Reverse Divisors and Magic Numbers

Eudes Antonio Costa[1] *and Ronaldo Antonio Santos*[2]

[1] *Federal University of Tocantins,* eudes@uft.edu.br
[2] *Federal University of Goiás,* rasantos@ufg.br

Abstract

The study examines the relationship between Ball's magic numbers and reverse divisors. These numbers are the source of beautiful and curious properties. Activities related to numbers can be a fun way to motivate mathematics students, while also enabling surprising analysis and connections.
Keywords: Magic, Number, Reverse, Divisor

1 Introduction

This paper presents the properties and connections between Ball's magic numbers and reverse divisor numbers. Numbers are endless sources of curious and surprising problems, and different properties and connections appeared when we carefully analyzed these classes of numbers.

First, following [SC22, WEB95], we present the definition of Ball's magic numbers their properties, and a characterization that allows for identification with lists of numbers formed only by zeros and ones.

We presented Ball's magic numbers as the first class, for example, 1089. Such a number arises when, from a three-digit number, 572, for example, we subtract its reverse 275, obtaining the number 297, which, added to its reverse 792, results in 1089. This sequence of calculations invariably results in 1089, regardless of the three-digit number taken at the beginning.

Section 3 presents the definitions and properties of the reverse divisors. Reverse divisors belong to a larger class of numbers called *permultiples*. Permutations of digits form numbers with multiples. The pair 142857 and 285714 is an example of a *permultiples pair*, with $285714 = 2 \times 142857$ (see [MOR98]). In reverse divisors, the permutation must obtain a reverse number. The pair 10989 and 98901 is an example of this, as $98901 = 9 \times 10989$; moreover, the permutation obtained results in the reverse of the first number. In this case, we say that 10989 is the reverse divisor of 98901, also known as *palintuples* (see more details in [HOL15]).

Magic numbers are reverse divisors. We presented the connections between these two classes in Section 4. We showed the numbers representing both Ball's magic numbers and reverse divisors. The sum of magic numbers that resulted in reverse divisors. Section 5 discusses the relationship between perfect squares and reverse divisors.

2 Magic numbers

Recreational activities where participants are asked to develop a sequence of calculations with the results given in advance are common in the literature. Ball magic numbers, as will be illustrated in Algorithm 1, are an example of such activities. The details and properties of these numbers were first introduced in [BAL26], followed by additional studies in [BAL05, BEI66, COS21, CM14, SC22, WEB95], among others.

Following these references, we will briefly present Ball magic numbers and some proprieties in this section, considering the representation at the base of 10.

For $n \geqslant 2$, let x_n be a positive non-palindromic number with n digits, that is, $x_n = a_{n-1} \dots a_1 a_0$. The number of n digits obtained by reversing the position of the digits of x_n is called the reverse number of x_n and is denoted by x'_n. Therefore, the reverse number of x_n is represented by the expression $x'_n = a_0 a_1 \dots a_{n-1}$, where a_i belongs to the set D, which is defined as $\{0, 1, \dots, 8, 9\}$, with $i = 0, \dots, n-1$, and $a_{n-1} \neq 0$ (base 10).

Now, we consider the following algorithm:

Algorithm 1. *: **Ball's magic number***

1. *Let x_n be a number;*
2. *Write the reverse x'_n ;*
3. *Find the difference(positive) $y_n = |x_n - x'_n|$;*
4. *Write the reverse y'_n;*
5. *Add to obtain the number $B = y_n + y'_n$.*

Remark 1. *Let x_n be a number with n digits; then, the numbers x'_n, y_n and y'_n are (considered) numbers of n digits, even if any digit is zero on the left.*

As defined in [COS21, SC22], the number $B \neq 0$ obtained from Algorithm 1 is called a **Ball's magic number**. When $x_n > x'_n$, we can write the magic number of the *Ball* as $B = (x_n - x'_n) + (x_n - x'_n)'$.

Example 1. *For $n = 2$, let be $x_2 = 71$ and $x'_2 = 17$, so $y_2 = 54$ and $y'_2 = 45$. We obtain $B = 54 + 45 = 99$. For $n = 3$, let be $x_3 = 843$ and $x'_3 = 348$, so we have $y_3 = 843 - 348 = 495$. Finally $B = 495 + 594 = 1089$. Therefore, 99 and 1089 are Ball's magic numbers.*

Some surprising results relate to Ball's magic numbers.

Theorem 1. *Every non-zero magic number of Ball B is a multiple of 99.*

We presented the proof in Section 2.1.

The quantities of possible Ball's magic numbers, given by Algorithm 1, depend on the number of digits in the initial numbers. For example, the sequence of possible

Ball's magic numbers, corresponding to initial numbers with 2, 3, 4, 5, 6, 7, ... digits, begins as 1, 1, 3, 3, 8, 8, ... This shows that for initial numbers with 2 digits, there is 1 possible Ball's magic number; for 3 digits, also 1 possible result; for 4 digits, 3 possible results, and so on.

Thus, the number $B(n)$ of possible *Ball* numbers associated with the number x_n with $n = 2k + 1$ digits (or $n = 2k$ digits) is the sum of the Fibonacci numbers i.e.,

Theorem 2. *Let x_{2n+1} be a natural number with $2n + 1$ digits; for all $n \geqslant 1$, the number of Ball $B(2n + 1)$ is*

$$F_2 + F_4 + \cdots + F_{2(n-1)} + F_{2n}$$

where F_j is the Fibonacci number at the j position, with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geqslant 2$.

This paper does not present the proof of Theorem 2. Interested readers can refer to demonstration in [COS21, SC22, WEB95].

2.1 Codes and Ball numbers

Webster [WEB95] showed an important characterization of Ball's magic numbers and, with this, proved Theorem 1 and the relationship between Ball's magic numbers and Fibonacci numbers. The relationship we show between Ball's magic numbers and reverse divisors is also strongly linked to this characterization. Thus, following [SC22, WEB95], we will summarize here this characterization.

Following our notation and Algorithm 1, let x_{n+1} be a number with $n+1$ digits and x'_{n+1} be its reverse. So,

$$\begin{aligned} x_{n+1} &= a_n a_{n-1} \dots a_{n-i} \dots a_i \dots a_1 a_0 \\ x'_{n+1} &= a_0 a_1 \dots a_i \dots a_{n-i} \dots a_{n-1} a_n \end{aligned}$$

with $a_n > a_0$. We have

$$\begin{aligned} x_{n+1} - x'_{n+1} &= (a_n - a_0)10^n + (a_{n-1} - a_1)10^{n-1} + \cdots + (a_{n-i} - a_i)10^{n-i} + \\ &\quad + \cdots + (a_i - a_{n-i})10^i + \cdots + (a_1 - a_{n-1})10 + (a_0 - a_n) \end{aligned}$$

The representation at the base 10 requires each coefficient to be greater than or equal to zero and less than 10. In some cases, for the coefficient to be positive, you need to shift 10 from the order i to $i - 1$ (the famous move of a ten or "borrow one").

Let us associate each number x_{n+1}, which has $n+1$ digits, with the number Z_{n+1}, called the code of x_{n+1}. This code Z_{n+1} consists of a sequence of $0's$ and $1's$ and contains the necessary information to pass from the initial number x_{n+1} to the resulting Ball's number B.

In this initial formulation, the construction of Z_{n+1} will be presented in a more intuitive manner; a formal definition will be explained later in the text.

We write the number $x_{n+1} = a_n \dots a_0$ with $a_n > a_0$, and below it, its reverse $x'_{n+1} = a_0 \dots a_n$, as shown below:

$$\begin{array}{rccc} & a_n & \dots & a_0 \\ - & a_0 & \dots & a_n \\ \hline & * & \dots & * \end{array}$$

Consider the i-th column from right to left ($i = 0, \dots, n$) by subtracting the number x'_{n+1} of x_{n+1}. The digit z_i is defined as follows: if the quantity 10 needs to be regrouped from column $(i+1)$-th to i-th, z_i will be 1; otherwise, it is 0. Thus, the code is given by $Z_{n+1} = z_0 z_1 \dots z_{n-1} z_n$.

For example, if $x_3 = 753$, we have:

$$\begin{array}{rccc} & \not{7}^{6} & \not{5}^{14} & \not{3}^{13} \\ - & 3 & 5 & 7 \\ \hline & 3 & 9 & 6 \\ \hline\hline & 0 & 1 & 1 \end{array}$$

and $Z_3 = 110$.

In general case, we write:

$$\begin{aligned} & x_{n+1} - x'_{n+1} \\ = \ & (a_n - a_0 - z_{n-1})10^n + (a_{n-1} - a_1 - z_{n-2} + 10z_{n-1})10^{n-1} + \dots \\ & + (a_{n-i} - a_i - z_{n-i-1} + 10z_{n-i})10^{n-i} + \dots + (a_i - a_{n-i} - z_{i-1} + 10z_i)10^i \\ & + \dots + (a_1 - a_{n-1} - z_0 + 10z_1)10 + (a_0 - a_n + 10z_0) \end{aligned}$$

We obtain the strings $z_0, \dots, z_n$ of $0's$ and $1's$. The number $z_0 \dots z_n$ with $n+1$ digits is called the code of x_{n+1} and is denoted by Z_{n+1}, that is, $Z_{n+1} = z_0 z_1 \dots z_{n-1} z_n$. Because we assume the following: $a_n > a_0$, $z_0 = 1$ and $z_n = 0$. The number of n digits obtained from Z_{n+1} by excluding the digit $z_n = 0$ (at the end) is denoted by $Z_{\overline{n+1}} = z_0 \dots z_{n-1}$ and is called the truncated code of x_{n+1}.

Formally we have:

Definition 1 (code). *Let $x_{n+1} = a_n a_{n-1} \dots a_1 a_0$ be a number, with $a_n > a_0$. The number $Z_{n+1} = z_0 \dots z_n$ is called the code related to x_{n+1} if $z_0 = 1$ and $z_n = 0$, and recursively*

$$z_i = \begin{cases} 1 & \text{if } a_i - a_{n-i} - z_{i-1} < 0 \\ 0 & \text{if } a_i - a_{n-i} - z_{i-1} \geqslant 0 \end{cases}$$

for $i = 0, \dots, n-1$ and $a_{-1} = 0$.

Notice that $0 \leqslant a_i - a_{n-i} - z_{i-1} + z_i 10 < 10$ for $i = 0, \dots, n-1$. Following code definition, we write

$$y_n = x_{n+1} - x'_{n+1} = \sum_{i=0}^{n} (a_i - a_{n-i} - z_{i-1} + z_i 10)10^i$$

Having introduced the requisite notation, we shall proceed to demonstrate the theorem.

Proof. Theorem 1

By hypothesis, we have that $a_n > a_0$; therefore, $z_0 = 1$. With this notation, we can write the magic number of *Ball* in base 10 as:

$$\begin{aligned}
B &= y_n + y'_n \\
&= \sum_{i=0}^{n} \big((a_i - a_{n-i} - z_{i-1} + 10z_i) + (a_{n-i} - a_i - z_{n-i-1} + 10z_{n-i})\big)10^i \\
&= \sum_{i=0}^{n} (-z_{i-1} + 10z_i - z_{n-i-1} + 10z_{n-i})10^i \\
&= -\sum_{i=0}^{n} z_{i-1}10^i + \sum_{i=0}^{n} z_i 10^{i+1} - \sum_{i=0}^{n} z_{n-i-1}10^i + \sum_{i=0}^{n} z_{n-i}10^{i+1} \\
&= -\sum_{i=0}^{n-1} z_i 10^{i+1} + \sum_{i=0}^{n-1} z_i 10^{i+1} - \sum_{i=1}^{n} z_{n-i}10^{i-1} + 10^2 \sum_{i=1}^{n} z_{n-i}10^{i-1} \\
&= (10^2 - 1)\sum_{i=1}^{n} z_{n-i}10^{i-1} \\
&= 99 \cdot (z_0 z_1 \ldots z_{n-1}) \qquad (1)
\end{aligned}$$

Therefore, the number of *Ball B* is a multiple of 99. □

Furthermore, all Ball numbers B are multiples of the truncated code $z_0z_1 \ldots z_{n-1}$. We provide several examples to illustrate these results.

Example 2. *Let $x_6 = 397862$ be a six-digit number. By subtracting $x'_6 = 268793$ from x_6, we obtain*

$$\begin{array}{cccccccc}
 & \not{3}^{2} & \not{9}^{8} & \not{7}^{17} & \not{8}^{7} & \not{6}^{15} & \not{2}^{12} \\
- & 2 & 6 & 8 & 7 & 9 & 3 \\
\hline
 & 1 & 2 & 9 & 0 & 6 & 9 \\
\hline\hline
 & 0 & 0 & 1 & 0 & 1 & 1
\end{array}$$

Notice that we regrouped ten (10) from the adjacent column on the left in columns 0, 1, and 3. Using the above notation, we have $z_0 = 1,\ z_1 = 1,\ z_2 = 0,\ z_3 = 1,\ z_4 = 0$ and $z_5 = 0$.

Thus, $Z_6 = 110100$ is a code associated with the number $x_6 = 397862$ and the truncated code is $Z_{\overline{6}} = 11010$, which is a divisor of the Ball's number B. In fact, for $x_6 = 397862$, we have Ball's number

$$B = 129069 + 960921 = 1089990 = 99 \times 11010 = 99 \times Z_{\overline{6}}$$

Example 3. *Consider the four-digit number $x_4 = accd$ in base 10 with $a > d$. By subtracting $x'_4 = dcca$ from x_4, we obtain*

DOI 10.2478/rmm-2025-0012

$$\begin{array}{ccccc} & \not{a}^{a-1} & \not{c}^{(c-1)+10} & \not{c}^{(c-1)+10} & \not{d}^{d+10} \\ - & \text{d} & \text{c} & \text{c} & \text{a} \\ \hline & \text{(a-d-1)} & 9 & 9 & \text{(d-a+10)} \\ \hline\hline & 0 & 1 & 1 & 1 \end{array}$$

We need to move 10 *from the adjacent column on the left in columns 0, 1, and 2. Using the above notation we have* $z_0 = 1,\ z_1 = 1,\ z_2 = 1,$ *and* $z_3 = 0$.

The code is $Z_4 = 1110$ *and the truncated code* $Z_{\overline{4}} = 111$ *is a divisor of* B. *Therefore, for any* $x_4 = accd$, *we obtain the Ball's number*

$$B = (x_4 - x_4') + (x_4 - x_4')' = 99 \times 111 = 99 \times Z_{\overline{4}}$$

Example 4. *According to [CM14, SC22] we have Ball's magic numbers and their respective codes for* $n \leqslant 6$ *as follows:*

n	B	*factorization*	*divisor of* B *(truncated code)*	*code*
2	99	99×1	1	10
3	1089	99×11	11	110
4	9999	99×101	101	1010
4	10890	99×110	110	1100
4	10989	99×111	111	1110
5	99099	99×1001	1001	10010
5	109890	99×1110	1110	11100
5	109989	99×1111	1111	11110
6	1089990	99×11010	11010	110100
6	1098900	99×11100	11100	111000
6	1099890	99×11110	11110	111000
6	999999	99×10101	10101	101010
6	991089	99×10011	10011	100110
6	990099	99×10001	10001	100010
6	1099989	99×11111	11111	111110
6	1090089	99×11101	11011	110110

From Theorem 2 that the quantity $B(2n+1)$ *of Ball's numbers is greater than* 2 *for* $n > 1$.

With the codes defined, we proceed following Webster [WEB95] to characterize them.

Proposition 3. *If* $a_n > a_0$ *then* $z_0 = 1$ *and* $z_n = 0$.

Furthermore,

Proposition 4 ([SC22, WEB95])**.** *For all* $i = 1, \ldots, n-1$, *we have:*

a) If $z_{i+1} = 1$ *and* $z_i = 0$, *then* $z_{n-i-1} = 0$.

b) If $z_{i+1} = 0$ *and* $z_i = 1$, *then* $z_{n-i-1} = 1$.

Proof. (a) We have $z_{i+1} = 1$; thus,

$$\begin{aligned} 0 \leqslant a_{i+1} - a_{n-(i+1)} - z_{i+1-1} &= a_{i+1} - a_{n-i-1} - z_i \\ &\overset{z_i=0}{=} a_{i+1} - a_{n-i-1} \end{aligned}$$

DOI 10.2478/rmm-2025-0012

So, $a_{n-i-1} - a_{i+1} > 0$, and hence, $a_{n-i-1} - a_{i+1} - z_{n-i-2} \geqslant 0$, that is, $z_{n-i-1} = 0$.

(b) For $z_{i+1} = 0$ we have $a_{i+1} - a_{n-i-1} - z_i \geqslant 0$.

As $z_i = 1$, we obtain $a_{i+1} - a_{n-i-1} - 1 \geqslant 0$, that is, $a_{n-i-1} - a_{i+1} \leqslant -1$ resulting in $a_{n-i-1} - a_{i+1} - z_{n-i-2} \leqslant -1 - z_{n-i-2} < 0$, and $z_{n-i-1} = 1$. □

The next result will demonstrate that the code associated with a given number is, in fact, unique.

Proposition 5. *A list of ones and zeros, $z_0 \dots z_n$, satisfying Propositions 3 and 4 is a code.*

Proof. Consider the number $c = z_0 \dots z_n$ formed by a list satisfying Propositions 3 and 4. The goal of this proof is to demonstrate that $c = z_0 \dots z_n$ is equal to the list $w = w_0 \dots w_n$, which is a known code defined below:

$$w_i = \begin{cases} 1 & \text{if } z_{n-i} - z_i - w_{i-1} < 0 \\ 0 & \text{if } z_{n-i} - z_i - w_{i-1} \geqslant 0 \end{cases}$$

As $z_0 = 1$ and $z_n = 0$ (Proposition 3), it follows that $w_0 = 1$ because $z_n - z_0 - w_{-1} < 0$ and $w_n = 0$ because $z_0 - z_n - w_{-1} \geqslant 0$.

The digit w_1 can be either zero or one. We assume that $w_1 = 1$. In this case, $z_{n-1} - z_1 - w_0 < 0$ or $z_{n-1} - z_1 < 1$. If $z_1 = 0$ then it is also z_{n-1}. However, this is a contradiction, because $z_0 = 1$ and $z_1 = 0$ imply $z_{n-1} = 1$ (Proposition 4). Thus, $z_1 = 1$.

On the other hand, if $w_1 = 0$ then $z_{n-1} - z_1 - w_0 \geqslant 0$ or $z_{n-1} - z_1 \geqslant 1$. This results implies that $z_1 = 0$.

Continuing this argument, we demonstrate that $w_i = z_i$ for all $i = 1 \dots n-1$. Therefore, c is a (unique) code. □

If the hypotheses of Proposition 4 are not verified, the symmetrical element can assume a value of zero or one. The following observations highlight this fact.

Remark 2. *For $i = 1, \dots, n-1$, we have:*

a) If $z_{i+1} = 1$ and $z_i = 1$, then z_{n-i-1} can be 0 or 1.

b) If $z_{i+1} = 0$ and $z_i = 0$, then z_{n-i-1} can be 0 or 1.

Furthermore, for codes with odd-digit quantities, we have

Remark 3. *Let $z_0 \dots z_{n-1} z_n z_{n+1} \dots z_{2n}$ be a code. So $z_n = z_{n-1}$. If they are different, for example, $z_{n-1} = 1$ and $z_n = 0$, it follows from Proposition 4 that $z_{2n-n} = z_n = 1$, which is contradictory. We obtain a similar contradiction when we assume $z_{n-1} = 0$ and $z_n = 1$.*

Let us examine whether particular sequences of ones and zeros can be regarded as codes.

Proposition 6. *For every $m \geqslant 0$, list $1\underbrace{00\dots00}_{m \text{ times}}10$ is a code with $m+3$ digits.*

Proof. The list has the form $1\underbrace{00\dots00}_{m \text{ times}}10$. We denote by

$$100\dots0010 = z_0 z_1 z_2 \dots z_{m+2}$$

The initial conditions $z_0 = 1$ and $z_{m+2} = 0$ are satisfied. In the symmetry condition, for $z_i = 0$ and $z_{i+1} = 0$ the symmetric element $z_{m+2-i-1}$ can be zero or one and is always satisfied. The case $z_0 = 1$ and $z_1 = 0$ implies that $z_{m+1} = 1$, the case $z_{m+1} = 1$ and $z_{m+2} = 0$ implies that $z_0 = 1$ and the case $z_m = 0$ and $z_{m+1} = 1$ implies that $z_1 = 0$. In all cases, the implications are true. We conclude that the symmetry conditions are satisfied, and the list is a code. □

A direct consequence of this

Corollary 1. *If $m > 0$ is an integer, then $B_m = 99 \times (10^m + 1)$ is the Ball's magic number.*

Proof. The result follows from Proposition 6 and the proof of Theorem 1 because $10^m + 1$ is a truncated code. □

Proposition 7. *Let $n \geqslant 1$ then, a list of ones an zeros of the form*

$$z_0 z_1 \dots z_{n-1} z_n = \underbrace{11\dots11}_{n\ times}0$$

is a code.

Proof. A list of ones and zeros constitutes a code if and only if the conditions $z_0 = 1$ and $z_n = 0$ are satisfied. We now proceed to examine the symmetry condition. In the event that $z_{i+1} = 1$ and $z_i = 1$, the symmetric element z_{n-i-1} may assume the values zero or one. It can be concluded that the aforementioned conditions will always be satisfied. In the case where $z_{n-1} = 1$ and $z_n = 0$, it can be deduced that $z_{n-n} = z_0 = 1$. This condition is therefore satisfied. It can be demonstrated that the list $z_0 z_1 \dots z_n = 11\dots110$ is a code. □

A direct consequence of this

Corollary 2. *If $n > 0$ is an integer, then $B_n = 99 \times \underbrace{11\dots11}_{n\ times}$ is the Ball's magic number.*

Proof. It follows from Theorem 1 and Proposition 7. □

Remark 4. *The truncated code $\underbrace{11\dots11}_{n\ times}$ formed only by digit one is known in the literature as a repunits number and uses the notation $R_n = \underbrace{11\dots1}_{n\ times}$, $n \geqslant 1$. Using decimal notation, we have the following general expression for the repunit:*

$$R_n = \frac{10^n - 1}{9} \quad \text{for all } n \geqslant 1$$

The properties related to repunits R_n, are given in [BEI66, CC15, CS20, CS22].

Remark 5. *We obtain the characterization of Ball's magic numbers and the corresponding codes by applying Algorithm 1 to the numbers in the form $x_{n+1} = a_n \dots a_0$, with $a_n > a_0$. Algorithm 1 can also be applied to the case $a_n = a_0$. In this case, the list began and ended at zero. More generally, from $x_{n+1} > x'_{n+1}$, with $a_n = a_0$, we* *have a list of the form* $\underbrace{00\ldots0}_{j\ \textit{times}} z_j z_{j+1}\ldots z_{n-j} \underbrace{00\ldots0}_{j\ \textit{times}}$*, where* j *is the smallest index which* $a_j > a_{n-j}$*. Thus, the list* $z_j \ldots z_{n-j}$ *is a code, and we refer to the complete list* $\underbrace{00\ldots0}_{j\ \textit{times}} z_j z_{j+1}\ldots z_{n-j} \underbrace{00\ldots0}_{j\ \textit{times}}$ *of the extended code.*

Proposition 8. *For all integer* $n \geqslant 1$*, the code* $R = \underbrace{11\ldots11}_{n\ \textit{times}}0$*, can be decomposed by adding code* A *and an extended code* C*.*

Proof. We want to write R as the addition of code A and extended code C; that is, we want to determine the codes $a_0a_1\ldots a_{n-1}a_n$ and $c_0c_1\ldots c_{n-1}c_n$ such that

$$\underbrace{11\ldots11}_{n\ \text{times}}0 = a_0a_1\ldots a_{n-1}a_n + c_0c_1\ldots c_{n-1}c_n$$

We choose a list $A = a_0a_1\ldots a_{n-1}a_n$ with $a_0 = 1$, $a_n = 0$, $a_1 = 0$ and $a_{n-1} = 1$, and also satisfy Propositions 3 and 4 for the indices $j = 2\ldots n-2$. The code A has the form:

$$10a_2a_3\ldots a_{n-2}10$$

We consider the auxiliary list $b_0b_1\ldots b_{n-1}b_n$ obtained from the process, where in A is one changes by zero and where is zero in one. Our auxiliary list has the following form:

$$01b_2\ldots b_{n-2}01$$

This is not a code; however, we show that the inner part (suppressing the extremes)

$$1b_2\ldots b_{n-2}0$$

is a code. Let us denote it by $c_0c_1\ldots c_{n-2}$, that is, $c_j = b_{j+1}$, $j = 0,\ldots,n-2$.

The first condition, $c_0 = 1$ and $c_{n-2} = 0$ is satisfied. We have demonstrated that this satisfies the symmetry condition. If $c_{i+1} = 1$ and $c_i = 0$, we must show that $c_{(n-2)-i-1} = 0$. In fact, $c_{i+1} = 1$ and $c_i = 0$ implies $b_{i+2} = 1$ and $b_{i+1} = 0$. This implies that $a_{i+2} = 0$ and $a_{i+1} = 1$. Because the list a_i satisfies the symmetry condition, we have $a_{n-i-2} = 1$. Thus, $b_{n-i-2} = 0$. But $b_{n-i-2} = c_{n-i-3}$, so $c_{n-i-3} = 0$. In a similar way, we prove symmetry in other cases. Finally, the list $C = c_0c_1\ldots c_{n-2}0$ is the extended code, and $A + C = R$. □

Example 5. *Let us decompose code* 11110 *by following the steps in Proposition 8: Consider the list* $a_0a_1a_2a_3a_4 = 10a_210$*. To be a code,* a_2 *must equal the digit on the left:* $a_2 = 0$*. The code has the form:*

$$a_0a_1a_2a_3a_4 = 10010$$

Applying the process: change one by zero and zero by one, we obtain $b_0b_1b_2b_3b_4 = 01101$*, which isn't a code. However, taking the central part, we have* $c_0c_1c_2 = 110$*; therefore,* $c_0c_1c_20 = 1100$ *is an extended code and* $A + C = 10010 + 1100 = 11110$*.*

Example 6. *For* $n = 5$*, Proposition 8 shows that* $11110 = 11100 + 10$ *or* $11110 = 10010 + 1100$*.*

Definition 2 (undulating)**.** *A natural number starting with* 1 *and alternating* 0 *and* 1 *successively is a number of the type smoothly undulating. We use the notation* $O(n)$ *to indicate the undulating number with* n *digits, and the set* $\{O_n\}_{n\geqslant 1}$ *is used to indicate the set of the type smoothly undulating.*

Example 7. *Undulations Number* $O(2) = 10$, $O(3) = 101$, $O(4) = 1010$, $O(5) = 10101$ *and* $O(11) = 10101010101$.

We can find the *undulating* numbers information in [CC21, PIC90].

In general, we have

Proposition 9. *For all* $n \geqslant 1$ *and* $O(n) \in \{O_n\}_{n\geqslant 1}$*:*

(a) if n *is even then* $11 \times O(n)$ *is a code;*

(b) if n *is odd then* $11 \times O(n)$ *is a truncated.*

Proof. (a) Consider n even. We have $11 \times O(n) = 10 \times O(n) + O(n) = 10 \times R_n$. It follows from Proposition 7 that a list of forms $R = 10 \times R_n$ is a code.

(b) If n is odd, then $11 \times O(n) = 10 \times O(n) + O(n) = O(n+1) + O(n) = R_n$, addition, from Proposition 7 we have R_n as a truncated code. □

Example 8. *Some examples of Proposition 9:* $11 \times O(1) = 11$*;* $11 \times O(2) = 110$*;* $11 \times O(3) = 1111$*;* $11 \times O(4) = 11110$ *e* $11 \times O(5) = 111111$.

In particular, we have

Proposition 10. *For all* $n > 1$ *even, the number* $O(n) \in \{O_n\}_{n\geqslant 1}$ *is a code.*

Proof. For $n = 2k$, we denote by $O(2k) = z_0 z_1 \dots z_{2k-1}$. Note that one appears at even positions and zero at odd positions. The first condition to be a code $z_0 = 1$ and $z_{2k-1} = 0$ (Proposition 3) is satisfied. To verify the symmetry condition (Proposition 4), given i even, $z_i = 1$ and $z_{i+1} = 0$ must imply $z_{2k-1-i-1} = 1$. However, this is true because $2k - i - 2$ is even. If i is odd, then $z_i = 0$ and $z_{i+1} = 1$, which implies $z_{2k-1-i-1} = 0$. This condition is also verified because $2k - i - 2$ is odd. We conclude that $O(2k)$ is a code. □

Corollary 3. *Let* $n > 0$ *be an integer; if* n *is odd, then* $O(n)$ *is a truncated code.*

Proof. In fact, if n is odd, then $n+1$ is even. According Proposition 10 $O(n+1)$ is a code, and $O(n)$ is a truncated code. □

The following proposition and its corollary present the relationship between Ball numbers generated by numbers $x_{n+1} = a_n \dots a_0$, with $a_n = a_0$ and the extended codes.

Proposition 11 ([CM14])**.** *If* B_0 *is a Ball number generated by* x_n *with* n *digits, then* $10B_0$ *is the Ball number generated by* x_{n+2} *with* $n+2$ *digits.*

Proof. Given a number $x_n = a_{n-1} \dots a_0$ with n digits, it follows from Theorem 1 that the Ball number B_0 generated by the number x_n is $99 \times Z_{\overline{n}}$, where $Z_{\overline{n}}$ is the truncated code associated with the number x_n, whose code we will denote by $Z_n = z_0 z_1 \dots z_n$.

To obtain a new magic number, we add the same digit $a \neq 0$ at the beginning and end of the number x_n and obtain $x_{n+2} = a a_{n-1} \dots a_0 a$ with $n+2$ digits.

DOI 10.2478/rmm-2025-0012

The extended code generated by x_{n+2} is $Z_{n+2} = 0z_0 \cdots z_n 0$, truncates which results in $Z_{\overline{n+2}} = z_0 \cdots z_n$. Again, from Theorem 1 it follows that the Ball number associated with the number x_{n+2} is $99 \times Z_{\overline{n+2}} = 99 \times 10 \times Z_{\overline{n}} = 10B_0$. □

From Proposition 11 that

Corollary 4. *If $z_0 \cdots z_n$ is a code, then $z_0 \cdots z_n 0$ is an extended code.*

3 Reverse divisor

In this section, we introduce and explore a specific subclass of *permultiple* numbers, called *palintuple* or *reverse divisors*, based on the fundamental definition of permultiples. We also investigate the intricate relationships between a number and the permutations of its digits, delving into their mathematical properties and the conditions under which these relationships hold.

Definition 3 (permulpiple)**.** *A number that is a multiple of one of its digits permutations is called a permultiple. The pair of numbers is called permultiple pair.*

Example 9 ([MOR98])**.** *The number 142857 when multiplied by two, three, four, five, or six, results in another number with the same digits cyclically permuted:*

$$
\begin{aligned}
142857 \cdot 2 &= 285714 \\
142857 \cdot 3 &= 428571 \\
142857 \cdot 4 &= 571428 \\
142857 \cdot 5 &= 714285 \\
142857 \cdot 6 &= 857142
\end{aligned}
$$

Therefore, a pair of numbers formed by 142857 and any one of the list: 285714, 428571, 571428, 714285 or 857142 is an example of a permultiple pair.

Example 10 ([HOL14, HOL15, WW12])**.** *The numbers 1089 and 9801 is a permultiple pairs, because $9801 = 9 \cdot 1089$. Similarly, 98901 is also a permultiple because $98901 = 9 \cdot 10989$.*

Example 11 ([BAR07])**.** *The numbers 102564 and 410256 are also permultiple pairs, as $410256 = 4 \cdot 102564$.*

Holt[HOL14, HOL15] studied several properties of a class of *permultiple* numbers, the *palintuples* (palindromes and multiples), in base 10, as in Example 10; we presented examples in other bases. Holt[HOL15] used the name *permultiple* and is a juxtaposition of permutations and multiple words. The palindromic number is equal to the reverse. *Palintuple* numbers are also known as *reverse divisors* because their digits are arranged in reverse order.

Definition 4 (reverse divisor). *Let x_n be a non-palindromic integer with $n \geqslant 2$ digits. We say that x_n is a reverse divisor if there is an integer $1 < k < 10$ such that $x'_n = k \cdot x_n$; that is, x_n divides its reverse x'_n. In this case, x'_n is a reverse multiple of x_n.*

We often say that x_n and x'_n are the reverse divisors.

Example 12 ([HOL14, HOL15, WW12]). *The number 1089 is a reverse divisor in the permultiple class because it divides the number 9801, because $9801 = 9 \cdot 1089$. Likewise, the numbers 10989 and 98901 are reverse divisors because $98901 = 9 \cdot 10989$.*

We have

Theorem 12 ([WW12]). *The number $10\ \underbrace{99\ldots99}_{n-4\ times}\ 89$ with $n > 3$ is the reverse divisor of $98\ \underbrace{99\ldots99}_{n-4\ times}\ 01$.*

Example 13 ([HOL14, HOL15, WW12]). *The numbers 21978 and 87912 are reverse divisors, because $87912 = 4 \cdot 21978$. Additionally, 2178 and 8712 are reverse divisors.*

We also have

Theorem 13 ([WW12]). *The number $21\ \underbrace{99\ldots99}_{n-4\ times}\ 78$, where $n > 3$ is a reverse divisor of $87\ \underbrace{99\ldots99}_{n-4\ times}\ 12$.*

We present the proofs of Theorems 12 and 13 in Subsections below. Webster [WW12] and others show that the two classes indicated in the above theorems contain all reverse divisors.

First, however, we call attention to the fact that the examples of reverse divisors presented in Examples 12 and 13 have more than 4 digits.

The following two results guarantee no reverse divisors with 2 or 3 digits.

Proposition 14. *There are no reverse divisors with 2 digits.*

Proof. We assume that $x_2 = 10 \cdot a + b$, where $a, b \in \{1,2,3,4,5,6,7,8,9\}$ is a reverse divisor. Without loss of generality, we can assume $b > a$, that is, $b = a + c$ with $c \neq 0$. So we have $x_2 = 10 \cdot a + b$ and $x'_2 = 10 \cdot b + a$.

Because x_2 is a divisor of x'_2, there exists an integer $1 < k < 10$ such that

$$k = \frac{10 \cdot b + a}{10 \cdot a + b} = \frac{10 \cdot b + 100 \cdot a - 99 \cdot a}{10 \cdot a + b} = \frac{10(10 \cdot a + b)}{10 \cdot a + b} - \frac{99 \cdot a}{10 \cdot a + b} = 10 - \frac{99 \cdot a}{11 \cdot a + c}$$

So,

$$10 - k = \frac{99 \cdot a}{11 \cdot a + c} \quad \text{and} \quad (10-k)c = 11\big(9a - a(10-k)\big)$$

Then, 11 divides $10 - k$ or 11 divides c. Contradiction.

Therefore, a number x_2 with two digits cannot be a reverse divisor.

□

In the same way

Proposition 15. *There are no reverse divisors with* 3 *digits.*

Proof. Let $a, b, c, d \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ such that the number $x_3 = 100 \cdot a + 10 \cdot b + c$ is a reverse divisor with $c = a + d$. We have $x_3 = 100 \cdot a + 10 \cdot b + c$ and $x_3' = 100 \cdot c + 10 \cdot b + a$.

Because x_3 is a divisor of x_3', there exists an integer $1 < k < 10$ such that

$$k = \frac{100 \cdot c + 10 \cdot b + a}{100 \cdot a + 10 \cdot b + c} = \frac{100 \cdot (a + d) + 10 \cdot b + (c - d)}{100 \cdot a + 10 \cdot b + c} = 1 + \frac{99 \cdot d}{100 \cdot a + 10 \cdot b + c}$$

that is,

$$k - 1 = \frac{99 \cdot d}{abc}, \quad \text{so} \quad (k - 1) \cdot abc = 9 \cdot 11 \cdot d$$

Because $k - 1 < 9$, it follows that neither 11 nor 9 divide $k - 1$; therefore, 11 and 3 must divide abc because $k - 1$ can be a multiple of 3. By inspection, we find no reverse divisors between multiples of 33.

Therefore, a number x_3 with three digits cannot be a reverse divisor. □

3.1 Proof of Theorem 12

Theorem 12 is a direct consequence of the two following result.

Proposition 16. *For every natural* $n > 3$ *we obtain* $11 \times (10^{n-2} - 1) = 10\ \underbrace{99 \ldots 99}_{n-4 \text{ times}}\ 89.$

Proof. This proof is by induction on n. For $n = 4$ we have $11 \times (10^2 - 1) = 1089$.

We assume that the result is true for some natural $k > 4$; that is,

$$11 \times (10^{k-2} - 1) = 10\ \underbrace{99 \ldots 99}_{k-4 \text{ times}}\ 89 \tag{2}$$

For $k + 1$, we have

$$\begin{aligned} 11 \times (10^{(k+1)-2} - 1) &= 11 \times (10 \times 10^{k-2} - 10 + 10 - 1) \\ &= 11 \times [(10 \times (10^{k-2} - 1) + (10 - 1)] \\ &= 10 \times (11 \times (10^{k-2} - 1) + 11 \times 9 \end{aligned} \tag{3}$$

By using the induction hypothesis, that is, Equation (2) in Equation (3), we have

$$\begin{aligned} 11 \times (10^{(k+1)-2} - 1) &= 10 \times (10\ \underbrace{99 \ldots 99}_{k-4 \text{ times}}\ 89) + 11 \times 9 \\ &= 10\ \underbrace{99 \ldots 99}_{k-4 \text{ times}}\ 890 + 99 \\ &= 10\ \underbrace{99 \ldots 99}_{k-4 \text{ times}}\ 989 \\ &= 10\ \underbrace{99 \ldots 99}_{(k+1)-4 \text{ times}}\ 89 \end{aligned}$$

Which completes the proof.

□

DOI 10.2478/rmm-2025-0012

Proposition 17. *For every natural* $n > 3$ *we obtain* $9\times 10\underbrace{99\ldots 99}_{n-4\ \textit{times}}89 = 98\underbrace{99\ldots 99}_{n-4\ \textit{times}}01.$

Proof. This proof is also by induction on n. For $n = 4$ we have $9 \times 1089 = 9801$.

From Proposition 16 we have $11 \times (10^{n-2} - 1) = 10\underbrace{99\ldots 99}_{n-4\ \text{times}}89$. Now, we admit that the result is valid for some natural $k > 4$; that is,

$$9 \times 11 \times (10^{k-2} - 1) = 98\underbrace{99\ldots 99}_{k-4\ \text{times}}01 \tag{4}$$

For $k + 1$, we have

$$\begin{aligned} 9 \times 11 \times (10^{(k+1)-2} - 1) &= 9 \times 11 \times (10 \times 10^{k-2} - 10 + 10 - 1) \\ &= 9 \times 11 \times [(10 \times (10^{k-2} - 1) + (10 - 1)] \\ &= 10 \times 9 \times (11 \times (10^{k-2} - 1) + 9 \times 11 \times 9 \end{aligned} \tag{5}$$

Using the induction hypothesis, Equation (4), in Equation (5), we have

$$\begin{aligned} 9 \times 11 \times (10^{(k+1)-2} - 1) &= 10 \times (98\underbrace{99\ldots 99}_{k-4\ \text{times}}01) + 9 \times 11 \times 9 \\ &= 98\underbrace{99\ldots 99}_{k-4\ \text{times}}010 + 891 \\ &= 98\underbrace{99\ldots 99}_{k-4\ \text{times}}901 \\ &= 98\underbrace{99\ldots 99}_{(k+1)-4\ \text{times}}01 \end{aligned}$$

Which completes the proof. ☐

Let us look into

Proof. of Theorem 12
For $n = 4$, $9801 = 9 \cdot 1089$ (Example 12).

For $n > 4$ let $Y = 11\times(10^{n-2}-1)$: from Proposition 16, we have $Y = 10\underbrace{99\ldots 99}_{n-4\ \text{times}}89$. Because $Y' = 98\underbrace{99\ldots 99}_{n-4\ \text{times}}01$, it follows from Proposition 17 that $Y' = 9 \cdot Y$. ☐

3.2 Proof of Theorem 13

The proof of Theorem 13 can be derived from that of the two following results.

Proposition 18. *For every natural* $n > 3$ *we obtain* $2\times 11\times(10^{n-2}-1) = 21\underbrace{99\ldots 99}_{n-4\ \textit{times}}78.$

Proof. We again apply induction on n. For $n = 4$ we have $2 \times 11 \times (10^2 - 1) = 2178$.

DOI 10.2478/rmm-2025-0012

From Proposition 16 we have $11 \times (10^{n-2} - 1) = 10 \underbrace{99\ldots99}_{n-4 \text{ times}} 89$, and admit that the result is valid for some natural $k > 4$; that is,

$$2 \times 11 \times (10^{k-2} - 1) = 2 \times 10 \underbrace{99\ldots99}_{k-4 \text{ times}} 89 = 21 \underbrace{99\ldots99}_{k-4 \text{ times}} 78 \tag{6}$$

For $k+1$, we have

$$\begin{aligned} 2 \times 11 \times (10^{(k+1)-2} - 1) &= 2 \times 11 \times (10 \times 10^{k-2} - 10 + 10 - 1) \\ &= 10[2 \times (11 \times (10^{k-2} - 1)] + 2 \times 11 \times 9 \end{aligned} \tag{7}$$

Using the induction hypothesis, Equation (6) in Equation (7), we have

$$\begin{aligned} 2 \times 11 \times (10^{(k+1)-2} - 1) &= 10[2 \times 10 \underbrace{99\ldots99}_{k-4 \text{ times}} 89] + 2 \times 11 \times 9 \\ &= 21 \underbrace{99\ldots99}_{k-4 \text{ times}} 780 + 198 \\ &= 21 \underbrace{99\ldots99}_{k-4 \text{ times}} 978 \\ &= 21 \underbrace{99\ldots99}_{(k+1)-4 \text{ times}} 78 \end{aligned}$$

which completes the proof. □

Proposition 19. *For every natural $n > 3$ we obtain*

$$4 \times 21 \underbrace{99\ldots99}_{n-4 \text{ times}} 78 = 87 \underbrace{99\ldots99}_{n-4 \text{ times}} 12$$

Proof. Again, we complete this proof using induction on n. For $n = 4$ we have $4 \times 2178 = 8712$.

From Proposition 18, we obtain $2 \times 11 \times (10^{n-2} - 1) = 21 \underbrace{99\ldots99}_{n-4 \text{ times}} 78$. Now, admit that the result is true for some natural $k > 4$, that is,

$$4 \times 21 \underbrace{99\ldots99}_{k-4 \text{ times}} 78 = 87 \underbrace{99\ldots99}_{k-4 \text{ times}} 12 \tag{8}$$

For $k+1$, we have

$$\begin{aligned} 4 \times 21 \underbrace{99\ldots99}_{(k+1)-4 \text{ times}} 78 &= 4 \times [2 \times 11 \times (10^{(k+1)-2} - 1)] \\ &= 4 \times 2 \times [11 \times (10 \times 10^{k-2} - 10 + 10 - 1)] \\ &= 10 \times 4 \times [2 \times (11 \times (10^{k-2} - 1)] + 4 \times 2 \times 11 \times 9 \end{aligned} \tag{9}$$

Using the induction hypothesis, Equation (8), and Proposition 18 in Equation (9), we have

$$\begin{aligned} 4 \times 21 \underbrace{99\ldots99}_{(k+1)-4 \text{ times}} 78 &= 10 \times (87 \underbrace{99\ldots99}_{k-4 \text{ times}} 12) + 792 \\ &= 87 \underbrace{99\ldots99}_{k-4 \text{ times}} 120 + 792 \\ &= 87 \underbrace{99\ldots99}_{k-4 \text{ times}} 912 \\ &= 87 \underbrace{99\ldots99}_{(k+1)-4 \text{ times}} 12 \end{aligned}$$

completes the proof. □

We can now prove Theorem 13.

Proof of Theorem 13. If $n = 4$, 2178 is the reverse divisor of 8712 (Example 13).

For $n > 4$ let $Y = 2 \times 11 \times (10^{n-2} - 1)$. From, Proposition 18, we have $Y = 21 \underbrace{99\ldots99}_{n-4 \text{ times}} 78$. Because $Y' = 87 \underbrace{99\ldots99}_{n-4 \text{ times}} 12$, it follows from Proposition 19 that $Y' = 4 \cdot Y$.

□

4 Magic reverse divisors

In this section, we identify the numbers that are both Ball's magic numbers and reverse divisors.

Theorem 20. *For every $n > 1$, the reverse divisor of $11 \times (10^n - 1)$ is a Ball's number.*

Proof. For all $n > 1$, we have that

$$11 \times (10^n - 1) = 11 \times \underbrace{99\ldots99}_{n \text{ times}} = 11 \times 9 \times \underbrace{11\ldots11}_{n \text{ times}} = 99 \times \underbrace{11\ldots11}_{n \text{ times}}$$

From Proposition 7 we have $\underbrace{11\ldots11}_{n \text{ times}}0$ as a code. Thus, the list $\underbrace{11\ldots11}_{n \text{ times}}$ is a truncated code. Therefore, $99 \times \underbrace{11\ldots11}_{n \text{ times}}$ is a Ball's magic number. □

From Theorem 20 and the results presented in Section 3.1, we find that each reverse divisor of form $11 \times (10^n - 1)$ has 9 as the reverse quotient.

The following result from Theorem 20:

Corollary 5. *If B is a magic number and reverse divisor, then B added to its reverse B' is $10B$.*

Proof. From Theorem 20, we obtain that $B = 99 \times \underbrace{11\ldots11}_{n \text{ times}} = 11 \times (10^n - 1)$.

Additionally, from Proposition 17 we have $B' = 9 \times 11 \times (10^n - 1)$ which completes the proof. □

DOI 10.2478/rmm-2025-0012

Corollary 6. *The reverse divisor* $22 \times (10^n - 1)$, $n > 1$, *is a magic number multiplied by* 2.

Proof. Note that $22 \times (10^n - 1) = 2 \times 11 \times (10^n - 1)$. □

The following results show that it is possible to write reverse divisors by adding two other magic numbers.

Theorem 21. *Let* $D = 11 \times (10^n - 1)$ *be a reverse divisor with* $n \geqslant 2$. *There are two magic numbers,* B_1 *and* B_2, *such that* $D = B_1 + B_2$.

Proof. It follows from Theorem 20 that the reverse divisor is $D = 99 \times \underbrace{11\ldots11}_{n \text{ times}}$. The corresponding code was $R = \underbrace{11\ldots11}_{n \text{ times}}0$. From Proposition 8 we can decompose code R as the sum of code A and the extended code C; that is, $R = A + C$. In this manner, the reverse divisor D is the sum of magic numbers because

$$
\begin{aligned}
D &= 99 \times \underbrace{11\ldots11}_{n \text{ times}} = 99 \times R \\
&= 99 \times (A + C) = 99 \times A + 99 \times C
\end{aligned}
$$

From Theorem 1 we obtain $B_1 = 99 \times A$ and $B_2 = 99 \times C$ as Ball's magic numbers. □

Example 14. *We can write the reverse divisor* $D = 1099989 = 99 \times 11111$ *as the sum of* $B_1 = 990099 = 99 \times 10001$ *and* $B_2 = 10989 = 99 \times 1110$, *both of which are magic numbers.*

Example 15. *The reverse divisor* $D = 109989 = 99 \times 1111$ *has a code* $11110 = 11100 + 10$. *So,*

$$
\begin{aligned}
109989 &= 99 \times 1111 = 99 \times (1110 + 1) \\
&= 99 \times 1110 + 99 \times 1 = 109890 + 99
\end{aligned}
$$

Alternatively, we can write the code as $11110 = 10010 + 1100$. *So,*

$$
\begin{aligned}
109989 &= 99 \times 1111 = 99 \times (1001 + 110) \\
&= 99 \times 1001 + 99 \times 110 = 99099 + 10890
\end{aligned}
$$

Example 16. *We also have* $1001001 \times 111 = 111111111$, *so* $99 \times 1001001 \times 111 = 99 \times 111111111$. *It follows that the reverse divisor* 99×111 *and magic number* 99×1001001 *are divisors of the reverse divisor* 99×111111111. *We can give a similar fact by* $101 \times 11 = 1111$ *because this leads to* $99 \times 101 \times 11 = 99 \times 1111$.

Example 16 allows us to conclude that by factoring repunits into codes or extended codes, we obtain magic, a divisor of reverse divisors. The following result provides reverse divisors with a special type of divisor:

Theorem 22. *A reverse divisor* $D = 11 \times (10^{2^n} - 1)$ *with* $n \geqslant 2$ *has at least* $n + 1$ *magic numbers as divisors.*

DOI 10.2478/rmm-2025-0012

Proof. Factoring D, we get

$$\begin{aligned} D = 11 \times 10^{2^n} - 1 &= 11(10-1)(10+1)(10^2+1)(10^4+1)\cdots(10^{2^{n-1}}+1) \\ &= 99(10+1)(10^2+1)(10^4+1)\cdots(10^{2^{n-1}}+1) \end{aligned}$$

It follows from Proposition 6 that $(10^{2^n}-1)$ is a code for every $m > 0$. Therefore, from Corollary 1, every $B_m = 11 \times (10^{2^m} - 1)$ is a divisor of D to $0 \leqslant m \leqslant n-1$. We conclude that magic numbers of the form $B_m = 99 \times (10^{2^m} + 1)$ with $0 \leqslant m \leqslant n-1$ are the divisors of D. Because D is a magic number, we have at least $n+1$ divisors. □

Example 17. *The number* $1099999989 = 11 \times (10^8 - 1)$ *is divisible by the magic numbers:* $990099 = 99 \times (10^4 + 1)$, $9999 = 99 \times (10^2 + 1)$ *and* $1089 = 99 \times (10 + 1)$.

The Ball magic number was 1089. It is a reverse divisor; some multiples are also reverse divisors, as shown below.

Proposition 23. *For all* $n > 1$ *and* $O(n) \in \{O_n\}_{n \geqslant 1}$, $X_n = 1089 \times O(n)$ *is the Ball's magic number.*

Proof. For all $n > 1$, $1089 \times O(n) = 99 \times 11 \times O(n)$.

From Proposition 9 the factor $11 \times O(n)$ is truncated or extended code. In both cases, Theorems 1 or Remark 5 guarantee that $1089 \times O(n)$ is the magic number. □

Theorem 24. *For all odd* $n > 1$, $X_n = 1089 \times O(n)$ *is the reverse divisor.*

Proof. Thus, $X_n = 99 \times 11 \times O(n)$. Proposition 9 indicates that $11 \times O(n)$ is one repunit. From Theorem 12 and Proposition 16 follows this result. □

Corollary 7. *For all odd* $n > 1$, $Y_n = 2178 \times O(n)$ *is the reverse divisor.*

Proof. From Theorem 13 it follows that 2178 is the reverse divisor. As $2178 = 2 \times 1089$, it follows from Theorem 24 that Y_n is also a reverse divisor. □

5 Reverse divisor and square number

In this section, we aim to determine that the product of two distinct reverse divisors cannot be a square number. Specifically, if R_n, $n \geqslant 1$, denotes a repunit (list with only ones) and $M \in \mathbb{N}$, we ultimately show that $R_n \times R_m \neq M^2$, for $m \neq n$.

Lemma 1 ([CS22])**.** *If* $n > 1$, $10^n - 1$ *is not a square number.*

Proof. For $n > 1$, 10^n is a multiple of 4, therefore, $10^n - 1$ leaves the remainder at 3 when divided by 4. The division of a square number by four admits only remainders 0 or 1. We conclude that $10^n - 1$ is not a square. □

Lemma 2 ([CS20, NM91])**.** *The number* R_n *with* $n > 1$ *is not a square.*

Proof. It follows from

$$\begin{aligned} 10^n - 1 &= 999\ldots99 = 9 \times 111\ldots11 \\ &= 9 \cdot R_n = 3^2 \cdot R_n \end{aligned}$$

Because $n \geqslant 2$, if R_n is a square number, then $10^n - 1$ would also have to be a square number. However, this is not the case, as can see in previous Lemma. □

DOI 10.2478/rmm-2025-0012

Remark 6. *A non-zero natural number formed by repeating the same digit (digit) is called a single digit, as in previous lemmas for $a = 1$ or 9. An interesting property of a single digit is that no single digit with two or more digits is a square number [CS22, NM91].*

According to [BEI66, CS20, CS22], for all $n \geqslant 1$, the repunits R_n can be expressed by the sequence

$$R_1 = 1 \text{ and } R_{n+1} = 10 \cdot R_n + 1 \text{ for } n > 1$$

We denote the greatest common divisor between two non-zero natural numbers a and b as (a, b). So,

Lemma 3. *Let m, n be natural numbers with $m \geqslant n > 0$ then we have $(R_m, R_n) = R_{(m,n)}$.*

Proof. By the Euclidean algorithm, there exist integers q, r_1 such that $m = qn + r_1$ with $0 \leqslant r_1 < n$. Thus,

$$(m, n) = (n, r_1)$$

We can apply the Euclidean algorithm again to obtain q_1 and r_2 such that $n = q_1 r_1 + r_2$,

$$(m, n) = (n, r_1) = (r_1, r_2)$$

We can continue applying the Euclidean algorithm until we reach a remainder of 0. We obtain a sequence r_1, r_2, $r_3, \ldots$, r_s, r_{s+1}, where $r_{s+1} = 0$. So,

$$(m, n) = (n, r_1) = (r_1, r_2) = \cdots = (r_s, r_{s+1}) = r_s$$

Since $R_m = 10^{r_1} R_n + R_{r_1}$, we have $(R_m, R_n) = (10^{r_1} R_n + R_{r_1}, R_n) = (R_n, R_{r_1})$. Then,

$$(R_m, R_n) = (R_n, R_{r_1}) = \cdots = (R_{r_s}, R_{r_{s+1}}) = (R_{r_s}, 0) = R_{(m,n)}$$

□

Lemma 4. *Let $m > n > 1$ be integers; then, $R_m \cdot R_n$ is not a square number.*

Proof. We denote this as $d = (m, n)$. From Lemma 3 that $(R_m, R_n) = R_d$. If $d = 1$ then R_m and R_n are coprime, so there is no common divisor, which means that $R_m \cdot R_n$ is not a square because R_m and R_n are not square numbers. If $d > 1$, R_d is the greatest common divisor of R_m and R_n. So, $R_m = R_d \cdot q_1$ and $R_n = R_d \cdot q_2$ with $q_1 \neq q_2$, $(q_1, q_2) = 1$ since $m \neq n$. We have $R_m \cdot R_n = (R_d)^2 \cdot q_1 \cdot q_2$, which implies that $R_m \cdot R_n$ is also not a square number because q_1 and q_2 are not square numbers. □

Proposition 25. *Let $m > n > 1$ be integers; then, $(10^m - 1)(10^n - 1)$ is not a square number.*

Proof. We can write $(10^m - 1) = 9 \cdot R_m$. Therefore, $(10^m - 1)(10^n - 1) = 9^2 \cdot R_m \cdot R_n$. It follows from Lemma 4 that $R_m \cdot R_n$ is not a square number, and thus $(10^m - 1)(10^n - 1)$ is not a square number. □

Finally, we have

Theorem 26. *The product of these two distinct reverse divisors is never a square.*

DOI 10.2478/rmm-2025-0012

Proof. From Theorem [WW12] that a reverse divisor has the form $11 \times (10^m - 1)$ or $22 \times (10^m - 1)$. From Proposition 25, we conclude that the product of distinct reverse divisors is never a square number. □

However, the product of the reverse divisor and its reverse is always a square, as in the case of $1089 \times 9801 = 3267^2$ (see [WW12]).

6 Conclusion

Throughout this study, we establish unexpected connections between magic numbers, reverse divisors, square numbers, repunits, and undulating numbers. Given the richness of mathematical structures, we are confident that further interesting properties involving these numbers can be derived.

References

[BAL26] Walter W. R. BALL. Mathematical recreations an essays. *The Macmillan Company*, 10, 1926.

[BAL05] Jhon BALL, editor. *Think of a Number*. DK Children, Great Britain, 2005.

[BAR07] Augusto M. A. BARROS. Qual a relação que existe entre os números 102564 e 410256? *Revista do Professor de Matemática*, 63:22–23, 2007.

[BEI66] A. H. BEILER, editor. *Recreations in the Theory of Numbers: The Queen of Mathematics Entertains*. New York, New: Dover Publications, 1966.

[CC15] Fernando S. CARVALHO and Eudes A. COSTA. Escrever o número $111 \cdots 111$ como produto de dois números. *Revista do Professor de Matemática*, 87, 2015. https://rpm.org.br/cdrpm/87/36.html#:~:text=111111%20%3D%203%20%C3%97%2037037.

[CC21] Eudes A. COSTA and Grieg A. COSTA. Existem números primos na forma 101... 101. *Revista do Professor de Matemática*, (103):21–22, 2021.

[CM14] Eudes A. COSTA and Elis G. C. MESQUITA. O número mágico m. *Revista da Olimpíada (IME-UFG)*, (9):33–43, 2014. https://files.cercomp.ufg.br/weby/up/1170/o/Eudes9.pdf.

[COS21] Eudes A. COSTA. Os números mágicos de ball e a sequência de fibonacci. *Revista Sergipana de Matemática e Educação Matemática*, 6(1):19–25, 2021.

[CS20] Eudes A. COSTA and Douglas C. SANTOS. Números repunidades: algumas propriedades e resolução de problemas. *Professor de Matemática Online*, 8(4):495–503, 2020.

[CS22] Eudes A. COSTA and Douglas C. SANTOS. Algumas propriedades dos números monodígitos e repunidades. *Revista de Matemática*, 2:47–58, 2022. https://periodicos.ufop.br/rmat/article/view/6827.

[HOL14] Benjamim V. HOLT. Some general results and open questions on palintiple numbers. *Integers*, 14(A42):1–13, 2014. https://www.emis.de/journals/INTEGERS/papers/o42/o42.pdf.

[HOL15] B. V. HOLT. On permutiples having a fixed set of digits. 2015. http://math.colgate.edu/~integers/r20/r20.pdf.

[MOR98] Carlos G. T. de A. MOREIRA. Números mágicos e contas de dividir. *Eureka*, (1):38–40, 1998. https://www.obmep.org.br/docs/Eureka.pdf.

[NM91] Herbert S. NIVEN, Ivan; ZUCKERMAN and Hugh L. MONTGOMERY, editors. *An introduction to the theory of numbers*. John Wiley & Sons, New York, 1991.

[PIC90] C. A. PICKOVER. Is there a double smoothly undulating integer? *Journal of Recreational Mathematics*, 22(1):52–53, 1990.

[SC22] Ronaldo A. SANTOS and Eudes A. COSTA. Números de ball generalizados. *Revista Sergipana de Matemática e Educação Matemática*, 7(1):61–85, 2022.

[WEB95] R. WEBSTER. A combinatorial problem with a fibonacci solution. *The Fibonacci Quarterly*, 33:26–31, 1995. https://www.fq.math.ca/Scanned/33-1/webster.pdf.

[WW12] R. WEBSTER and G. WILLIAMS. On the trail of reverse divisors: 1089 and all that follow. *Mathematical Spectrum*, 45:96–102, 2012. http://users.mct.open.ac.uk/gw3285/publications/reverse-divisors.pdf.