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Perfect matching cuts partitioning a graph into complementary subgraphs*

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Abstract

In PARTITION INTO COMPLEMENTARY SUBGRAPHS (COMP-SUB) we are given a graph $G = (V, E)$, and an edge set property Π, and asked whether G can be decomposed into two graphs, H and its complement $\overline{H}$, for some graph H, in such a way that the edge cut $[V(H), V(\overline{H})]$ satisfies the property Π. Motivated by previous work, we consider COMP-SUB(Π) when the property $\Pi = \mathscr{PM}$ specifies that the edge cut of the decomposition is a perfect matching. We prove that COMP-SUB$(\mathscr{PM})$ is GI-hard when the graph G is C_5-free or G is $\{C_{k\geq 7}, \overline{C}_{k\geq 7}\}$-free. On the other hand, we show that COMP-SUB$(\mathscr{PM})$ is polynomial-time solvable on $hole$-free graphs and on P_5-free graphs. Furthermore, we present characterizations of COMP-SUB$(\mathscr{PM})$ on chordal, distance-hereditary, and extended P_4-laden graphs.

Keywords: Graph partitioning, complementary subgraphs, perfect matching, matching cut, graph isomorphism.

Math. Subj. Class. (2020): 05C70, 05C85, 05C76, 05C51, 05C69

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1 Introduction

Finding graph partitions with some special properties has been a topic of extensive research. Several combinatorial problems can be viewed as partition problems, such as VERTEX COLORING and CLIQUE COVER. In addition, many graph classes, e.g. bipartite and split graphs, can also be defined through a partition of its vertex set. In particular, the class of *complementary prisms* [11] are defined over complementary parts. The *complementary prism* $G\overline{G}$ of a graph G arises from the disjoint union of a labeled graph G and its complement $\overline{G}$ by adding the edges of a perfect matching between vertices with same label in G and $\overline{G}$. Studies concerning the computational complexity of classical graph problems restricted to the class of complementary prisms graphs can be found in [4, 10].

We say that a graph $G = (V, E)$ is decomposed into two graphs G_1 and G_2 if $V(G)$ can be partitioned into V_1 and V_2, where $G[V_1] \simeq G_1$ and $G[V_2] \simeq G_2$. The edge cut $[V_1, V_2]$ is called the edge cut of this decomposition.

As a generalization of complementary prisms, Nascimento, Souza and Szwarcfiter [17] introduced the problem defined as follows.

PARTITION INTO COMPLEMENTARY SUBGRAPHS (COMP-SUB)

Instance: A graph $G = (V, E)$, and an edge set property Π.

Question: Can G be decomposed into two graphs, H and its complement $\overline{H}$, for some graph H, in such a way that the edge cut M of the decomposition satisfies the property Π?

For short, we abbreviate PARTITION INTO COMPLEMENTARY SUBGRAPHS with the edge set property Π as COMP-SUB(Π). We write $G \in$ COMP-SUB(Π) to denote that G is a yes-instance of COMP-SUB(Π) and we call $(H, \overline{H})$ as a *complementary decomposition* of G.

The COMP-SUB(Π) problem also finds motivation in parameterized complexity. Recognizing whether a graph has a complementary decomposition can be useful for solving problems in FPT-time, as pointed out in [17]. Nascimento, Souza and Szwarcfiter [17] considered the cases where the edge cut M is empty or induces a complete bipartite graph. They also presented some remarks when Π is a general edge set property. In particular, when M is empty, they make some links between COMP-SUB(Π) and the GRAPH ISOMORPHISM problem, from which they show that COMP-SUB(Π) is GI-hard.

It is known that the recognition of complementary prisms can be done in polynomial time [5]. This implies that, when the property Π is a perfect matching M between corresponding vertices in H and $\overline{H}$, the COMP-SUB(Π) problem is polynomial-time solvable. So, a natural question is the study of COMP-SUB(Π) when Π specifies that M is any perfect matching. In this context, two related problems arise: MATCHING CUT [13, 18] and PERFECT MATCHING CUT [12]. A *(perfect) matching cut* is a partition of vertices of a graph into two parts such that the set of edges crossing between the parts forms a (perfect) matching. Considering $\Pi = \mathscr{PM}$ as the property of being a perfect matching, COMP-SUB($\mathscr{PM}$) can be seen as a variant of PERFECT MATCHING CUT with the additional restriction that the two parts must induce complementary subgraphs. Note that studies regarding matchings satisfying particular constraints have received wide attention in the literature (c.f. [9, 14, 15, 20, 21]).

Motivated by Nascimento, Souza and Szwarcfiter [17], in this paper we deal with COMP-SUB(Π), when $\Pi = \mathscr{PM}$ considers M as a perfect matching. We show that

COMP-SUB($\mathcal{PM}$) is GI-hard when the graph G is C_5-free or G is $\{C_{k\geq 7}, \overline{C}_{k\geq 7}\}$-free. On the other hand, we present polynomial time algorithms able to solve COMP-SUB($\mathcal{PM}$) when the input graph G is *hole*-free or P_5-free. In addition, we characterize graphs $G \in$ COMP-SUB($\mathcal{PM}$) when G is chordal, distance-hereditary, or extended P_4-laden. Although extended P_4-laden graphs generalize cographs, we also show a simpler characterization for cographs.

The paper is organized as follows. Section 2 contains some fundamental concepts and an auxiliary result. Sections 3 and 4 contains our results on some *cycle*-free graphs and graphs with few P_4's, respectively. Further discussions are presented in Section 5.

2 Preliminaries

We consider only finite, simple, and undirected graphs, and we use standard terminology and notation. See [1] for graph-theoretic terms not defined here.

Let G be a graph. For a vertex $v \in V(G)$, we denote its *open neighborhood* by $N_G(v)$, and its *closed neighborhood*, denoted by $N_G[v] := N_G(v) \cup \{v\}$. For a set $U \subseteq V(G)$, let $N_G(U) = \bigcup_{v\in U} N_G(v) \setminus U$, and $N_G[U] = N_G(U) \cup U$.

The *degree* of a vertex $v \in V(G)$ on a set $U \subseteq V(G)$, denoted by $d_U(v)$, is $d_U(v) = |N_G(u) \cap U|$. If $U = V(G)$, we simply write $d_G(u)$. We say that $v \in V(G)$ is an *isolated* (resp. *universal*) vertex if $d_G(v) = 0$ (resp. $d_G(v) = |V(G)| - 1$). We define the *distance*, $\mathrm{dist}_G(u, v)$, between two vertices u and v of a graph G as the length of the shortest path between u and v.

The subgraph of G *induced* by U, denoted by $G[U]$, is the graph whose vertex set is U and whose edge set consists of all the edges in $E(G)$ that have both endvertices in U.

Let G be a graph. A set $U \subseteq V(G)$ is called a *clique* (resp. *independent set*) if the vertices in U are pairwise adjacent (resp. nonadjacent). We denote by K_n a *complete graph*, I_n an independent set, P_n a *path graph*, and C_n a *cycle graph* on n vertices. Let r be a positive integer. An *r-partite graph* is one whose vertex set can be partitioned into r subsets, in such a way that no edge has both ends in the same subset. An r-partite graph is *complete* if any two vertices in different subsets are adjacent. When r is not specified, we simply say *(complete) multipartite*. A *split graph* G is one whose vertex set admits a partition $V(G) = C \cup I$ into a clique C and an independent set I. The *complement* $\overline{G}$ of a graph G is the graph defined by $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

Let $P = v_1 v_2 \dots v_n$ be a path. We call $v_2, \dots, v_{n-1}$ as *inner vertices* of P. Two or more paths in a graph are *independent* if none of them contains an inner vertex of another. A graph G is *ℓ-connected* if any two of its vertices can be joined by ℓ independent paths. A 2-connected graph is called *biconnected*.

A vertex v in a graph G is a *cutvertex* or *cutpoint*, if $G\setminus\{v\}$ is disconnected. A maximal connected subgraph without a cutpoint is a *block*. The *block-cutpoint tree* of a graph G is a bipartite graph whose vertex set consists of the set of cutpoints of G and the set of blocks of G. A cutpoint is adjacent to a block whenever the cutpoint belongs to the block in G.

Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic*, denoted as $G \simeq H$, if and only if there is a bijection, called *isomorphism function*, $\varphi\colon V \to V'$ such that $uv \in E$ if and only if $\varphi(u)\varphi(v) \in E'$, for every $u, v \in V$. A graph G is *self-complementary* if $G \simeq \overline{G}$. The GRAPH ISOMORPHISM problem receives as input two graphs G and G' and asks whether $G \simeq G'$. We denote by GI the class of problems that admit a polynomial-time reduction to GRAPH ISOMORPHISM.

A problem Q is GI-*complete* if the two conditions are satisfied: (i) Q is a member of GI; and (ii) Q is GI-*hard*, that is, for every problem $Q' \in$ GI, Q' is polynomially reducible to Q.

We denote the set of positive integers $\{1, \dots, k\}$ by $[k]$. Let G and $G_1, \dots, G_k$ be graphs. We say that G is $\{G_1, \dots, G_k\}$-*free* if G does not contain G_i as an induced subgraph, for every $i \in [k]$.

A *module* of a graph is a set X of vertices such that for each vertex $x \notin X$, either every member of X is adjacent to x, or no member of X is adjacent to x [16].

Let G and H be two graphs such that $V(G) \cap V(H) = \emptyset$. The *disjoint union* of G and H, denoted by $G \cup H$, is the graph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. The *join* of G and H, denoted by $G + H$, is the graph with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H) \cup \{uv : u \in V(G) \text{ and } v \in V(H)\}$.

Let G be a graph and $\mathscr{C}$ a class of graphs. A set $S \subseteq V(G)$ is a $\mathscr{C}$-*modulator* if $G \setminus S$ belongs to $\mathscr{C}$. We define the *distance* of G to class $\mathscr{C}$ as the size of a minimum S which is a $\mathscr{C}$-modulator.

Let G be a graph that has a complementary decomposition (G_1, G_2) with perfect matching cut $M = \{u_1v_1, \dots, u_nv_n\}$, where $u_i \in V(G_1)$ and $v_i \in V(G_2)$, $i \in [n]$. We say that u_i (resp. v_i) is the *corresponding vertex* of v_i (resp. u_i), for every $i \in [n]$. For $X \subseteq V(G_1)$, we call $X^{G_2} = \{v_i \in V(G_2) : u_i \in X\}$ as the *corresponding set* of X over G_2. Similarly, for $X \subseteq V(G_2)$, we call $X^{G_1} = \{u_i \in V(G_1) : v_i \in X\}$ as the *corresponding set* of X over G_1.

Next, we present an auxiliary result, defined for COMP-SUB($\mathscr{PM}$) with a restriction on the graphs of the decomposition. A *cograph* is a P_4-free graph.

Lemma 2.1. *Let G be a graph. The problem of determining whether G can be decomposed into two graphs, G_1, and its complement G_2, such that G_1 is a cograph and the edge cut of the decomposition is a perfect matching, can be solved in polynomial time.*

Proof. Let $\mathscr{C}$ be the class of cographs and G a $2n$-vertex graph. Suppose that G is decomposable into complementary subgraphs G_1 and G_2, such that $G_1 \in \mathscr{C}$ and the edge cut M of the decomposition is a perfect matching.

Since $\mathscr{C}$ is closed under complement, we have that $G_2 \in \mathscr{C}$. Given that GRAPH ISOMORPHISM is linear-time solvable on cographs [7], we perform a brute force algorithm to check every *relevant* partition $V(G_1), V(G_2)$ of $V(G)$. For that, we propose Algorithm 1, explained in sequel.

Algorithm 1: PARTITION-INTO-COMPLEMENTARY-COGRAPHS(G)

Input: A graph G.
Output: Whether G admits a complementary decomposition such that the edge cut of the decomposition is a perfect matching.

forall $x_1, x_2, y_1, y_2 \in V(G)$ **do**
    $V(G_1) := N_G[\{x_1, x_2\}] \setminus \{y_1, y_2\}$
    $V(G_2) := V(G) \setminus V(G_1)$
    $M := \{xy \in E(G) : x \in V(G_1), y \in V(G_2)\}$
    if *M is a perfect matching* ***and*** *G_1 is a cograph* ***and*** *G_2 is a cograph* ***and*** $G_1 \simeq \overline{G}_2$ **then**
        return yes
return no

We know that a cograph is connected if and only if its complement is disconnected [8]. Consequently, if a complementary decomposition (G_1, G_2) exists, then either G_1 or G_2 is disconnected, say G_2. Then G_1 can be obtained by a join between the corresponding connected components of G_2. Thus, there exist two adjacent vertices $x_1, x_2 \in V(G_1)$, such that $N_{G_1}[\{x_1, x_2\}] = V(G_1)$. Furthermore, the edge set M of the decomposition implies that there exist $y_1, y_2 \in V(G_2)$ such that $x_1y_1, x_2y_2 \in M$.

By the above arguments, it is possible to find $V(G_1)$ by means of $N_G[\{x_1, x_2\}]$ except for two vertices $y_1, y_2 \in N_G[\{x_1, x_2\}]$ that must belong to $V(G_2)$. This way, $V(G_2)$ is obtained by $\{y_1, y_2\} \cup \{v \in V(G) : v \notin N_G[\{x_1, x_2\}]\}$. Once found $V(G_1), V(G_2)$, and M, we test whether M is a perfect matching and whether G_1 and G_2 are cographs. If so, we compute $\overline{G}_2$ and then we check isomorphism between G_1 and $\overline{G}_2$.

The correctness of the algorithm follows from the fact that all the possible relevant partitions (for the emergence of the cographs, if any) are considered.

Now, we show that Algorithm 1 runs in polynomial time.

For enumerating every 4-tuple of vertices $x_1, x_2, y_1, y_2 \in V(G)$ it is required $O(n^4)$ time. After, in $O(n+m)$ time we can check whether M is a perfect matching, as well as checking whether G_1 and G_2 are cographs. Finally, for computing $\overline{G}_2$ and checking isomorphism between G_1 and $\overline{G}_2$ is also required $O(n+m)$ time. Therefore, the running time of Algorithm 1 takes $O(n^5 + n^4m)$ time. □

3 Results on some C_k-free graphs

We begin by showing two hardness results, in Theorems 3.1 and 3.2.

Theorem 3.1. COMP-SUB($\mathcal{PM}$) *is* GI*-hard on* $\{C_{k\geq 7}, \overline{C}_{k\geq 7}\}$*-free graphs.*

Proof. Given that GRAPH ISOMORPHISM is GI-hard on split graphs [6], we show a polynomial-time reduction from such a problem to COMP-SUB($\mathcal{PM}$).

Note that a split graph is connected if and only if it does not contain isolated vertices. Therefore, we may assume that the instances of GRAPH ISOMORPHISM on split graphs are pairs of connected split graphs.

Let A and B be connected split graphs such that $|V(A)| = |V(B)| = n$, for some $n \geq 3$. From an instance (A, B) of GRAPH ISOMORPHISM, we construct an instance G of COMP-SUB($\mathcal{PM}$).

Let G arise from the disjoint union between A, $\overline{B}$, K_n, and I_n. Denote K_n by K and I_n by I. We make every vertex in $V(A)$ adjacent to every vertex in $V(K)$. Furthermore, we add an arbitrary perfect matching between $V(A)$ and $V(I)$ and between $V(K)$ and $V(\overline{B})$. An example of graph G follows in Figure 1. Aditionally, let $H_1 = G[V(A) \cup V(K)]$ and $H_2 = G[V(\overline{B}) \cup V(I)]$. Clearly, the construction can be done in polynomial time.

We first show that G is $\{C_{k\geq 7}, \overline{C}_{k\geq 7}\}$-free.

Claim 1. *Let G be the graph obtained from the construction. It holds that G is a $\{C_{k\geq 7}, \overline{C}_{k\geq 7}\}$-free graph.*

Proof of Claim 1. We prove that (I) G is $C_{k\geq 7}$-free, and (II) G is $\overline{C}_{k\geq 7}$-free.

(I) Suppose by contradiction that G contains a $C_{k\geq 7}$, denoted as C, as induced subgraph. We may assume that k is minimum.

By construction, H_1 and H_2 are split graphs and it is clear that H_1 and H_2 are $C_{\ell+4}$-free, for every $\ell \geq 0$. Then $V(C) \not\subseteq V(H_1)$ and $V(C) \not\subseteq V(H_2)$. So, we assume that

$V(C) \cap V(H_1) \neq \emptyset$ and $V(C) \cap V(H_2) \neq \emptyset$. Since I is a set of vertices with degree one in G, we have that $V(C) \cap I = \emptyset$. So, we may suppose that $V(C) \cap V(\overline{B}) \neq \emptyset$ and since C is a cycle, $|V(C) \cap V(\overline{B})| \geq 2$. Since $\overline{B}$ is split, we have that $|V(C) \cap V(\overline{B})| \leq 4$.

Since C is a cycle and K is a complete graph, C must contain exactly two vertices from K and no vertex of A. Then, $|V(C)| \geq 7$ implies that $|V(C) \cap V(\overline{B})| \geq 5$, a contradiction.

(II) Suppose by contradiction that G contains a $\overline{C}_{k\geq 7}$, denoted as D, as induced subgraph. Let $V(D) = \{d_1, \ldots, d_\ell\}$, for some $\ell \geq 7$, and $E(D) = \{d_i d_j : 1 \leq i < j \leq \ell\} \setminus (\{d_i d_{i+1} : 1 \leq i \leq \ell - 1\} \cup \{d_\ell d_1\})$.

By definition of D, $\{d_1, d_2, d_4, d_5\}$ induces a C_4. Then, since H_1 and H_2 are split graphs, $V(D) \not\subseteq V(H_1)$ and $V(D) \not\subseteq V(H_2)$. So, we assume that $V(D) \cap V(H_1) \neq \emptyset$ and $V(D) \cap V(H_2) \neq \emptyset$. Then, there exists $i, j \in [\ell]$ such that $d_i \in V(H_1)$, $d_j \in V(H_2)$ and $d_i d_j \in E(D)$.

Without loss of generality, suppose that $i = 1$. Since $\{d_1, d_3, d_5\}$ induces a K_3, we may assume that $\{d_1, d_3, d_5\} \subseteq V(H_1)$. Thus, $d_1 d_j \in E(D)$, for some $j \in \{4, 6, \ldots, \ell - 1\}$. Notice that, if $j = 4$ (resp. $j \geq 6$), then $\{d_1, d_4, d_6\}$ (resp. $\{d_1, d_3, d_j\}$) induces a K_3 which intersects both $V(H_1)$ and $V(H_2)$, a contradiction. Therefore G is $\overline{C}_{k\geq 7}$-free. □

In what follows, we prove that (A, B) is a *yes*-instance of GRAPH ISOMORPHISM if and only if G is a *yes*-instance of COMP-SUB($\mathscr{PM}$).

Suppose that $A \simeq B$. Since $I_n = \overline{K}_n$, $\overline{\overline{B}} \simeq A$, and there is no edge between a vertex in I and a vertex in $V(\overline{B})$, it is easy to see that H_1 and $\overline{H}_2$ are isomorphic. Therefore, G is a *yes*-instance of COMP-SUB($\mathscr{PM}$).

Let (V', V'') be a partition of $V(G)$ into complementary parts such that $[V', V'']$ is a perfect matching. Since I is a set of vertices with degree one in G and A is connected, it holds that either ($I \subset V'$ and $V(A) \subset V''$) or ($V(A) \subset V'$ and $I \subset V''$). Suppose that $V(A) \subset V'$. This implies that $V' = V(A) \cup K$ and $V'' = V(\overline{B}) \cup I$. Since $G[V']$ and $G[V'']$ are complementary, we have that $G[V'] \simeq \overline{G[V'']}$. Hence, due to the automorphism of universal vertices, it holds that $A \simeq B$. □

See in Figure 1 an example of the construction presented in Theorem 3.1.

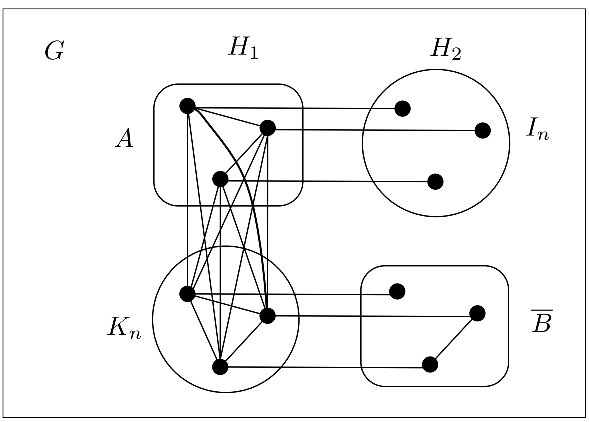

Figure 1: Graph G constructed for Theorem 3.1.

Our next result shows that COMP-SUB($\mathscr{PM}$) is also GI-hard on C_5-free graphs. Notice that the construction of G presented in Theorem 3.1 allows the existence of C_5 as an induced subgraph. For instance, if $\overline{B}$ contains an induced $P_3 = v_1v_2v_3$, the corresponding vertices of v_1 and v_3 (in K) union with v_1, v_2, v_3 induce a C_5 in G. Hence, we adapt that construction to ensure a C_5-free graph.

Theorem 3.2. COMP-SUB($\mathscr{PM}$) *is* GI*-hard on* C_5*-free graphs.*

Proof. As in the proof of Theorem 3.1, we reduce GRAPH ISOMORPHISM on split graphs to COMP-SUB($\mathscr{PM}$).

Let A and B be split graphs. We may assume that A and B does not contain isolated vertices, A and B are not complete graphs, and $|V(A)| = |V(B)| = n$, for some $n \geq 3$. From an instance (A, B) of GRAPH ISOMORPHISM, we construct an instance G of COMP-SUB($\mathscr{PM}$) as follows.

- First, let H be a graph arising from the disjoint union of the graphs $X_1 = A$, $X_2 = B$, $Y_1 = \overline{B}$, and $Y_2 = \overline{A}$.
- Add to $E(H)$ all the edges between a vertex in T and a vertex in R, for $\{T, R\} \in \{\{X_1, Y_1\}, \{Y_1, Y_2\}, \{Y_2, X_2\}\}$. Figure 2 contains an example of graph H.
- Create a graph H' by a copy of H. To distinguish vertices from H and H' we let $v' \in V(H')$ be the *corresponding vertex* of $v \in V(H)$.
- Let the graph G arise from the disjoint union of H and H' by the addition of the edges $vv' \in E(G)$, for every $v \in V(H)$.

Clearly, the construction can be done in polynomial time. Next, we show that G is indeed a C_5-free graph.

Claim 1. *Let G be the graph obtained from the construction. It holds that G is a C_5-free graph.*

Proof of Claim 1. By contradiction, suppose that G has a C_5, denoted as C, as an induced subgraph. If C has edges crossing from H to H' then they should be exactly two edges, because C is 2-connected and has size five. However, for every $u, v \in V(H)$, we have $uv \in E(H)$ if and only if $u'v' \in E(H')$, which implies that C cannot have edges crossing from H to H' due to the order of C. Thus, either $V(C) \subseteq V(H)$ or $V(C) \subseteq V(H')$. Since $H \simeq H'$, we assume that $V(C) \subseteq V(H)$.

Recall that A and B are split graphs, which implies that X_i, Y_i, for every $i \in [2]$ are also split graphs. Hence, X_i and Y_i, for every $i \in [2]$, are C_5-free graphs. Consequently, there exists T, R such that $\{T, R\} \in \{\{X_1, Y_1\}, \{Y_1, Y_2\}, \{Y_2, X_2\}\}$, with the property that $V(C) \cap V(T) \neq \emptyset$ and $V(C) \cap V(R) \neq \emptyset$. Thus, for such a pair (R, T), the join $R + T$ implies that $|V(C) \cap V(R)|, |V(C) \cap V(T)| \leq 2$ and $|V(C) \cap (V(R) \cup V(T))| = 3$.

Since $|V(C) \cap (V(R) \cup V(T))| = 3$ and $|V(C)| = 5$, there exists $Z \in \{X_1, X_2, Y_1, Y_2\} \setminus \{R, T\}$ such that either the join $Z + R$ or $Z + T$ contains a subgraph of C, for which $V(C) \cap V(Z) \neq \emptyset$, say $Z + R$. Since $d_C(v) = 2$, for every $v \in V(C)$, we have that $|V(C) \cap V(R)| = 2$, hence, $|V(C) \cap V(Z)| = 1$ and $V(C) \cap (\{V(A), V(\overline{A}), V(B), V(\overline{B})\} \setminus \{R, T, Z\}) = \emptyset$. This implies that $|V(C) \cap V(H)| \leq 4$, a contradiction. □

Now, assume that $A \simeq B$. We show that HH' is a partition of G such that $H' \simeq \overline{H}$ and $[V(H), V(H')]$ is a perfect matching cut of G, implying that G is a *yes*-instance of COMP-SUB($\mathscr{PM}$). The construction of G implies that $[V(H), V(H')]$ is a perfect matching cut of G. Thus, it remains to show that $H' \simeq \overline{H}$. To this end, recall that P_4 is self-complementary, and the graph obtained by contracting into a single vertex each of X_i, Y_i, $i \in [2]$, is a P_4.

Note that H has the shape

$$X_1–Y_1–Y_2–X_2$$

and its complement $\overline{H}$ must have shape

$$\overline{Y}_1–\overline{X}_2–\overline{X}_1–\overline{Y}_2,$$

where R–T means a join between the vertices of R and T, and R, T are induced subgraphs. Also, if $A \simeq B$ then $X_1 \simeq X_2$ and $Y_1 \simeq Y_2$, consequently $\overline{X}_1 \simeq \overline{X}_2 \simeq Y_1 \simeq Y_2$ and $\overline{Y}_1 \simeq \overline{Y}_2 \simeq X_1 \simeq X_2$. Therefore,

$$H' \simeq X_1–Y_1–Y_2–X_2 \simeq \overline{Y}_1–\overline{X}_2–\overline{X}_1–\overline{Y}_2 \simeq \overline{H}.$$

For the converse, we suppose that G is a *yes*-instance of COMP-SUB($\mathscr{PM}$).

Let (V^*, V^{**}) be a partition of $V(G)$ into complementary parts such that $[V^*, V^{**}]$ is a perfect matching. By definition of perfect matching cut, it is clear that $[V^*, V^{**}] \subseteq \{wz \in E(G) : N_G(w) \cap N_G(z) = \emptyset\}$. Also, due to the joins of the construction, it holds that $\{wz \in E(G) : N_G(w) \cap N_G(z) = \emptyset\} = [V(H), V(H')]$. Since $|V^*| = |V^{**}|$ it follows that $(V^*, V^{**}) = (V(H), V(H'))$ and $H \simeq \overline{H}$.

Let $g : V(H) \to V(\overline{H})$ be an isomorphism function between H and $\overline{H}$. By performing a modular decomposition we find the maximal modules X_1, X_2, Y_1, Y_2 in H, and $\overline{X}_1, \overline{X}_2, \overline{Y}_1, \overline{Y}_2$ in $\overline{H}$.

By construction, for every $x \in X_1 \cup X_2$, $1 \leq d_H(x) \leq 2n - 1$, and, for every $y \in Y_1 \cup Y_2$, $2n \leq d_H(y) \leq 3n - 2$. Those degrees imply that g maps $V(X_1) \cup V(X_2)$ to $V(\overline{Y}_1) \cup V(\overline{Y}_2)$, and $V(Y_1) \cup V(Y_2)$ to $V(\overline{X}_1) \cup V(\overline{X}_2)$. By the uniqueness of the decomposition into maximal modules [16], we obtain that either $V(X_1) \mapsto V(\overline{Y}_1)$ or $V(X_1) \mapsto V(\overline{Y}_2)$.

If $V(X_1) \mapsto V(\overline{Y}_1)$, then

$$\begin{aligned} V(Y_1) &\mapsto V(\overline{X}_2), \\ V(Y_2) &\mapsto V(\overline{X}_1), \text{ and} \\ V(X_2) &\mapsto V(\overline{Y}_2). \end{aligned}$$

Since $H \simeq \overline{H}$, the mapping $V(X_1) \mapsto V(\overline{Y}_1)$ implies that $A \simeq X_1 \simeq \overline{Y}_1 \simeq \overline{\overline{B}} \simeq B$. Otherwise, $V(X_1) \mapsto V(\overline{Y}_2)$ implies that

$$\begin{aligned} V(Y_1) &\mapsto V(\overline{X}_1), \\ V(Y_2) &\mapsto V(\overline{X}_2), \text{ and} \\ V(X_2) &\mapsto V(\overline{Y}_1). \end{aligned}$$

Then, the mapping $V(Y_2) \mapsto V(\overline{X}_2)$ implies that $\overline{A} \simeq Y_2 \simeq \overline{X}_2 \simeq \overline{B}$. Therefore, $A \simeq B$. $\square$

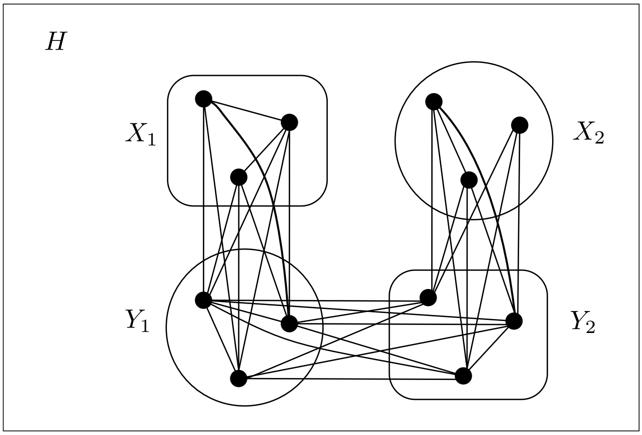

Figure 2: Graph H constructed for Theorem 3.2.

See in Figure 2 an example of graph H constructed for Theorem 3.2, where $A = P_3$ and $B = K_2 \cup K_1$.

Despite the hardness results presented in Theorems 3.1, and 3.2, next we show that COMP-SUB($\mathscr{PM}$) can be solved in polynomial time on $hole$-free graphs. Recall that a $hole$ is a cycle on 5 or more vertices.

Theorem 3.3. COMP-SUB($\mathscr{PM}$) *is polynomial-time solvable on* $hole$*-free graphs.*

Proof. Let G be a $hole$-free graph having $2n$-vertices. We assume that n is at least 5; otherwise, the problem can be solved in $O(1)$ time.

Suppose that $G \in$ COMP-SUB($\mathscr{PM}$), then G is decomposable into complementary subgraphs G_1 and G_2, such that the edge cut M of the decomposition is a perfect matching.

Recall that G_1 or G_2 is a connected graph. Thus, we assume that G_1 is connected.

We go through the proof by analysing the structure of the graphs of the decomposition by means of their connectivity (Claims 1 and 2), and we conclude by showing how to find that decomposition when it exists.

Claim 1. *Let G_1 be a connected graph with at least five vertices and $F \subseteq V(G_1)$. If $G[F]$ is biconnected, then $G[F^{G_2}]$ is a cluster graph.*

Proof of Claim 1. Suppose, by contradiction, that $G[F^{G_2}]$ is not a cluster graph and let $v_1v_2v_3$ be a P_3 in $G[F^{G_2}]$. Since $G[F]$ is 2-connected, there exist two independent paths between any two vertices in F. Consider $u_1, u_2, u_3 \in F$ as the corresponding vertices of v_1, v_2, v_3, respectively. Let P and P' be two independent paths between u_1 and u_3 in F. Since P and P' are independent, u_2 does not belong to P or P', say P. Then $P \cup \{v_1, v_2, v_3\}$ induces a $hole$ in G, a contradiction. □

Next, we see more on the structure of G_1 and G_2.

Claim 2. *Let G_1 be a connected graph having at least five vertices. If G_1 is non-biconnected, then either there is $S \subset V(G_1)$ with $|S| \leq 2$ such that $G_1 \setminus S$ is biconnected; or there is $S' \subset V(G_2)$ with $|S'| \leq 2$ such that $G_2 \setminus S'$ is biconnected.*

Proof of Claim 2. Suppose that G_1 is non-biconnected and let T be a block-cut-point tree of G_1. Let $\mathscr{B} = \{B_1, \dots, B_s\}$ and $\mathscr{C} = \{c_1, \dots, c_t\}$ be the sets of blocks and cutpoints in G_1, respectively. The proof is divided in two cases: (I) $|\mathscr{B}| \geq 2$, $|\mathscr{C}| = 1$; and (II) $|\mathscr{B}| \geq 2$, $|\mathscr{C}| \geq 2$.

Recall that if $|\mathscr{B}| = 1$, then $|\mathscr{C}| = 0$ and G_1 is biconnected.

(I) Suppose that $|\mathscr{B}| \geq 2$ and $|\mathscr{C}| = 1$. Let $\mathscr{C} = \{c\}$. We have that $G_1 \setminus \{c\}$ is the disjoint union $(B_1 \setminus \{c\}) \cup \cdots \cup (B_s \setminus \{c\})$. This implies that $\overline{G}_1 \setminus \{\overline{c}\}$ is the join $(\overline{B}_1 \setminus \{\overline{c}\}) + \cdots + (\overline{B}_s \setminus \{\overline{c}\})$.

- If $s \geq 3$ then $G_2 \setminus \{\overline{c}\} = (\overline{B}_1 \setminus \{\overline{c}\}) + \cdots + (\overline{B}_s \setminus \{\overline{c}\})$ is biconnected.
- If $s = 2$, $|\overline{B}_1 \setminus \{\overline{c}\}| \geq 2$, and $|\overline{B}_2 \setminus \{\overline{c}\}| \geq 2$ then $G_2 \setminus \{\overline{c}\} = (\overline{B}_1 \setminus \{\overline{c}\}) + (\overline{B}_2 \setminus \{\overline{c}\})$ is also biconnected.
- If $s = 2$ and $|\overline{B}_1 \setminus \{\overline{c}\}| = 1$ then $|\overline{B}_2 \setminus \{\overline{c}\}| \geq 2$. Otherwise, G_1 (and G_2) has only three vertices. Thus, B_2 is a block of G_1 with size $|V(G_1)| - 1$, and $S = B_1 \setminus \{c\}$ is as required.

(II) Now, consider that $|\mathscr{B}| \geq 2$ and $|\mathscr{C}| \geq 2$. Let $B, B' \in \mathscr{B}$ be two distinct leaves in T and $c, c' \in \mathscr{C}$ be two distinct cutpoints such that $Bc, B'c' \in E(T)$.

Let $D = V(G_1) \setminus (B \cup B')$. Since B (resp. B') is a leaf in T, we have that $V(B) \setminus \{c\}$ (resp. $V(B') \setminus \{c'\}$) is not adjacent to $B' \cup D$ (resp. $B \cup D$). This implies that $\overline{G}_1 \setminus \{\overline{c}, \overline{c}'\}$ is the join $(\overline{B} \setminus \{\overline{c}\}) + (\overline{B}' \setminus \{\overline{c}'\}) + \overline{D}$.

- If $D \neq \emptyset$, we have that $(\overline{B} \setminus \{\overline{c}\}) + (\overline{B}' \setminus \{\overline{c}'\}) + \overline{D}$ is biconnected. Thus, $G_2 \setminus \{\overline{c}, \overline{c}'\}$ is biconnected as required.
- If $D = \emptyset$, $|B \setminus \{c\}| \geq 2$, and $|B' \setminus \{c'\}| \geq 2$, then $G_2 \setminus \{\overline{c}, \overline{c}'\} = (\overline{B} \setminus \{\overline{c}\}) + (\overline{B}' \setminus \{\overline{c}'\})$ is also biconnected.
- If $D = \emptyset$ and $|B \setminus \{c\}| = 1$, then $|B' \setminus \{c'\}| \geq 2$. Otherwise, G_1 (and G_2) has only four vertices. Thus, $G_1 \setminus B$ is biconnected (notice that $|B| = 2$).

This completes the proof of Claim 2. □

By Claim 1, if G_1 is biconnected, then G_2 is a cluster graph. Since $G_1 \simeq \overline{G}_2$, we have that G_1 is a complete multipartite graph. Hence, G_1 and G_2 are cographs and, by Lemma 2.1, we can find the complementary partition of G in polynomial time.

Now, if G_1 is non-biconnected, recall that by Claim 2, either there is $S \subset V(G_1)$ with $|S| \leq 2$ such that $G_1 \setminus S$ is biconnected; or there is $S' \subset V(G_2)$ with $|S'| \leq 2$ such that $G_2 \setminus S'$ is biconnected.

Thus, there is a fixed number of vertices (at most 2) such that removing from G_1 or G_2 leaves a biconnected graph. We deal with the case that there exist $c, c' \in V(G_2)$ such that $G_2 \setminus \{c, c'\}$ is biconnected. The approach for the other case is similar.

If there exist $c, c' \in V(G_2)$ such that $G_2 \setminus \{c, c'\}$ is 2-connected, by Claim 1 (dual), we have that the graph induced by $(V(G_2) \setminus \{c, c'\})^{G_1}$ is a cluster graph. Then G_1 and $\overline{G}_2$ have distance to cluster equal to 2. We proceed by Algorithm 2.

Algorithm 2: PARTITION-INTO-COMPLEMENTARY-SUBGRAPHS(G)

Input: A graph G.
Output: Whether G is partitionable into two complementary graphs G_1 and G_2 such that G_1 and $\overline{G}_2$ have distance to cluster equal to 2 and the edge cut of the decomposition is a perfect matching.

1 **forall** $x_1, \ldots, x_4, y_1, \ldots, y_4 \in V(G)$ **do**
2 $\quad V(G_2) := N_G[\{y_1, \ldots, y_4\}] \setminus \{x_1, \ldots, x_4\}$
3 $\quad V(G_1) := V(G) \setminus V(G_2)$
4 $\quad M := \{xy \in E(G) : x \in V(G_1), y \in V(G_2)\}$
5 $\quad$ **if** M *is a perfect matching* **then**
6 $\quad\quad$ **forall** *cluster-modulator* S_1 *of* G_1*, such that* $|S_1| \leq 2$ **do**
7 $\quad\quad\quad$ **forall** *cluster-modulator* S_2 *of* $\overline{G}_2$*, such that* $|S_2| = |S_1|$ **do**
8 $\quad\quad\quad\quad$ **forall** *mapping* $f\colon S_1 \mapsto S_2$ **do**
9 $\quad\quad\quad\quad\quad$ **if** f *can be extended to an isomorphism from* G_1 *to* $\overline{G}_2$ **then**
10 $\quad\quad\quad\quad\quad\quad$ **return** yes
11 **return** no

Since G_2 has distance to complete multipartite equal to 2, there exist four vertices $y_1, \ldots, y_4 \in V(G_2)$ such that $N_{G_2}[\{y_1, \ldots, y_4\}] = V(G_2)$. Then, if a complementary decomposition (G_1, G_2) exists, we have that $|N_G[\{y_1, \ldots, y_4\}]| = n + 4$. Thence, it is possible to find $V(G_2)$ which is $N_G[\{y_1, \ldots, y_4\}]$ except for four vertices $x_1, \ldots, x_4 \in N_G[\{y_1, \ldots, y_4\}]$. We put $x_1, \ldots, x_4$ in $V(G_1)$ as well as the remaining vertices $\{v \in V(G) : v \notin N_G[\{y_1, \ldots, y_4\}]\}$. Given $V(G_1), V(G_2)$, and M, we check whether M is a perfect matching. If so, we compute $\overline{G}_2$ and we proceed to the step of finding cluster-modulators S_1 for G_1 and S_2 for $\overline{G}_2$, that are done by Lines 6–7. In a naive manner, all the possible pair of modulators can be found in $O(n^4)$, but we show how to find them in a more efficient way.

We first find a $P_3 = w_1w_2w_3$ in G_1. We know that at least one vertex in $\{w_1, w_2, w_3\}$ must be included in a cluster-modulator for G_1. Then, for every $w \in \{w_1, w_2, w_3\}$ we put $w \in S_1$ and we branch by searching (if any) for a $P_3 = w'_1w'_2w'_3$ in $G_1 \setminus \{w\}$. Again, given that at least one vertex in $\{w'_1, w'_2, w'_3\}$ must be included in a cluster-modulator for G_1, for every $w' \in \{w_1, w_2, w_3\}$ we put $w' \in S_1$. If $G_1 \setminus S_1$ is a cluster graph, we proceed to finding, in the same manner, a cluster-modulator S_2 for $\overline{G}_2$. Note that this is basically a bounded search tree algorithm for finding cluster vertex deletion sets.

Given a pair of modulators S_1 and S_2 such that $|S_1| = |S_2|$, and a mapping from S_1 to S_2, the final task is checking if such a mapping can be extended to an isomorphism between G_1 and $\overline{G}_2$. Note that, by the bounded search tree technique, the number of pairs of modulators and mappings that must be considered is bounded by a constant.

Recall that G_1 (resp. $\overline{G}_2$) is a disjoint union of complete graphs $H_1 \cup \cdots \cup H_p$ (resp. $H'_1 \cup \cdots \cup H'_p$), for some $p \geq 2$, with the addition of two vertices w, w' (resp. z, z') arbitrarily adjacent to $H_1 \cup \cdots \cup H_p$ (resp. $H'_1 \cup \cdots \cup H'_p$). With this structure, an isomorphism from G_1 to $\overline{G}_2$ can be determined as follows.

For a mapping $w \mapsto z$, $w' \mapsto z'$, we can map H_i to H'_j, $i, j \in [p]$, if and only if

$|V(H_i)| = |V(H'_j)|$ and

$|N_{H_i}(w) \setminus N_{H_i}(w')| = |N_{H'_j}(z) \setminus N_{H'_j}(z')|$ and

$|N_{H_i}(w') \setminus N_{H_i}(w)| = |N_{H'_j}(z') \setminus N_{H'_j}(z)|$ and

$|N_{H_i}(w) \cap N_{H_j}(w')| = |N_{H'_i}(z) \cap N_{H'_j}(z')|$.

Therefore, each mapping $w \mapsto z$, $w' \mapsto z'$ defines "types" of cliques, from which the mapping can be extended to an isomorphism from G_1 to $\overline{G}_2$ if and only if G_1 and $\overline{G}_2$ have the same number of cliques per type.

Next, we analyse the running time of Algorithm 2.

First, in Line 1, we check every 8-tuple of vertices in $V(G)$ to separate those $x_1, \ldots, x_4 \in V(G_1)$ and $y_1, \ldots, y_4 \in V(G_2)$, which requires $O(n^8)$ time. Lines 2–4 define $V(G_1), V(G_2)$, and M, which run in $O(n + m)$ time. Checking whether M is a perfect matching (Line 5) can be done in $O(n + m)$ time.

Recall that a P_3 in G can be found in $O(n + m)$ time. By the method previously described, Line 6 can be done by finding a $P_3 = w_1w_2w_3$ in G_1; for every $w \in \{w_1, w_2, w_3\}$ finding a $P_3 = w'_1w'_2w'_3$ in $G_1 \setminus \{w\}$ in G_1; and finally, for every $w' \in \{w_1, w_2, w_3\}$, checking whether $G_1 \setminus S_1 = \{w, w'\}$ is a cluster graph. This produces a ternary search tree with height equal to 2. Hence with 9 leaf nodes, that are at most 9 possible cluster-modulators $\{w, w'\}$ for G_1. This requires a running time of $O(m + n)$. For every of those possible cluster-modulators for G_1 we proceed to finding every cluster-modulator for $\overline{G}_2$ (Line 7) by the same method. This gives an amount of at most 81 possible 4-tuples w, w', z, z' that must be checked, hence Lines 6–8 run in $O(n + m)$ time.

Finally, for Line 9, checking whether an isomorphism from G_1 to $\overline{G}_2$ can be extended from f can be done by checking sizes of cliques and neighborhoods, which can be done in $O(n + m)$ time.

Therefore, the overall running time of Algorithm 2 is of order $O(n^8(n+m)) = O(n^9 + n^8m)$. □

Next, it follows a characterization for COMP-SUB($\mathscr{PM}$) in the class of distance hereditary graphs, which is a subclass of *hole*-free graphs. A *distance-hereditary* graph is a $\{domino, house, gem, hole\}$-free graph. A *domino*, a *house*, and a *gem* are depicted in Figure 3. For the next result, let ϱ be the graph in Figure 3.

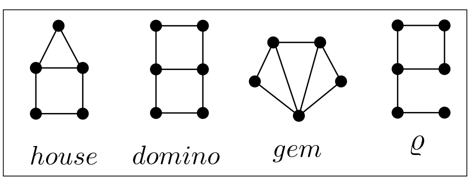

Figure 3: Some small subgraphs.

Proposition 3.4. *Let G be a distance-hereditary graph of order $2n$. It holds that $G \in$ COMP-SUB($\mathscr{PM}$) if and only if $G \in \{K_n\overline{K}_n, \varrho\}$.*

Proof. Let G be a distance-hereditary graph. Clearly, $K_n\overline{K_n} \in$ COMP-SUB($\mathscr{PM}$). Let ϱ be the graph with vertex set $V(\varrho) = \{u_1, u_2, u_3, v_1, v_2, v_3\}$ denoted as in Figure 4. Let $V_1 = \{u_1, u_2, u_3\}$, and $V_2 = \{v_1, v_2, v_3\}$. Clearly $G[V_1] \simeq \overline{G[V_2]}$, then $\varrho \in$ COMP-SUB($\mathscr{PM}$).

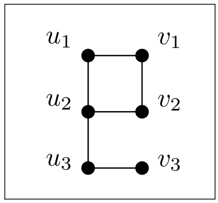

Figure 4: Graph ϱ.

Next, suppose that $G \in$ COMP-SUB($\mathscr{PM}$). By definition, there exist a complementary decomposition (G_1, G_2) of G such that the edge cut M of the decomposition is a perfect matching. Let $M = \{u_1v_1, \ldots, u_nv_n\}$ where $u_i \in V(G_1)$ and $v_i \in V(G_2)$, for every $i \in [n]$. We may assume that G_1 is connected. We begin by showing that G_2 is a cluster graph with no induced K_3.

Claim 1. *If G_1 is connected, then G_2 is a $\{P_3, K_3\}$-free graph.*

Proof of Claim 1. Suppose, by contradiction, that G_2 contains (I) an induced P_3 or (II) an induced K_3.

(I) Let $i, j, k \in [n]$ be pairwise distinct, such that $v_iv_j, v_jv_k \in E(G_2)$ and $v_iv_k \notin E(G_2)$. Since G_1 is connected, there exists a (u_p, u_q)-path in G_1, for every $p, q \in \{i, j, k\}$. Since G is *hole*-free, it follows that $\text{dist}_{G_1}(u_i, u_j) \leq 1$. Since G_1 is connected, u_i and u_j lie in the same connected component of G_1, then $u_iu_j \in E(G_1)$. With a similar argument, we obtain that $u_ju_k \in E(G_1)$. Again, since G is *hole*-free, $u_iu_k \notin E(G_1)$. However, the set $\{u_i, u_j, u_k, v_i, v_j, v_k\}$ induces a *domino* in G, a contradiction.

(II) Let $i, j, k \in [n]$ be pairwise distinct, such that $\{v_i, v_j, v_k\}$ induces a K_3 in G_2. In this case, we have that each of $\{u_i, u_j, u_k\}$ must lie in distinct connected components of G_1, otherwise $\{u_i, u_j, u_k, v_i, v_j, v_k\}$ induces a *house* in G. Thus, we have a contradiction to the connectedness of G_1. □

By Claim 1, G_2 is a $\{P_3, K_3\}$-free graph, that is, $G_2 \simeq pK_1 \cup qK_2$, for some $p, q \geq 0$. Since $G_1 \simeq \overline{G_2}$, we have that $G_1 = A + B$, where A is a complete graph $K_p = \overline{pK_1}$ and B is a complete multipartite graph $\overline{qK_2}$. For the rest of the proof, we consider the possible cases for q. We deal with $q \geq 2$ in Claim 2. In the sequence, we address the cases $q = 1$ and $q = 0$.

Claim 2. *If $q \geq 2$, then for every edge $e \in E(G_1)$, e belongs to a K_3 or a C_4 in G_1.*

Proof of Claim 2. Suppose that $q \geq 2$. Recall that $G_1 = A + B$ where $A = K_p$ and $B = \overline{qK_2}$. Let B' be a C_4 subgraph of B with $V(B') = \{u_1, \ldots, u_4\}$ and $E(B') = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$. Let $e = xy \in E(G_1)$. If $x, y \in V(B')$, the conclusion that e belongs to a C_4 is immediate.

If $x, y \in V(G_1) \setminus V(B')$, then $e = xy \in E(A)$. The join $A + B$ implies that u_i is a common neighbor of x and y, for every $i \in [4]$. Then e belongs to a K_3 induced by $\{x, y, u_i\}$ in G_1.

Otherwise, we may assume that $x \in V(G_1) \setminus V(B')$ and $y \in V(B')$. Let $w \in N_{B'}(y)$. Since x is adjacent to every vertex in B', we have that $e = xy$ belongs to a K_3 induced by $\{x, y, w\}$ in G_1. □

Let $q \geq 2$. By Claim 2, every edge $e \in E(G_1)$ belongs to a K_3 or a C_4 in G_1. Recall that $|E(G_2)| = q \geq 2$, then let $i, j \in [n]$ such that $v_iv_j \in E(G_2)$. Given that G_1 is connected and G is $hole$-free, we conclude that $u_iu_j \in E(G_1)$. By Claim 2, we have that u_iu_j belongs to a K_3 or a C_4 in G_1. In the former, G contains an induced $house$, a contradiction. In the latter, G contains an induced $domino$, also a contradiction.

Next, let $q = 1$. Since G_1 is connected, we have that $p \geq 1$. If $p = 1$, then $G_1 \simeq P_3$ and $G = \varrho$. If $p \geq 2$, then G_1 is isomorphic to a K_{p+2} minus one edge. It is easy to see that every $e \in E(G_1)$ belongs to a K_3 in G_1. Since $|E(G_2)| = q = 1$, let $i, j \in [n]$ such that $v_iv_j \in E(G_2)$. Again, given that G_1 is connected and G is $hole$-free, we conclude that $u_iu_j \in E(G_1)$. But u_iu_j belongs to a K_3 in G_1, consequently G contains an induced $house$, a contradiction.

Finally, let $q = 0$. Clearly, $p = n$, $G_1 = A = K_n$, and $G = K_n\overline{K}_n$. □

We close this section with a characterization of COMP-SUB($\mathscr{PM}$) on chordal graphs. Recall that a chordal graph is a $C_{k\geq 4}$-free graph.

Proposition 3.5. *Let G be a chordal graph of order $2n$. It holds that $G \in$ COMP-SUB($\mathscr{PM}$) if and only if $G = K_n\overline{K}_n$.*

Proof. Let G be a chordal graph. Clearly, $K_n\overline{K}_n \in$ COMP-SUB($\mathscr{PM}$).

Consider that $G \in$ COMP-SUB($\mathscr{PM}$). There exist a complementary decomposition (G_1, G_2) of G such that the edge cut M of the decomposition is a perfect matching. Let $M = \{u_1v_1, \ldots, u_nv_n\}$ where $u_i \in V(G_1)$ and $v_i \in V(G_2)$, for every $i \in [n]$.

Suppose, by contradiction, that $G \neq K_n\overline{K}_n$. We may assume that there exists $i, j, k, l \in [n]$ such that $u_iu_j \in E(G_1)$ and $v_kv_l \in E(G_2)$. Since M is a perfect matching and G is chordal, we have that $v_iv_j \notin E(G_2)$ and $u_ku_l \notin E(G_1)$. Then, we obtain that $u_pv_q \in E(G_1)$ if and only if $v_pv_q \notin E(G_2)$, for every distinct $p, q \in [n]$. Furthermore, the chordality of G implies that there exists no path between v_i, v_j in G_2, and no path between u_k, u_l in G_1. Consequently both G_1 and its complement G_2 are disconnected, a contradiction. □

4 Results on some P_k-free graphs

In this section, we still consider $\Pi = \mathscr{PM}$ as the property that considers M as a perfect matching. We begin by showing how to solve COMP-SUB($\mathscr{PM}$) in polynomial time when the input graph G is P_5-free.

Theorem 4.1. COMP-SUB($\mathscr{PM}$) *is polynomial-time solvable on P_5-free graphs.*

Proof. Let G be a $2n$-vertex P_5-free graph. Recall that if $G \in$ COMP-SUB($\mathscr{PM}$), then G is decomposable into complementary subgraphs G_1 and G_2, such that the edge cut M of the decomposition is a perfect matching. Since G is P_5-free, the existence of M implies that G_1 and G_2 are P_4-free, that is, G_1 and G_2 are cographs. Then, the conclusion follows by applying Lemma 2.1. □

A graph is *extended P_4-laden* if every induced subgraph with at most six vertices that contains more than two induced P_4's is $\{2K_2, C_4\}$-free. Extended P_4-laden graphs generalize cographs, P_4-sparse, P_4-lite, P_4-laden and P_4-tidy graphs, and they were considered under the perspective of partitioning. For instance, Bravo et al. [3] show that partitioning an extended P_4-laden graph into at most k independent sets and at most ℓ cliques is linear-time solvable for $k, \ell \geq 1$, and Bravo et al. [2] show a linear time algorithm for recognizing graphs that can be partitionable into a clique and a forest. In addition, Pedrotti and De Mello [19] describe a linear-time algorithm that lists the minimal separators of extended P_4-laden graphs.

Another related result to partitioning is implied by considering that extended P_4-laden graphs are P_6-free. The result on 3-colorability by Randerath and Schiermeyer [22] implies that the problem of partitioning a graph into 3 independent sets is polynomial-time solvable on extended P_4-laden graphs.

We present in Proposition 4.2 a characterization concerned to COMP-SUB($\mathscr{PM}$) on extended P_4-laden graphs.

Proposition 4.2. *Let G be an extended P_4-laden graph of order $2n$. It holds that $G \in$ COMP-SUB($\mathscr{PM}$) if and only if $G = K_n\overline{K}_n$.*

Proof. Let $G = K_n\overline{K}_n$. We analyse the subgraphs of G with at most 6 vertices to show that G is an extended P_4-laden graph. Let G' be a subgraph of G such that $|V(G')| \leq 6$. If G' is a subgraph of K_n or $\overline{K}_n$, it is clear that G' does not have induced P_4's. Then, we suppose that $V(G')$ intersects both $V(K_n)$ and $V(\overline{K}_n)$. Notice that two induced P_4's arise in G' only if $|V(G') \cap V(\overline{K}_n)| \geq 3$ and $|V(G') \cap V(K_n)| \geq 3$. Since G is a split graph, G' is also a split graph. This implies that G' is $\{2K_2, C_4\}$-free and hence, G is extended P_4-laden.

Now, we show that $G \in$ COMP-SUB($\mathscr{PM}$) implies that $G = K_n\overline{K}_n$. Suppose that $G \in$ COMP-SUB($\mathscr{PM}$), and, by contradiction, that $G \neq K_n\overline{K}_n$. Since $G \in$ COMP-SUB($\mathscr{PM}$) there exists a complementary decomposition (G_1, G_2) of G, such that the edge cut M of the decomposition is a perfect matching. Let $M = \{u_1v_1, \ldots, u_nv_n\}$ where $u_i \in V(G_1)$ and $v_i \in V(G_2)$, for every $i \in [n]$. We assume that G_1 is connected.

Given that $G \neq K_n\overline{K}_n$, let $u_1u_2u_3$ be an induced P_3 in G_1 and $G' = G[\{u_i, v_i : i \in [3]\}]$.

Since $u_iv_i \in E(G')$, for every $i \in [3]$, we have that $\{u_1, v_1, u_3, v_3\}$ induces a $2K_2$ in G'. Then, we may suppose that $v_1v_3 \in E(G')$. Notice that $\{u_2, v_2, v_1, v_3\}$ induces a $2K_2$ in G', then we consider that $v_1v_2 \in E(G')$ or $v_2v_3 \in E(G')$. In both possibilities we have an induced C_4 in G', by $\{u_1, v_1, u_2, v_2\}$ in the first, and by $\{u_2, v_2, u_3, v_3\}$ in the latter, a contradiction. □

Our last result characterizes cographs *yes*-instances of COMP-SUB($\mathscr{PM}$). Recall that a *cograph* is a P_4-free graph.

Proposition 4.3. *Let G be a cograph of order $2n$. Then, $G \in$ COMP-SUB($\mathscr{PM}$) if and only if $G = K_2$.*

Proof. Let G be a cograph. Trivially $K_2 \in$ COMP-SUB($\mathscr{PM}$).

Suppose that $G \in$ COMP-SUB($\mathscr{PM}$), and, by contradiction, that $G \neq K_2$. Let (G_1, G_2) be a complementary decomposition of G, in which the edge cut M of the decomposition is a perfect matching. Let $M = \{u_1v_1, \ldots, u_nv_n\}$ where $u_i \in V(G_1)$ and $v_i \in V(G_2)$, for every $i \in [n]$.

Let $i, j \in [n]$ such that $u_i u_j \in E(G_1)$. We know that $u_i v_j \notin E(G)$ whenever $i \neq j$. Since G is P_4-free, we obtain that $u_i u_j \in E(G_1)$ if and only if $v_i v_j \in E(G_2)$, for every distinct $i, j \in [n]$. This implies that $G_1 \simeq G_2$, then $G_1 \simeq \overline{G}_1$, i.e., G_1 is self-complementary. Since a cograph is connected if and only if its complement is disconnected [8], we conclude that G_1 cannot be self-complementary, a contradiction. □

5 Concluding remarks

We have considered COMP-SUB($\mathscr{PM}$) problem when $\mathscr{PM}$ states the edge cut of the decomposition as a perfect matching. We have presented polynomial-time algorithms for solving COMP-SUB($\mathscr{PM}$) when the input graph G is *hole*-free or P_5-free and we have shown characterizations on chordal, distance-hereditary, and extended P_4-laden graphs.

Concerning complexity results, despite its resemblance with the NP-complete problem PERFECT MATCHING CUT, we show that COMP-SUB($\mathscr{PM}$) is GI-hard when the given input graph G is C_5-free or $\{C_{k\geq 7}, \overline{C}_{k\geq 7}\}$-free.

We remark that our results by Theorem 3.1, Theorem 3.3, and Proposition 3.5 address the cases when G is a $C_{k\geq \ell}$-free graph, for every $\ell \geq 4$, except for $\ell = 6$. Then, we leave the following open question.

Question 5.1. Can COMP-SUB($\mathscr{PM}$) on $C_{k\geq 6}$-free graphs be solved in polynomial time?

We also leave the complexity of COMP-SUB($\mathscr{PM}$) on P_6-free graphs open. Furthermore, we still do not know whether COMP-SUB($\mathscr{PM}$) is GI-complete.

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