A Short Note on Gentzen´s LJ and NJ Systems Isomorphism

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Abstract: We state a new intuitionistic sequent calculus and use it to clarify Gentzen´s *NJ* and *LJ* isomorphism, it contains new negation rules which are immediate readings of what seems to be good and sound natural deduction rules.

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We define the *LJ´* sequent calculus as follows. It has the initial sequents of *LJ* ([Gen35], pp. 83-85) and all the rules, excepting rules for negation and thinning on the succedent. Instead, it has the following two negation rules:

$$\frac{\Gamma\to A}{\neg A,\Gamma\to B}\ \neg\text{-}IA' \qquad\qquad \frac{A,\Gamma\to B}{\Gamma\to\neg A}\ \neg\text{-}IS'$$

Rule ¬-*IS´* can only be applied under a restriction: *B* must be a propositional variable not occurring in *G* or *A*. In other words, it should not occur in the lower sequent of ¬-*IS´*.

LJ´ is an interesting system. It is equivalent to the original *LJ*. *Hauptsatz* holds for it. Its negation rules are immediate transliteration of what seems to be good and sound natural deduction rules for negation. If that is correct, the traditional way in which intuitionists define negation – by means of absurd constant, saying that there is no introduction rule for absurd, as is done in, for example, [Mar96] –, can be jeopardized.

Theorem 1: *(i)* For all sequents *G®A*, *G®A* is provable in *LJ* iff it is provable in *LJ´*. *(ii)* For all sequents *G®* , *G®* is provable in *LJ* iff *G®B* is provable in *LJ´*, such that *B* is a propositional variable not occurring in *G*. *(iii)* If *p* is a cutless proof of *G®A* (*G®*) in *LJ*, then there is way to obtain a cutless proof of *G®A* (*G®B*, where *B* is like it was in *(ii)*,) in *LJ´*.

Demonstration: *(i)* and *(ii)* by induction in the length of the proof of *G®A* (*G®*). First, let´s prove it from right to left. Suppose given a proof of *G®A* in *LJ´*. It suffices to show how to obtain ¬-*IA´* and ¬-*IS´* in *LJ* (other cases are straightforward). Suppose *G=ØC,D* and *A=B*:

$$\dfrac{\begin{array}{c}M\\ \Delta\to C\end{array}}{\neg C,\Delta\to B}\ \neg\text{-}IA' \;\Rightarrow\; \dfrac{\dfrac{\begin{array}{c}\textbf{(by induction)}\\ M\\ \text{D ® }\mathbf{C}\end{array}}{\neg C,\Delta\to}\ \neg\text{-}IA}{\neg C,\Delta\to B}\ \text{Thinning}$$

Suppose *A=ØC*:

$$\dfrac{\begin{array}{c}M\\ C,\Gamma\to B\end{array}}{\Gamma\to\neg C}\ \neg\text{-}IS' \;\Rightarrow\; \dfrac{\dfrac{\begin{array}{c}\textbf{(by induction)}\\ M\ [B/(B\wedge\neg B)]\\ \mathbf{C,G\text{ ® }B}\ [B/(B\wedge\neg B)]\end{array}\qquad \begin{array}{c}M\ \text{(easy in } LJ)\\ B\wedge\neg B\to\end{array}}{C,\Gamma\to}\ \text{Cut}}{\Gamma\to\neg C}\ \neg\text{-}IS$$

The expression [*B/(BÙØB)*] is used to indicate substitution of all occurrences of *B* (which doesn´t occur in *C* or *G*) by *BÙØB*. In second place, to prove it from left to right, suppose there is a proof *p* of *G®A* (*G®*) in *LJ*. First subcase, if *G®* , there are two relevant subsubcases (other subsubcases are straightforward). Suppose *G=ØC,D* and suppose that *B* is a propositional variable not occurring in *G*:

(by induction)

M ⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀ M
Δ→C ⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀ D®**C**
———— ¬-*IA* ⇒ ———— ¬-*IA´*
¬C,Δ→ ⠀⠀⠀⠀⠀⠀⠀⠀⠀ ¬C,Δ→B

Suppose *B* is a propositional variable not occurring in *D* and suppose *C* is a propositional variable not occurring in *G*:

(by induction)

M ⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀ M [B/C]
Δ→ ⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀ D®**B** [B/C]
—— r ⇒ ———— r´
Γ→ ⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀ Γ→C

Rule *r*, above, could not be a thinning in the succedent, neither ¬-*IA*, nor ¬-*IS*. So, *LJ´* has a rule that exactly matches *r*. We denoted it by *r´*. Applying *r´* over *G* we obtain *D*. Now the second subcase, if *p* proves *G®A*, we have two important subsubcases (others are straightforward). Suppose *A=ØC* and suppose that *B* is a propositional variable not occurring in *G* or *C*:

(by induction)

M ⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀ M
Γ,C→ ⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀ G,**C**®**B**
———— ¬-*IS* ⇒ ———— ¬-*IS´*
Γ→¬C ⠀⠀⠀⠀⠀⠀⠀⠀⠀ Γ→¬C

Suppose *B* is a propositional variable not occurring in *G*. Suppose, also, that no individual parameter in *A* is the *eingenvariable* of an ∀-*IS* or ∃-*IA* in the proof given by induction (if there were such parameters, new parameters should be substituted for them in the proof):

M
Γ→ ⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀ **(by induction)**
——— Thinning ⇒ ⠀ M [B/A]
Γ→A ⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀⠀ G®**B** [B/A]

So *(i)* and *(ii)* are proven. Also *(iii)*. In the proof of *(ii)* no Cut rule was used. Then, any proof in *LJ´* obtained by transformation on an *LJ* proof, will be cutless if the *LJ* proof was cutless. *QED*

Corolary 1 – *Hauptsatz* holds for *LJ´*.

Demonstration: Using theorem 1, because *hauptsatz* holds for *LJ* (cf. [Gen35] p. 87).

The *LJ´* system has no sequents with empty succedent. As such, *LJ´* can be readily related to a new natural deduction system *NJ´*. The system *NJ´* is obtained from *NJ* (cf. [Gen35] pp. 76-79) leaving out Gentzen´s rules for negation and absurdity and adding introduction and elimination rules for negation. Negation introduction and negation elimination in *NJ´* correspond exactly to ¬-*IS´* and ¬-*IA´* in *LJ´*[1], respectively. As Gentzen showed that *LJ* and *NJ* are equivalent ([Gen35], s. 5), the trip is "round": *NJ* and *NJ´* are equivalent.

Many authors, as Curry ([Cur63] chapters 5, 6 and 7) and Prawitz ([Pra65] Appendix A), noticed that natural deduction and sequent calculus are structurally similar, interpreting sign ® as a symbol for logical consequence. Raggio [Rag88] stresses the relationship:

> *As is well known, Gentzen´s thesis begins building a system of natural deduction for quantification theory. ... by means of this system of natural deduction Gentzen showed that the connectives – except negation – and the quantifiers can be analyzed by introduction and elimination rules in such a way that the latter are the inverses of the former. ... in order to prevent and clarify certain difficulties connected with negation, Gentzen, in the second part of his thesis, developed a new system: the logic of sequences. In this second system the natural rules of elimination become rules of introduction in the antecedent. In this new system negation loses "as by magic" all its problems.*

If he is right, theorem *1* above explains why "negation loses as by magic all its problems" in *LJ* sequent calculus. Even admitting empty succedents, ¬-*IS* and ¬-*IA* in *LJ* are expressing what would be negation elimination and negation introduction for natural deduction. Thinning, over an empty succedent, is expressing *ex falso quodlibet*. It is difficult to associate any natural deduction construction to the empty succedent[2]. However, as we saw, via *LJ´*, these constructions in natural deduction correspond to *LJ* sequents.

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[1] See [San04].

[2] Tennant [Ten99] had done that, but at the cost of multiplying the number of natural deduction rules.