GENERALIZED SOLUTIONS TO THE KdV HIERARCHY IN 2-DIMENSION

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Dedicated to Hebe Biagioni, adviser and friend

Abstract

We obtain results of existence and uniqueness in the algebra $\mathcal{G}_2((0,T)\times \mathbf{R}^2)$ for the Cauchy problem to the Korteweg-de Vries hierarchy in $\mathbf{R}^2$. In particular, we show a result on the existence for initial rough data such as the distribution δ-Dirac and its derivatives. Our results can be extended to equations in $\mathbf{R}^n$ with minor changes on the initial data.

1 Introduction

In this work we extend for KdV hierarchy in the 2-dimensional space the results obtained in [17] for the KdV hierarchy in $\mathbf{R}$. We obtain results of existence and uniqueness of solutions in $\mathcal{G}_2((0,T)\times\mathbf{R}^2)$, for the Cauchy problem for KdV hierarchy in $\mathbf{R}^2$. Our results are general, since they allow singular initial data such as the distribution δ -Dirac and its derivatives. The space $\mathcal{G}_2(\Omega)$, see section 1.1, is a variant of the algebra of the generalized functions established by J. F. Colombeau in [8] (see also [1] and [19]).

In 1-dimensional space other equations such as Korteweg de Vries, third member of Lax hierarchy, Olver, Benney, Fisher, Benjamin-Bona-Mahony (BBM),

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Smith (S), Benjamin-Ono (BO), Burgers, cubic Schrödinger (NLS) and modified Korteweg-de Vries (mKdV) were studied in the same context, as shown in [2], [3], [4], [5], [6], [16] and [17].

The KdV hierarchy in $\mathbf{R}$ is given by a family of equations of the form

$$u_t = \frac{\partial}{\partial x} G_m(u), \tag{1.1}$$

where $u : (0, \infty) \times \mathbf{R} \to \mathbf{R}$, $(t, x) \to u(t, x)$ is the function to be determinated, and for each $m \in \mathbf{Z}^+$, $G_m(u)$ is the gradient in u of a functional $F_m(u)$, which is constant along solutions of the KdV equation

$$u_t = uu_x + u_{xxx}, \tag{1.2}$$

i.e.,

$$\frac{d}{d\lambda} F_m(u + \lambda v)|_{\lambda=0} = (G_m(u), v) = \int_{\mathbf{R}} G_m(u(x)) v(x) dx, \quad v \in L^2(\mathbf{R}).$$

Recent results of well-posedeness in Sobolev spaces with negative index were obtained for the equation (1.2) in [7]. In [20], results of well-posedeness in weighted Sobolev spaces, were obtained for the family (1.1).

The family of functionals $\{F_m(u)\}|_{m \in \mathbb{Z}^+}$ in consideration, was exhibited by Miura, Gardner, Kruskal and Zabusky in [14] and [18]. They show the existence of infinitely many functionals of the form $F_m(u) = \int_{\mathbf{R}} f_m(u) dx$, where each $f_m(u)$ is a polynomial of degree $m + 1$ in the x-derivatives of u up to order $m - 1$, as seen in [22]. An alternative proof of the existence of these functionals was presented by Lax in [15].

The generalization of equation (1.2) to the space $\mathbf{R}^n$, as seen in [21], consists in considering $(t, x) \in \mathbf{R}^{1+n}$ and equation

$$u_t = u \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} + \sum_{i,j=1}^{n} \frac{\partial^3 u}{\partial x_i \partial x_j^2}. \tag{1.3}$$

The proof of our main results, Theorem 2.2, depends on the existence of many conserved functionals of the equation (1.3). We can observe directly three of them, which are

$$F_0(u) = \int_{\mathbf{R}^n} 3u dx, \;\; F_1(u) = \int_{\mathbf{R}^n} \frac{1}{2} u^2 dx \;\; \text{and} \;\; F_2(u) = \int_{\mathbf{R}^n} (\frac{1}{6} u^3 - \frac{1}{2} |\nabla u|^2) dx,$$

whose respective gradients are given by

$$G_0(u) = 3, \quad G_1(u) = u \quad \text{ and } \; G_2(u) = \frac{1}{2}u^2 + \Delta u, \tag{1.4}$$

where Δ and ∇ are usual operators.

In [11] and [15] it is published the formula due to Lenart, valid for real case,

$$LG_m = \frac{\partial}{\partial x} G_{m+1},$$

where L is the operator $\frac{\partial^3}{\partial x^3} + \frac{2}{3}u\frac{\partial}{\partial x} + \frac{1}{3}u_x$. The generalization of this operator to the case $\mathbf{R}^n$ is given by

$$L = \sum_{\alpha_3} D^{\alpha_3} + \frac{2}{3}u \sum_{\alpha_1} D^{\alpha_1} + \frac{1}{3}\sum_{\alpha_1} D^{\alpha_1}u, \;\text{ where } \alpha_i \in \mathbf{N}^n, \;\; |\alpha_i| = i.$$

Note that L is antisymmetric ($L^* = -L$). The Lenart relation is now given by

$$LG_m = \sum_{\alpha_1} D^{\alpha_1} G_{m+1}, \tag{1.5}$$

where $G_m(u)$ for $m = 0, 1, 2$ is given by (1.4) and for $m \geq 3$, it can be determined by following [15], with minor changes. In fact, it is possible to prove that $G_m(u)$ is, for every m, the gradient of a functional

$$F_m(u) = \int_{\mathbf{R}^n} f_m(u)dx, \tag{1.6}$$

which is a conserved quantity to equation (1.3). In the expression (1.6) $f_m(u)$ is a polynomial in u and its derivatives in the variable x.

It is easy to verify that (1.5) holds for $m = 0$ and 1. By following the ideas of Lax [15], it is possible to exhibit an infinite sequence of such functionals and after rescaling, obtain that f_m is given by

$$\begin{aligned} f_m(u) = \; & \textstyle\sum_{\alpha_{m-1}} (D^{\alpha_{m-1}}u)^2 + \sum_{\alpha_{m-2}} c_{\alpha_{m-2}} u(D^{\alpha_{m-2}}u)^2 + c_{\alpha_0}u^{m+1} \\ & +Q_m(u, D^{\alpha_1}u, ..., D^{\alpha_{m-3}}u), \end{aligned} \tag{1.7}$$

where c_{α_i} are appropriate constants and Q_m is a polynomial in the x-derivatives of u up to order $m-3$. The terms of f_m are given by a product

$$\prod_{i=0}^{m-1} (D^{\alpha_i}u)^{l_{\alpha_i}}, \tag{1.8}$$

where α_i and l_{α_i} satisfy the relation

$$\sum_{i=0}^{m-1}(1+\frac{i}{2})l_{\alpha_i} = m+1, \tag{1.9}$$

see [18] for the real case. Thus we obtain a sequence G_m of gradients of functionals F_m which are constant values along the solutions of equation (1.3). The hierarchy of the KdV equation in dimension n, as seen in [21], is given by the family of equations

$$u_t = \sum \partial^{\alpha_1} G_m(u). \tag{1.10}$$

For the case $m = 3$ and $n = 2$, we obtain the KdV of fifth order in dimension two, i.e.,

$$\begin{aligned} u_t \;=\; & \frac{5}{6}u^2u_x + \frac{5}{6}u^2u_y + \frac{10}{3}u_xu_{xx} + \frac{4}{3}u_xu_{xy} + \frac{4}{3}u_yu_{xy} + \frac{4}{3}u_xu_{yy} + \\ & \frac{4}{3}u_yu_{xx} + \frac{10}{3}u_yu_{yy} + \frac{5}{3}uu_{xxx} + \frac{5}{3}uu_{xxy} + \frac{5}{3}uu_{xyy} + \frac{5}{3}uu_{yyy} + \\ & u_{xxxxx} + u_{xxxxy} + 2u_{xxxyy} + 2u_{xxyyy} + u_{xyyyy} + u_{yyyyy}. \end{aligned}$$

We observe from (1.7) that the polynomial f_m has degree $m+1$ and order $m-1$. Also from (1.9) we have that $G_m(u)$ is a polynomial of degree m and order $2m-2$. Thus, the equation (1.10) is a nonlinear equation of order $2m-1$ and degree m. By using equation (1.5) it is possible to show that each functional $F_m(u)$ is also a constant along solutions of equation (1.10).

1.1 The $G_2(\Omega)$ Algebra and some Definitions

We study the Cauchy problem for the 2D-KdV hierarchy in the Colombeau algebra $\mathcal{G}_2(\Omega)$, a particular case of the algebras $\mathcal{G}_{p,q}(\Omega)$, defined in [4].

Let $I = (0,1)$ and $\Omega \subset \mathbf{R}^n$ be an open set. We set

$$\begin{aligned} \mathcal{E}_2[\Omega] = (H^\infty(\Omega))^I = \; & \{\widehat{u} : I \to H^\infty(\Omega),\, \varepsilon \in I \mapsto \widehat{u}_\varepsilon \in H^\infty(\Omega), \\ & \text{real valued for all } \varepsilon > 0\}, \end{aligned}$$

where $H^\infty(\Omega) = \cap_{k\in\mathbf{Z}} H^k(\Omega)$.

We set

$$\mathcal{E}_{M,2}[\Omega] = \{\widehat{u} \in \mathcal{E}_2[\Omega] \text{ such that for all } k \in \mathbf{Z}^+, \exists\, N > 0 \text{ such that}$$
$$\|\widehat{u}_\varepsilon\|_k = \mathcal{O}(\varepsilon^{-N}), \text{ as } \varepsilon \to 0\}, \tag{1.11}$$
$$\mathcal{N}_2[\Omega] = \{\widehat{u} \in \mathcal{E}_{M,2}[\Omega] \text{ such that for all } k \in \mathbf{Z}^+, \text{ and } M > 0,$$
$$\|\widehat{u}_\varepsilon\|_k = \mathcal{O}(\varepsilon^{M}), \text{ as } \varepsilon \to 0\},$$

where $\|\cdot\|_k$ is the usual norm of the Sobolev space $H^k(\Omega)$. The elements of $\mathcal{E}_{M,2}[\Omega]$ are denoted by $\widehat{u}$, $\widehat{v}, \cdots$ and are called moderate. The set $\mathcal{N}_2[\Omega]$ is called null space.

If Ω has the cone property, then, see [4]:

(i) $\mathcal{E}_{M,2}[\Omega]$ is an algebra with partial derivatives,

(ii) $\mathcal{N}_2[\Omega]$ is an ideal of $\mathcal{E}_{M,2}[\Omega]$ which is invariant under derivatives,

(iii) If $\Omega = \mathbf{R}^n$ and $\widehat{u} \in \mathcal{E}_2[\Omega]$, then for all $\varepsilon > 0$ $\lim_{|x|\to\infty} \widehat{u}_\varepsilon(x) = 0$.

The set $\mathcal{G}_2(\Omega)$, defined by

$$\mathcal{G}_2(\Omega) = \frac{\mathcal{E}_{M,2}[\Omega]}{\mathcal{N}_2[\Omega]},$$

is also an algebra; its elements, denoted by u, $v, \cdots$ are called *generalized functions* in Ω. The multiplication in $\mathcal{G}_2(\Omega)$ is defined on representatives i.e., if u, $v \in \mathcal{G}_2(\Omega)$ then the product uv is the class of $\widehat{u}\widehat{v}$, where $\widehat{u}$ and $\widehat{v}$ are the respective representatives of u and v, It is easy to see that this product doesn't depend on the representatives. We then have

Theorem 1.1. *([4]) (i) There is a derivative operator in $\mathcal{G}_2(\Omega)$ which is linear and is induced by the derivative in $\mathcal{E}_{M,2}[\Omega]$, i.e., if $u \in \mathcal{G}_2(\Omega)$ and $\alpha \in \mathbf{N}^n$ then $D^\alpha u := cl(D^\alpha \widehat{u})$ in $\mathcal{G}_2(\Omega)$, where $\widehat{u} \in \mathcal{E}_{M,2}[\Omega]$ is a representative of u, and $cl(D^\alpha \widehat{u}) \in \mathcal{G}_2(\Omega)$ is the class of $D^\alpha \widehat{u}$.*
(ii) There is an embedding of $H^{-\infty}(\mathbf{R}^n) = \cup_{s\in\mathbf{R}} H^s(\mathbf{R}^n)$ into $\mathcal{G}_2(\mathbf{R}^n)$ obtained in the following way: we fix $\rho \in S(\mathbf{R}^n)$ such that

$$\int_{\mathbf{R}^n} \rho(x)dx = 1 \text{ and } \int_{\mathbf{R}^n} x^\alpha \rho(x)dx = 0,\ \forall \alpha \in \mathbf{N}^n, |\alpha| \geq 1.$$

Let $\iota : w \to (w * \rho_\varepsilon)_\varepsilon$, where $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$. This defines a linear injection of $H^{-\infty}(\mathbf{R}^n)$ into $\mathcal{E}_{M,2}[\mathbf{R}^n]$, which induces an embedding $H^{-\infty}(\mathbf{R}^n)$ into $\mathcal{G}_2(\mathbf{R}^n)$,(so we can look $H^\infty(\mathbf{R}^n)$ as a subalgebra of $\mathcal{G}_2(\mathbf{R}^n)$).

Next we give some definitions:

Definition 1.1. *For $u \in \mathcal{G}_2((0,T)\times\mathbf{R}^n)$, the restriction of u to $\{0\}\times\mathbf{R}^n$ is the class of $\widehat{u}_\varepsilon(0,\cdot)$ in $\mathcal{G}_2(\mathbf{R}^n)$, where $\widehat{u}_\varepsilon$ is a representative of u. We denote this class by $u|_{\{t=0\}}$.*

Definition 1.2. *We say that $u \in \mathcal{G}_2(\Omega)$ is of r-(log)$^{\frac{1}{j}}$-type, $2 \le r \le \infty$, $j \ge 1$, if it has a representative $\widehat{u} \in \mathcal{E}_{M,2}[\Omega]$ such that*

$$\|\widehat{u}_\varepsilon\|_{L^r} = \mathcal{O}(|\log\varepsilon|^{\frac{1}{j}}), \ \ as\ \varepsilon \to 0.$$

Notice that if $\widehat{u} \in \mathcal{E}_{M,2}[\Omega]$, then (1.11) holds with the $W^{k,r}(\Omega)$-norm.

We also observe a nonlinear property of generalized functions: if $F \in O_M(\mathbf{R}^l)$ i.e., F is a smooth function and together with all its derivatives grow at most like some power of $|x|$ as $|x| \to \infty$, we can define $F(u_1, u_2, \cdots, u_l) \in \mathcal{G}_2(\Omega)$ for $u_i \in \mathcal{G}_2(\Omega)$, $i = 1, \cdots, l$, (see [1]).

Definition 1.3. *Let $P(u, \partial^\alpha u)$ be a polynomial in u and its derivatives. We say that u is a solution to the problem*

$$\begin{aligned} &u_t = P(u,\partial^\alpha u) \ in\ \mathcal{G}_2((0,T)\times\mathbf{R}^n) \\ &u|_{\{t=0\}} = g \ \ in\ \ \mathcal{G}_2(\mathbf{R}^n), \end{aligned}$$

if for every representative $\widehat{u} \in \mathcal{E}_{M,2}[(0,T)\times\mathbf{R}^n]$ of u and $\widehat{g} \in \mathcal{E}_{M,2}[\mathbf{R}^n]$ of g, there are $\widehat{N} \in \mathcal{N}_2[(0,T)\times\mathbf{R}^n]$ and $\widehat{\eta} \in \mathcal{N}_2[\mathbf{R}^n]$ such that

$$\begin{aligned} &\widehat{u}_t = P(\widehat{u},\partial^\alpha\widehat{u}) + \widehat{N}, \ in\ (0,T)\times\mathbf{R}^n \\ &\widehat{u}|_{\{t=0\}} = \widehat{g}\ + \widehat{\eta}, \ in\ \mathbf{R}^n. \end{aligned}$$

For more details on generalized functions, see [1], [8], [9] and [19].

2 Existence of generalized solutions to the hierarchy in $\mathbf{R}^2$

In this section we shall establish a result on the existence of solutions in $\mathcal{G}_2((0,T)\times\mathbf{R}^2)$ for the Cauchy problem

$$\begin{array}{l} u_t = \partial_{x_1} G_m(u) + \partial_{x_2} G_m(u), \ \ in \ \mathcal{G}_2((0,T)\times\mathbf{R}^2) \\ u|_{\{t=0\}} = g \ \ in \ \ \mathcal{G}_2(\mathbf{R}^2). \end{array} \tag{2.12}$$

The following result, due Saut [21, Theorem 2], is used in the proof of the Theorem 2.2.

Theorem 2.1. *Let $g \in H^m(\mathbf{R}^n)$ and $T > 0$ finite. Then the Cauchy problem*

$$\begin{array}{l} u_t = \sum_{\alpha_1} D^{\alpha_1} G_m(u), \\ u|_{\{t=0\}} = g \end{array} \tag{2.13}$$

has a solution in $L^\infty((0,T) : H^m(\mathbf{R}^n))$.

Remark 2.1. *It follows from this result and Sobolev's embedding theorem, that if $g \in H^j(\mathbf{R}^n)$, $j \in \mathbf{Z}$, then $u \in C^\infty((0,T)\times\mathbf{R}^n)$.*

In the proof of the Theorem 2.2, we need the following lemma

Lemma 2.1. *If u is a solution to the problem (2.13), for the case $n = 2$, then for every $k \in \mathbf{N}$ there is a polynomial P_{k+2} in the variables $\|D^{\alpha_l}\widehat{g}_\varepsilon\|_0$, $l \le k+2$, of degree $k+2$, such that*

$$\|D^{\alpha_k} u\|_{L^\infty} \le P_{k+2}\,(\|g\|_0\,, \|D^{\alpha_1} g\|_0\,, \cdots, \|D^{\alpha_{k+2}} g\|_0). \tag{2.14}$$

A similar estimate also holds for $\|D^{\alpha_{k+1}} u(t,\cdot)\|_0$, i.e.,

$$\|D^{\alpha_{k+1}} u(t,\cdot)\|_0 \le \widetilde{P}_{k+2}\,(\|g\|_0\,, \|D^{\alpha_1} g\|_0\,, \cdots, \|D^{\alpha_{k+2}} g\|_0). \tag{2.15}$$

Proof. To simplify the notations, we drop $\mathbf{R}^2$ in the integral; also the constants are taken equal one. The conservation law

$$F_1(u) = \frac{1}{2}\int u^2(x,t)dx,$$

gives that $\|u(t)\|_0 = \|g\|_0$. Therefore, it follows from the Gagliardo-Nirenberg's inequality (2.16), (see [10]),

$$\begin{aligned} \|D^\alpha v\|_{L^p} &\leq C \left\|D^\beta v\right\|_{L^q}^{\theta} \|v\|_{L^r}^{1-\theta}, \\ \frac{1}{p} - \frac{|\alpha|}{n} &= \theta(\frac{1}{q} - \frac{|\beta|}{n}) + (1-\theta)\frac{1}{r}, \theta \in [\frac{|\alpha|}{|\beta|}, 1], \end{aligned} \tag{2.16}$$

with $p = \infty,\ n = 2,\ q = r = 2,\ \alpha = 0,\ |\beta| = 2,\ \theta = \frac{1}{2}$ and Young's inequality (2.17)

$$ab \leq \delta\frac{a^p}{p} + k(\delta)\frac{b^q}{q}, \quad \frac{1}{p} + \frac{1}{q} = 1, \tag{2.17}$$

with $p = q = 2$, that

$$\|u(t)\|_{L^\infty} \leq \|g\|_0 + \sum_{\alpha_2} \|D^{\alpha_2} u(t)\|_0 . \tag{2.18}$$

To estimate $\sum_{\alpha_2} \|D^{\alpha_2} u(t)\|_0$, we use the conservation law

$$F_3(u) = \int [\sum_{\alpha_2} (D^{\alpha_2} u)^2 + \sum_{\alpha_1} c_{\alpha_1} u (D^{\alpha_1} u)^2 + cu^4] dx,$$

from which we obtain

$$\begin{aligned} \int \sum_{\alpha_2} (D^{\alpha_2} u)^2 dx &= -\sum_{\alpha_1} \int c_{\alpha_1} u (D^{\alpha_1} u)^2 dx - c\int u^4 dx + \\ &\int \sum_{\alpha_2} (D^{\alpha_2} g)^2 dx + \sum_{\alpha_1} \int c_{\alpha_1} g (D^{\alpha_1} g)^2 + c\int g^4 dx. \end{aligned}$$

Thus we have the following result

$$\begin{aligned} \sum_{\alpha_2} \|D^{\alpha_2} u(t)\|_0^2 \leq \|u(t)\|_{L^\infty} \int \sum_{\alpha_1} (D^{\alpha_1} u)^2 dx + \|g\|_{L^\infty} \int \sum_{\alpha_1} (D^{\alpha_1} g)^2 dx + \\ c\|u(t)\|_{L^\infty}^2 \int u^2 dx + c\|g\|_{L^\infty}^2 \int g^2 dx. \end{aligned} \tag{2.19}$$

To estimate the term $\int \sum_{\alpha_1} (D^{\alpha_1} u)^2 dx$, we use the conservation law

$$F_2(u) = \int [\sum_{\alpha_1} (D^{\alpha_1} u)^2 + cu^3] dx,$$

so we obtain $\int \sum_{\alpha_1} (D^{\alpha_1} u)^2 = \int \sum_{\alpha_1} (D^{\alpha_1} g)^2 + c\int g^3 - c\int u^3$,

and therefore $\displaystyle\int \sum_{\alpha_1} (D^{\alpha_1} u)^2 \leq \sum_{\alpha_1} \|D^{\alpha_1} g\|_0^2 + c\|g\|_{L^\infty} \int g^2 + c\|u(t)\|_{L^\infty} \int u^2,$

or

$$\int \sum_{\alpha_1} (D^{\alpha_1} u)^2 \leq \sum_{\alpha_1} \|D^{\alpha_1} g\|_0^2 + (\|g\|_0 + \sum_{\alpha_2} \|D^{\alpha_2} g\|_0) \|g\|_0^2 + \|u(t)\|_{L^\infty} \|g\|_0^2 . \tag{2.20}$$

By replacing (2.20) in (2.19) and using (2.18), we obtain the estimate

$$\begin{aligned} \sum_{\alpha_2} \|D^{\alpha_2} u(t)\|_0^2 \leq\ & c(\|g\|_0 + \sum_{\alpha_2} \|D^{\alpha_2} u(t)\|_0) \sum_{\alpha_1} \|D^{\alpha_1} g\|_0^2 + c(\|g\|_0 + \\ & \sum_{\alpha_2} \|D^{\alpha_2} u(t)\|_0)(\|g\|_0 + \sum_{\alpha_2} \|D^{\alpha_2} g\|_0) \|g\|_0^2 + c(\|g\|_0 + \\ & \sum_{\alpha_2} \|D^{\alpha_2} u(t)\|_0)(\|g\|_0 + \sum_{\alpha_2} \|D^{\alpha_2} u(t)\|_0) \|g\|_0^2 + c(\|g\|_0 + \\ & \sum_{\alpha_2} \|D^{\alpha_2} g\|_0) \sum_{\alpha_1} \|D^{\alpha_1} g\|_0^2 + c(\|g\|_0 + \sum_{\alpha_2} \|D^{\alpha_2} u(t)\|_0) \\ & \|g\|_0^2 + c(\|g\|_0 + \sum_{\alpha_2} \|D^{\alpha_2} g\|_0) \|g\|_0^2 . \end{aligned}$$

By using repeatedly Young's inequality (2.17) with $p = q = 2$, we obtain an estimate to $\sum_{\alpha_2} \|D^{\alpha_2} u(t)\|_0$ polynomial type in $\|g\|_0$ and $\|D^\alpha g\|_0$ $|\alpha| = 1, 2$ of degree 2, which replaced in (2.18), gives

$$\|u(t)\|_{L^\infty} \leq P_2 (\|g\|_0 , \|D^{\alpha_1} g\|_0 , \|D^{\alpha_2} g\|_0),$$

and therefore

$$\|u\|_{L^\infty} \leq P_2 (\|g\|_0 , \|D^{\alpha_1} g\|_0 , \|D^{\alpha_2} g\|_0).$$

Coming back to Eq. (2.20), we obtain a similar estimate for $\sum_{\alpha_1} \|D^{\alpha_1} u\|_0$, i.e.,

$$\sum_{\alpha_1} \|D^{\alpha_1} u\|_0 \leq \widetilde{P}_2 (\|g\|_0 , \|D^{\alpha_1} g\|_0 , \|D^{\alpha_2} g\|_0).$$

In the next, we use similar arguments, the conservation laws and induction to complete the proof. Given an integer ℓ, we have from (2.16) and (2.17), the estimate

$$\sum_{\alpha_\ell} \|D^{\alpha_\ell} u(t)\|_{L^\infty} \leq \sum_{\alpha_\ell} \|D^{\alpha_\ell} u(t)\|_0 + \sum_{\alpha_{\ell+2}} \|D^{\alpha_{\ell+2}} u(t)\|_0 . \tag{2.21}$$

Taking into consideration results (2.14) and (2.15) for $1 \le k \le \ell - 1$, we prove for $k = \ell$. By using the conservation law

$$\begin{aligned} F_{\ell+3}(u) &= \int \{ \sum_{\alpha_{\ell+2}} (D^{\alpha_{\ell+2}} u)^2 + \sum_{\alpha_{\ell+1}} c_{\alpha_{\ell+1}} u (D^{\alpha_{\ell+1}} u)^2 + \\ & c_{\alpha_0} u^{\ell+4} + Q_{\ell+3}(u, D^{\alpha_1} u, ..., D^{\alpha_\ell} u) \} dx, \end{aligned}$$

we obtain the equality

$$\begin{aligned} \sum_{\alpha_{\ell+2}} \|D^{\alpha_{\ell+2}} u(t)\|_0^2 &= -\sum_{\alpha_{\ell+1}} c_{\alpha_{\ell+1}} \int u (D^{\alpha_{\ell+1}} u)^2 dx - c_{\alpha_0} \int u^{\ell+4} dx - \\ & \int Q_{\ell+3}(u, D^{\alpha_1} u, ..., D^{\alpha_\ell} u) dx + \\ & \sum_{\alpha_{\ell+2}} \int (D^{\alpha_{\ell+2}} g)^2 dx + \sum_{\alpha_{\ell+1}} c_{\alpha_{\ell+1}} \int g (D^{\alpha_{\ell+1}} g)^2 dx + \\ & c_{\alpha_0} \int g^{\ell+4} dx + \int Q_{\ell+3}(g, D^{\alpha_1} g, ..., D^{\alpha_\ell} g) dx. \end{aligned}$$

And therefore we obtain the estimate

$$\begin{aligned} \sum_{\alpha_{\ell+2}} \|D^{\alpha_{\ell+2}} u(t)\|_0^2 &\le \|u\|_{L^\infty} \sum_{\alpha_{\ell+1}} \|D^{\alpha_{\ell+1}} u(t)\|_0^2 + \|u\|_{L^\infty}^{\ell+2} \|u\|_0^2 + \\ & \left| \int Q_{\ell+3}(u, D^{\alpha_1} u, ..., D^{\alpha_\ell} u) dx \right| + \\ & \sum_{\alpha_{\ell+2}} \left\| D^{\ell+2} g \right\|^2 + \|g\|_{L^\infty} \sum_{\alpha_{\ell+1}} \|D^{\alpha_{\ell+1}} g\|_0^2 + \\ & \|g\|_{L^\infty}^{\ell+2} \|g\|_0^2 + \left| \int Q_{\ell+3}(g, D^{\alpha_1} g, ..., D^{\alpha_\ell} g) dx \right|. \quad (2.22) \end{aligned}$$

To estimate the factor $\|D^{\alpha_{\ell+1}} u(t)\|_0^2$, we use the conservation law

$$\begin{aligned} F_{\ell+2}(u) &= \int \{ \sum_{\alpha_{\ell+1}} (D^{\alpha_{\ell+1}} u)^2 + \sum_{\alpha_\ell} c_{\alpha_\ell} u (D^{\alpha_\ell} u)^2 + \\ & c_{\alpha_0} u^{\ell+3} + Q_{\ell+2}(u, D^{\alpha_1} u, ..., D^{\alpha_{\ell-1}} u) \} dx, \end{aligned}$$

from which we obtain the estimate,

$$\begin{aligned}\sum_{\alpha_{\ell+1}} \|D^{\alpha_{\ell+1}}u(t)\|_0^2 \;\leq\;& \|u\|_{L^\infty} \sum_{\alpha_\ell} \|D^{\alpha_\ell}u(t)\|_0^2 + \|u\|_{L^\infty}^{\ell+1} \|u\|_0^2 + \\ & |\int Q_{\ell+2}(u, D^{\alpha_1}u, ..., D^{\alpha_{\ell-1}}u)dx| + \\ & \sum_{\alpha_{\ell+1}} \|D^{\alpha_{\ell+1}}g\|^2 + \|g\|_{L^\infty} \sum_{\alpha_\ell} \|D^{\alpha_\ell}g\|_0^2 + \\ & \|g\|_{L^\infty}^{\ell+1} \|g\|_0^2 + |\int Q_{\ell+2}(g, D^{\alpha_1}g, ..., D^{\alpha_{\ell-1}}g)dx| \quad (2.23)\end{aligned}$$

By using induction hypothesis, we obtain that the term $\|u\|_{L^\infty} \sum_{\alpha_\ell} \|D^{\alpha_\ell}u(t)\|_0^2$ can be bounded by a polynomial in $\|g\|_0, \|D^{\alpha_1}g\|_0, \cdots, \|D^{\alpha_{\ell+1}}g\|_0$ of degree $2+2(\ell+1) = 2\ell+4$.

In order to estimate the integral $\int Q_{\ell+3}(u, D^{\alpha_1}u, ..., D^{\alpha_\ell}u)dx$ in (2.22), we use the fact that each term of $Q_{\ell+3}$ has degree ≥ 3 and is of the form, $\prod_{i=0}^{\ell}(D^{\alpha_i}u)^{l_{\alpha_i}}$, where $\sum_{i=0}^{\ell}(1+\frac{i}{2})l_{\alpha_i} = \ell+4$, as seen in (1.8). Thus we can write each term of $\int Q_{\ell+3}dx$ in the form

$$\int u^{l_{\alpha_0}}(D^{\alpha_1}u)^{l_{\alpha_1}} \cdots (D^{\alpha_i}u)^{l_{\alpha_i}-1} \cdots (D^{\alpha_j}u)^{l_{\alpha_j}-1} \cdots (D^{\alpha_\ell}u)^{l_{\alpha_\ell}} D^{\alpha_i}u D^{\alpha_j}u dx,$$

where $0 \leq i, j \leq \ell-1$. The expression above can be estimated by

$$\begin{aligned}&\|u\|_{L^\infty}^{l_{\alpha_0}} \|D^{\alpha_1}u\|_{L^\infty}^{l_{\alpha_1}} \cdots \|D^{\alpha_i}u\|_{L^\infty}^{l_{\alpha_i}-1} \cdots \|D^{\alpha_j}u\|_{L^\infty}^{l_{\alpha_j}-1} \cdots \\ &\|D^{\alpha_\ell}u\|_{L^\infty}^{l_{\alpha_\ell}} \|D^{\alpha_i}u(t)\|_0 \|D^{\alpha_j}u(t)\|_0 .\end{aligned}$$

By induction hypothesis, we have that $\|D^{\alpha_i}u\|_{L^\infty}$, $1 \leq i \leq \ell-1$, can be bounded by a polynomial $P^*_{i+2}(\|g\|_0, \|D^{\alpha_1}g\|_0, \cdots, \|D^{\alpha_{i+2}}g\|_0)$, therefore $\int \prod_{i=0}^{\ell}(D^{\alpha_i}u)^{l_{\alpha_i}}dx$ can be bounded by a polynomial in $\|g\|_0, \|D^{\alpha_1}g\|_0, \cdots, \|D^{\alpha_{\ell+1}}g\|_0$ whose degree is given by

$$\begin{aligned}&2l_{\alpha_0} + 3l_{\alpha_1} + \cdots + (i+2)(l_{\alpha_i}-1) + \cdots + (j+2)(l_{\alpha_j}-1) \\ &+ \cdots + (\ell-1+2)(l_{\alpha_\ell}-1) + (i+1) + (j+1),\end{aligned}$$

which equals to $2\ell+4$. From where we obtain that $\int Q_{\ell+3}(u, D^{\alpha_1}u, \cdots, D^{\alpha_\ell}u)dx$ can be estimated by a polynomial in $\|g\|_0, \|D^{\alpha_1}g\|_0, \cdots, \|D^{\alpha_{\ell+1}}g\|_0$ of degree

$2\ell+4$. Analogously $\int Q_{\ell+2}dx$ in (2.23) can be estimated; the same result holds for $\int Q_{\ell+3}(g, D^{\alpha_1}g, \cdots, D^{\alpha_\ell}g)dx$ in (2.22) and $\int Q_{\ell+2}(g, D^{\alpha_1}g, \cdots, D^{\alpha_{\ell-1}}g)dx$ in (2.23). By replacing (2.23) in (2.22), we obtain

$$\|D^{\alpha_{\ell+2}}u(t)\|_0 \leq \widetilde{P}_{\ell+3}(\|g\|_0, \|D^{\alpha_1}g\|_0, \cdots, \|D^{\alpha_{\ell+2}}\|_0), \tag{2.24}$$

which replaced in (2.23) yields (2.15). Finallly, inequalities (2.21) and (2.24) yield, the estimate (2.14).

□

Theorem 2.2. *Let $g \in \mathcal{G}_2(\mathbf{R}^2)$ and $T > 0$ finite. Then the Cauchy problem (2.12) has a solution in $\mathcal{G}_2((0,T) \times \mathbf{R}^2)$.*

Proof: Let $\widehat{g}$ be a representative of g in $\mathcal{E}_{M,2}[\mathbf{R}^2]$. Since $\widehat{g}_\varepsilon \in H^\infty(\mathbf{R}^2)$, for all $\varepsilon > 0$, by Theorem 2.1 and remark 2.1, there is a solution $\widehat{u}_\varepsilon$ in $C^\infty([0,T] \times \mathbf{R}^n) \cap L^\infty([0,T], H^\infty(\mathbf{R}^n))$ to the problem

$$\begin{array}{l} \partial_t \widehat{u}_\varepsilon = \partial_{x_1} G_m(\widehat{u}_\varepsilon) + \partial_{x_2} G_m(\widehat{u}_\varepsilon) \ \text{ in } (0,T) \times \mathbf{R}^2 \\ \widehat{u}_\varepsilon(0) = \widehat{g}_\varepsilon \ \text{ in } \mathbf{R}^2. \end{array} \tag{2.25}$$

Hence, given $\alpha \in \mathbf{N}^{1+2}$, it follows from the equation in (2.25) that $D^\alpha \widehat{u}_\varepsilon(t, \cdot) \in L^2(\mathbf{R}^2)$ for all $t \in (0,T)$ and $\varepsilon > 0$. Since

$$\|D^\alpha \widehat{u}_\varepsilon\|_0 \leq \sqrt{T}\, sup_t \|D^\alpha \widehat{u}_\varepsilon(t, \cdot)\|_0, \tag{2.26}$$

we have $\widehat{u}_\varepsilon \in H^\infty((0,T) \times \mathbf{R}^2)$. In particular the map $\widehat{u} : I \times (0,T) \times \mathbf{R}^2 \to \mathbf{R}$, $(\varepsilon, t, x) \mapsto \widehat{u}_\varepsilon(t,x)$ belongs to $\mathcal{E}_2[(0,T) \times \mathbf{R}^2]$. The proof of $\widehat{u} \in \mathcal{E}_{M,2}[(0,T) \times \mathbf{R}^2]$ is a consequence of inequality (2.26) and of the lemma 2.1.

□

3 Uniqueness Result

The following result establishes the uniqueness of solutions to the problem (2.12).

Theorem 3.1. *If u and its x-derivatives up to order* $3m-3$ *are of* ∞*-(log)*$^{\frac{1}{2(m-1)^2}}$*-type, then there is at most one solution* $u \in \mathcal{G}_2((0,T)\times \mathbf{R}^2)$ *to the problem (2.12).*

Proof: Let $u, v \in \mathcal{G}_2((0,T)\times\mathbf{R}^2)$ be two solutions of (2.12) as stated above, i.e., there are respective representatives $\widehat{u} = \widehat{u}_\varepsilon(\cdot,\cdot)$ and $\widehat{v} = \widehat{v}_\varepsilon(\cdot,\cdot)$ in $\mathcal{E}_{M,2}[(0,T)\times \mathbf{R}^2)]$ such that $\widehat{u} = (\widehat{u}_\varepsilon)_\varepsilon$, $\widehat{v} = (\widehat{v}_\varepsilon)_\varepsilon$ satisfy the condition: their x-derivatives up to order $3m-3$ satisfy the estimate $\|\widehat{u}_\varepsilon\|_{L^\infty} = \mathcal{O}(|\log\varepsilon|^{\frac{1}{2(m-1)^2}})$, as $\varepsilon \to 0$. There are $\widehat{N} \in \mathcal{N}_2((0,T)\times\mathbf{R}^2)$ and $\widehat{\eta} \in \mathcal{N}_2(\mathbf{R}^2)$ such that

$$\begin{array}{l} \partial_t(\widehat{u}_\varepsilon - \widehat{v}_\varepsilon)(t,x) = \sum_1^2 \partial_{x_i}(G_m(\widehat{u}_\varepsilon) - G_m(\widehat{v}_\varepsilon))(t,x) + \widehat{N}_\varepsilon(t,x) \\ (\widehat{u}_\varepsilon - \widehat{v}_\varepsilon)(0,x) = \widehat{\eta}_\varepsilon(x). \end{array} \tag{3.27}$$

By changing representatives, we may assume $\widehat{\eta} = 0$. For simplicity, we drop the ε and hat in our notations. Thus, setting $w = u - v$, problem (3.27) can be written in the form

$$\begin{array}{l} \partial_t w = \sum_1^2 \partial_{x_i}(G_m(u) - G_m(v)) + N \\ w|_{t=0} = 0. \end{array} \tag{3.28}$$

By [12, proposition 3.4(ii)] see also [13], it is sufficient show that

$$\|w\|_0 = \mathcal{O}(\varepsilon^q), \quad \text{as} \;\; \varepsilon \to 0.$$

After multiplying equation in (3.28) by w and integrating with respect to x and t, it results, after integration by parts, in

$$\frac{1}{2}\int_0^t \int_{\mathbf{R}^2} \partial_t w^2 dxdt = -\int_0^t \int_{\mathbf{R}^2} \sum_1^2 \partial_{x_i} w(G_m(u) - G_m(v))dxdt + \int_0^t \int_{\mathbf{R}^2} wNdxdt. \tag{3.29}$$

In what follows, we drop $\mathbf{R}^2$ and $dxdt$ in the integrals. Remind that $G_m(u)$ is the gradient of the functional $F_m(u)$ in u, where $F_m(u)$ has the form (1.6), and each term of $f_m(u)$ is of type (1.8). In particular the terms

$$\sum_{\alpha_{m-1}} (D^{\alpha_{m-1}}u)^2 \text{ and } \sum_{\alpha_{m-2}} (D^{\alpha_{m-2}}u)^2 u \;\; \text{of} \;\; f_m \;\; \text{leads to}$$

$$2(-1)^{m-1} \sum_{\alpha_{m-1}} D^{2\alpha_{m-1}} u \;\text{ and }\; \sum_{\alpha_{m-2}} [2(-1)^{m-2} D^{\alpha_{m-2}}(uD^{\alpha_{m-2}}u) + (D^{\alpha_{m-2}}u)^2],$$

respectively, in G_m. The first expression does not contribute in (3.29), and the contribution of the second expression is given by

$$|\int_0^t \int \sum_1^2 \partial_{x_i} w \sum_{\alpha_{m-2}} [2(-1)^{m-2} D^{\alpha_{m-2}}(uD^{\alpha_{m-2}}u - vD^{\alpha_{m-2}}v) + (D^{\alpha_{m-2}}u)^2 - (D^{\alpha_{m-2}}v)^2]|$$

$$\begin{aligned} &= |\int_0^t \int 2(-1)^{m-1} \sum_{\alpha_{m-1}} D^{\alpha_{m-1}} w \sum_{\alpha_{m-2}} (uD^{\alpha_{m-2}}w + wD^{\alpha_{m-2}}v) + \\ &\quad \int_0^t \int (-1)^{m-1} \sum_{l=1}^{2} \partial_{x_i} w \sum_{\alpha_{m-2}} D^{\alpha_{m-2}} w (D^{\alpha_{m-2}}u + D^{\alpha_{m-2}}v)|. \end{aligned}$$

Which, after using the inequality (2.17), can be estimated by

$$c\int_0^t [\|w(s)\|_0^2 + \sum_{\alpha_1} \|D^{\alpha_1} w(s)\|_0^2 + \sum_{\alpha_{m-2}} \|D^{\alpha_{m-2}} w(s)\|_0^2 + \sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}} w(s)\|_0^2].$$

Which, after using of the inequality (2.16), can be estimated by

$$c\int_0^t [\|w(s)\|_0^2 + \sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}} w(s)\|_0^2]ds,$$

where $c = 2(\|u\|_{L^\infty} + \sum_{\alpha_{m-2}} (\|D^{\alpha_{m-2}}u\|_{L^\infty} + \|D^{\alpha_{m-2}}v\|_{L^\infty}))$.
The term $u^{m+1}$of $f_m(u)$ gives in $G_m(u)$ the term $(m+1)u^m$, whose contribution in (3.29) is given by

$$\begin{aligned} & |\int_0^t \int \sum_1^2 \partial_{x_i} w(u^m - v^m)| \\ = \ & c|\int_0^t \int \sum_1^2 \partial_{x_i} ww(u^{m-1} + u^{m-2}v + \cdots + uv^{m-2} + v^{m-1})| \\ \leq \ & c\int_0^t (\|w(s)\|_0^2 + \sum_{\alpha_1} \|D^{\alpha_1} w(s)\|_0^2)\, ds, \end{aligned}$$

where $c = \|u\|_{L^\infty}^{m-1} + \|u\|_{L^\infty}^{m-2} \|v\|_{L^\infty} + \cdots + \|u\|_{L^\infty} \|v\|_{L^\infty}^{m-2} + \|v\|_{L^\infty}^{m-1}$.

In a similar fashion the other terms in (3.29) can be estimated giving, after summing up the results and substituting into (3.29), the estimate

$$\|w(t)\|_0^2 \leq \|w\|_0 \|N\|_0 + c\int_0^t (\|w(s)\|_0^2 + \sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}} w(s)\|_0^2)ds, \qquad (3.30)$$

where c is a polynomial in $\|D^{\alpha_k}u\|_{L^\infty}$, $\|D^{\alpha_k}v\|_{L^\infty}$, $k=0,1,2,\cdots,m-2$, with degree at most $m-1$.

After multiplying equation in (3.28) by $G_m(u)-G_m(v)$ and integrating in x and t it results in

$$\int_0^t\int \partial_t w(G_m(u)-G_m(v)) = \int_0^t\int N(G_m(u)-G_m(v)). \tag{3.31}$$

The term $2(-1)^{m-1}\sum_{\alpha_{m-1}} D^{2\alpha_{m-1}}u$ of $G_m(u)$, gives on the left-hand side of (3.31) the contribution

$$2(-1)^{m-1}\int_0^t\int \partial_t w\sum_{\alpha_{m-1}} D^{2\alpha_{m-1}}w = \int_0^t\int \partial_t\sum_{\alpha_{m-1}}(D^{\alpha_{m-1}}w)^2 = \sum_{\alpha_{m-1}}\|D^{\alpha_{m-1}}w\|_0^2 .$$

Therefore equation (3.31) can be written in the form

$$\sum_{\alpha_{m-1}}\|D^{\alpha_{m-1}}w\|_0^2 = \sum_{\lambda\le m-2} J_{\alpha_\lambda}(w(t)) + \int_0^t\int N(G_m(u)-G_m(v)), \tag{3.32}$$

where the $J'_{\alpha_\lambda}s$ are contributions in the left hand side of (3.31) obtained as it follows: we consider the following cases:

(a) term $(m+1)u^m$ of G_m , whose contribution in (3.31) is

$$J_{\alpha_0} = \int_0^t\int \partial_t ww(u^{m-1}+u^{m-2}v+\cdots+uv^{m-2}+v^{m-1}).$$

Integration by parts in the variable t gives

$$\begin{aligned} J_{\alpha_0} \;\le\;& c\left\|u^{m-1}+u^{m-2}v+\cdots+uv^{m-2}+v^{m-1}\right\|_{L^\infty}\int w^2 \\ &+c\left\|\partial_t(u^{m-1}+u^{m-2}v+\cdots+uv^{m-2}+v^{m-1})\right\|_{L^\infty}\int_0^t\int w^2. \end{aligned}$$

Since the right hand side of the equation in the problem (2.12) has degree m $\partial_t(u^{m-1}+u^{m-2}v+\cdots+uv^{m-2}+v^{m-1})$ has degree $2m-2$ and we obtain the estimate

$$J_{\alpha_0} \le c\{\|w(t)\|_0^2 + \int_0^t \|w(s)\|_0^2\,ds\},$$

where c is a polynomial in $\left\|D^k u\right\|_{L^\infty}$, $\left\|D^k v\right\|_{L^\infty}$, $k=0,1,2,\cdots,2m-1$, of degree $2m-2$.

(b) Taking into consideration the term

$$\sum_{\alpha_{m-2}}[2(-1)^{m-2}D^{\alpha_{m-2}}(uD^{\alpha_{m-2}}u)+(D^{\alpha_{m-2}}u)^2]$$

of G_m, we have the following contribution in (3.31)

$$\begin{aligned} J_{\alpha_{m-2}} &= \int_0^t\int \partial_t w\sum_{\alpha_{m-2}}2(-1)^{m-2}D^{\alpha_{m-2}}(uD^{\alpha_{m-2}}u-vD^{\alpha_{m-2}}v) \\ &\quad +\int_0^t\int \partial_t w\sum_{\alpha_{m-2}}[(D^{\alpha_{m-2}}u)^2-(D^{\alpha_{m-2}}v)^2] \\ &= \int_0^t\int \partial_t w\sum_{\alpha_{m-2}}[2(-1)^{m-2}D^{\alpha_{m-2}}(uD^{\alpha_{m-2}}w+wD^{\alpha_{m-2}}v)] \\ &\quad +\int_0^t\int \partial_t w\sum_{\alpha_{m-2}}D^{\alpha_{m-2}}w(D^{\alpha_{m-2}}u+D^{\alpha_{m-2}}v). \end{aligned}$$

By using integration by parts in x, we reach

$$\begin{aligned} J_{\alpha_{m-2}} &\le |\int_0^t\int 2\sum_{\alpha_{m-2}}D^{\alpha_{m-2}}\partial_t w(uD^{\alpha_{m-2}}w+wD^{\alpha_{m-2}}v) \\ &\quad +\int_0^t\int \partial_t w\sum_{\alpha_{m-2}}D^{\alpha_{m-2}}w(D^{\alpha_{m-2}}u+D^{\alpha_{m-2}}v)|. \end{aligned}$$

Therefore

$$\begin{aligned} J_{\alpha_{m-2}} &\le |\int_0^t\int \sum_{\alpha_{m-2}}[\partial_t(D^{\alpha_{m-2}}w)^2u+2D^{\alpha_{m-2}}\partial_t wwD^{\alpha_{m-2}}v] \\ &\quad +\int_0^t\int \sum_{\alpha_{m-2}}\partial_t wD^{\alpha_{m-2}}w(D^{\alpha_{m-2}}u+D^{\alpha_{m-2}}v)|. \end{aligned}$$

By integration by parts in t, we obtain

$$\begin{aligned} J_{\alpha_{m-2}} &\le |\int\sum_{\alpha_{m-2}}(D^{\alpha_{m-2}}w)^2u-\int_0^t\int\sum_{\alpha_{m-2}}(D^{\alpha_{m-2}}w)^2\partial_t u+ \\ &\quad 2\int\sum_{\alpha_{m-2}}D^{\alpha_{m-2}}wwD^{\alpha_{m-2}}v-2\int_0^t\int\sum_{\alpha_{m-2}}D^{\alpha_{m-2}}w\partial_t(wD^{\alpha_{m-2}}v). \\ &\quad +\int_0^t\int\sum_{\alpha_{m-2}}\partial_t wD^{\alpha_{m-2}}w(D^{\alpha_{m-2}}u+D^{\alpha_{m-2}}v)|. \end{aligned}$$

Using the equality

$$\partial_t w D^{\alpha_{m-2}} w D^{\alpha_{m-2}} u = D^{\alpha_{m-2}} w \partial_t w D^{\alpha_{m-2}} v + \partial_t w (D^{\alpha_{m-2}} w)^2,$$

we obtain

$$\begin{aligned} J_{\alpha_{m-2}} \leq & \ |\int \sum_{\alpha_{m-2}} (D^{\alpha_{m-2}} w)^2 u - \int_0^t \int \sum_{\alpha_{m-2}} (D^{\alpha_{m-2}} w)^2 \partial_t u + \\ & 2 \int \sum_{\alpha_{m-2}} D^{\alpha_{m-2}} w w D^{\alpha_{m-2}} v + \int_0^t \int \sum_{\alpha_{m-2}} \partial_t w (D^{\alpha_{m-2}} w)^2 - \\ & 2 \int_0^t \int \sum_{\alpha_{m-2}} D^{\alpha_{m-2}} w w \partial_t D^{\alpha_{m-2}} v|. \end{aligned}$$

Thus, we have

$$\begin{aligned} J_{\alpha_{m-2}} \leq & \ \|u\|_{L^\infty} \sum_{\alpha_{m-2}} \|D^{\alpha_{m-2}} w(t)\|_0^2 + \|\partial_t u\|_{L^\infty} \int_0^t \sum_{\alpha_{m-2}} \|D^{\alpha_{m-2}} w(s)\|^2 \, ds \\ & +2 \|D^{\alpha_{m-2}} v\|_{L^\infty} \sum_{\alpha_{m-2}} (\|w(t)\|_0^2 + \|D^{\alpha_{m-2}} w(t)\|_0^2) \\ & + \|\partial_t w\|_{L^\infty} \int_0^t \sum_{\alpha_{m-2}} \|D^{\alpha_{m-2}} w(s)\|_0^2 \, ds \\ & + \|\partial_t D^{\alpha_{m-2}} v\|_{L^\infty} \int_0^t \sum_{\alpha_{m-2}} (\|w(s)\|_0^2 + \|D^{\alpha_{m-2}} w(s)\|_0^2) ds. \end{aligned}$$

Therefore, we have the following estimate

$$\begin{aligned} J_{\alpha_{m-2}} \leq & \ c\{(\|w(t)\|_0^2 + \sum_{\alpha_{m-2}} \|D^{\alpha_{m-2}} w(t)\|_0^2) \\ & + \int_0^t (\|w(s)\|_0^2 + \sum_{\alpha_{m-2}} \|D^{\alpha_{m-2}} w(s)\|_0^2) ds\}. \end{aligned}$$

Since equation in the problem (2.12) has order $2m-1$, we have that c is a polynomial in $\|D^{\alpha_k} u\|_{L^\infty}$ and $\|D^{\alpha_k} v\|_{L^\infty}$, $k = 0, 1, 2, \cdots, 3m-3$ of degree m.

Similarly the other terms in (3.32) can be estimated, giving

$$J_{\alpha_\lambda} \leq c\{\|w(t)\|_0^2 + \sum_{\alpha_\lambda} \|D^{\alpha_\lambda} w(t)\|_0^2 + \int_0^t (\|w(s)\|_0^2 + \sum_{\alpha_\lambda} \|D^{\alpha_\lambda} w(s)\|_0^2) ds\},$$

where λ is an integer such that $0 < \lambda < m-2$ and c is a polynomial in $\|D^{\alpha_k}u\|_{L^\infty}$ and $\|D^{\alpha_k}v\|_{L^\infty}$, $k = 0, 1, 2, \cdots, 3m-3$, of degree $\leq 2m-3$.

By combining these results with (3.32) we obtain, after we use (2.16), the following estimate

$$\sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}}w(t)\|_0^2 \leq c\{\|w(t)\|_0^2 + \sum_{\alpha_{m-2}} \|D^{\alpha_{m-2}}w(t)\|_0^2 +$$

$$\int_0^t (\|w(s)\|_0^2 + \sum_{\alpha_{m-2}} \|D^{\alpha_{m-2}}w(s)\|_0^2)ds\} + \|N\|_0 \|G_m(u) - G_m(v)\|_0 . \qquad (3.33)$$

Inequality (2.16), with $\alpha = m-2,\ \beta = m-1,\ \theta = \frac{m-2}{m-1}$ and $p = q = r = 2$, gives

$$\|D^{\alpha_{m-2}}w\|_0^2 \leq c\,\|D^{\alpha_{m-1}}w\|_0^{2\frac{m-2}{m-1}}\,\|w\|_0^{\frac{2}{m-1}} .$$

Applying inequality (2.17) with $p = \frac{m-1}{m-2}$, we obtain

$$\|D^{\alpha_{m-2}}w(t)\|_0^2 \leq \delta\,\|D^{\alpha_{m-1}}w(t)\|_0^2 + k(\delta)\,\|w(t)\|_0^2 , \qquad (3.34)$$

where $k(\delta) = \delta^{2-m}$ and $\delta > 0$ (arbitrary). By substituting (3.34) in the estimate (3.33), we obtain

$$\sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}}w(t)\|_0^2 \leq c\{\|w(t)\|_0^2 + k(\delta)\,\|w(t)\|_0^2 + \delta \sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}}w(t)\|_0^2\} +$$

$$c\int_0^t (\|w(s)\|_0^2 + \sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}}w(s)\|_0^2)ds + \|N\|_0 \|G_m(u) - G_m(v)\|_0 .$$

Choosing δ conveniently, we obtain

$$\begin{aligned}\sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}}w(t)\|_0^2 \ \leq\ & c\,\|w(t)\|_0^2 + c\int_0^t (\|w(s)\|_0^2 + \sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}}w(s)\|_0^2)ds \\ & + \|N\|_0 \|G_m(u) - G_m(v)\|_0 ,\end{aligned}$$

where c is a polynomial in $\left\|D^k u\right\|_{L^\infty}$ and $\left\|D^k v\right\|_{L^\infty}$, $k = 0, 1, 2, \cdots, 3m-3$, of degree $2(m-1)^2$. By using (3.30), we find

$$\begin{aligned}\sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}}w(t)\|_0^2 \ \leq\ & c\,\|w\|_0 \|N\|_0 + c\int_0^t (\|w(s)\|_0^2 + \sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}}w(s)\|_0^2)ds \\ & + \|N\|_0 \|G_m(u) - G_m(v)\|_0 .\end{aligned}$$

Adding this inequality to (3.30) it results in

$$\|w(t)\|_0^2 + \sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}} w(t)\|_0^2 \leq c\int_0^t (\|w(s)\|_0^2 + \sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}} w(s)\|_0^2) ds$$
$$+c(\|N\|_0 \|G_m(u) - G_m(v)\|_0 + \|N\|_0 \|w\|_0).$$

By using Gronwall's Lemma, we obtain

$$\|w(t)\|_0^2 + \sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}} w(t)\|_0^2 \leq \alpha e^{cT}, \tag{3.35}$$

where $\alpha = c(\|N\|_0 \|G_m(u) - G_m(v)\|_0 + \|N\|_0 \|w\|_0)$ and c is a polynomial in $\|D^{\alpha_k} u\|_{L^\infty}$ and $\|D^{\alpha_k} v\|_{L^\infty}$, $k = 0, 1, 2, \cdots, 3m-3$, of degree $2(m-1)^2$.

The solutions u, v and its x-derivatives up to order $3m-3$ are of ∞-$(\log)^{\frac{1}{2(m-1)^2}}$-type, i.e. their representatives u and v can be chosen such that

$$\|D^{\alpha_k} u_\varepsilon\|_{L^\infty}, \|D^{\alpha_k} v_\varepsilon\|_{L^\infty} = \mathcal{O}(|\log \varepsilon|^{\frac{1}{2(m-1)^2}}), \text{ as } \varepsilon \to 0,\ k = 0, 1, 2, \cdots, 3m-3.$$

From (3.35) and since $N \in \mathcal{N}_2((0,T) \times \mathbf{R})$, we have

$$\|w(t)\|_0^2 + \sum_{\alpha_{m-1}} \|D^{\alpha_{m-1}} w(t)\|_0^2 \leq c\varepsilon^q, \forall q \text{ and } \varepsilon > 0, \text{ small enough.} \tag{3.36}$$

We have then that $\sup_{0 \leq t \leq T} \|w_\varepsilon(t)\|_0^2 = \mathcal{O}(\varepsilon^q)$ as $\varepsilon \to 0$, $\forall q$; and therefore

$$\|w_\varepsilon\|_0 = \mathcal{O}(\varepsilon^q), \quad \text{as} \quad \varepsilon \to 0. \tag{3.37}$$

Remark 3.1. *A consequence of the Lemma 2.1 is that if $g \in \mathcal{G}_2(\mathbf{R}^2)$, together with x-derivatives up to order $3m-1$ are of 2-$(\log)^{\frac{1}{2(3m-1)(m-1)^2}}$-type, then the solutions given in Theorem 2.2, together with their x-derivatives up to order $3m-3$ are of ∞-$(\log)^{\frac{1}{2(m-1)^2}}$-type.*

We observe that our results extend for equations in $\mathbf{R}^n$.

Remark 3.2. *By following the same technique, we can prove the following result: let $g \in \mathcal{G}_2(\mathbf{R}^n)$ and $T > 0$ finite. Then the Cauchy problem*

$$u_t = \sum_{\alpha_1} D^{\alpha_1} G_m(u), \ \ \text{in } \mathcal{G}_2((0,T) \times \mathbf{R}^n)$$
$$u|_{\{t=0\}} = g \ \ \text{in } \ \mathcal{G}_2(\mathbf{R}^n),$$

has a solution in $\mathcal{G}_2((0,T) \times \mathbf{R}^n)$. Moreover, if g and its derivatives up to order $3(m-1)+n$ are of 2-$(\log)^{\frac{1}{2(3m-1)(m-1)^2}}$-type, then there is an unique solution in $\mathcal{G}_2((0,T) \times \mathbf{R}^n)$.

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