Revista Sergipana de Matemática e Educação Matemática

https://doi.org/10.34179/revisem.v10i1.21000

REVERSE DIVISORS ARE BALL NUMBERS

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Abstract

This study investigates the relationships between Ball magic numbers and reverse divisors. Specifically, we extend the set of numbers that are simultaneously reverse divisors and Ball magic numbers. Webster and Williams [16] classify reverse divisors into three types: basic of the first and second types, and non-basic ones. The basic of the first type basics are numbers that begin with 10 and end with 89, with all intermediate digits being 9, for example, 1099989. The basic of the second type is double that of a first type basic, such as 2199978. Non-basic reverse divisors are obtained by concatenating magic reverse divisors. Costa and Santos [6] proved that a first type basic reverse divisor is a Ball's magic number. This work demonstrated that a non-basic reverse divisor, obtained by concatenating first-type basic reverse divisors, is also a Ball's magic number.

Keywords: reverse divisor, magic Ball number.

1 Introduction

The magic numbers, reverse divisors, and the relationships between them continue to fascinate many readers. The starting point is always a number $x_n = a_{n-1}a_{n-2}\dots a_1a_0$ and its reverse $x'_n = a_0\dots a_{n-1}$, where $n \geq 2$ is a natural number, $a_i \in D = \{0, 1, \dots, 8, 9\}$, with $i = 0, \cdots, n-1$ and $a_{n-1} \neq 0$ (base 10). The number x_n is a palindrome when $x_n = x'_n$.

An example that involves operations with a number and its reverse.

Example 1.1. Let the number $x_3 = 321$ and its reverse $x'_3 = 123$. Denote $y_3 = x_3 - x'_3 = 198$ whose reverse $y'_3 = 891$. Adding the results, we obtain $B_3 = y_3 + y'_3 = 1089$. The number 1089 has the incredible property that $9 \cdot 1089 = 9801$, meaning $B'_3 = 9 \cdot B_3$.

Can any pattern be deduced from this example? Will this procedure always lead us to a number with this incredible property? Can numbers with this property be generated in this way? Let us explore this last question slightly further.

The number 10891089 has the incredible property that $9 \times 10891089 = 98019801$. Repeating the procedure from Example 1.1, 321321 generates 10891089? The answer is **no**; let us see why:

$$321321 - 123123 = 198198 \text{ and } 198198 + 891891 = 1090089\ .$$

Question 1.2. Is there a number x that generates 10891089? If so, what is this number?

The procedure used in Example 1.1 characterizes Ball's magic numbers. Specifically, we have:

Algorithm 1.3. : **Ball's Magic Number**

1. Consider a non-palindromic number x_n;
2. Write its reverse x'_n ;
3. Find the (positive) number $y_n = |x_n - x'_n|$;
4. Write its reverse y'_n;
5. Obtain the number $B = y_n + y'_n$.

Given the number x_n with n digits, we note that the numbers x'_n, y_n, and y'_n are considered numbers with n digits, even if any digit is zero.

The number B, resulting from Algorithm 1.3, is called the **Ball magic number**, or simply the **Ball number**. Since the number x_n is not a palindrome, we can assume $x_n > x'_n$, and in a simplified manner, we obtain the Ball number by $B = (x_n - x'_n) + (x_n - x'_n)'$.

The numbers 99 and 1089 are examples of Ball magic numbers. In addition to Rouse Ball [2], among the works dedicated to the study of Ball numbers, we highlight Acheson[1], Beiler [3], and Webster [15]. In Portuguese, we also have Costa [4] or Santos and Costa [14] as references.

The incredible property mentioned in Example 1.1 is related to reverse divisors. The definition follows below.

Definition 1.4. Let x_n be a non-palindromic number with $n \geq 2$ digits. The number x_n is a reverse divisor if x_n divides its reverse x'_n. We say that x_n and x'_n form a pair of reverse divisors.

Example 1.5. [2, 8, 9, 16] The number 1089, from Example 1.1, is a reverse divisor since it divides 9801. Similarly, the pair 10989 and 98901 form a reverse divisor pair, because $98901 = 9 \cdot 10989$.

Remark 1.6. Let x_n be a non-palindromic integer with $n \geq 2$ digits, if x_n is both a Ball number and a reverse divisor simultaneously, we say that x_n is a Ball (or magic) reverse divisor.

The numbers 1089 and 10989 are examples of Ball reverse divisors. While, 10891089 is a reverse divisor, and we do not know if it is Ball number. In Webster and Williams [16], two basic classes of reverse divisors are exhibited, namely:

Theorem 1.7. *[16] [Tipo 1] Every number of the form* $10\underbrace{99\ldots99}_{n-4\ times}89$ *is a reverse divisor of* $98\underbrace{99\ldots99}_{n-4\ times}01$ *, where* $n > 3$ *is a natural number.*

Theorem 1.8. *[16] [Tipo 2] Every number of the form* $21\underbrace{99\ldots99}_{n-4\ times}78$ *is a reverse divisor of* $87\underbrace{99\ldots99}_{n-4 times}12$ *, where* $n > 3$ *is a natural number.*

Costa and Santos demonstrate in [6] that the reverse divisors given in Theorem 1.7 (Type 1) are Ball numbers, and those in Theorem 1.8 (Type 2) are double a Ball number.

In this work, we show that the reverse divisors of **type 3**, Theorem 4.10, constitute another class of reverse divisors beyond the two basic classes, which are numbers formed by the **symmetric concatenation** of reverse divisors of **type 1**, as presented by Webster and Williams in [16], and they are also magic Ball numbers.

2 Ball numbers and Codes

The characterization of magic Ball numbers is essential for the study of their relationships with reverse divisors. According to Webster [15], in this section, we present the main results related to Ball numbers. Given a number x_{n+1} and its reverse x'_{n+1}, that is,

$$\begin{aligned} x_{n+1} &= a_n a_{n-1} \ldots a_{n-i} \ldots a_i \ldots a_1 a_0 \ , \\ x'_{n+1} &= a_0 a_1 \ldots a_i \ldots a_{n-i} \ldots a_{n-1} a_n \ , \end{aligned}$$

with $a_n > a_0$. Applying Algorithm 1.3, we obtain:

$$\begin{aligned} x_{n+1} - x'_{n+1} &= (a_n - a_0)10^n + (a_{n-1} - a_1)10^{n-1} + \cdots + (a_{n-i} - a_i)10^{n-i} + \\ &\quad + \cdots + (a_i - a_{n-i})10^i + \cdots + (a_1 - a_{n-1})10 + (a_0 - a_n) \ . \end{aligned}$$

In base 10, the representation requires that each coefficient be greater than or equal to zero and less than 10. In some cases, to ensure that the coefficient is positive, it may be necessary to shift 10 from the order i to $i-1$ (the famous "borrow one" operation).

For each number with $n+1$ digits $x_{n+1} = a_n \dots a_0$, we associate a number Z_{n+1} called the code x_{n+1}. This code Z_{n+1} consists of a sequence of $0s$ and $1s$, ends with a digit 0, and encodes the necessary information to obtain the resulting Ball number B from the initial number x_{n+1}. We will present the construction of Z_{n+1} informally. Write the number $x_{n+1} = a_n \dots a_0$, where we take $a_n > a_0$, and below it, its reverse $x'_{n+1} = a_0 \dots a_n$, as shown below:

$$\begin{array}{cccccc} & a_n & \dots & a_i & \dots & a_0 \\ - & a_0 & \dots & a_{n-i} & \dots & a_n \\ \hline & * & \dots & * & \dots & * \end{array}$$

In the i-th column from right to left ($i = 0, \dots, n$), we have the number $a_i - a_{n-i}$. To ensure that the result stays between 0 and 9, we define a digit z_i as follows: if a quantity of 10 (grouping or base) needs to be shifted from the $(i+1)$-th column to the i-th column, z_i will be 1; otherwise, it is 0. With the above definition, we can write the difference between x_{n+1} and x'_{n+1} as follows:

$$\begin{aligned} & x_{n+1} - x'_{n+1} \\ = \ & (a_n - a_0 - z_{n-1})10^n + (a_{n-1} - a_1 - z_{n-2} + z_{n-1}10)10^{n-1} + \dots \\ & + (a_{n-i} - a_i - z_{n-i-1} + z_{n-i}10)10^{n-i} + \cdots + (a_i - a_{n-i} - z_{i-1} + z_i 10)10^i + \\ & \cdots + (a_1 - a_{n-1} - z_0 + z_1 10)10 + (a_0 - a_n + z_0 10) \ . \end{aligned}$$

In this way, we ensure that each of the coefficients is between 0 and 9, and we obtain a sequence $z_0, \dots, z_n$ of 0's and 1's. The number $z_0 \dots z_n$ with $n+1$ digits is called the code of x_{n+1} and is denoted by Z_{n+1}, i.e., $Z_{n+1} = z_0 \dots z_n$.

Since we assume that $a_n > a_0$, $z_0 = 1$, and $z_n = 0$. The number of n digits obtained from Z_{n+1} by excluding the digit $z_n = 0$ (at the end) is denoted by $Z_{\overline{n+1}} = z_0 \dots z_{n-1}$ and is called the truncated code of x_{n+1}. More precisely, we have:

Definition 2.1. Let's set $z_{-1} = z_n = 0$ and recursively for z_i, $i = 0, \ldots, n-1$,

$$z_i = \begin{cases} 1; & \text{if } a_i - a_{n-i} - z_{i-1} < 0 \ ; \\ 0; & \text{if } a_i - a_{n-i} - z_{i-1} \geq 0 \ ; \end{cases}$$

and thus, it follows that $0 \leq a_i - a_{n-i} - z_{i-1} + z_i 10 < 10$ for $i = 0, \ldots, n-1$.

From Definition 2.1, we can write:

$$y_n = x_{n+1} - x'_{n+1} = \sum_{i=0}^{n} (a_i - a_{n-i} - z_{i-1} + z_i 10) 10^i \ .$$

Theorem 2.2. *[4, 14, 15] Consider the number $x_{n+1} = a_n \ldots a_0$, with $a_n > a_0$. The magic Ball number B generated by x_{n+1} is non-zero and written in the form $99 \cdot (z_0 z_1 \ldots z_{n-1})$.*

Proof. By hypothesis, we have $a_n > a_0$, so $z_0 = 1$. With this notation, we can write a magic number of *Ball* in base 10, as follows:

$$\begin{aligned} B &= y_n + y'_n \\ &= \sum_{i=0}^{n} \big((a_i - a_{n-i} - z_{i-1} + z_i 10) + (a_{n-i} - a_i - z_{n-i-1} + z_{n-i} 10) \big) 10^i \\ &= \sum_{i=0}^{n} (-z_{i-1} + z_i 10 - z_{n-i-1} + z_{n-i} 10) 10^i \\ &= -\sum_{i=0}^{n} z_{i-1} 10^i + \sum_{i=0}^{n} z_i 10^{i+1} - \sum_{i=0}^{n} z_{n-i-1} 10^i + \sum_{i=0}^{n} z_{n-i} 10^{i+1} \\ &= -\sum_{i=0}^{n-1} z_i 10^{i+1} + \sum_{i=0}^{n-1} z_i 10^{i+1} - \sum_{i=1}^{n} z_{n-i} 10^{i-1} + 10^2 \sum_{i=1}^{n} z_{n-i} 10^{i-1} \\ &= (10^2 - 1) \sum_{i=1}^{n} z_{n-i} 10^{i-1} \\ &= 99 \cdot (z_0 z_1 \ldots z_{n-1}) \ . \end{aligned} \tag{2.1}$$

Therefore, the number of *Ball* B is a multiple of both 99 and the truncated code $z_0 z_1 \ldots z_{n-1}$. □

The example below illustrates the process of generating codes.

Example 2.3. [6] Consider the six-digit number $x_6 = 397862$. Subtracting $x_6' = 268793$ from x_6, we have:

$$\begin{array}{cc|ccccc}
 & 3 & \cancel{9}^{8} & \cancel{7}^{17} & \cancel{8}^{7} & \cancel{6}^{15} & \cancel{2}^{12} \\
- & 2 & 6 & 8 & 7 & 9 & 3 \\
\hline
 & 1 & 2 & 9 & 0 & 6 & 9 \\
\hline\hline
 & 0 & 0 & 1 & 0 & 1 & 1
\end{array}$$

The digit 1 is placed in the columns that required the shifting of ten (10) from the adjacent column to the left. In the remaining positions, the digit 0 is placed, that is,

$$z_0 = 1, \; z_1 = 1, \; z_2 = 0, \; z_3 = 1, \; z_4 = 0 \text{ e } z_5 = 0 \; .$$

Therefore, we have the code $Z_6 = 110100$ and the truncated code $Z_{\overline{6}} = 11010$, which is a divisor of Ball number B. Thus, the Ball number generated by $x_6 = 397862$ can be written as:

$$B = 99 \times Z_{\overline{6}} = 99 \times 11010 = 1089990 \; .$$

To improve understanding of the codes, Table 1 displays the magic number of Ball and the code that generates it, where $n \leq 5$ indicates the number of digits.

Table 1: Magic Number and Code

n	No.	factorization	truncated code	code
2	99	99×1	1	10
3	1089	99×11	11	110
4	990	99×10	10	100
4	9999	99×101	101	1010
4	10890	99×110	110	1100
4	10989	99×111	111	1110
5	10890	99×110	110	1100
5	99099	99×1001	1001	10010
5	109890	99×1110	1110	11100
5	109989	99×1111	1111	11110

Fonte: [14]

With the inclusion of $z_n = 0$, following [15], the list of zeros (0's) and ones (1's) is called codes and is characterized by Propositions 2.4 and 2.5, that is,

Proposition 2.4. [6, 15] If $z_0 \ldots z_n$ is a code and $a_n > a_0$, then $z_0 = 1$ and $z_n = 0$.

Proposition 2.5. [6, 14, 15] If $z_0 \ldots z_n$ is a code. For all $i = 1, \ldots, n-1$, we have:

(a) If $z_{i+1} = 1$ and $z_i = 0$, then $z_{n-i-1} = 0$.

(b) If $z_{i+1} = 0$ and $z_i = 1$, then $z_{n-i-1} = 1$.

Conversely, we have,

Proposition 2.6. [6, 15] A list of ones(1s) and zeros(0s), $z_0 \ldots z_n$, satisfying propositions 2.4 and 2.5, is a code.

Now, the process of obtaining new magic numbers is restricted to obtaining new codes. Let us, therefore, analyze when a specific sequence or list of ones(1s) and zeros(0s) is a code.

In the following results, we find the verification of some lists with their respective references.

Proposition 2.7. [6] If $z_n \cdots z_0$ is a code, then $z_n \cdots z_0 0$ is also a code.

Proposition 2.8. [6] Every list with $m > 0$ elements 1s and one 0 in the form $\underbrace{11\ldots11}_{m\text{ times}}0$ is a code.

Proposition 2.9. [6] For every $m \geq 0$, a list in the form $1\underbrace{00\ldots00}_{m\text{ times}}10$ is a code.

The truncated code $\underbrace{11\ldots11}_{n\text{ times}}$ formed only by the digit 1 is known in the literature as *repunit* numbers (repetition of the unit) and is denoted by $R_n = \underbrace{11\ldots1}n$, times, $n \geq 1$. A *repunit* number can be written as:

$$R_n = \frac{10^n - 1}{9}, \quad \text{for all } n \geq 1.$$

Some properties related to *repunit* numbers R_n can be found, for example, in [3, 5, 12] and references.

Another curious list related to codes is the smoothly undulating $um-zero$ numbers denoted by UZ. These are lists formed only by the digits 1 and 0 in alternating sequence, starting with 1.

We will use the notation uz_n where n indicates the number of digits in the smoothly undulating number $uz_n \in UZ$, for $n > 1$. For example: $uz_2 = 10$, $uz_3 = 101$, $uz_4 = 1010$, $uz_5 = 10101$, and $uz_{11} = 10101010101$. Some properties related to smoothly undulating numbers can be found, for example, in [13] and references.

Proposition 2.10. [6] For all $n \geq 1$ and $uz_n \in UZ$, $11 \times uz_n$ is a code for even n and a truncated code for odd n.

Proposition 2.11. [6] For every $n > 1$:
(a) If n is even, then uz_n is a code.
(b) If n is odd, then uz_n is a truncated code.

Another particular code, presented in a previous work, follows below.

Proposition 2.12. [6] A code of the form $R_n = \underbrace{11\ldots11}_{n \text{ times}}0$ can be decomposed by adding two codes, A and C.

Remark 2.13. Theorem 2.2 guarantees that every number in the form $99 \cdot (z_0 z_1 \ldots z_{n-1})$ is a Ball number provided that $z_0 z_1 \ldots z_{n-1}0$ is a code, as follows:

1. If B_k is a Ball number, then $10 \cdot B_k$ is also a Ball number (Proposition 2.7).
2. For every $n > 0$, numbers in the form $99 \times \underbrace{11\ldots11}_{n \text{ times}}$ are Ball numbers (Proposition 2.8).
3. For every $m > 0$, numbers in the form $99 \times (10^{2^m} + 1)$ are Ball numbers (Proposition 2.9).
4. For every $n > 1$ and $uz_n \in UZ$, $1089 \times uz_n$ is a Ball number (Proposition 2.10).

3 Code with symmetric concatenation

The study proposed in this work requires the construction of new codes. In the following propositions, we show how to use **symmetric concatenation** to obtain new codes.

Proposition 3.1. Let $n \in \mathbb{N}^*$, $c_n = z_0 \ldots z_n$ be a code, and $V_r = \underbrace{00\ldots00}_{r \text{ times}}$ with $r \geq 0$. The list $c_n V_r c_n$ is a code.

Proof. We denote the list $c_n V_r c_n$ by $w_0 w_1 \ldots w_{2n+r+1}$. In this list, $w_0 = z_0 = 1$ and $w_{2n+r+1} = 0$, thus satisfying Proposition 2.4. Since c_n is a code, Proposition 2.5 is satisfied, i.e., $z_i z_{i+1} = 01$ in c_n implies that $z_{n-(i+1)} = 0$, and $z_i z_{i+1} = 10$ implies that $z_{n-(i+1)} = 1$. We must verify that the same symmetry condition is satisfied for the list $w_0 w_1 w_2 \ldots w_{2n+r+1}$.

- For $i < n$, $w_i w_{i+1} = z_i z_{i+1}$ and the symmetric position is $w_{2n+r+1-(i+1)} = z_{n-(i+1)}$. The symmetry condition in the code c_n ensures that the condition of Proposition 2.5 is fulfilled.
- For $n \leq i \leq n+r-1$, $w_i w_{i+1} = 00$. In this case, the symmetric position $w_{2n+r+1-(i+1)} = 0$ fulfills the symmetry condition.
- For $i = n+r$, $w_i w_{i+1} = 01$, the symmetric position is $w_{2n+r+1-(i+1)} = w_{2n+r+1-(n+r+1)} = w_n = 0$. It fulfills Proposition 2.5.
- For $n+r < i \leq 2n+r$, $w_i w_{i+1} = z_{i-(n+r+1)} z_{i-(n+r+1)+1}$, the symmetric position is $w_{2n+r-(i+1)} = z_{2n+r-(i+1)}$. By letting $j = i-(n+r)$, the symmetric position of $z_j z_{j+1}$ is $z_{n-(j+1)} = z_{2n+r-(i+1)}$. Thus, the symmetry of the code c_n ensures that the condition of Proposition 2.5 is again fulfilled.

We conclude that the list $w_0 w_1 ... w_{2n+r+1}$ satisfies the conditions of Propositions 2.4 and 2.5, thus being a **new code**.

□

With similar arguments to those used in Proposition 3.1, we can replace V_r with a code and obtain a new code. See the following proposition.

Proposition 3.2. If, for $n, m \in \mathbb{N}^*$, the lists $c_n = z_0 \dots z_n$ and $d_m = x_0 \dots x_m$ are codes, then the list $c_n d_m c_n$ is also a code.

Proof. We denote the list $c_n d_m c_n = w_0 w_1 ... w_{2n+m+2}$. In this list, $w_0 = z_0 = 1$ and $w_{2n+m+2} = x_m = 0$, thus satisfying Proposition 2.4. Since c_n and d_m are codes, these lists satisfy Proposition 2.5. We must verify that the same symmetry condition is satisfied for the list $w_0 w_1 w_2 ... w_{2n+m+2}$.

For $i < n$, $w_i w_{i+1} = z_i z_{i+1}$. The symmetric position is $w_{2n+m+2-(i+1)} = z_{n-(i+1)}$. The symmetry condition in the code c_n ensures that the condition of Proposition 2.5 is fulfilled.

For $i = n$, $w_i w_{i+1} = 01$, the symmetric position is

$$w_{2n+m+2-(i+1)} = w_{2n+m+2-(n+1)} = w_{n+m+1} = x_m = 0 \ .$$

It fulfills Proposition 2.5.

For $n+1 \leq i \leq n+m$, $w_i w_{i+1} = x_{i-(n+1)} x_{i+1-(n+1)}$. In this case, the symmetric position $w_{2n+m+2-(i+1)} = x_{n+m+1-(i+1)}$ fulfills the symmetry condition.

For $n+1 \leq i \leq n+m$, $w_i w_{i+1} = x_{i-(n+1)} x_{i+1-(n+1)}$. In this case, the symmetric position $w_{2n+m+2-(i+1)} = x_{n+m+1-(i+1)}$ fulfills the symmetry condition.

For $n+m+1 < i \leq 2n+m+1$, $w_i w_{i+1} = z_{i-(n+m+2)} z_{i-(n+m+2)+1}$, the symmetric position is $w_{2n+m+2-(i+1)} = z_{2n+m+2-(i+1)}$. By letting $j = i-(n+m+2)$, the symmetric position of $z_j z_{j+1}$ is $z_{n-(j+1)} = z_{2n+m+2-(i+1)}$. Thus, the symmetry of the code c_n ensures that the condition of Proposition 2.5 is again fulfilled.

We conclude that the list $w_0 w_1 ... w_{2n+m+2}$ satisfies the conditions of Propositions 2.4 and 2.5, thus being a new code.

□

Recursively, we can generalize the construction of new codes.

Theorem 3.3. *Let c_1, c_2, ..., c_n, d_m be codes, and $V_r = \underbrace{00\ldots00}_{r \text{ times}}$ with $r \geq 0$. Then, the symmetric concatenation $c_n c_{n-1} ... c_1 V' c_1 ... c_{n-1} c_n$ is a code, where $V' = V_r$ or $V' = d_m$.*

Proof. For $n = 1$, Proposition 3.1 or 3.2 ensures that $c_1 V' c_1$ is a code. Suppose the list is a code for a number k of codes, that is, $c_k c_{k-1} ... c_1 V' c_1 ... c_{k-1} c_k$. We must show that $c_{k+1} c_k ... c_1 V' c_1 ... c_k c_{k+1}$ is also a code, with c_{k+1} being a code. From the induction hypothesis, it follows that $c_k c_{k-1} ... c_1 V' c_1 ... c_{k-1} c_k$ is a code, and therefore Proposition 3.2 ensures that $c_{k+1} c_k c_{k-1} ... c_1 V' c_1 ... c_{k-1} c_k c_{k+1}$ is also a code. We conclude that the list $c_n c_{n-1} ... c_1 V' c_1 ... c_{n-1} c_n$ is a code for all $n \geq 1$.

□

Finally, the main result of this section is

Theorem 3.4. *Let c_1, c_2, ..., c_n, d_m be codes and $V_r = \underbrace{00\ldots00}_{r \text{ times}}$ with $r \geq 0$. Then, the symmetric concatenation with 0,*

$$c_n 0 c_{n-1} 0 ... c_2 0 c_1 V' c_1 0 c_2 ... c_{n-1} 0 c_n$$

is a truncated code, where $V' = V_r$ or $V' = d_m$.

Proof. We must show that $c_n 0 c_{n-1} 0 ... c_2 0 c_1 V' c_1 0 c_2 ... c_{n-1} 0 c_n 0$ is a code. To do this, let $c'_j = c_j 0$, which is a code by Proposition 2.7. Now, it follows from Theorem 3.3 that $c'_n c'_{n-1} ... c'_1 V' c'_1 ... c'_{n-1} c'_n$ is a code. □

Remark 3.5. For all $i = 1, 2, ..., n$, let c_i be a code, by Proposition 2.2, $c_{i0} = c_i 0$ also be a code. In connection with Theorem 3.4, we will have that

$$c'_{n0} c'_{(n-1)0} ... c'_{10} V' c'_{10} ... c'_{(n-1)0} c'_{n0}$$

is a code.

4 The magic reverse divisors

In this section, we show that the symmetric concatenation of magic reverse divisors is also a magic reverse divisor, Theorem 4.10. For this purpose, we use the results of Webster and Williams [16] on the symmetric concatenation of reverse divisors.

We saw in Example 1.5, the reverse divisors with more than three digits. In Costa and Santos [6], Propositions 14 and 15, it is shown that there are no reverse divisors with 2 or 3 digits.

Remark 4.1. Recently, Weisgerber[17] revisits the famous essay **A Mathematician's Apology**, from 1940, by the English mathematician Godfrey Harold Hardy (1877-1947), which provides a justification for a serious study of mathematics Hardy argues that mathematics must be justified as a creative art, the 'beauty is the first test: there is no permanent place in the world for ugly mathematics' [7, 1940/2012, p. 85]. According to Weisgerber, "The general criteria Hardy identifies and discusses in his essay by which a mathematician's patterns must be judged are beauty and seriousness"[17, p. 2]. At the center of this discussion is Hardy's example: 8712 and 9801 are the only four-digit numbers that are integer multiples of their reverse divisors.

Regarding the results involving reverse divisors, we emphasize that Theorem 1.7 is a direct consequence of the following results, and the proof can be consulted in [6].

Proposition 4.2. [6] For every natural number $n > 3$, we have that

$$11 \times (10^{n-2} - 1) = 10\ \underbrace{99\ldots99}_{n-4 \text{ times}}\ 89\ .$$

Proposition 4.3. [6] For every natural number $n > 3$, we have that

$$9 \times 10\ \underbrace{99\ldots99}_{n-4 \text{ times}}\ 89 = 98\ \underbrace{99\ldots99}_{n-4 \text{ times}}\ 01\ .$$

The following results prove Theorem 1.8 and their proofs can also be found in [6].

Proposition 4.4. [6] For every natural number $n > 3$, we have

$$2 \times 11 \times (10^{n-2} - 1) = 21\ \underbrace{99\ldots99}_{n-4 \text{ times}}\ 78\ .$$

Proposition 4.5. [6] For every natural number $n > 3$, we have

$$4 \times 21\ \underbrace{99\ldots99}_{n-4 \text{ times}}\ 78 = 87\ \underbrace{99\ldots99}_{n-4 \text{ times}}\ 12\ .$$

Therefore, the reverse divisors of type 1 are Ball numbers.

Theorem 4.6. *[6] Every reverse divisor of the form* $11 \times (10^n - 1)$ *is a Ball magic number, for* $n > 1$.

The next result is an immediate consequence of Theorem 4.6. Furthermore, reverse divisors of type 2 are also Ball numbers.

Corollary 4.7. [6] Every reverse divisor of the form $22 \times (10^n - 1)$ is twice a magic number, for $n > 1$.

As we stated, Theorem 4.6 ensures that every reverse divisor of the form $11 \times (10^n - 1)$, the reverse divisors given in Theorem 1.7, are Ball magic numbers. Similarly, Corollary 4.7 ensures that every reverse divisor of the form $22 \times (10^n - 1)$ is twice a Ball magic number. It is worth noting that Webster and Williams [16] also show that numbers of the first type have 9 as their reverse quotient, while those of the second type have 4 as their reverse quotient.

Another consequence of Theorem 4.6 that can be easily verified.

Corollary 4.8. [6] If B is a magic number and also a reverse divisor, then B added to its reverse B' is $10B$.

Now we will demonstrate that Question 1.2 has an affirmative answer, that is, the number 10891089 is a reverse divisor, and also a Ball number.

Firstly, given $k_1, k_2, ..., k_t, r$ positive integers, $B_{k_1}, B_{k_2}, ..., B_{k_t}, V_r$ where B_j are Ball numbers for $j \in k_1, k_2, ..., k_t$ and $V_r = \underbrace{00\ldots00}_{r \text{ times}}$ with $r \geq 0$. Consider a number of the form $B_{k_1}B_{k_2}...B_{k_t}V_rB_{k_t}...B_{k_2}B_{k_1}$. Now, we have a more general question

Question 4.9. Is a number in the form $B_{k_1}B_{k_2}...B_{k_t}V_rB_{k_t}...B_{k_2}B_{k_1}$ a reverse divisor and a Ball number?

In order to answer that question, let us use Webster's result[16].

Theorem 4.10. *Given* $k_1, k_2, ..., k_t$ *positive integers,* $B_{k_1}, B_{k_2}, ..., B_{k_t}$ *reverse divisors of type 1 (Theorem 1.7). Consider* $V_r = \underbrace{00\ldots00}_{r \text{ times}}$ *with* $r \geq 0$. *Every number in the form* $B_{k_1}B_{k_2}...B_{k_t}VB_{k_t}...B_{k_2}B_{k_1}$ *is a reverse divisor, where* $V = V_r$ *or* V *is a reverse divisor of type 1, with* r *digits.*

Proof. We have that

$$\begin{array}{rl} & B_{k_1}B_{k_2}...B_{k_t}VB_{k_t}...B_{k_2}B_{k_1} \\ = & B_{k_1} \times 10^{k_1+2(k_2+...+k_t)+r} + B_{k_2} \times 10^{k_2+2(k_3+...+k_t)+r} + ... + \\ & B_{k_t} \times 10^{k_1+k_2+...+k_t+r} + B_{k_t} \times 10^{k_1+k_2+...+k_{t-1}} + ... + B_{k_2} \times 10^{k_1} + B_{k_1}\ . \end{array} \tag{4.1}$$

It follows that every reverse divisor $B_s = 9B'_s$, for $s \in k_1, k_2, ..., k_t$ and $V = 9V'$. Thus, for each s, substituting into Equation 4.1, we will have that

$$\begin{array}{rl} & B_{k_1}B_{k_2}...B_{k_t}VB_{k_t}...B_{k_2}B_{k_1} \\ = & B_{k_t} \times 10^{k_1+k_2+...+k_t+r} + B_{k_t} \times 10^{k_1+k_2+...+k_{t-1}} + ... + B_{k_2} \times 10^{k_1} + B_{k_1} \\ = & 9B'_{k_t} \times 10^{k_1+k_2+...+k_t+r} + 9B'_{k_t} \times 10^{k_1+k_2+...+k_{t-1}} + ... + 9B'_{k_2} \times 10^{k_1} + 9B'_{k_1} \\ = & 9\big(B'_{k_t} \times 10^{k_1+k_2+...+k_t+r} + B'_{k_t} \times 10^{k_1+k_2+...+k_{t-1}} + ... + B'_{k_2} \times 10^{k_1} + B'_{k_1}\big) \\ = & 9 \times B'_{k_1}B'_{k_2}...B'_{k_t}V'B'_{k_t}...B'_{k_2}B'_{k_1}\ . \end{array}$$

Therefore, it follows that $B_{k_1}B_{k_2}...B_{k_t}VB_{k_t}...B_{k_2}B_{k_1}$ is a reverse divisor with a reverse quotient of 9. ☐

Example 4.11. Consider the pairs of ball numbers $B_1 = 1089 = 99 \times 11$ and $B_2 = 109989 = 99 \times 1111$. Let $B = B_1B_2B_1$ be the symmetric concatenation of B_1 and B_2, i.e., $B = \mathbf{1089}109989\mathbf{1089}$, observe that, $\mathbf{1089}109989\mathbf{1089} = 1089 \times 10^{10} + 109989 \times 10^4 + 1089$. Thus,

$$\begin{array}{rcl} b & = & 1089 \times 10^{10} + 109989 \times 10^4 + 1089) \\ & = & 99 \times 11 \times 10^{10} + 99 \times 1111 \times 10^4 + 99 \times 11 \\ & = & 99 \times (11 \times 10^{10} + 1111 \times 10^4 + 11) \\ & = & 99 \times (11\mathbf{00}1111\mathbf{00}11)\ . \end{array}$$

See that **11** is a truncated code and $c_1 = 110$ is a code, the same way **1111** is a truncated code and $c_2 = 1110$ is a code. According to Theorem 3.4, $c'_1c'_2c_1$ is a truncated code, where $c'_i = c_i0$ for $i = 1, 2$. Therefore, the number b is a Ball number.

The main result of this work is

Theorem 4.12. *Let $B_{k_1}, B_{k_2}, ..., B_{k_t}$ reverse divisors of type 1, where $k_1, k_2, ..., k_t$ are positive integers (Theorem 1.7), and let $V_r = \underbrace{00\ldots00}_{r\ times}$ with $r \geq 0$. Every number in the form $B_{k_1}B_{k_2}...B_{k_t}VB_{k_t}...B_{k_2}B_{k_1}$ is a Ball number, where $V = V_r$ or $V = \underbrace{99\ldots99}_{n\ times}$, with n even, or V is a reverse divisor of type 1.*

Proof. It follows from Theorem 4.6 that every reverse divisor B_{k_s} is a Ball number, for $s \in 1, 2, ..., t$. Furthermore, we have

$$B_{k_s} = 99 \times z_{0_s} \ldots z_{k_s} = 99 \times c_{k_s} \ . \tag{4.2}$$

Where $z_{0_s} \ldots z_{k_s}$ is the truncated code associated with the Ball number B_{k_s}.

And according to [13, Proposition 2.2], for n even

$$\underbrace{99 \ldots 99}_{n \text{ times}} = 99 \times \underbrace{1010 \ldots 101}_{n-1 \text{ times}} = 99 \times uz_{n-1} \ . \tag{4.3}$$

From Proposition 2.10, the number $c = \underbrace{1010 \ldots 101}_{n-1 \text{ times}}$ is also a truncated code.

We still have that

$$\begin{aligned} & B_{k_1}B_{k_2}...B_{k_t}VB_{k_t}...B_{k_2}B_{k_1} \\ = \ & B_{k_1} \times 10^{k_1+2(k_2+...+k_t)+r} + B_{k_2} \times 10^{k_2+2(k_3+...+k_t)+r} + ... + \\ & B_{k_t} \times 10^{k_1+k_2+...+k_t+r} + B_{k_t} \times 10^{k_1+k_2+...+k_{t-1}} + ... + B_{k_2} \times 10^{k_1} + B_{k_1} \ . \end{aligned} \tag{4.4}$$

For each s, substituting the Equations 4.2 and 4.3 into Equation 4.4, we have:

$$B_{k_1}B_{k_2}...B_{k_t}V_rB_{k_t}...B_{k_2}B_{k_1} = 99 \times c'_{k_1}c'_{k_2}...c'_{k_t}c'c'_{k_t}...c'_{k_2}c_{k_1} \ .$$

Where each $c'_{k_j} = c_{k_j}0$ or $c'_{k_j} = c_{k_j}00$. It follows from Theorem 3.4 that $c'_{k_1}c'_{k_2}...c'_{k_t}V'c'_{k_t}...c'_{k_2}c'_{k_1}$ is a code. Therefore, from Theorem 2.2, it follows that $B_{k_1}B_{k_2}...B_{k_t}VB_{k_t}...B_{k_2}B_{k_1}$ is a Ball number.

□

Note that Theorems 4.10 and 4.12 affirmatively answer Question 4.9. Thus, the reverse divisors of the third type, Theorem 4.10, which were obtained by concatenating reverse divisors of the first type, Theorem 1.7, are also Ball magic numbers.

Example 4.13. Consider the pairs of reverse divisors $n_1 = 1089$, $n'_1 = 9801$ and $n_2 = 109989$, $n'_2 = 989901$. Let $n = n_1n_2n_1$ be the symmetric concatenation of n_1 and n_2, i.e., $n = \mathbf{1089}109989\mathbf{1089}$, observe that, $\mathbf{1089}109989\mathbf{1089} = 1089 \times 10^{6+4} + 109989 \times 10^4 + 1089$. Thus,

$$\begin{aligned} 9n &= 9(1089 \times 10^{10} + 109989 \times 10^4 + 1089) \\ &= 9801 \times 10^{10} + 989901 \times 10^4 + 9801) \\ &= \mathbf{9801}989901\mathbf{9801} \ . \end{aligned}$$

Therefore, the numbers $n = n_1n_2n_1$ and $n' = n'_1n'_2n'_1$ are reverse divisors.

Example 4.14 shows that not every symmetric concatenation of Ball numbers results in a reverse divisor.

Example 4.14. Consider the pairs of Ball numbers $B_1 = 1089 = 99 \times 11$ and $B_2 = 9999 = 99 \times 101$. Let $B = B_1 B_2 B_1$ be the symmetric concatenation of B_1 and B_2, i.e., $b =$ **1089**9999**1089**, observe that, **1089**9999**1089** $= 1089 \times 10^8 + 9999 \times 10^4 + 1089$. Thus,

$$\begin{aligned} 9B &= 9(1089 \times 10^8 + 9999 \times 10^4 + 1089) \\ &= 9801 \times 10^8 + 89991 \times 10^4 + 9801 \\ &= 980999919801. \end{aligned}$$

The numbers B and $B' = 9B$ **are not** reverse divisors.

However, the following result shows that every symmetric concatenation of Ball numbers is also a Ball number. Remember, the Example 4.14 exhibits a symmetric concatenation of Ball numbers that are reverse divisors, which also results in a new reverse divisor (Theorem 4.15).

Theorem 4.15. *Given* $k_1, k_2, ..., k_t$ *positive integers,* $B_{k_1}, B_{k_2}, ..., B_{k_t}$ *Ball numbers. Consider* $V_r = \underbrace{00\ldots00}_{r\ times}$ *with* $r \geq 0$. *Every number in the form*

$$B_{k_1} B_{k_2} ... B_{k_t} V B_{k_t} ... B_{k_2} B_{k_1}$$

is a Ball number, where $V = V_r$ *or* V *is a Ball number.*

Proof. We have that each Ball number is of the form

$$B_{k_s} = 99 \times z_{0_s} \ldots z_{k_s} = 99 \times c_{k_s} \ . \tag{4.5}$$

Where $z_{0_s} \ldots z_{k_s}$ is the code associated with the Ball number B_{k_s}.

We also have that

$$\begin{aligned} & B_{k_1} B_{k_2} ... B_{k_t} V B_{k_t} ... B_{k_2} B_{k_1} \\ = \ & B_{k_1} \times 10^{k_1 + 2(k_2 + ... + k_t) + r} + B_{k_2} \times 10^{k_2 + 2(k_3 + ... + k_t) + r} + ... \\ & + B_{k_t} \times 10^{k_1 + k_2 + ... + k_t + r} + B_{k_t} \times 10^{k_1 + k_2 + ... + k_{t-1}} + ... + B_{k_2} \times 10^{k_1} + B_{k_1} \ . \end{aligned} \tag{4.6}$$

For each s, substituting Equation 4.5 into Equation 4.6, we have:

$$
\begin{aligned}
& B_{k_1}B_{k_2}...B_{k_t}VB_{k_t}...B_{k_2}B_{k_1} \\
= \; & 99(c_{k_1} \times 10^{k_1+k_2+...+k_t+r} + c_{k_t} \times 10^{k_1+k_2+...+k_{t-1}} + ... + c_{k_2} \times 10^{k_1} + c_{k_1}) \\
= \; & 99 \times c'_{k_1}c'_{k_2}...c'_{k_2}c_{k_1} \ .
\end{aligned}
$$

Where each $c'_{k_j} = c_{k_j}0$ or $c'_{k_j} = c_{k_j}00$. It follows from Theorem 3.4 that $c'_{k_1}c'_{k_2}...c'_{k_t}V'c'_{k_t}...c'_{k_2}c'_{k_1}$ is a code. Therefore, it follows from Theorem 2.2 that $B_{k_1}B_{k_2}...B_{k_t}VB_{k_t}...B_{k_2}B_{k_1}$ is a Ball number. □

5 Conclusion

Throughout history, we have seen the interest that integers and their properties stimulate within the mathematical community. Magic Ball numbers and reverse divisors are examples that reinforce this thesis. According to Hardy [7], the fascinating discovery of reverse divisor numbers has shown us how even simple patterns can deeply engage mathematicians. Hardy's focus on the beauty of mathematics reflects the aesthetic allure found in these patterns.

The elegance and simplicity of mathematical relationships often lead mathematicians to discover and appreciate the inherent beauty of the subject. This paper reveals not only the beauty but also the intriguing nature of reverse divisors, Ball numbers, and their relationships.

With the potential to expand in various directions, this work explores possibilities for advancement in both the educational applications and theoretical aspects related to the studied numbers. From an educational perspective, magic Ball numbers and reverse divisors naturally arouse curiosity among students, making them suitable for creating more attractive and engaging materials and lessons. In terms of advancing and structuring the theoretical properties involving such numbers, the work extends the set of reverse divisors by including the possibility of numbers in the form 1099...9989 at the center of symmetric concatenations, based on the formulation provided by Webster and Williams[16]. It demonstrates that the symmetric concatenation of codes forms a new code, thereby expanding the set of numbers that are simultaneously reverse divisors and magic Ball numbers.

Acknowledgments

This work was partially supported by the PROPESQ-UFT.

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Received 05 November 2024
Revised 05 December 2024
Accepted 27 February 2025