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# A Subgradient-Like Algorithm for Solving Vector Convex Inequalities

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**Abstract** In this paper, we propose a strongly convergent variant of Robinson's subgradient algorithm for solving a system of vector convex inequalities in Hilbert spaces. The advantage of the proposed method is that it converges strongly, when the problem has solutions, under mild assumptions. The proposed algorithm also has the following desirable property: the sequence converges to the solution of the problem, which lies closest to the starting point and remains entirely in the intersection of three balls with radius less than the initial distance to the solution set.

**Keywords** Projection methods · Strong convergence · Subgradient algorithm · Vector convex functions

## 1 Introduction

We propose an algorithm for solving a special vector optimization problem on a convex set, where the optimality is defined via a conic inclusion in Hilbert spaces. This problem is called a vector convex inequality and consists in finding a particular element of a feasible convex set, which solves a special inclusion, i.e., the image of the solution by the vector convex map belongs to a closed and convex pointed cone. When the cone is the Pareto cone, the problem becomes the convex feasibility problem, which has been well studied and has many applications in optimization theory,

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approximation theory, image reconstruction, and so on; see [1–3]. An excellent survey of projection methods for solving convex feasibility problems can be found in [4].

The proposed method is subgradient-like iterative algorithm and generates a sequence that converges strongly to the solution closest to the starting point. The new method is related to Polyak's and Robinson's subgradient methods; see [1, 5]. It uses an idea, which is similar to that exposed in [6–8] with the same goal, upgrading weak to strong convergence. Strong convergence is forced by combining a subgradient iteration with a simple projection step, onto the intersection of the feasible set with suitable halfspaces containing the solution set.

A popular strategy for solving vector optimization problems, in particular the system of convex inequalities, is the scalarization approach. The most widely used scalarization technique is the weighting method. This procedure may lead to unbounded numerical problems, which, therefore, may lack minimizers; see [9–13]. Another disadvantage of this approach is that the choice of the parameters is not known in advance, leaving the modeler and the decision-maker with the burden of choosing them; see [14]. Instead of that, our method explores, strongly, the vectorial structure of the system of convex inequalities.

Many important real-world problems in economics and engineering are modeled in infinite-dimensional spaces. These include optimal control and structural design problems and the problem of minimal area surface with obstacles, among others. On the other hand, even when we have to solve infinite-dimensional problems, numerical implementations of algorithms are certainly applied to finite-dimensional approximations of these problems. Nevertheless, it is important to have convergence theory for the infinite-dimensional case, in order to guarantee robustness and stability with respect to the discretization schemes employed to obtain finite-dimensional approximations. This issue is closely related to the so-called Mesh Independence Principle [15–17]. This principle relies on infinite-dimensional convergence to predict the convergence properties of a discretized finite-dimensional method. Furthermore, Mesh Independence Principle provides theoretical background for the design of refinement strategies. We suggest the reader to see [18], where a variety of applications are described. A strong convergence principle for Fejér-monotone methods in Hilbert spaces is extensively analyzed in [19].

As Robinson [5] said, a useful extension of his algorithm would be to define it in the Hilbert space setting. He observed that the convergence analysis of his method could be carried out under very strong and undesirable hypothesis, as, for example, that the solution set has nonempty interior. In this paper, we will show that our algorithm converges strongly to the solution of the problem that lies closest to the starting point, without any additional assumption. We emphasize that this last special feature is interesting even in finite-dimensional spaces and is useful in many specific applications, e.g., in image reconstruction [20–22].

## 2 Preliminary

We start describing our notation. $\mathcal{H}$ is a real Hilbert space, $\mathcal{L}(\mathcal{H}, \mathbb{R}^m)$ is the set of all linear continuous operators from $\mathcal{H}$ onto $\mathbb{R}^m$, and $\mathcal{H}^*$ is the dual space of $\mathcal{H}$, i.e., the set of all linear continuous operators from $\mathcal{H}$ onto $\mathbb{R}$. The inner product in $\mathcal{H}$ is denoted by $\langle\cdot,\cdot\rangle$, and by $\|\cdot\|$ the norm determined by this inner product. The closed ball centered at $x \in \mathcal{H}$ with radius $\rho$ is denoted by $B[x,\rho]$, i.e., $B[x,\rho] = \{y \in \mathcal{H} : \|y-x\| \le \rho\}$. The set $C$ is a closed and convex subset of $\mathcal{H}$. For an element $x \in \mathcal{H}$, we define the orthogonal projection of $x$ onto $C$, $P_C(x)$, as the unique point in $C$ such that $\|P_C(x) - y\| \le \|x - y\|$ for all $y \in C$.

Let $K$ be a closed, convex, and pointed cone in $\mathbb{R}^m$. The partial order $\preceq$ ($\prec$) induced in $\mathbb{R}^m$ by $K$ is defined as $x \preceq y$ ($x \prec y$) if and only if $y - x \in K$ ($y - x \in \operatorname{int}(K)$). The partial orders $\succeq$ and $\succ$ are defined in a similar way. The positive dual cone $K^*$ of $K$ is defined by $w \in K^*$ if and only if $\langle w, x\rangle \ge 0$ for all $x \in K$. The vector function $F : \mathcal{H} \to \mathbb{R}^m$ is called convex if, for all $x, y \in \mathcal{H}$ and $\alpha \in [0,1]$,

$$F\bigl(\alpha x + (1-\alpha)y\bigr) \preceq \alpha F(x) + (1-\alpha)F(y).$$

We recall that $F$ is a convex function if and only if

$$\operatorname{epi}(F) = \bigl\{(x,v) \in \mathcal{H} \times \mathbb{R}^m : F(x) \preceq v\bigr\}$$

is a convex set; see [23].

In this paper, we are interested in the problem of finding $x^* \in C$ such that

$$F\bigl(x^*\bigr) \preceq 0, \tag{1}$$

which is called a system of convex inequalities; see [5]. The solution set of this problem is denoted by $S^*$.

From now on $F$ is a convex function. Hence, for all $w \in K^*$, the mapping $\langle w, F\rangle : \mathcal{H} \to \mathbb{R}$, defined by $\langle w, F\rangle(x) = \langle w, F(x)\rangle$, is convex; see Proposition 6.2 of [12]. In analogy with the scalar case, the subdifferential set is defined as

$$\partial F(x) := \bigl\{U \in \mathcal{L}\bigl(\mathcal{H}, \mathbb{R}^m\bigr) : F(y) \succeq F(x) + U(y-x)\ \forall y\bigr\}$$

for any $x \in \mathcal{H}$. The elements of $\partial F(x)$ are called subgradients of $F$ at $x$. If the dimension of $\mathcal{H}$ is finite and $x$ is in the interior, or in the relative interior, of the domain of $F$, then $\partial F(x) \neq \emptyset$. This is an immediate consequence of Theorem 4.6 of [23]. A fundamental fact is that convex functions defined on finite dimensional spaces are locally Lipschitz; see [23].

Let $\varphi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a scalar convex function. Define

$$E(\varphi) := \bigl\{(y,v) \in \mathcal{H} \times \mathbb{R} : y \in \operatorname{dom}(\varphi),\ v > \varphi(y)\bigr\}.$$

Observe that $E(\varphi) = \operatorname{int}(\operatorname{epi}(\varphi))$ if and only if $E(\varphi)$ is open. The following proposition establishes the relationship between locally boundedness of $\varphi$ and the openness of $E(\varphi)$.

**Proposition 2.1** *Let $\varphi$ be convex function. The set $E(\varphi)$ is nonempty and open, considering the norm $\|(x,t)\|_1 = \|x\| + |t|$, if and only if* dom $(\varphi)$ *is open and $\varphi$ is locally bounded from above at each point of its domain.*

*Proof* Assume that $E(\varphi)$ is nonempty and open. So, for any point $(\hat{x},\hat{t}) \in E(\varphi)$, there exists $\delta > 0$ such that $B[(\hat{x},\hat{t}),\delta] \subset E(\varphi)$. Then, $\|y-\hat{x}\| < \delta$ implies that $\varphi(y) \le \hat{t}+\delta$. We conclude that $\varphi$ is locally bounded above at $\hat{x}$ and that $\operatorname{dom}(\varphi)$ is open.

Conversely, if we assume that $\operatorname{dom}(\varphi)$ is open and that $\varphi$ is locally bounded above at $\hat{x}$, then there exist $\delta > 0$ and $M$ such that $\hat{x}+\delta u \in \operatorname{dom}(\varphi)$ and $\varphi(\hat{x}+\delta u) < M$ for any $u$ with $\|u\| \le 1$. By the convexity of $\varphi$ and taking $\varphi(\hat{x}) < \hat{t} < M$ and

$$\begin{aligned}\rho &:= \bigl[\hat{t}-\varphi(\hat{x})\bigr]\sin\biggl(\frac{\pi}{2}-\arctan\biggl(\frac{M-\varphi(\hat{x})}{\delta}\biggr)\biggr)\\ &= \bigl[\hat{t}-\varphi(\hat{x})\bigr]\cos\biggl(\arctan\biggl(\frac{M-\varphi(\hat{x})}{\delta}\biggr)\biggr),\end{aligned}$$

we get that $B[(\hat{x},\hat{t}),\rho] \subset E(\varphi)$, establishing that $(\hat{x},\hat{t})$ is in the interior of $E(\varphi)$, i.e., $E(\varphi)$ is open. □

It is known that convex lower semicontinuous functions are locally bounded; see [24]. Then, with the last proposition, we have proved that if $\varphi$ is convex lower semicontinuous and $\operatorname{dom}(\varphi)$ is open, then $E(\varphi)$ is also open. The following lemma establishes the existence of subgradients of $\varphi$.

**Lemma 2.1** *Let $\varphi$ be a locally bounded above and convex function with open* $\operatorname{dom}(\varphi)$. *Then, $\partial\varphi(x)$ is nonempty for all $x \in$* $\operatorname{dom}(\varphi)$.

*Proof* Take $x \in \operatorname{dom}(\varphi)$. The set $E(\varphi)$ is nonempty since $(x,t) \in E(\varphi)$ for any $t > \varphi(x)$. The convexity of $\varphi$ implies that $E(\varphi)$ is convex. By the last proposition, $E(\varphi)$ is open because $\varphi$ is locally bounded above by hypothesis. Observe that $(x,\varphi(x)) \notin E(\varphi)$. By Lemma 1.3 of [24], there exists $\xi \in (\mathcal{H}\times\mathbb{R})^*$ such that $\xi(y,v) > \xi(x,\varphi(x))$ for all $(y,v) \in E(\varphi)$. Since $\xi(y,v) = \xi(y,0)+v\xi(0,1)$, there exists $z \in \mathcal{H}^*$ such that $\xi(y,v) = \langle z,y\rangle + kv$, where $k = \xi(0,1)$. In particular,

$$\langle z,x\rangle + kv > \langle z,x\rangle + k\varphi(x)$$

for any $v > \varphi(x)$. Then, $k > 0$. Thus, the above inequality becomes

$$v > \varphi(x) + \biggl\langle -\frac{1}{k}z, y-x\biggr\rangle$$

for any $(y,v) \in E(\varphi)$. Therefore,

$$\varphi(y) \ge \varphi(x) + \biggl\langle -\frac{1}{k}z, y-x\biggr\rangle$$

because $\varphi(y) = \inf\{v \in \mathbb{R}\colon (y,v) \in E(\varphi)\}$. We conclude that $-\frac{1}{k}z \in \partial\varphi(x)$, i.e., $\partial\varphi(x) \neq \emptyset$. □

If $\operatorname{dom}(\varphi)$ is the entire space, and $\varphi$ is a continuous convex functional, then the result given in Lemma 2.1 can be seen in Theorem 3.26 of [25]. The following example shows that there exist convex functions that are not locally bounded above and that, nevertheless, have subgradients.

*Example 2.1* Let

$$V := \left\{ u = (u_n)_{n\geq 1} \colon \sum_{n=1}^{\infty} n^2 u_n^2 < \infty \right\}.$$

Then $V$ is a Hilbert space with inner product

$$(u, v) = \sum_{n=1}^{\infty} n^2 u_n v_n$$

and norm $\|u\| = \sqrt{(u,u)} = \sum_{n=1}^{\infty} n^2 u_n^2$. The dual of $V$ is identified with the space

$$V^* = \left\{ \alpha = (\alpha_n)_{n\geq 1} \colon \sum_{n=1}^{\infty} \frac{1}{n^2} \alpha_n^2 < \infty \right\}.$$

The scalar product $\langle \cdot, \cdot \rangle_{V^*,V}$ is given by

$$\langle \alpha, v \rangle_{V^*,V} = \sum_{n=1}^{\infty} \alpha_n v_n,$$

and the Riesz–Fréchet isomorphism $T \colon V \to V^*$ is given by

$$u = (u_n)_{n\geq 1} \mapsto Tu = \left(n^2 u_n\right)_{n\geq 1}.$$

Observe that the series $e_n$ that have the $n$th terms equal one and the other are null, $n = 1, 2, \ldots$, are in $V$ and in $V^*$. Define $\mu \colon \mathbb{N} \times V \to \mathbb{R}$ by $\mu(n, x) = n x_n^2$ and $\eta \colon V \to \mathbb{R}$ by $\eta(x) = \max\{\mu(n, x) \colon n \in \mathbb{N}\}$. The domain of the function $\eta$ is the entire space $V$ because $\lim_{n\to\infty} n x_n^2 = 0$. The functions $\mu(i, \cdot) \colon V \to \mathbb{R}$, $i = 1, 2, \ldots$, are closed and convex. Therefore, the function $\eta$ is closed and convex.

Fix $\bar{x} \in V$ such that $\bar{x}_n > 0$ and $n\bar{x}_n^2 < 1$ for all $n \geq 1$. Then, $\eta(\bar{x}) < 1$. Take any $L > 0$, any $0 < \delta < 1$, and any $n_0 \in \mathbb{N}$ such that $n_0 > 4\frac{L+1}{\delta^2}$. Observe that $y = \bar{x} + \frac{\delta}{2} e_n \in B[\bar{x}, \delta]$ and that, if $n > n_0$, then

$$n\left(\bar{x}_n + \frac{\delta}{2}\right)^2 > n_0\left(\bar{x}_n + \frac{\delta}{2}\right)^2 > n_0 \frac{\delta^2}{4} > L + 1 > 1.$$

Therefore,

$$\eta(y) = n\left(\bar{x}_n + \frac{\delta}{2}\right)^2$$

and

$$\left|\eta(y) - \eta(\bar{x})\right| = n\left(\bar{x}_n + \frac{\delta}{2}\right)^2 - \eta(\bar{x}) > n\frac{\delta^2}{4} - \left(\eta(\bar{x}) - n\bar{x}_n^2\right) > L > L\delta > L\|y - \bar{x}\|$$

for any $n > n_0$. Hence, the function $\eta$ is not locally bounded at $x$, not locally Lipschitz at $x$, and not lower semicontinuous. Nevertheless, $\partial\eta(x) \neq \emptyset$ for each $x$ in dominium of $\eta$. Indeed, fix $n \in \mathbb{N}$ and $x \in V$. Observe that $2nx_n e_n \in V^*$. Since $ny_n^2 \geq nx_n^2 + 2nx_n(y_n - x_n)$, then

$$\mu(n, y) \geq \mu(n, x) + \langle 2nx_n e_n, y - x\rangle$$

for any $y \in V$. Henceforth,

$$\eta(y) \geq \mu(n, y) \geq \mu(n, x) + \langle 2nx_n e_n, y - x\rangle = \eta(x) + \langle 2nx_n e_n, y - x\rangle$$

for any $n$ such that $\eta(x) = \mu(n, x)$, i.e., $2nx_n e_n \in \partial\eta(x)$ for any $n \in \mathbb{N}$ such that $\eta(x) = \mu(n, x)$.

We say that $G \subset K^* \cap B[0, 1]$ is a generator of $K^*$if each element from $K^*$ can be expressed as a linear combination with nonnegative coefficients of elements of $G$, i.e., $K^* = \mathrm{co}(\mathrm{conv}(G))$. In the following lemma, we show how to find elements of the subdifferential set of a convex function.

**Lemma 2.2** *Assume that $G = \{w_1 w_2, \ldots, w_s\}$ is a finite generator of $K^*$, $F : \mathcal{H} \to \mathbb{R}^m$ is convex and $\partial\langle w_i, F\rangle(x) \neq \emptyset$, $i = 1, \ldots, s$, for all $x$. Then $\partial F(x) \neq \emptyset$ for any $x$.*

*Proof* Take $x \in \mathcal{H}$ and $u_i \in \partial\langle w_i, F\rangle(x)$, $i = 1, 2, \ldots, s$. Define $\tilde{U} \in \mathcal{L}(\mathbb{R}^m, \mathcal{H})$ such that $\tilde{U}(w_i) = u_i$, $i = 1, 2, \ldots, s$, and $U \in \mathcal{L}(\mathcal{H}, \mathbb{R}^m)$, the adjoint operator of $\tilde{U}$. Since, for any $w \in K^*$, there exist $\lambda_i \geq 0$, $i = 1, 2, \ldots, s$, such that $w = \sum_{i=1}^s \lambda_i w_i$, we get, for all $y \in \mathcal{H}$, that

$$\begin{aligned}
&\big\langle w, F(y) - F(x) - U(y - x)\big\rangle \\
&\quad = \left\langle \sum_{i=1}^s \lambda_i w_i, F(y) - F(x) - U(y - x)\right\rangle \\
&\quad = \sum_{i=1}^s \lambda_i \big\{\big\langle w_i, F(y)\big\rangle - \big\langle w_i, F(x)\big\rangle - \big\langle w_i, U(y - x)\big\rangle\big\} \\
&\quad = \sum_{i=1}^s \lambda_i \big\{\big\langle w_i, F(y)\big\rangle - \big\langle w_i, F(x)\big\rangle - \big\langle \tilde{U}(w_i), y - x\big\rangle\big\} \\
&\quad = \sum_{i=1}^s \lambda_i \big\{\big\langle w_i, F(y)\big\rangle - \big\langle w_i, F(x)\big\rangle - \langle u_i, y - x\rangle\big\} \\
&\quad \geq 0.
\end{aligned}$$

Since $K$ is a closed and convex cone, the last inequality implies

$$F(y) \succeq F(x) + U(y - x).$$

Therefore, $U$ belongs to $\partial F(x)$. □

The proof of the existence of subgradient of $F$ at $x \in \mathrm{dom}(F)$ is constructive, in the sense that we show how to find an element of $\partial F(x)$. Other sufficient conditions for the existence of subgradients of convex vector functions have appeared in the literature; see [26, 27]. Under any one of these conditions, we can prove the well-definedness of the proposed algorithm. In the convergence analysis, the above-mentioned sufficient conditions will be replaced by the hypothesis of the existence of subdifferentials of $F$ at every point in its domain.

Ending this section, in the following two propositions, we mention two well-known facts.

**Proposition 2.2** *The solution set of Problem* (1) *is convex and closed.*

*Proof* Immediate. □

**Proposition 2.3** *For all $x, y \in \mathcal{H}$ and all $z \in C$, $\langle x - P_C(x), z - P_C(x)\rangle \leq 0$.*

*Proof* See Theorem 5.2 of [24]. □

## 3 The Subgradient-Like Algorithm

**Algorithm A**
*Initialization step.* Take $x^0 \in C$ and $U^0 \in \partial F(x^0)$.
*Iterative step.* Given $x^k$ and $U^k \in \partial F(x^k)$, define

$$H_k := \{x \in \mathcal{H} : F(x^k) + U^k(x - x^k) \preceq 0\} \tag{2}$$

and

$$W_k := \{x \in \mathcal{H} : \langle x - x^k, x^0 - x^k\rangle \leq 0\}. \tag{3}$$

Compute

$$x^{k+1} := P_{C \cap W_k \cap H_k}(x^0). \tag{4}$$

If $x^{k+1} = x^k$, then stop.

Observe that, by (2)–(3), $W_k$ and $H_k$ are convex and closed sets for each $k$. Thus, $C \cap H_k \cap W_k$ is a convex and closed set for each $k$. So, if $C \cap H_k \cap W_k$ is nonempty, then by (4) the next iterate, $x^{k+1}$, is well defined.

## 4 Convergence Analysis

In this paper, we assume that $S^*$ is nonempty. From now on, $\{x^k\}$ is the sequence generated by Algorithm A. Now we establish some useful properties.

**Lemma 4.1** $S^* \subseteq C \cap H_k \cap W_k$ *for all $k$.*

*Proof* We proceed by induction. By definition, $S^* \subseteq C$. Take $x^* \in S^*$. By the convexity of $F$ and (2), we have $0 \succeq F(x^*) \succeq F(x^0) + U^0(x^* - x^0)$ with $U^0 \in \partial F(x^0)$, and thus $x^* \in H_0$. Since $W_0 = \mathcal{H}$, $S^* \subseteq C \cap H_0 \cap W_0$. Assume that $S^* \subseteq C \cap H_\ell \cap W_\ell$, for $\ell \le k$. Henceforth, $x^{k+1} = P_{C\cap H_k \cap W_k}(x^0)$ is well defined. Take $x^* \in S^*$. Clearly, $x^* \in C$. Since $U^{k+1} \in \partial F(x^{k+1})$, we get

$$0 \succeq F\left(x^*\right) \succeq F\left(x^{k+1}\right) + U^{k+1}\left(x^* - x^{k+1}\right). \tag{5}$$

It follows from (5) that $x^* \in H_{k+1}$. On the other hand,

$$\left\langle x^* - x^{k+1}, x^0 - x^{k+1}\right\rangle = \left\langle x^* - P_{C\cap H_k\cap W_k}\left(x^0\right), x^0 - P_{C\cap H_k\cap W_k}\left(x^0\right)\right\rangle \le 0, \tag{6}$$

using the induction hypothesis and Lemma 2.3 in the above inequality. Inequality (6) implies that $x^* \in W_{k+1}$ and hence, $S^* \subseteq C \cap H_{k+1} \cap W_{k+1}$. □

**Corollary 4.1** *Algorithm A is well defined.*

*Proof* By the previous lemma, $\emptyset \neq S^* \subseteq C \cap H_k \cap W_k$ for all $k$. Then, by (4), given $x^0$, the sequence $\{x^k\}$ is computable. □

The next proposition validates the stop criterium.

**Proposition 4.1** *If Algorithm A stops at iterate $k$, then $x^k$ belongs to $S^*$.*

*Proof* Assume that $x^{k+1} = x^k$. Since $x^k \in W_k$, by (4) we get that $F(x^k) \preceq 0$, i.e., $x^k \in S^*$. □

In the following lemma we establish that $\{x^k\}$ is bounded.

**Lemma 4.2** *The sequence $\{x^k\}$ is bounded. Furthermore,*

$$\left\{x^k\right\} \subset B\left[x^0, \rho\right] \cap B\left[x^*, \rho\right] \cap B\left[\frac{x^0 + x^*}{2}, \frac{\sqrt{2}}{2}\rho\right], \tag{7}$$

*where $x^* = P_{S^*}(x^0)$ and $\rho = \operatorname{dist}(x^0, S^*)$.*

*Proof* Lemma 4.1 says that $S^* \subseteq C \cap W_k \cap H_k$ for all $k$, and by the definition of $x^{k+1}$, see (4), it is true that

$$\left\|x^{k+1} - x^0\right\| \le \left\|z - x^0\right\| \tag{8}$$

for all $k$ and all $z \in S^*$. Henceforth, taking in (8) $z = x^*$, we have

$$\left\|x^{k+1} - x^0\right\| \le \left\|x^* - x^0\right\| = \rho \tag{9}$$

for all $k$. Hence, $\{x^k\}$ is bounded. Furthermore, by Proposition 2.3,

$$0 \ge \left\langle x^* - x^{k+1}, x^0 - x^{k+1}\right\rangle = \frac{1}{2}\left(\left\|x^* - x^{k+1}\right\|^2 - \left\|x^* - x^0\right\|^2 + \left\|x^{k+1} - x^0\right\|^2\right)$$

for all $k$, obtaining

$$\|x^{k+1}-x^*\| \leq \|x^0-x^*\| = \rho. \tag{10}$$

Using Lemma 4.1, Proposition 2.3, and the definition of $x^k$, we get

$$\begin{aligned}\left\|\frac{x^0+x^*}{2}-x^k\right\|^2 &= \frac{1}{4}\left\{\|x^0-x^*\|^2+2\langle x^0-x^k, x^*-x^k\rangle+\|x^*-x^k\|^2\right\}\\ &\leq \frac{1}{4}\left\{\|x^0-x^*\|^2+\|x^*-x^k\|^2\right\}.\end{aligned}$$

Henceforth, by (9) and (10) we get

$$\left\|\frac{x^0+x^*}{2}-x^k\right\| \leq \frac{\sqrt{2}}{2}\rho.$$

The last inequality, (9) and (10) establish (7). □

Until now, it was been proved that Algorithm A is well defined, in the sense that all needed computations can be done, that the generated sequence is bounded, and that in the case it is finite, it ends at a solution of Problem (1). Our goal now is to prove that all cluster points of $\{x^k\}$ are in $S^*$. To do that, we need that the following hypotheses would be fulfilled. We assume hereinafter that $\partial F(x)$ is bounded on bounded sets, i.e., $\cup_{x\in B}\partial F(x)$ is bounded for any bounded subset $B$ of $\mathcal{H}$. We emphasize that this assumption holds trivially in finite-dimensional spaces; see [23]. Furthermore, it has been considered in the literature for the convergence analysis of many classical methods for solving some important problems as, for example, equilibrium problems, variational inequalities, and scalar optimization problems in infinite-dimensional spaces; see, for instance, [1, 28–30]. Furthermore, from now on, the functions $\langle w, F\rangle : \mathcal{H} \to \mathbb{R}$ with $w \in K^*$ are lower semicontinuous. It is well known that lower semicontinuous functions are locally bounded in Hilbert spaces, and therefore, by Lemma 2.1, the subdifferential set, at points in the interior of its domain, is nonempty.

**Lemma 4.3** *All weak cluster points of* $\{x^k\}$ *belong to* $S^*$.

*Proof* Since $x^{k+1} \in W_k$,

$$0 \geq \langle x^{k+1}-x^k, x^0-x^k\rangle = \frac{1}{2}\left(\|x^{k+1}-x^k\|^2-\|x^{k+1}-x^0\|^2+\|x^k-x^0\|^2\right).$$

Thus,

$$0 \leq \|x^{k+1}-x^k\|^2 \leq \|x^{k+1}-x^0\|^2-\|x^k-x^0\|^2,$$

establishing that $\{\|x^k-x^0\|\}$ is a nondecreasing sequence. It follows from Lemma 4.2 that $\{\|x^k-x^0\|\}$ is bounded and thus, is a convergent sequence. Therefore,

$$\lim_{k\to\infty}\|x^{k+1}-x^k\| = 0. \tag{11}$$

Let $\bar{x}$ be a weak cluster point of $\{x^k\}$ and $\{x^{j_k}\}$ be a convergent subsequence to $\bar{x}$. Since $x^{k+1} \in H_k$, we have that

$$F\left(x^{j_k}\right) + U^{j_k}\left(x^{j_k+1} - x^{j_k}\right) \preceq 0. \tag{12}$$

By assumption, the sequence $\{U^{j_k}\}$ is bounded. So, there exists $\Lambda > 0$ such that $\|U^{j_k}\| \le \Lambda$ for all $k$. Since $\{x^{j_k+1} - x^{j_k}\}$ converges strongly to zero by (11), for all $\varepsilon > 0$, there exists $K(\varepsilon) \in \mathbb{N}$ such that for all $k \ge K(\varepsilon)$, $\|x^{j_k+1} - x^{j_k}\| \le \frac{\varepsilon}{\Lambda}$. Then,

$$\left\|U^{j_k}\left(x^{j_{k+1}} - x^{j_k}\right)\right\| \le \left\|U^{j_k}\right\| \left\|x^{j_{k+1}} - x^{j_k}\right\| \le \varepsilon$$

for all $k \ge K(\epsilon)$. Establishing that the sequence $\{U^{j_k}(x^{j_{k+1}} - x^{j_k})\}$ converges strongly to zero. By taking limits in (12), we obtain that

$$0 \ge \lim_{k\to\infty}\left\langle w, F\left(x^{j_k}\right)\right\rangle \ge \liminf_{k\to\infty}\left\langle w, F\left(x^k\right)\right\rangle \tag{13}$$

for any $w \in K^*$. Since the function $\langle w, F(x)\rangle$ is weakly lower semicontinuous for all $w \in K^*$, using (13), we get

$$0 \ge \left\langle w, F(\bar{x})\right\rangle$$

for all $w \in K^*$. The above inequality implies that

$$F(\bar{x}) \preceq 0.$$

So, $\bar{x} \in S^*$. □

Finally, we are ready to prove the strong convergence of the sequence $\{x^k\}$ generated by Algorithm A to the solution that lies closest to $x^0$.

**Theorem 4.1** *Define $x^* = P_{S^*}(x^0)$. Then $\{x^k\}$ converges strongly to $x^*$.*

*Proof* By Proposition 2.2, $S^*$ is closed and convex. Therefore, $x^*$, the orthogonal projection of $x^0$ onto $S^*$, exists. By the definition of $x^{k+1}$ we have that

$$\left\|x^{k+1} - x^0\right\| \le \left\|z - x^0\right\| \quad \forall z \in H_k \cap W_k \cap C. \tag{14}$$

Since $x^* \in S^* \subseteq H_k \cap W_k \cap C$ for all $k$, it follows from (14) that

$$\left\|x^k - x^0\right\| \le \left\|x^* - x^0\right\| \tag{15}$$

for all $k$. By Lemma 4.2, $\{x^k\}$ is bounded, and, by Lemma 4.3, each of its weak cluster points belongs to $S^*$. Let $\{x^{i_k}\}$ be any weakly convergent subsequence of $\{x^k\}$, and let $\hat{x} \in S^*$ be its weak limit. Observe that

$$\begin{aligned}\left\|x^{i_k} - x^*\right\|^2 &= \left\|x^{i_k} - x^0 - \left(x^* - x^0\right)\right\|^2 \\ &= \left\|x^{i_k} - x^0\right\|^2 + \left\|x^* - x^0\right\|^2 - 2\left\langle x^{i_k} - x^0, x^* - x^0\right\rangle \\ &\le 2\left\|x^* - x^0\right\|^2 - 2\left\langle x^{i_k} - x^0, x^* - x^0\right\rangle,\end{aligned}$$

where the inequality follows from (15). By the weak convergence of $\{x^{i_k}\}$ to $\hat{x}$, we obtain

$$\limsup_{k\to\infty} \|x^{i_k} - x^*\|^2 \le 2\big(\|x^* - x^0\|^2 - \langle \hat{x} - x^0, x^* - x^0\rangle\big). \tag{16}$$

Applying Proposition 2.3 with $K = S^*$, $x = x^0$, and $z = \hat{x} \in S^*$ and taking into account that $x^*$ is the projection of $x^0$ onto $S^*$, we have that

$$\langle x^0 - x^*, \hat{x} - x^*\rangle \le 0.$$

Rewriting the above inequality, we obtain

$$\begin{aligned} 0 &\ge -\langle \hat{x} - x^*, x^* - x^0\rangle \\ &= -\langle x^0 - x^*, x^* - x^0\rangle - \langle \hat{x} - x^0, x^* - x^0\rangle \\ &\ge \|x^* - x^0\|^2 - \langle \hat{x} - x^0, x^* - x^0\rangle. \end{aligned}$$

Combining the above inequality with (16), we conclude that $\{x^{i_k}\}$ converges strongly to $x^*$. Thus, we have shown that every weakly convergent subsequence of $\{x^k\}$ converges strongly to $x^*$. Hence, the whole sequence $\{x^k\}$ converges strongly to $x^* \in S^*$. □

*Remark 4.1* The proposed algorithm is of subgradient-type; therefore in Sect. 2, we discussed about the existence of subgradients of convex vector functions. This matter is completely solved in the following three cases: when the domain is a subset of finite dimensional spaces; see [23], when the domain is an entire Banach space; see [25], and, in more general settings, when the cone $K$ satisfies some conditions; see [26, 27]. Our results in Lemmas 2.1 and 2.2 are a minor contribution; they are not contained in the first case because they are true in Hilbert spaces, in the second case because the domain of the functions could be a proper convex open subset of the space, and in the third case because the continuity of the functions is assumed. We emphasize with Example 2.1 that all known conditions are only sufficient.

## 5 Final Remarks

A modification of Robinson's subgradient algorithm for finding one solution of a vector convex inequality, forcing strong convergence in Hilbert spaces, has been proposed. Our modification consists in the adding of one linear constraint to perform the projection step, improving the convergence features of the original algorithm.

Concerning the complexity of the projection step (4), the presence of the halfspaces does not entail any significant additional computational cost comparing with the computation of the projection onto $C$ itself. Even though we are working in infinite-dimensional spaces, projection onto an intersection of halfspaces demands to solve a system of linear equations.

Clearly, the proposed method is especially effective when applied to nonsimple feasible set. In this case, adding linear constraints to perform the projection step, will not increase the cost. Actually, if the constraints are nonlinear, projections onto $C \cap W_k \cap H_k$ may, sometimes, turn out to be easier than onto the feasible set. Since we are adding only one halfspace in the projection step, the computational cost of the proposed algorithm is similar to Robinson's method; see [5].

We emphasize that given the starting point $x^0$, the sequence generated by the proposed algorithm converges to the closest solution of the system of inequalities. That is,

$$\min \|x - x^0\| \quad \text{s.a. } F(x) \preceq_K 0,$$

is solved. As a subject of future research, there remains the use of such a kind of algorithm to solve more general problems; see [31].

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