Research Article

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On generating properties of the weak commutativity of *p*-groups, *p* odd

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Abstract: In the present paper, we examine the generating properties of Sidki's weak commutativity group. More precisely, if G and G^φ are two isomorphic groups, the weak commutativity group $\chi(G)$ is the group generated by G and G^φ subject to the relations $[g, g^\varphi] = 1$ for all $g \in G$. Here we provide bounds for the number of generators of some subgroups of $\chi(G)$ when G is a p-group of odd order and either G is powerful or $D(G)$ is abelian.

Keywords: Metabelian groups, extra-special p-groups, weak commutativity, powerful p-groups

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1 Introduction

Let G be a group and let G^φ be a copy of the group G, where $\varphi : G \to G^\varphi$ is the isomorphism given by $g \mapsto g^\varphi$. The *weak commutativity group* $\chi(G)$ is defined as follows:

$$\chi(G) = \langle G \cup G^\varphi \mid [g, g^\varphi] = 1 \text{ for all } g \in G\rangle.$$

The group $\chi(G)$, also known as Double Sidki of G, was introduced by Sidki in [17] to determine some finiteness conditions in Group Theory. Nonetheless, its interest comes from its connection with other group constructions and homology. To better understand this, we recall the main properties of some subgroups of the group $\chi(G)$. First of all, the subgroups $L(G) = \langle g^{-1}g^\varphi \mid g \in G\rangle$ and $D(G) = [G, G^\varphi] = \langle [g, h^\varphi] \mid g, h \in G\rangle$ commute. The subgroup $W(G) = L(G) \cap D(G)$ is central in $L(G)D(G)$ and it contains the normal subgroup $R(G) = [G, L(G), G^\varphi] \leq W(G)$. As a result, the group $W(G)/R(G)$ is isomorphic to the Schur multiplier $M(G)$ of a group G, and $D(G)/R(G)$ is isomorphic to the non-abelian exterior square $G \wedge G$. For more details about the group $\chi(G)$ and related constructions, we refer to [3, 5, 16, 17].

The weak commutativity group $\chi(G)$ preserves many invariants of the group G. Indeed, if G is a finite π-group (π is the set of primes), nilpotent, solvable, polycyclic-by-finite or finitely presented, virtually nilpotent, of growth type, then so is the group $\chi(G)$ (see [6, 7, 13, 14, 16, 17]). On the other hand, the study of quantitative aspects of the group $\chi(G)$, when G is a finite p-group, first appears in the seminal works [11, 16, 17]. These works provide some bounds to the order, the exponent and the nilpotency class of the group $\chi(G)$ and its subsections. Recently, the authors [1, 2, 4] deepened the study of $\chi(G)$, when G is a finite p-group. In the present paper, we deal with generating properties of the group $D(G)$ and its subgroups, when G belongs to two major classes of finite p-groups of odd order.

To set the notation, let p be a prime and G be a finite p-group. We denote by $d(G)$ the minimal number of generators of G, i.e., $[G : \Phi(G)] = p^{d(G)}$, where $\Phi(G)$ is the Frattini subgroup of G. First, we consider p to be an odd prime and the class of p-groups G such that $D(G)$ is abelian. Following [16], we point out that $D(G)$ is abelian

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if and only if G' centralizes $D(G)$. As a consequence, G' is necessarily abelian, that is, G is a metabelian p-group. Due to the interplay between G' and $D(G)$, a first idea is to study the minimal number of generators of $D(G)$ in terms of the rank of the abelian group G'.

Theorem 1.1. *Let p be an odd prime and let G be a finite p-group with $G' \leq C_{\chi(G)}(D(G))$. If G' has rank k and $d(G) = d$, then $d(D(G)) \leq \frac{d(d+2k-1)}{2}$.*

It is clear that if the derived subgroup G' is cyclic, then $D(G)$ is abelian, since $D(G)$ is isomorphic to G' modulo its central subgroup $W(G)$. An interesting subclass of p-groups with cyclic derived subgroups is that of extra-special p-groups, whose derived subgroup has order p (see Remark 2.5 for more information). Indeed, we obtain sharper information about the structure and the order of $D(G)$ when G belongs to this class.

Theorem 1.2. *Let p be an odd prime and let G be a finite extra-special p-group. Then $D(G)$ is an elementary abelian p-group of rank at most $\binom{d+1}{2}$.*

As a consequence, we give a more detailed description of the subgroup $D(G)$ depending on the exponent of the extra-special group G.

Corollary 1.3. *Let p be an odd prime and let G be an extra-special p-group of order p^{2n+1}.*
(a) *If $n = 1$, then $R(G) = 1$ and*
  (1) *$D(G) = C_p \times C_p \times C_p$ if $\exp(G) = p$,*
  (2) *$D(G) = C_p$ if $\exp(G) = p^2$.*
(b) *If $n > 1$ and $\exp(G) = p$, then $|R(G)| \leq p^{2n}$ and $|D(G)| \leq p^{2n^2+n}$.*
(c) *If $\exp(G) = p^2$, then $|R(G)| = 1$ and $|D(G)| = p^{2n^2-n}$.*

It is worth to mention that Theorem 1.1 (for rank 1), Theorem 1.2 and Corollary 1.3 (b) cannot be improved. For instance, if G is an extra-special p-group of order 3^5 with exponent $\exp(G) = p$, then the group $D(G)$ is elementary abelian of rank 10 and the bounds obtained in Corollary 1.3 (b) are attained.

The second class of odd order p-groups we considered is that of powerful p-groups. A finite p-group G is said to be *powerful* if $p > 2$ and $G' \leq G^p$, or $p = 2$ and $G' \leq G^4$. A related concept is that of *powerfully embedded* subgroup, that is, a subgroup N of G is powerfully embedded in G if $[N, G] \leq N^p$ for $p > 2$, or $[N, G] \leq N^4$ for $p = 2$. More information about finite powerful p-groups can be found in [15]. In [15], Lubotzky and Mann showed that if G is a powerful p-group with $d = d(G)$, then the Schur Multiplier $M(G)$ and the exterior square $G \wedge G$ can be generated by at most $\binom{d}{2}$ generators. As a consequence, for a powerful p-group G they presented bounds to the order of the Schur Multiplier $M(G)$ and the exterior square $G \wedge G$ in terms of the number of generators $d(G)$ and $\exp(G)$.

Regarding the weak commutativity group, if p is an odd prime and G is a powerful p-group, the group $\chi(G)$ may not be a powerful group. Indeed, the group $G = C_9 \times C_3$, which is the second group of order 27 in the SmallGroup library [18], is powerful, but $\chi(G)$ is not. On the other hand, the subgroups $\chi(G)'$ and $D(G)$ are always powerful embedded in $\chi(G)$ (see [4]). Thus we will take advantage of this to better describe the subgroups $D(G)$ and $\chi(G)'$ when G is a finite powerful p-group. We start bounding the numbers $d(D(G))$ and $d(\chi(G)')$ in terms of $d(G)$.

Theorem 1.4. *Let d be a positive integer and p be an odd prime. Assume that G is a powerful p-group with $d(G) = d$. Then $d(\chi(G)') \leqslant 3\binom{d}{2}$ and $d(D(G)) \leqslant \binom{d}{2}$.*

We point out that the bound for $d(D(G))$ obtained in the previous result coincides with the one obtained in [15] for $M(G)$ and $G \wedge G$. In particular, it is the best possible as the group $G = [729, 132]$, which is the 132nd group of order 729 in the SmallGroup library [18], is such that $d(D(G)) = 3$ and $d(\chi(G)') = 9$. Now, the powerful structure of the group G and Theorem 1.4 yield the following bounds for the order of $D(G)$ and $\chi(G)'$.

Theorem 1.5. *Let d and e be positive integers. Let p be an odd prime. Assume that G is a powerful p-group with $d(G) = d$ and exponent $\exp(G) = p^e$.*
(a) *If $p = 3$, then $|\chi(G)'| \leqslant 3^{(e+1)\binom{d}{2}+2d(e-1)}$ and $|D(G)| \leqslant 3^{(e+1)\binom{d}{2}}$.*
(b) *If $p \geqslant 5$, then $|\chi(G)'| \leqslant p^{e\binom{d}{2}+2d(e-1)}$ and $|D(G)| \leqslant p^{e\binom{d}{2}}$.*

The bound obtained in Theorem 1.5 (b) is somehow sharp. Indeed, if $p \geq 5$, then $G = C_p \times C_p$ is such that $D(G) = \chi(G)' = C_p$. On the other hand, for $p = 3$ we have no examples attaining the obtained bound in Theorem 1.5 (a).

We conclude with some application of the powerful theory for the group $\chi(G)$. We say that a p-group is *special of rank k* if $G' = Z(G)$ is an elementary abelian p-group of rank k. As a consequence, $\Phi(G) = G'$ and, if G has order p^n, then $d(G) = n - k$.

Theorem 1.6. *Let p be an odd prime and let G be a special p-group of order p^n and rank k. If $G' = G^p$, then $|D(G)| \leq p^{\frac{(n-k)(n+k-1)}{2}}$.*

In general, if G is a special p-group of rank k, then the group $D(G)$ does not need to be abelian. For instance, the group $G = [243, 41]$, which is the 41st group of order 243 in the SmallGroup library [18], is a special p-group of rank 2 in which $D(G)$ is non-abelian.

The paper is organized as follows. In Section 2, we collect results of general nature that are later used in the proofs of our main theorems. More precisely, we report on some properties of p-groups and on the structure of the lower central terms of the group $\chi(G)$. Section 3 is devoted to the study of finite p-groups G such that the subgroup $D(G)$ is abelian. Finally, in Section 4, we obtain bounds to the order and number of generators of $D(G)$, when G is a finite powerful p-group, $p \geqslant 3$.

2 Preliminaries

Our notation and terminology for groups is standard. For arbitrary elements $x, y \in G$, we write $x^y = y^{-1}xy$ for the conjugate of x by y; the commutator of x and y is then $[x, y] = x^{-1}y^{-1}xy$ and our commutators are left normed: $[x, y, z] = [[x, y], z]$.

The following theorem is known as P. Hall's collection formula (see [9, Theorem 2.6] for more details).

Theorem 2.1. *Let G be a p-group and let x, y be the elements of G. Then for any $k \geq 0$ we have*

$$[x, y]^{p^k} \equiv [x^{p^k}, y] \pmod{\gamma_2(L)^{p^k}\gamma_p(L)^{p^{k-1}}\gamma_{p^2}(L)^{p^{k-2}}\cdots\gamma_{p^k}(L)},$$

where $L = \langle x, [x, y]\rangle$.

The next result is the basic fact about finite p-groups (see [10]).

Lemma 2.2. *Let G be a finite p-group and let N, M be normal subgroups of G. If $N \leq M[N, G]N^p$, then $N \leq M$.*

The following results concern the powerful p-groups and can be found in [15]. They will be crucial tools in the proof of Theorem 1.6.

Proposition 2.3. *Let G be a p-group and let X be a subset of G such that the normal closure $\langle X\rangle^G$ of X in G is powerfully embedded in G. Then $\langle X\rangle^G = \langle X\rangle$.*

Proposition 2.4. *Let G be a powerful p-group generated by the elements $g_1, \dots, g_d \in G$. Then $G = \langle g_1\rangle \cdots \langle g_d\rangle$.*

Now, we recall that a finite p-group G is called *extra-special* if the derived subgroup G' and the center $Z(G)$ coincide and have order p. Every finite extra-special p-group has order p^{2n+1} for some positive integer n.

Let p be an odd prime and consider the following presentations:

$$M(p) = \langle a, b \mid a^p = b^p = 1,\ [a, b]^a = [a, b] = [a, b]^b\rangle,$$
$$N(p) = \langle a, b \mid a^{p^2} = b^p = 1,\ a^b = a^{1+p}\rangle.$$

Remark 2.5. If G is an extra-special p-group of order p^{2n+1}, p is odd, then either G has exponent p and it is the central product of n groups of type $M(p)$, or G has exponent p^2 and it is the central product of $n - 1$ groups of type $M(p)$ and a group of type $N(p)$. Therefore, an extra-special group can be presented using at most one generator of order p^2.

The following is [12, Theorem 3.3.6].

Theorem 2.6. *Let p be an odd prime and let G be an extra-special p-group of order p^{2n+1}.*
(i) *If $n = 1$ and $\exp(G) = p$, then $M(G) \simeq C_p \times C_p$.*
(ii) *If $n = 1$ and $\exp(G) = p^2$, then $M(G)$ is trivial.*
(iii) *If $n > 1$, then $M(G)$ is an elementary abelian p-group of order p^{2n^2-n-1}.*

We conclude this section with some properties of the weak commutativity group. The following basic properties are consequences of the defining relations of $\chi(G)$ and the commutator rules (see [17, Proposition 4.1.13] and [16, Lemma 2.1] for more details).

Lemma 2.7. *The following relations hold in $\chi(G)$ for all $x, y, y_i, z, z_i \in G$:*
(a) $[x, y^\varphi] = [x^\varphi, y]$,
(b) $[x, y^\varphi]^{z^\varphi} = [x, y^\varphi]^z$,
(c) $[x, y^\varphi]^{\omega(z_1^{\varepsilon_1},\dots,z_n^{\varepsilon_n})} = [x, y^\varphi]^{\omega(z_1,\dots,z_n)}$ *for $\varepsilon_i \in \{1, \varphi\}$ and any word $\omega(z_1, \dots, z_n) \in G$,*
(d) $[x^\varphi, y_1, \dots, y_n, x] = [x, y_1, \dots, y_n, x^\varphi]$,
(e) $[x, y^\varphi, y, \dots, y] = [x, y, y^\varphi, \dots, y] = \cdots = [x, y, y, \dots, y^\varphi]$.

Let N be a normal subgroup of a finite group G and set $\overline{G}$ for the quotient group G/N. Then the canonical epimorphism $\pi : G \to \overline{G}$ gives rise to an epimorphism $\tilde{\pi} : \chi(G) \to \chi(\overline{G})$ such that $g \mapsto \overline{g}$, $g^\varphi \mapsto \overline{g^\varphi}$, where $\overline{G^\varphi} = G^\varphi/N^\varphi$ is identified with $\overline{G}^\varphi$. We will denote by $\tilde{\pi}_D$ the restriction of $\tilde{\pi}$ to $D(G)$.

Lemma 2.8 ([17, Propositions 4.1.12 (i) and 4.1.13 (i)–(iii)]). *With the above notation we have:*
(a) $\ker(\tilde{\pi}_D) = [N, G^\varphi] \trianglelefteq \chi(G)$,
(b) *the sequences*

$$1 \to [N, G^\varphi] \to D(G) \to D(\overline{G}) \to 1$$

and

$$1 \to [N, G^\varphi] \cap R(G) \to R(G) \to R(\overline{G}) \to 1$$

are exact, where $R(\overline{G}) = [\overline{G}, L(\overline{G}), \overline{G}^\varphi] \leq D(\overline{G}) = [\overline{G}, \overline{G}^\varphi] \leq \chi(\overline{G})$.

The next lemma is taken from [1].

Lemma 2.9. *Let G be a finite group and let A be an abelian normal subgroup of G. Then $[A, G^\varphi]$ is nilpotent of class at most 2. Moreover, if $\exp(A)$ is odd, then $\exp([A, G^\varphi])$ divides $\exp(A)$.*

We denote by $\gamma_i(G)$ the ith term of the lower central series of a group G. When $G' \leq C_{\chi(G)}(D(G))$, the lower central series of $\chi(G)$ is completely described. Indeed, the following is [16, Proposition 3.2.3].

Proposition 2.10. *Let G be a finite p-group of odd order with nilpotency class c such that $G' \leq C_{\chi(G)}(D(G))$. Then:*
(i) $[\gamma_j(\chi(G)), G^\varphi, G] \leq [\gamma_{j+1}(G), G^\varphi]$ *for every $j \geq 1$,*
(ii) $\gamma_{j+1}(\chi(G)) = [\gamma_j(\chi(G)), G^\varphi]\gamma_{j+1}(G)\gamma_{j+1}(G^\varphi)$ *for every $j \geq 1$,*
(iii) *the nilpotency class of $\chi(G))$ is at most $c + 1$.*

The next result will be needed in the proof of the main results of Section 4 (see [4, Theorem C]).

Lemma 2.11. *Let p be an odd prime and let G be a powerful p-group.*
(a) *If $k \geqslant 2$, then the k-th term of the lower central series $\gamma_k(\chi(G))$ and $D(G)$ are powerfully embedded in $\chi(G)$.*
(b) *If $p = 3$, then $\exp(\chi(G))$ divides $3 \cdot \exp(G)$. If $p \geq 5$, then $\exp(\chi(G)) = \exp(G)$.*

3 p-groups whose derived subgroup centralizes $D(G)$

In this section, we prove Theorems 1.1, 1.2 and Corollary 1.3. All of these results assume $G' \leqslant C_{\chi(G)}(D(G))$ or, equivalently, subgroup $D(G)$ is abelian. These groups form a subclass of metabelian groups, which contains, for instance, all groups with a cyclic derived subgroup. Now we prove Theorem 1.1.

Proof of Theorem 1.1. Let $X = \{g_1, \dots, g_d\}$ be a set of generators for G and let $Y = \{h_1, \dots, h_k\}$ be a set of generators of G'. By [16, Lemma 3.2.1 (iii)], $[D(G), G] \le [G', G^\varphi]$. Thus we have

$$D(G) = \langle [g_i, g_j^\varphi] \mid 1 \le i < j \le d \rangle^{\chi(G)} = \langle [g_i, g_j^\varphi] \mid 1 \le i < j \le d \rangle [G', G^\varphi].$$

Denote by A the subgroup generated by the set $\{[y, x^\varphi] \mid x \in X,\ y \in Y\}$. Now, we will prove that $A = [G', G^\varphi]$.

Claim 1. *For any $g \in G$, we have $[G', g^\varphi] = \{[y, g^\varphi] \mid y \in G'\}$. In particular, $[G', g_i^\varphi] \le A$ for any $1 \le i \le d$.*

Indeed, since G' centralizes $D(G)$, it is sufficient to observe that

$$[y_1 y_2, g^\varphi] = [y_1, g^\varphi]^{y_2} [y_2, g^\varphi] = [y_1, g^\varphi][y_2, g^\varphi].$$

Claim 2. *For any $r \ge 2$, we have $[\gamma_r(G), G^\varphi] \le A[\gamma_{r+1}(G), G^\varphi]$.*

Indeed, let $\alpha \in \gamma_r(G)$ and $g \in G$. Then g can be written as a product of elements of X and their inverses. If $g \in X$, the statement is clear by Claim 1. Therefore, assume that there exist $u \in G$ and $x \in X$ such that $g = ux$. Then

$$[\alpha, (ux)^\varphi] = [\alpha, x^\varphi][\alpha, u^\varphi][\alpha, u^\varphi, x^\varphi].$$

By Proposition 2.10 (i), we have $[\gamma_r(G), G^\varphi, G^\varphi] \le [\gamma_{r+1}(G), G^\varphi]$, and applying Claim 1 it follows that

$$[\alpha, (ux)^\varphi] \in A[\gamma_{r+1}(G), G^\varphi].$$

To conclude, assume that G has nilpotency class c. Then applying c times Claim 2 yields $[G', G^\varphi] \le A$. Therefore, $d(D(G)) \le \binom{d}{2} + dk = \frac{d(d+2k-1)}{2}$. □

As an immediate consequence of Lemma 2.9 and Proposition 2.10, we have the following:

Lemma 3.1. *Let G be a p-group, p odd, such that $G' \le C_{\chi(G)}(D(G))$. Then the exponent of $\gamma_3(\chi(G))$ divides the exponent of G'.*

To investigate the minimal number of generators for the group $D(G)$, we will need the following remark.

Remark 3.2. Let p be an odd prime and let G be a p-group of class 2. Let $X = \{g_1, \dots, g_d\}$ be a set of generators of G. If $G' = \langle c \rangle$, then:

(a) the subgroup $[G', G^\varphi]$ is generated by the set

$$\{[c^\varphi, g_k] \mid 1 \le k \le d\},$$

(b) the subgroup $D(G)$ is generated by the sets

$$\{[g_i, g_j^\varphi], [c^\varphi, g_k] \mid 1 \le i < j \le d \text{ and } 1 \le k \le d\}.$$

Proof. From $D(G) = [G, G^\varphi]$ and $G = \langle X \rangle$, we have

$$D(G) = \langle [g_i, g_j^\varphi] \mid g_i, g_j \in X \rangle^{\chi(G)} = \langle [g_i, g_j^\varphi] \mid 1 \le i < j \le d \rangle^G.$$

By Proposition 2.10, $\gamma_3(\chi(G))$ is central in $\chi(G)$. Therefore, if $u, v \in X$, we have $[g_i, g_j^\varphi, uv] = [g_i, g_j^\varphi, v][g_i, g_j^\varphi, u]$. Moreover, $[g_i, g_j^\varphi] = [g_i^\varphi, g_j]$, as well as $[g_i, g_i^\varphi] = 1$. Now the assert follows. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $X = \{g_1, \dots, g_d\} = \{a_1, b_1, \dots, a_n, b_n\}$ be a set of generators of G. From the fact that $D(G)$ is isomorphic with G' modulo the central subgroup $W(G)$ and G' is cyclic, it follows that $D(G)$ is abelian. This implies that $G' \le C_{\chi(G)}(D(G))$, and Theorem 1.1 yields that $D(G)$ has a rank at most $\frac{d(d+1)}{2}$. Therefore, we only need to prove that $\exp(D(G))$ is p.

By Remark 3.2, the group $D(G)$ is generated by the set $\{[g_i, g_j^\varphi], [c, g_k^\varphi] \mid 1 \le i < j \le d \text{ and } 1 \le k \le d\}$. Moreover, Lemma 3.1 implies that $\gamma_3(\chi(G))$ has an exponent dividing the exponent of G'. Thus $[c, g_k^\varphi]^p = 1$ for any i, j, k. By Remark 2.5, we can assume that for any $1 \le i < j \le d$, at least one between g_i and g_j has order p. Without loss of generality, assume that g_i has order p. Then, since p is odd, p divides $\binom{p}{2}$ and we have

$$1 = [g_i^p, g_j^\varphi] = [g_i, g_j^\varphi]^p [g_i, g_j^\varphi, g_i]^{\binom{p}{2}} = [g_i, g_j^\varphi]^p.$$ □

Corollary 3.3. *Let p be an odd prime and let G be an extra-special p-group of exponent p^2. Then $\chi(G)$ has class* 2 *and its exponent equals the exponent of G.*

Proof. Using the notation of the previous result, we now prove that $\gamma_3(\chi(G))$ is trivial. By Proposition 2.10 (ii), it suffices to show that $[G', G^\varphi]$ is trivial.

Set $G' = \langle c\rangle$. Moreover, since $[G', G^\varphi] = \langle [c, g_j^\varphi] \mid 1 \leqslant j \leqslant d\rangle$, it suffices to prove that $[c, g_j^\varphi] = 1$ for any $1 \leqslant j \leqslant d$. As G has exponent p^2, we have an element g_i such that $g_i^p = c$. By the previous result, the subgroup $D(G)$ is elementary abelian. Now, we have

$$[c, g_j^\varphi] = [g_i^p, g_j^\varphi] = [g_i, g_j^\varphi]^p [g_i, g_j^\varphi, g_i]^{\binom{p}{2}} = 1.$$

In particular, $R(G) \leq \gamma_3(\chi(G)) = [G', G^\varphi] = 1$. Consequently, $\chi(G)$ has nilpotency class 2 and is a regular p-group. Thus $\exp(\chi(G))$ equals $\exp(G)$. □

Remark 3.4. For a finite extra-special p-group G, p odd, with $\exp(G) = p^2$, the group $\chi(G)$ has class 2 and $\exp(\chi(G)) = \exp(G)$. In particular, $R(G)$ is trivial.

Now, we finish this section with the proof of Corollary 1.3.

Proof of Corollary 1.3. To prove (a), assume that G has order p^3. By Theorem 1.2, $D(G)$ is an elementary abelian p-group of rank at most 3. If G has exponent p, Theorem 2.6 yields that $W(G)/R(G)$ is isomorphic to $C_p \times C_p$. Since $D(G)/W(G)$ has order p, we conclude that $R(G)$ is trivial and $D(G) \simeq C_p \times C_p \times C_p$. If $\exp(G) = p^2$, then Remark 3.4 implies $R(G) = 1$ and by Theorem 2.6 we obtain that $W(G)/R(G)$ is trivial as well. Therefore, $D(G)$ is a cyclic group of order p.

Now assume that $n > 1$. In order to prove (b), let $\exp(G) = p$. Combining Proposition 2.10 (ii) and Remark 3.2, $\gamma_3(\chi(G))$ can be generated by elements of the form $[c, g^\varphi]$ where c is a generator of G' and g is a generator of G. Therefore, evoking Remark 3.2, we can easily see that $D(G)$ is generated by $(2n-1)n + 2n = 2n^2 + n$ elements. Observing that $D(G)/W(G)$ has order p and $W(G)/R(G)$ has rank $2n^2 - n - 1$ by Theorem 2.6, it follows that $R(G)$ has rank at most $2n^2 + n - 1 - (2n^2 - n - 1) = 2n$.

In order to prove (c), assume that G has exponent p^2. Then, by Remark 3.4, $R(G)$ is trivial. Therefore, by Remark 3.2, we can conclude that $D(G)$ can be generated by $(2n-1)n$ elements. Actually, $2n^2 - n$ is the rank of $D(G)$ as $W(G)/R(G)$ has rank $2n^2 - n - 1$ by Theorem 2.6. □

Following Remark 3.2, it seems reasonable to bound the number of generators of $D(G)$ in terms of the number of generators of the group G and its nilpotency class. For instance, if G has nilpotency class 2, generated by the set $\{g_1, \dots, g_d\}$, and G' centralizes $D(G)$, then $\chi(G)$ has class 3 and a set of generators for $D(G)$ is $\{[g_i, g_j^\varphi], [g_i, g_j^\varphi, g_k] \mid 1 \leq i < j \leq d \text{ and } 1 \leq k \leq d\}$, thus $D(G)$ can be generated by $\frac{d^3-d}{2}$. However, this bound reveals in some sense that our machinery is failing in attaining a sharp bound, since GAP experiments produced smaller numbers.

4 Powerful p-groups, p odd

In this section, we prove Theorems 1.4, 1.5 and 1.6. Combining the ideas of Lubotzky and Mann [15] in the context of powerful p-groups, we deduce some bounds to the order and number of generators of $D(G)$ and the derived subgroup $\chi(G)'$.

Proof of Theorem 1.4. Recall that p is an odd prime and that G is a finite powerful p-group with $d(G) = d$. Set $G = \langle g_1, \dots, g_d\rangle$. We need to prove that $d(\chi(G)') \leqslant 3\binom{d}{2}$ and $d(D(G)) \leqslant \binom{d}{2}$.

Consider $\mathcal{C}_1 = \{[g_i, g_j], [g_i^\varphi, g_j^\varphi], [g_i, g_j^\varphi] \mid 1 \leqslant i < j \leqslant d\}$ and $\mathcal{C}_2 = \{[g_i, g_j^\varphi] \mid 1 \leqslant i < j \leqslant d\}$. We deduce that the derived subgroup $\chi(G)' = \langle \mathcal{C}_1\rangle^{\chi(G)}$ and $D(G) = \langle \mathcal{C}_2\rangle^{\chi(G)}$. By Lemma 2.11 (a), $D(G)$ and $\chi(G)'$ are powerfully embedded in $\chi(G)$ and so, $\chi(G)'$ and $D(G)$ are powerful. From this we deduce that $\chi(G)'$ and $D(G)$ are generated by $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, by Proposition 2.3. It follows that the number of generators $d(\chi(G)') \leqslant 3\binom{d}{2}$ and $d(D(G)) \leqslant \binom{d}{2}$. □

We are now in a position to prove Theorem 1.5.

Proof of Theorem 1.5. Recall that p is an odd prime and that G is a finite powerful p-group with $d(G) = d$ and $\exp(G) = p^e$.

By Lemma 2.11 (a), $\chi(G)'$ and $D(G)$ are powerfully embedded in $\chi(G)$ and so, $\chi(G)'$ and $D(G)$ are powerful p-groups. Now, Theorem 1.4 yields $d(D(G)) \leqslant \binom{d}{2}$ and $d(\chi(G)') \leqslant 3\binom{d}{2}$. More precisely, with the notation of the prove of Theorem 1.4,

$$\chi(G)' = \langle [g_i, g_j], [g_i^\varphi, g_j^\varphi], [g_i, g_j^\varphi] \mid 1 \leqslant i < j \leqslant d\rangle$$

and

$$D(G) = \langle [g_i, g_j^\varphi] \mid 1 \leqslant i < j \leqslant d\rangle.$$

We first examine the case $p = 3$. By Lemma 2.11 (b), the exponent $\exp(\chi(G))$ divides 3^{e+1}. From the above and Proposition 2.4, it follows that $|D(G)| \leqslant 3^{(e+1)\binom{d}{2}}$. Since G is powerful, we deduce that $G' \leqslant G^p = \{g^p \mid g \in G\}$ and, consequently, $|G'| \leqslant 3^{d(e-1)}$. As a consequence, we have

$$|\chi(G)'| = |D(G)| \cdot |G'|^2 \leqslant 3^{(e+1)\binom{d}{2}+2d(e-1)}.$$

Now, assume that $p \geqslant 5$. By Lemma 2.11 (b), the exponent $\exp(\chi(G)) = \exp(G)$. Arguing as above, we conclude that the derived subgroup $|\chi(G)'| \leqslant p^{e\binom{d}{2}+2d(e-1)}$. □

Lemma 4.1. *Let p be an odd prime and let G be a powerful p-group. Then $[G', G^\varphi]$ is powerfully embedded in $\chi(G)$.*

Proof. First, we prove that $[G^p, G^\varphi] \leqslant D(G)^p$. Let $x, y \in G$. By Theorem 2.1,

$$[x^p, y^\varphi] \equiv [x, y^\varphi]^p \pmod{\gamma_2(L)^p\gamma_p(L)},$$

where $L = \langle x, [x, y^\varphi]\rangle$. Note that $\gamma_2(L) \leqslant [D(G), G] \leqslant D(G)$. Therefore, we have $\gamma_2(L)^p \leqslant D(G)^p$. On the other hand, $\gamma_p(L) \leqslant [D(G),{}_{p-1}\chi(G)] \leqslant [D(G),{}_2\chi(G)]$. Consequently, $[x^p, y^\varphi] \in D(G)^p[D(G),{}_2\chi(G)]$ for any $x, y \in G$. Thus, $[D(G), \chi(G)] \leqslant D(G)^p[D(G),{}_2\chi(G)]$ and by Lemma 2.2 we obtain that $[D(G), \chi(G)] \leqslant D(G)^p$, as required.

Consequently,

$$[[G', G^\varphi], \chi(G)] \leqslant [[G^p, G^\varphi], \chi(G)] \leqslant [D(G)^p, G^\varphi].$$

Now we prove that $[D(G)^p, G^\varphi] \leq [D(G), G^\varphi]^p$. Let $\alpha \in D(G)$ and $y \in G$. Then, by Theorem 2.1, it follows that

$$[\alpha^p, y^\varphi] \equiv [\alpha, y^\varphi]^p \pmod{\gamma_2(K)^p\gamma_p(K)},$$

where $K = \langle \alpha, [\alpha, y^\varphi]\rangle$. Note that

$$\gamma_2(K) \leqslant [D(G), G^\varphi, D(G)] \leqslant [D(G), \chi(G), \chi(G)] \leqslant [D(G), \chi(G)],$$

thus $\gamma_2(K)^p \leqslant [D(G), \chi(G)]^p$. On the other hand, we have

$$\begin{aligned}\gamma_p(K) &\leq [D(G), G^\varphi, D(G),{}_{p-2}\chi(G)]\\ &\leq [G', G^\varphi, D(G),{}_{p-2}\chi(G)]\\ &\leq [D(G)^p, D(G),{}_{p-2}\chi(G)]\\ &\leq [D(G)^p,{}_{p-1}\chi(G)] \leq [D(G)^p, \chi(G), \chi(G)].\end{aligned}$$

Considering all the elements $\alpha \in D(G)$ and $y \in G$, we deduce that

$$\begin{aligned}[D(G)^p, G^\varphi] &= [D(G)^p, \chi(G)]\\ &\leq [D(G), G^\varphi]^p[D(G)^p, \chi(G), \chi(G)]\\ &= [D(G), \chi(G)]^p[D(G)^p, \chi(G), \chi(G)].\end{aligned}$$

Therefore,

$$[D(G)^p, \chi(G)] \leq [D(G), \chi(G)]^p[D(G)^p, \chi(G), \chi(G)].$$

Now, applying Lemma 2.2 with $N = [D(G)^p, \chi(G)]$ and $M = [D(G), \chi(G)]^p$, the result follows. □

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. By Lemma 2.8 (b), we have the exact sequence

$$1 \to [G', G^{\varphi}] \to D(G) \to D(G/G') \to 1.$$

Therefore, it is sufficient to estimate the size of $[G', G^{\varphi}]$ and $D(G/G')$.

Assume that G is generated by the set $X = \{g_1, \dots, g_{n-k}\}$ and G' is generated by the set $Y = \{y_1, \dots, y_k\}$. Then we have

$$[G', G^{\varphi}] = \langle [y, x^{\varphi}] \mid x \in X, y \in Y \rangle^{\chi(G)}.$$

On the other hand, by Lemma 4.1, the group $[G', G^{\varphi}]$ is powerfully embedded in $\chi(G)$. Therefore, applying Proposition 2.3, we have

$$[G', G^{\varphi}] = \langle [y, x^{\varphi}] \mid x \in X, y \in Y \rangle$$

and $[G', G^{\varphi}]$ can be generated by $(n-k)k$ elements. Moreover, by Lemma 3.1, $\exp([G', G^{\varphi}])$ divides $\exp(G')$. Thus

$$|[G', G^{\varphi}]| \leq p^{(n-k)k}. \tag{4.1}$$

Now, observe that G/G' is an elementary abelian p-group of rank $n-k$. Thus, by [17, Theorem 4.2.1 (iii)], $R(G/G') = 1$ and, consequently, $D(G/G')$ is isomorphic to the Schur multiplier $M(G/G')$. In particular,

$$|D(G/G')| = |M(G/G')| = p^{\frac{(n-k)(n-k-1)}{2}}. \tag{4.2}$$

Now combining (4.1) and (4.2), we obtain the desired result. □

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