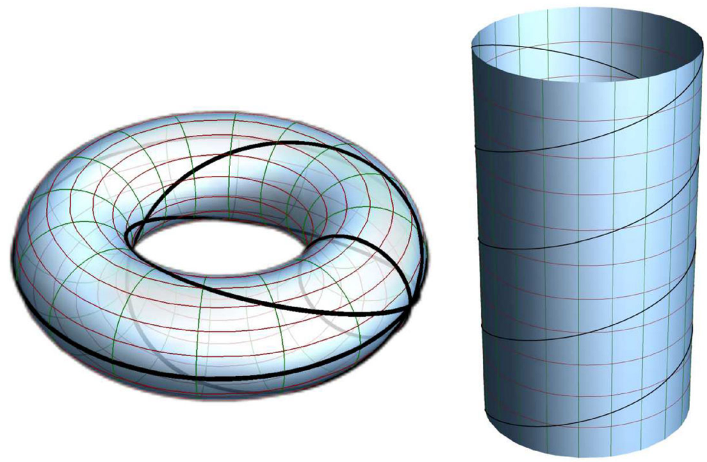

Fig. 3 On the torus, we have drawn two loxodromic curves that coincide with the Villarceau circles (see [15]) and also two circles which are principal lines. In the case of the cylinder of revolution, its loxodromic curves coincide with its geodesics

6 Characterization of Dupin's cyclides.

A Dupin cyclide in $\mathbb{R}^3$ is a surface whose lines of curvature consist of circles or straight lines. Classical examples include circular cylinders, circular cones, and tori of revolution. Indeed, every Dupin cyclide can be obtained from one of these three canonical surfaces via inversion in a sphere in $\mathbb{R}^3$. First introduced by Dupin in 1822, these surfaces have been studied extensively by differential geometers ever since. Since 1990, they have also received significant attention from researchers in computer-aided geometric design (CGAD) and related fields (see [5,8,33]).

Theorem 6 *A necessary and sufficient condition for both families of curvature lines to be skew curves is that the surface is a Dupin cyclide, including as particular cases the torus, the cone of revolution, and the cylinder of revolution.*

Proof Equation (14) is the differential equation of the skew curves in a principal chart (the curves $u = const$, $v = const$ are the curvature lines of the surface considered). For the curves $u = const$ to be skew curves, we have the condition:

$$2G^4(Ee_u - E_u e) = 0,$$

Since $F = 0$, we have $EG \neq 0$. Therefore,

$$(Ee_u - E_u e) = 0.$$

Similarly, for the curves $v = const$ to be skew curves, we have:

$$(Gg_v - G_v g) = 0.$$

In this parametrization $k_1 = \frac{e}{E}$, and thus $\frac{\partial k_1}{\partial u} = \frac{Ee_u - E_u e}{E^2}$. Therefore, the two previous conditions imply that

$$\frac{\partial k_1}{\partial u} = \frac{\partial k_2}{\partial v} = 0.$$

These last two equalities characterize the Dupin cyclides. □

From Corollary 1, we know that on the cone, cylinder, and the torus of revolution, the skew curves coincide with the loxodromic curves. Liouville's theorem states that any Dupin cyclide can be obtained by an inversion of a cone, cylinder, or torus of revolution. Combining these results with the fact that skew curves are invariant under inversion, we can readily show that on any Dupin cyclide, the skew curves coincide with loxodromic curves. The following theorem characterizes Dupin cyclides with this property.

Theorem 7 *The only surfaces for which skew curves intersect the lines of curvature at a constant angle are the Dupin cyclides; this class includes, as special cases, the torus, the right circular cone, and the right circular cylinder.*

Proof From the differential equation (14) that defines skew curves, together with the classical existence theorem for geodesic curves (see Chapter 3 of [4]), it follows that for every non-umbilical point $p \in S$ and for every tangent direction w at p there exists a skew curve passing through p whose initial tangent is w. By Theorem 3, we know that the skew curves satisfy the next differential equation

$$EG(k_1 - k_2)\alpha' + E\sqrt{G}\frac{\partial k_2}{\partial v}\cos(\alpha) + G\sqrt{E}\frac{\partial k_1}{\partial u}\sin(\alpha) = 0. \tag{35}$$

If $\alpha = \lambda$ is constant, then $\alpha' \equiv 0$ and (35) reduces to

$$E\sqrt{G}\,\frac{\partial k_2}{\partial v}\cos\lambda + G\sqrt{E}\,\frac{\partial k_1}{\partial u}\sin\lambda = 0.$$

Since, by the remark above, this relation must hold for every tangent direction at each non-umbilical point p, then

$$\frac{\partial k_1}{\partial u} = 0, \qquad \frac{\partial k_2}{\partial v} = 0,$$

over the entire surface, which, as we already know, is a characterization of Dupin cyclides. On the other hand, on a cyclide surface

$$\frac{\partial k_1}{\partial u} = 0, \qquad \frac{\partial k_2}{\partial v} = 0, \qquad k_1 \neq k_2,$$

With these two conditions, the equation (35) assumes the next form

$$EG\alpha' = 0.$$

Therefore, the general solution is given by $\alpha = const$, which concludes the proof. The global behavior of the loxodromic curves in the torus has been well studied; see Fig. 4. □

The following result is a weaker version of the previous theorem.

Theorem 8 *The necessary and sufficient condition for a surface to be a Dupin cyclide is that at least two distinct families of skew curves intersect its lines of curvature at a constant angle, and the two families are nowhere tangent to each other at their intersection points.*

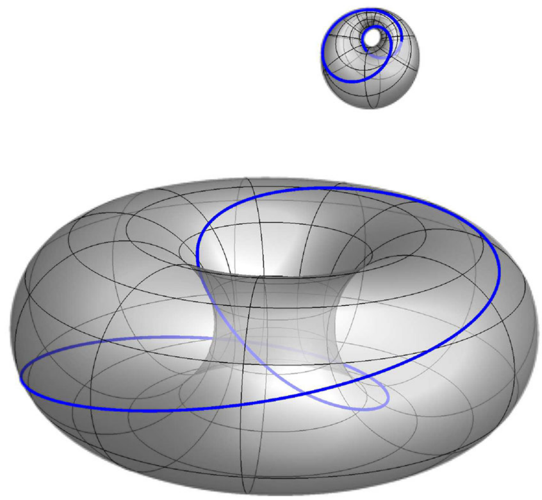

Fig. 4 A Dupin cyclide obtained under inversion of a torus of revolution. The blue curve on the torus is a skew curve, which is transformed into another skew curve on the resulting Dupin cyclide after the inversion

Proof Assume that at least two distinct families of skew curves intersect the lines of principal curvature of the surface at a constant angle. Let θ_1 and θ_2 denote the angles at which these two families intersect the principal curvature lines. Then, according to Equation (15), we have

$$\begin{aligned} -\sin(\theta_1)\frac{\frac{\partial k_1}{\partial u}}{\sqrt{E}(k_1-k_2)} - \cos(\theta_1)\frac{\frac{\partial k_2}{\partial v}}{\sqrt{G}(k_1-k_2)} &= 0, \\ -\sin(\theta_2)\frac{\frac{\partial k_1}{\partial u}}{\sqrt{E}(k_1-k_2)} - \cos(\theta_2)\frac{\frac{\partial k_2}{\partial v}}{\sqrt{G}(k_1-k_2)} &= 0, \end{aligned} \tag{36}$$

where, $\theta_1 \neq \theta_2 \neq 0$ and $\theta_1 \neq \theta_2 \neq \frac{\pi}{2}$. The previous system can be understood as a linear and homogeneous system in $\frac{\frac{\partial k_1}{\partial u}}{\sqrt{E}(k_1-k_2)}$ and $\frac{\frac{\partial k_2}{\partial v}}{\sqrt{G}(k_1-k_2)}$. Thus,

$$\begin{vmatrix} -\sin(\theta_1) & -\cos(\theta_1) \\ -\sin(\theta_2) & -\cos(\theta_2) \end{vmatrix} = \sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2) = \sin(\theta_1-\theta_2).$$

Since the two families are nowhere tangent at their intersection points, we have $\theta_1 - \theta_2 \neq k\pi$ for every $k \in \mathbb{Z}$, and therefore $\sin(\theta_1-\theta_2) \neq 0$. Consequently,

$$\frac{\frac{\partial k_1}{\partial u}}{\sqrt{E}(k_1-k_2)} = \frac{\frac{\partial k_2}{\partial v}}{\sqrt{G}(k_1-k_2)} = 0.$$

□

7 *D*-curves and skew curves

Among surface curves governed by second-order differential equations, geodesics represent the most prominent class. However, the literature also examines the *D*-curves, which are characterized as extremals of total geodesic torsion; that is, they correspond to critical points of the functional torsion. For classical results on *D*-curves, see [6,7,9,26,27,30,31,34]; for more recent work, see [13,14].

In a principal parametrization, the total geodesic torsion functional is given by

$$\mathcal{T}_g = \int \tau_g ds = \int \frac{(Ge - Eg)u'v'}{\sqrt{GE}(Eu'^2 + Gv'^2)} ds. \tag{37}$$

The differential equation of D-curves is given by

$$A(u'v'' - v'u'')u'v' + bu'^5 + cu'^3v'^2 + du'^2v'^3 + zv'^5 = 0, \tag{38}$$

where,

$$\begin{aligned}
A &= 6(Ge - gE)EG,\\
b &= 2EG(eE_u - Ee_u),\\
c &= 3(EG_u - GE_u)(Ge - gE) + 2G^2(eE_u - Ee_u),\\
d &= 3(EG_v - GE_v)(Ge - gE) + 2E^2(gG_v - Gg_v),\\
z &= 2EG(gG_v - Gg_v).
\end{aligned}$$

Note that the functional integrand (37) does not depend solely on the point on the surface; it also depends on the direction (u', v'). L. Santaló [30,31] demonstrated that on Dupin cyclides, D-curves are the loxodromic curves. By combining Santaló's theorem and Theorem 7, we deduce that on Dupin cyclides, the D-curves and the skew curves coincide. This is noteworthy because on Dupin cyclides, loxodromic curves constitute a natural family of curves; that is, there exists a function $\Omega(u, v)$, depending solely on the point (u, v), such that the loxodromic curves coincide with the extremal curves of the functional

$$\int \Omega(u, v) ds.$$

This property, however, does not always hold (see [17,21]).

With this context in mind, it is natural to ask on which surfaces the skew curves coincide with the D-curves.

Theorem 9 *The only surfaces for which the skew curves coincide with the D-curves are the Dupin cyclides.*

Proof From the differential equation of the skew curves (14), we will isolate the second-order term $(u'v'' - v'u'')$ and substitute it in the differential equation for the D-curves (38). This results in the following equation

$$\begin{aligned}
&2E^2G^2(e_uE - eE_u)u'^5 + 6E^4\,(g_vG - G_vg)\,u'^4v' + 8EG^3(e_uE - eE_u)u'^3v'^2 +\\
&+ 8E^3G(g_vG - G_vg)u'^2v'^3 + 6G^4\,(e_uE - eE_u)\,u'v'^4 + 2E^2G^2(g_vG - G_vg)v'^5 = 0.
\end{aligned}$$

This relation must be an identity with respect to u' and v', since fixing the point u, v must hold for any pair of values u' and v'. As we are in a parametrization by principal curvature lines with $EG \neq 0$; we have that

$$e_uE - eE_u = 0, \quad g_vG - G_vg = 0.$$

Since $k_1 = e/E$ and $k_2 = g/G$ it follows that $(k_1)_u = (k_2)_v = 0$. In a principal chart, this characterizes the Dupin cyclides, which completes the proof. □

8 On others cases of integration

We now exhibit two classical surfaces examples where the differential equations governing skew curves admit explicit integration by quadratures. First, we examine cylindrical surfaces, followed by the case of general conical surfaces. In each instance, we find a first integral.

8.1 General cylinder

Considering the cylinder in terms of its lines of curvature, that is, its circular cross-sections (for $v = const$) and its straight generatrices (for $u = const$), where u and v represent the arc lengths along each of these curves, we obtain:

$$\begin{aligned} E &= 1, & G &= 1, & F &= 0, \\ e &= k(u), & f &= 0, & g &= 0. \end{aligned}$$

where $k(u)$ is a curvature of right section. Assuming u as a parameter, with $v = v(u)$, and substituting into differential equation (14)

$$k'(u)v'(u) + v''(u)k(u) + v'(u)^3 k'(u) = 0.$$

This differential equation admits a first integral

$$1 + v'(u)^2 - C_1 k^2(u) v'(u)^2 = 0,$$

where C_1 is a first constant of integration. Hence we deduce

$$v = \pm \int \frac{1}{\sqrt{C_1 k^2(u) - 1}} + C_2,$$

where C_2 is a second constant of integration. Note that in the case of the revolution cylinder, where the curvature k is constant, the solution curves are given by $v = C_1 u + C_2$, and the corresponding skew curves are helices (see Fig. 3).

8.2 General cone

Theorem 10 *The skew curves on a cone free of umbilical points (without flat points) can be integrated by quadrature. The function*

$$I(u, v, \alpha) = \sin(\alpha) k_g, \tag{39}$$

is a first integral.

Proof The cone can be parametrized by $X(u, v) = v\lambda(u)$, where λ is a spherical curve such that $|\lambda| = 1$ and $|\lambda'| = 1$. The coefficients of the first fundamental form and the principal curvatures are given by

$$\begin{aligned} E &= v^2, & F &= 0, & G &= 1, \\ k_1(u, v) &= k_g(u), & f &= 0, & k_2 &= 0. \end{aligned}$$

Substituting these values into the differential equation for the skew curves (15), we obtain the following system:

$$\begin{aligned} u' &= \frac{\cos(\alpha)}{v}, \\ v' &= \sin(\alpha), \\ \alpha' &= -\sin(\alpha)\frac{\frac{\partial k_1}{\partial u}}{\sqrt{E}(k_1 - k_2)}. \end{aligned} \tag{40}$$

Thus,

$$-\frac{\cos(\alpha)}{\sin(\alpha)}d\alpha = \frac{\frac{\partial k_g}{\partial u}}{k_g}du.$$

Therefore,

$$\sin(\alpha)k_g = const. \tag{41}$$

This implies that

$$I(u, v, \alpha) = \sin(\alpha)k_g, \tag{42}$$

is a first integral for the differential equation of the skew curves on cones. □

Let us examine some consequences of this first integral. With the first integral and the first fundamental form of our cone,

$$ds^2 = v^2du^2 + dv^2,$$

we obtain that

$$v = A \exp\left(\pm \int_0^s (k_g^2 C - 1)^{-\frac{1}{2}} du\right), \tag{43}$$

are the solutions of the skew curves, where A and C are constants. The constant A is determined by fixing the initial point of the extremal curve, as for $s = 0$ it follows that $A = v(0)$. The constant C is determined based on the angle it makes with the generator of the cone at the initial point. Indeed, note that α is given by

$$\tan(\alpha) = \frac{v'}{v} = \pm\left(k_g^2 C - 1\right)^{-\frac{1}{2}}. \tag{44}$$

Recalling that k_g corresponds to the value of the generator passing through the initial point, we isolate C by squaring the equation, causing the double sign to disappear, and C is determined by a unique value. From equation (44), we deduce that a skew curve on a conical surface intersects all generators with the same value k_g from the directrix curve at a constant angle. Therefore, if one of these curves intersects the same generator multiple times, the angle of these intersections remains constant. Conversely, for a skew curve to intersect all generators at a constant angle, the curvature k_g must be constant, implying that the conical surface is a cone of revolution.

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The authors declare that they have no competing interests relevant to the content of this article.

Author details
[1]Departamento de Matemática, Universidad Del Cauca, Popayán, Cauca, Colombia, [2]Instituto de Matemática e Estatística, Universidade Federal de Goiás, Goiânia, Goiás, Brazil .

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