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Lines of principal curvature near singular end points of surfaces in $\mathbb{R}^3$

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Abstract.

In this paper are studied the nets of principal curvature lines on surfaces embedded in Euclidean 3−space near their end points, at which the surfaces tend to infinity.

This is a natural complement and extension to smooth surfaces of the work of Garcia and Sotomayor (1996), devoted to the study of principal curvature nets which are structurally stable –do not change topologically– under small perturbations on the coefficients of the equations defining algebraic surfaces.

This paper goes one step further and classifies the patterns of the most common and stable behaviors at the ends, present also in generic families of surfaces depending on one-parameter.

§1. Introduction

A *surface* of *smoothness class* C^k in Euclidean (x, y, z)-space $\mathbb{R}^3$ is defined by the variety $A(\alpha)$ of zeros of a real function α of class C^k in $\mathbb{R}^3$. The exponent k ranges among the positive integers as well as on the symbols ∞, ω (for analytic) and $a(n)$ (for algebraic of degree n).

In the class $C^{a(n)}$ of algebraic surfaces of degree n, we have $\alpha = \sum \alpha_h$, $h = 0, 1, 2, ..., n$, where α_h is a homogeneous polynomial of degree h with real coefficients: $\alpha_h = \sum a_{ijk} x^i y^j z^k$, $i + j + k = h$.

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The space $\mathbb{R}^3$ will be endowed with the Euclidean metric $ds^2 = dx^2 + dy^2 + dz^2$ also denoted by $<,>$, and with the positive orientation induced by the volume form $\Omega = dx \wedge dy \wedge dz$.

An *end point* or *point at infinity* of $A(\alpha)$ is a point in the unit sphere $\mathbb{S}^2$, which is the limit of a sequence of the form $p_n/|p_n|$, for p_n tending to infinity in $A(\alpha)$.

The *end locus*, $E(\alpha)$, of $A(\alpha)$ is the collection of its end points. This set is a geometric measure of the non-compactness of the surface and describes how it tends to infinity.

A surface $A(\alpha)$ is said to be *regular at* $p \in E(\alpha)$ if in a neighborhood of p, $E(\alpha)$ is a regular smooth curve in $\mathbb{S}^2$. Otherwise, p is said to be a *critical end point* of $A(\alpha)$.

For the class $a(n)$, $E(\alpha)$ is contained in the algebraic curve $E_n(\alpha) = \{p \in \mathbb{S}^2;\ \alpha_n(p) = 0\}$. The regularity of $E(\alpha)$ is equivalent to that of $E_n(\alpha)$.

The gradient vector field of α, will be denoted by $\nabla\alpha = \alpha_x\partial/\partial x + \alpha_y\partial/\partial y + \alpha_z\partial/\partial z$, where $\alpha_x = \partial\alpha/\partial x$, etc.

The zeros of this vector field are called *critical points* of α; they determine the set $C(\alpha)$. The regular part of $A(\alpha)$ is the smooth surface $S(\alpha) = A(\alpha) \setminus C(\alpha)$. When $C(\alpha)$ is disjoint from $A(\alpha)$, the surface $S(\alpha) = A(\alpha)$ is called *regular.* The *orientation* on $S(\alpha)$ will be defined by taking the gradient $\nabla\alpha$ to be the *positive normal.* Thus $A(-\alpha)$ defines the same surface as $A(\alpha)$ but endowed with the opposite orientation on $S(-\alpha)$.

The Gaussian normal map N, of $S(\alpha)$ into the sphere $\mathbb{S}^2$, is defined by the unit vector in the direction of the gradient: $N_\alpha = \nabla\alpha/|\nabla\alpha|$. The eigenvalues $-k^1_\alpha(p)$ and $-k^2_\alpha(p)$ of the operator $DN_\alpha(p)$, restricted to $T_pS(\alpha)$, the tangent space to the surface at p, define the principal curvatures, $k^1_\alpha(p)$ and $k^2_\alpha(p)$ of the surface at the point p. It will be always assumed that $k^1_\alpha(p) \le k^2_\alpha(p)$.

The points on $S(\alpha)$ at which the principal curvatures coincide, define the set $U(\alpha)$ of *umbilic points* of the surface $A(\alpha)$. On $S(\alpha) \setminus U(\alpha)$, the eigenspaces of DN_α, associated to $-k^1_\alpha$ and $-k^2_\alpha$ define line fields $L_1(\alpha)$ and $L_2(\alpha)$, mutually orthogonal, called respectively *minimal* and *maximal principal line fields* of the surface $A(\alpha)$. The smoothness class of these line fields is C^{k-2}, where $k-2 = k$ for $k = \infty$, ω and $a(n)-2 = \omega$.

The maximal integral curves of the line fields $L_1(\alpha)$ and $L_2(\alpha)$ are called respectively the lines of minimal and maximal principal curvature, or simply the *principal lines* of $A(\alpha)$.

What was said above concerning the definition of these lines is equivalent to require that they are non trivial solutions of Rodrigues' differential equations:

$$(1) \qquad DN_\alpha(p)dp + k^i_\alpha(p)dp = 0, \qquad < N(p), dp >= 0, \;\; i = 1, 2.$$

where $p = (x, y, z)$, $\alpha(p) = 0$, $dp = dx\partial/\partial x \;+ dy\partial/\partial y \;+ dz\partial/\partial z$. See [22, 23].

After elimination of k^i_α, $i = 1, 2$, the first two equations in (1) can be written as the following single implicit quadratic equation:

$$(2) \qquad < DN_\alpha(p)dp \wedge N(p), dp >= [DN_\alpha(p)dp, N_\alpha(p), dp] = 0.$$

The left (and mid term) member of this equation is the *geodesic torsion* in the direction of dp. In terms of a local parametrization $\overline{\alpha}$ introducing coordinates (u, v) on the surface, the equation of lines of curvature in terms of the coefficients (E, F, G) of the first and (e, f, g) of the second fundamental forms is, see [22, 23],

$$(3) \qquad [Fg - Gf]dv^2 + [Eg - Ge]dudv + [Ef - Fe]du^2 = 0.$$

The net $F(\alpha) = (F_1(\alpha), F_2(\alpha))$ of orthogonal curves on $S(\alpha) \setminus U(\alpha)$, defined by the integral foliations $F_1(\alpha)$ and $F_2(\alpha)$ of the line fields $L_1(\alpha)$ and $L_2(\alpha)$, will be called *the principal net* on $A(\alpha)$.

The study of families of principal curves and their umbilic singularities on immersed surfaces was initiated by Euler, Monge, Dupin and Darboux, to mention only a few. See [2, 18] and [14, 22, 23] for references.

Recently this classic subject acquired new vigor by the introduction of ideas coming from Dynamical Systems and the Qualitative Theory of Differential Equations. See the works [14], [5], [7], [13] of Gutierrez, Garcia and Sotomayor on the structural stability, bifurcations and genericity of principal curvature lines and their umbilic and critical singularities on compact surfaces.

The scope of the subject was broadened by the extension of the works on structural stability to other families of curves of classical geometry. See [12], for the asymptotic lines and [8, 9, 10, 11] respectively for the arithmetic, geometric, harmonic and general mean curvature lines. Other pertinent directions of research involving implicit differential equations arise from Control and Singularity Theories, see Davydov [3] and Davydov, Ishikawa, Izumiya and Sun [4].

In [6] the authors studied the behavior of the lines of curvature on algebraic surfaces, i.e. those of $C^{a(n)}$, focusing particularly their generic and stable patterns at end points. Essential for this study was the operation of compactification of algebraic surfaces and their equations (2) and (3) in $\mathbb{R}^3$ to obtain compact ones in $\mathbb{S}^3$. This step is reminiscent of the Poincaré compactification of polynomial differential equations [19].

In this paper the study in [6] will be extended to the broader and more flexible case of C^k-smooth surfaces.

As mentioned above, in the case of algebraic surfaces studied in [6], the ends are the algebraic curves defined by the zeros, in the Equatorial Sphere $\mathbb{S}^2$ of $\mathbb{S}^3$, of the highest degree homogeneous part α_n of the polynomial α. Here, to make the study of the principal nets at ends of smooth surfaces tractable by methods of Differential Analysis, we follow an inverse procedure, going from compact smooth surfaces in $\mathbb{S}^3$ to surfaces in $\mathbb{R}^3$. This restriction on the class of surfaces studied in this paper is explained in Subsection 1.1.

The new results of this paper on the patterns of principal nets at end points are established in Sections 2 and 3. Their meaning for the Structural Stability and Bifurcation Theories of Principal Nets is discussed in Section 4, where a pertinent problem is proposed. The essay [21] presents a historic overview of the subject and reviews other problems left open.

1.1. Preliminaries

Consider the space $\mathcal{A}_c^k$ of real valued functions α^c which are C^k-smooth in the three dimensional sphere $\mathbb{S}^3 = \{|p|^2+|w|^2 = 1\}$ in $\mathbb{R}^4$, with coordinates $p = (x, y, z)$ and w. The meaning of the exponent k is the same as above and $a(n)$ means polynomials of degree n in four variables of the form $\alpha^c = \sum \alpha_h w^h$, $h = 0, 1, 2, ..., n$, with α_h homogeneous of degree h in (x, y, z).

The *equatorial* sphere in $\mathbb{S}^3$ will be $\mathbb{S}^2 = \{(p, w) : \; |p| = 1, \; w = 0\}$ in $\mathbb{R}^3$. It will be endowed with the positive orientation defined by the outward normal. The northern hemisphere of $\mathbb{S}^3$ is defined by $\mathbb{H}^+ = \{(p, w) \in \mathbb{S}^3 : \; w > 0\}$.

The surfaces $A(\alpha)$ considered in this work will be defined in terms of functions $\alpha^c \in \mathcal{A}_c^k$ as $\alpha = \alpha^c \circ \mathbb{P}$, where $\mathbb{P}$ is the *central projection* of $\mathbb{R}^3$, identified with the tangent plane at the north pole $\mathbb{T}_w^3$, onto $\mathbb{H}^+$, defined by:

$$\mathbb{P}(p) = (p/(|p|^2 + 1)^{1/2}, \;\; 1/(|p|^2 + 1)^{1/2}).$$

For future reference, denote by $\mathbb{T}_y^3$ the tangent plane to $\mathbb{S}^3$ at the point $(0, 1, 0, 0)$, identified with $\mathbb{R}^3$ with orthonormal coordinates (u, v, w), with w along the vector $\omega = (0, 0, 0, 1)$. The central projection $\mathbb{Q}$ of $\mathbb{T}_y^3$ to $\mathbb{S}^3$ is such that $\mathbb{P}^{-1} \circ \mathbb{Q} : \mathbb{T}_y^3 \to \mathbb{T}_w^3$ has the coordinate expression $(u, v, w) \to (u/w, v/w, 1/w)$.

For $m \leq k$, the following expression defines uniquely, the functions involved:

$$\alpha^c(p,w) = \sum w^j \alpha^c_j(p) \ + \ o(|w|^m), \, j = 0, \, 1, \, 2, ..., m. \tag{4}$$

In the algebraic case ($k = a(n)$) studied in [6], $\alpha^c = \sum w^{n-h}\alpha_h$, $h = 0, 1, 2, .., n$, where the obvious correspondence $\alpha_h = \alpha^c_{n-h}$ holds.

The end points of $A(\alpha)$, $E(\alpha)$, are contained in $E(\alpha^c) = \{\alpha^c_0(p) = 0\}$.

At a *regular end point* p of $E(\alpha)$ it will be required that α^c_0 has a regular zero, i.e. one with non-vanishing derivative i.e. $\nabla\alpha^c_0(p) \neq 0$. At regular end points, the end locus is oriented by the positive unit normal $\nu(\alpha) = \nabla\alpha^c_0/|\nabla\alpha^c_0|$. This defines the positive unit tangent vector along $E(\alpha)$, given at p by $\tau(\alpha)(p) = p\wedge\nu(\alpha)(p)$. An end point is called *critical* if it is not regular.

A regular point $p \in E(\alpha)$ is called a *biregular end point* of $A(\alpha)$ if the geodesic curvature, k_g, of the curve $E(\alpha)$ at p, considered as a spherical curve, is different from zero; it is called an *inflexion end point* if k_g is equal to zero.

When the surface $A(\alpha)$ is regular at infinity, clearly $E(\alpha) = E(\alpha^c) = \{\alpha^c_0(p) = 0\}$.

The analysis in sections 2 and 3 will prove that there is a natural *extension* $F_c(\alpha) = (F_{c1}(\alpha), F_{c2}(\alpha))$ of the net $(\mathbb{P}(F_1(\alpha)), \mathbb{P}(F_2(\alpha))$ to $A_c(\alpha) = \{\alpha^c = 0\}$, as a net of class C^{k-2}, whose singularities in $E(\alpha)$ are located at the *inflexion* and *critical* end points of $A(\alpha)$. This is done by means of special charts used to extend the quadratic differential equations that define $(\mathbb{P}(F_1(\alpha)), \mathbb{P}(F_2(\alpha))$ to a full neighborhood in $A_c(\alpha)$ of the arcs of biregular ends. The differential equations are then extended to a full neighborhood of the singularities. See Lemma 2, for regular ends, and Lemma 4, for critical ends.

The main contribution of this paper consists in the resolution of singularities of the extended differential equations, under suitable genericity hypotheses on α^c. This is done in sections 2 and 3. It leads to eight patterns of principal nets at end points. Two of them – elliptic and hyperbolic inflexions– have also been studied in the case of algebraic surfaces [6].

§2. Principal Nets at Regular End Points

Lemma 1. *Let p be a regular end point of $A(\alpha)$, $\alpha = \alpha^c \circ \mathbb{P}$. Then there is a mapping $\overline{\alpha}$ of the form*

$\overline{\alpha}(u,w) = (x(u,w), y(u,w), z(u,w)),\ w > 0$, *defined by*

$$x(u,w) = \frac{u}{w}, \qquad y(u,w) = \frac{h(u,w)}{w}, \qquad z(u,w) = \frac{1}{w}. \tag{5}$$

which parametrizes the surface $A(\alpha)$ *near* p*, with*

$$\begin{aligned} h(u,w) =& k_0 w + \frac{1}{2}au^2 + buw + \frac{1}{2}cw^2 \\ &+\frac{1}{6}(a_{30}u^3 + 3a_{21}u^2 w + 3a_{12}uw^2 + a_{03}w^3) \\ &+\frac{1}{24}(a_{40}u^4 + 4a_{31}u^3 w + 6a_{22}u^2 w^2 \\ &+4a_{13}uw^3 + a_{04}w^4) + h.o.t \end{aligned} \tag{6}$$

Proof. With no lost of generality, assume that the regular end point p is located at $(0,1,0,0)$, the unit tangent vector to the regular end curve is $\tau = (1,0,0,0)$ and the positive normal vector is $\nu = (0,0,1,0)$. Take orthonormal coordinates u, v, w along $\tau, \nu, \omega = (0,0,0,1)$ on the tangent space, $\mathbb{T}^3_y$ to $\mathbb{S}^3$ at p. Then the composition $\mathbb{P}^{-1} \circ \mathbb{Q}$ writes as $x = u/w,\ y = v/w,\ z = 1/w$.

Clearly the surface $A(\alpha^c)$ near p can be parametrized by the central projection into $\mathbb{S}^3$ of the graph of a C^k function of the form $v = h(u,w)$ in $\mathbb{T}^3_y$, with $h(0,0) = 0$ and $h_u(0,0) = 0$. This means that the surface $A(\alpha)$, with $\alpha = \alpha^c \circ \mathbb{P}^{-1}$ can be parametrized in the form (5) with h as in (6).

Q.E.D.

Lemma 2. *The differential equation (3) in the chart* $\overline{\alpha}$ *of Lemma 1, multiplied by* $w^8\sqrt{EG - F^2}$*, extends to a full domain of the chart* (u,w) *to one given by*

$$\begin{aligned} &Ldw^2 + Mdudw + Ndu^2 = 0, \\ L =& -b - a_{21}u - a_{12}w - (c + a_{22})uw \\ &-(b + \frac{1}{2}a_{31})u^2 - \frac{1}{2}a_{13}w^2 + h.o.t. \\ M =& -a - a_{30}u - a_{21}w - \frac{1}{2}(2a + a_{40})u^2 \\ &-a_{31}uw + \frac{1}{2}(2c - a_{22})w^2 + h.o.t. \\ N =& w[au + bw + a_{30}u^2 + 2a_{21}uw + a_{12}w^2 + h.o.t.] \end{aligned} \tag{7}$$

where the coefficients are of class C^{k-2}.

Proof. The coefficients of first fundamental form of $\overline{\alpha}$ in (5) and (6) are given by:

$$\begin{aligned} E(u,w) =& \frac{1+h_u^2}{w^2} \\ F(u,w) =& \frac{h_u(wh_w - h) - u}{w^3} \\ G(u,w) =& \frac{1+u^2+(wh_w-h)^2}{w^4} \end{aligned}$$

The coefficients of the second fundamental form of $\overline{\alpha}$ are:

$$\begin{aligned} e(u,w) =& \frac{h_{uu}}{w^4\sqrt{EG-F^2}}, \\ f(u,w) =& \frac{h_{uw}}{w^4\sqrt{EG-F^2}}, \\ g(u,w) =& \frac{h_{ww}}{w^4\sqrt{EG-F^2}} \end{aligned}$$

where $e = [\overline{\alpha}_{uu}, \overline{\alpha}_u, \overline{\alpha}_w]/|\overline{\alpha}_u \wedge \overline{\alpha}_w|$, $f = [\overline{\alpha}_{uw}, \overline{\alpha}_u, \overline{\alpha}_w]/|\overline{\alpha}_u \wedge \overline{\alpha}_w|$ and $g = [\overline{\alpha}_{ww}, \overline{\alpha}_u, \overline{\alpha}_w]/|\overline{\alpha}_u \wedge \overline{\alpha}_w|$.

The differential equation of curvature lines (3) is given by $Ldw^2 + Mdudw + Ndu^2 = 0$, where $L = Fg - Gf$, $M = Eg - Ge$ and $N = Ef - Fe$.

These coefficients, after multiplication by $w^8\sqrt{EG-F^2}$, keeping the same notation, give the expressions in (7). Q.E.D.

The differential equation (7) is non-singular, i.e., defines a regular net of transversal curves if $a \neq 0$. This will be seen in item a) of next proposition. Calculation expresses a as a non-trivial factor of k_g.

The singularities of equation (7) arise when $a = 0$; they will be resolved in item b), under the genericity hypothesis $a_{30}b \neq 0$.

Proposition 1. *Let $\overline{\alpha}$ be as in Lemma 1. Then the end locus is parametrized by the regular curve $v = h(u,w)$, $w = 0$.*

a) *At a biregular end point, i.e., regular and non inflexion, $a \neq 0$, the principal net is as illustrated in Fig. 1, left.*

b) *If p is an inflexion, bitransversal end point, i.e., $\beta(p) = a_{30}b \neq 0$, the principal net is as illustrated in Fig. 1, hyperbolic $\beta < 0$, center, and elliptic $\beta > 0$, right.*

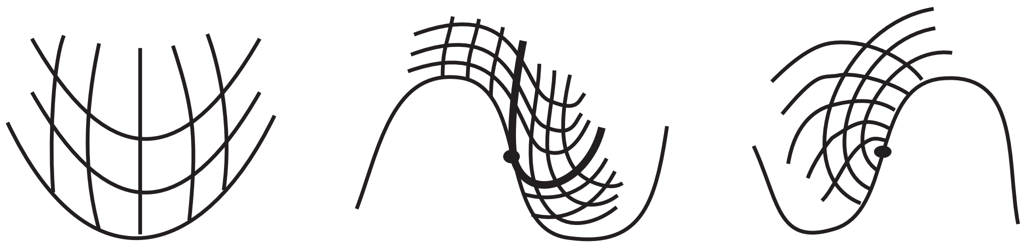

Fig. 1. Curvature lines near regular end points: biregular (left) and inflexions: hyperbolic (center) and elliptic (right).

Proof. Consider the implicit differential equation

$$\mathcal{F}(u,w,p) = -(b + a_{21}u + a_{12}w + h.o.t.)p^2 \quad -(a + a_{30}u + a_{21}w + h.o.t.)p \quad +w(au + bw + h.o.t.) = 0. \tag{8}$$

The Lie-Cartan line field tangent to the surface $\mathcal{F}^{-1}(0)$ is defined by $X = (\mathcal{F}_p, p\mathcal{F}_p, -(\mathcal{F}_u + p\mathcal{F}_w))$ in the chart $p = dw/du$ and $Y = (q\mathcal{F}_q, \mathcal{F}_q, -(q\mathcal{F}_u + \mathcal{F}_w))$ in the chart $q = du/dw$. Recall that the integral curves of this line field projects to the solutions of the implicit differential equation (8).

If $a \neq 0$, $\mathcal{F}^{-1}(0)$ is a regular surface, $X(0) = (-a, 0, 0) \neq 0$ and $Y(0) = (b, a, 0)$. So by the Flow Box theorem the two principal foliations are regular and transversal near 0. This ends the proof of item a).

If $a = 0$, $\mathcal{F}^{-1}(0)$ is a quadratic cone and $X(0) = 0$. Direct calculation shows that

$$DX(0) = \begin{pmatrix} -a_{30} & -a_{21} & -2b \\ 0 & 0 & 0 \\ 0 & 0 & a_{30} \end{pmatrix}$$

Therefore 0 is a saddle point with non zero eigenvalues $-a_{30}$ and a_{30} and the associated eigenvectors are $e_1 = (1, 0, 0)$ and $e_2 = (b, 0, -a_{30})$.

The saddle separatrix tangent to e_1 is parametrized by $w = 0$ and has the following parametrization $(s, 0, 0)$. The saddle separatrix tangent to e_2 has the following parametrization:

$$u(s) = s + O(s^3),\ w(s) = -\frac{a_{30}}{b}\frac{s^2}{2} + O(s^3),\ p(s) = -\frac{a_{30}}{b}s + O(s^3).$$

If $a_{30}b < 0$ the projection $(u(s), w(s))$ is contained in the semiplane $w \geq 0$. As the saddle separatrix is transversal to the plane $\{p = 0\}$ the phase portrait of X is as shown in the Fig. 2 below. The projections of the integral curves in the plane (u, w) shows the configurations of the principal lines near the inflexion point.

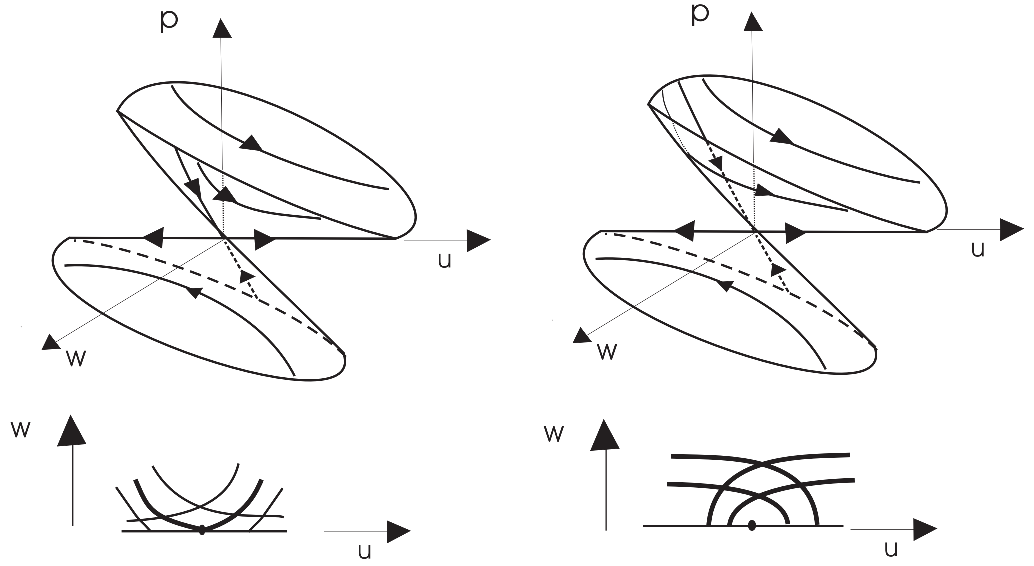

Fig. 2. Phase portrait of X near the singular point of saddle type

Q.E.D.

Proposition 2. *Let $\overline{\alpha}$ be as in Lemma 1. Suppose that, contrary to the hypothesis of Proposition 1, $a = 0$, $a_{30} = 0$, but $ba_{40} \neq 0$ holds.*

The differential equation (7) of the principal lines in this case has the coefficients given by:

$$
\begin{aligned}
L(u,w) =& -[b + a_{21}u + a_{12}w + \frac{1}{2}(2b + a_{31})u^2 \\
& + (c + a_{22})uw + \frac{1}{2}a_{13}w^2 + h.o.t.] \\
M(u,w) =& -[a_{21}w + \frac{1}{2}a_{40}u^2 \\
& + a_{31}uw + \frac{1}{2}(a_{22} - 2c)w^2 + h.o.t.] \\
N(u,w) =& w^2(b + 2a_{21}u + a_{12}w + h.o.t.)
\end{aligned}
\tag{9}
$$

The principal net is as illustrated in Fig. 3.

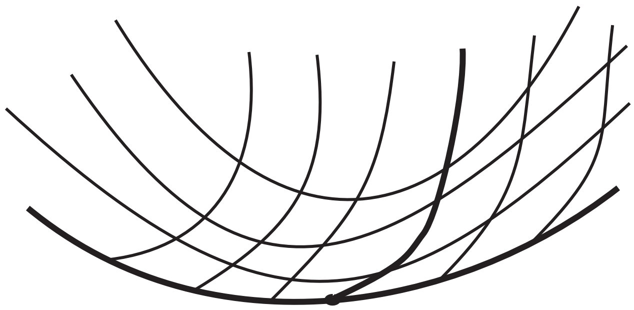

Fig. 3. Curvature lines near a hyperbolic-elliptic inflexion end point

Proof. From equation (7) it follows the expression of equation (9) is as stated. In a neighborhood of 0 this differential equation factors in to the product of two differential forms $X_+(u,w) = A(u,w)dv - B_+(u,w)du$ and $X_-(u,w) = A(u,w)dv - B_-(u,w)du$, where $A(u,w) = 2L(u,w)$ and $B_\pm(u,w) = M(u,w) \pm \sqrt{(M^2-4LN)(u,w)}$. The function A is of class C^{k-2} and the functions $B_\pm$ are Lipschitz. Assuming $a_{40} > 0$, it follows that $A(0) = -2b \neq 0$, $B_-(u,0) = 0$ and $B_+(u,0) = a_{40}u^2 + h.o.t.$ In the case $a_{40} < 0$ the analysis is similar, exchanging B_- with B_+.

Therefore, outside the point 0, the integral leaves of X_+ and X_- are transversal. Further calculation shows that the integral curve of X_+ which pass through 0 is parametrized by $(u, -\frac{a_{40}}{6b}u^3 + h.o.t.)$.

This shows that the principal foliations are extended to regular foliations which however fail to be a net a single point of cubic contact. This is illustrated in Fig. 3 in the case $a_{40}/b < 0$. The case $a_{40}/b > 0$ is the mirror image of Fig. 3. Q.E.D.

Proposition 3. *Let $\overline{\alpha}$ be as in Lemma 1. Suppose that, contrary to the hypothesis of Proposition 1, $a = 0$, $b = 0$, but $a_{30} \neq 0$ holds.*

The differential equation of the principal lines in this chart is given by:

$$\begin{aligned} &-[a_{21}u + a_{12}w + \frac{1}{2}a_{31}u^2 + (c + a_{22})uw + \frac{1}{2}a_{13}w^2 + h.o.t.]dw^2 \\ (10)\quad &-[a_{30}u + a_{21}w + \frac{1}{2}a_{40}u^2 + a_{31}uw + \frac{1}{2}(a_{22} - 2c)w^2 + h.o.t.]dudw \\ &+w[(a_{30}u^2 + 2a_{21}uw + a_{12}w^2) + h.o.t.]du^2 = 0. \end{aligned}$$

a) *If $(a_{21}^2 - a_{12}a_{30}) < 0$ the principal net is as illustrated in Fig. 4 (left).*

b) *If* $(a_{21}^2 - a_{12}a_{30}) > 0$ *the principal net is as illustrated in Fig. 4(right).*

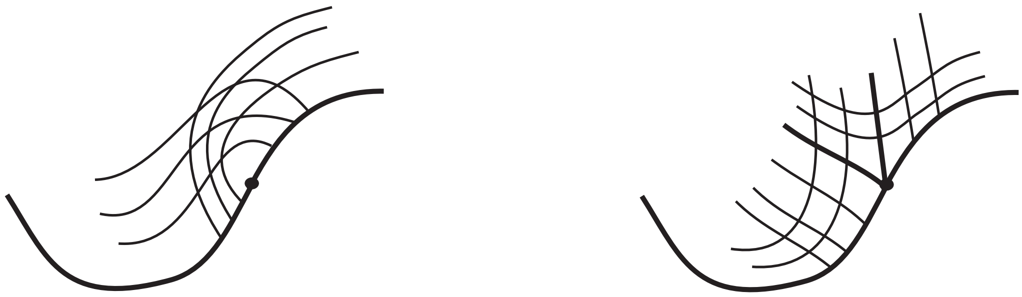

Fig. 4. Curvature lines near an umbilic-inflexion end point

Proof. Consider the Lie-Cartan line field defined by

$$X = (\mathcal{F}_p, p\mathcal{F}_p, -(\mathcal{F}_u + p\mathcal{F}_w))$$

on the singular surface $\mathcal{F}^{-1}(0)$, where

$$\begin{aligned}\mathcal{F}(u,w,p) = &- [a_{21}u + a_{12}w + h.o.t.]p^2 - [a_{30}u + a_{21}w + h.o.t.]p \\ &+ w[(a_{30}u^2 + 2a_{21}uw + a_{12}w^2) + h.o.t.] = 0.\end{aligned}$$

The singularities of X along the projective line (axis p) are given by the polynomial equation $p(a_{30} + 2a_{21}p + a_{12}p^2) = 0$. So X has one, respectively, three singularities, according to $a_{21}^2 - a_{12}a_{30}$ is negative, respectively positive. In both cases all the singular points of X are hyperbolic saddles and so, topologically, in a full neighborhood of 0 the implicit differential equation (10) is equivalent to a Darbouxian umbilic point D_1 or to a Darbouxian umbilic point of type D_3. See Fig. 5 and [14, 17].

In fact,

$$DX(0,0,p) = \begin{pmatrix} -2a_{21}p - a_{30} & -2a_{12}p - a_{21} & 0 \\ -p(2a_{21}p + a_{30}) & -p(2a_{12}p + a_{21}) & 0 \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

where, $A_{31} = p((c+a_{22})p^2+2a_{31}p+a_{40})$, $A_{32} = p[a_{13}p^2+(2a_{22}-c)p+a_{31}]$ and $A_{33} = 4a_{21}p + a_{30} + 3a_{12}p^2$.

The eigenvalues of $DX(0,0,p)$ are $\lambda_1(p) = -(a_{30} + 3a_{21}p + 2a_{12}p^2)$, $\lambda_2(p) = 4a_{21}p + a_{30} + 3a_{12}p^2$ and $\lambda_3 = 0$.

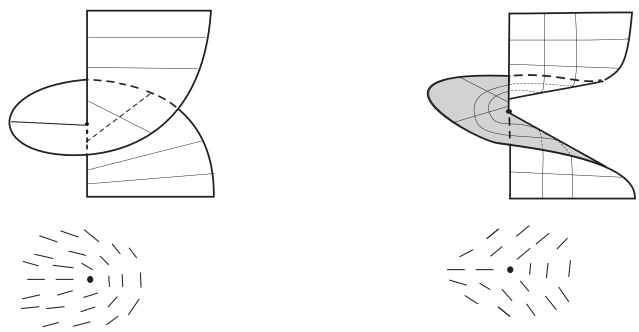

Fig. 5. Resolution of a singular point by a Lie-Cartan line field

Let p_1 and p_2 be the roots of $r(p) = a_{30} + 2a_{21}p + a_{12}p^2 = 0$.
Therefore, $\lambda_1(p_i) = a_{21}p_i + a_{30}$ and $\lambda_2(p_i) = -2(a_{21}p_i + a_{30})$.
As

$$r(-\frac{a_{30}}{a_{21}}) = \frac{a_{30}(a_{12}a_{30} - a_{21}^2)}{a_{21}^2} \neq 0,$$

it follows that $\lambda_1(p_i)\lambda_2(p_i) < 0$ and $\lambda_1(0)\lambda_2(0) = -a_{30}^2 < 0$. So the singularities of X are all hyperbolic saddles. If $a_{21}^2 - a_{12}a_{30} < 0$, X has only one singular point $(0,0,0)$. If $a_{21}^2 - a_{12}a_{30} > 0$, X has three singular points $(0,0,0)$, $(0,0,p_1)$ and $(0,0,p_2)$.

In the first case in a full neighborhood of $(0,0)$ the principal foliations have the topological type of a D_1 Darbouxian umbilic point. In the region $w > 0$ the behavior is as shown in Fig. 4 (left). In the second case the principal foliations have the topological type of a D_3 Darbouxian umbilic point and so the behavior in the finite region $w > 0$ is as shown in Fig. 4 (right). Q.E.D.

§3. Principal Nets at Critical End Points

Let p be a critical end point of the surface $A(\alpha)$, $\alpha = \alpha^c \circ \mathbb{P}$. Without lost of generality assume that the point p is located at $(0,1,0,0)$ and that the surface $\alpha^c = 0$ is given by the graph of a function $w = h(u,v)$, where h vanishes together with its first partial derivatives at $(0,0)$ and the u and v are the principal axes of the quadratic part of its second order jet.

Through the central projection $\mathbb{Q}$, the coordinates (u,v,w) can be thought to be orthonormal in the tangent space $\mathbb{T}_p^3$ to $\mathbb{S}^3$ at p, with w along $\omega = (0,0,0,1)$, u along $(1,0,0,0)$ and v along $(0,1,0,0)$.

Lemma 3. *Let p be a critical end point of the surface $A(\alpha)$, $\alpha = \alpha^c \circ \mathbb{P}$. Then there is a mapping $\overline{\alpha}$ of the form*

$$\overline{\alpha}(u,v) = (x(u,v), y(u,v), z(u,v))$$

defined by

$$x(u,v) = \frac{u}{h(u,v)}, \quad y(u,v) = \frac{v}{h(u,v)}, \quad z(u,v) = \frac{1}{h(u,v)} \tag{11}$$

which parametrizes the surface $A(\alpha)$ near p. The function h is as follows.

i) *If p is a definite critical point of h, then*

$$\begin{aligned} h(u,v) =& (a^2u^2 + b^2v^2) + \frac{1}{6}(a_{30}u^3 + 3a_{21}u^2v + 3a_{12}uv^2 + a_{03}v^3) \\ &+ \frac{1}{24}(a_{40}u^4 + 6a_{31}u^3v + 4a_{22}u^2v^2 + 6a_{13}uv^3 + a_{04}v^4) + h.o.t. \end{aligned} \tag{12}$$

ii) *If p is a saddle critical point of h, then*

$$\begin{aligned} h(u,v) =& (-au + v)v + \frac{1}{6}(a_{30}u^3 + 3a_{21}u^2v + 3a_{12}uv^2 + a_{03}v^3) \\ &+ \frac{1}{24}(a_{40}u^4 + 6a_{31}u^3v + 4a_{22}u^2v^2 + 6a_{13}uv^3 + a_{04}v^4) + h.o.t. \end{aligned} \tag{13}$$

Proof. The map $x = u/w,\ y = v/w,\ z = 1/w$ from $\mathbb{T}^3_y$ to $\mathbb{T}^3_w$, expresses the composition $\mathbb{P}^{-1} \circ \mathbb{Q}$.

Therefore the surface $A(\alpha)$, with $\alpha = \alpha^c \circ (\mathbb{P})^{-1}$ can be parametrized with the functions x, y, z as is stated in equation (11).

The function h takes the form given in equation (12) if it is definite positive. If it is a non degenerate saddle, after a rotation of principal axes, h can be written in the form given in equation (13). Q.E.D.

Lemma 4. *The differential equation (3) in the chart $\overline{\alpha}$ of Lemma 3, multiplied by $h^8\sqrt{EG - F^2}$, extends to a full domain of the chart (u,v) to one given by*

$$\begin{aligned} &Ldw^2 + Mdudw + Ndu^2 = 0, \\ L =& h^8\sqrt{EG - F^2}(Fg - Gf), \\ M =& h^8\sqrt{EG - F^2}(Eg - Ge), \\ N =& h^8\sqrt{EG - F^2}(Ef - Fe). \end{aligned} \tag{14}$$

where the coefficients are of class C^{k-2}. Here (E, F, G) and (e, f, g) are the coefficients of the first and second fundamental forms of the surface in the chart $\overline{\alpha}$.

Proof. The first fundamental form of the surface parametrized by $\overline{\alpha}$, equation (11), in Lemma 3 is given by:

$$\begin{aligned} E(u,v) =& \frac{(h-uh_u)^2+(v^2+1)h_u^2}{h^4} \\ F(u,v) =& \frac{-h(uh_u+vh_v)+(u^2+v^2+1)h_uh_v}{h^4} \\ G(u,v) =& \frac{(h-vh_v)^2+(u^2+1)h_v^2}{h^4} \end{aligned}$$

The coefficients of the second fundamental form are given by :

$$\begin{aligned} e(u,v) =& -\frac{h_{uu}}{h^4\sqrt{EG-F^2}} \\ f(u,v) =& -\frac{h_{uv}}{h^4\sqrt{EG-F^2}} \\ g(u,v) =& -\frac{h_{vv}}{h^4\sqrt{EG-F^2}} \end{aligned}$$

where $e = [\overline{\alpha}_{uu}, \overline{\alpha}_u, \overline{\alpha}_v]/|\overline{\alpha}_u \wedge \overline{\alpha}_v|$, $f = [\overline{\alpha}_{uv}, \overline{\alpha}_u, \overline{\alpha}_v]/|\overline{\alpha}_u \wedge \overline{\alpha}_v|$ and $g = [\overline{\alpha}_{vv}, \overline{\alpha}_u, \overline{\alpha}_v]/|\overline{\alpha}_u \wedge \overline{\alpha}_v|$.

Therefore the differential equation of curvature lines, after multiplication by $h^8|\overline{\alpha}_u \wedge \overline{\alpha}_v|$ is as stated. Q.E.D.

3.1. Differential Equation of Principal Lines around a Definite Critical End Point

Proposition 4. *Suppose that 0 is a critical point of h given by equation (12), with $a > 0$, $b > 0$ (local minimum).*

In polar coordinates $u = br\cos\theta$, $v = ar\sin\theta$ the differential equation (14) is given by $Ldr^2 + Mdrd\theta + Nd\theta^2 = 0$, where:

$$\begin{aligned} L =& l_0 + l_1 r + h.o.t, \\ M =& m_0 + m_1 r + h.o.t, \\ N =& r^2(\frac{1}{2}n_0 + \frac{1}{6}n_1 r + \frac{1}{24}n_2 r^2 + h.o.t.) \end{aligned} \tag{15}$$

with $m_0 = M(\theta, 0) = -8a^7b^7 \neq 0$ and the coefficients $(l_0, l_1, m_1,\ n_0, n_1, n_2)$ are trigonometric polynomials with coefficients depending on the fourth order jet of h at $(0,0)$, expressed in equations (16) to (19).

Proof. Introducing polar coordinates $u = br\cos\theta$, $v = ar\sin\theta$ in the equation (14), where h is given by equation (12), it follows that the differential equation of curvature lines near the critical end point 0, is given by $Ldr^2 + Mdrd\theta + Nd\theta^2 = 0$, where: $m_0 = M(\theta,0) = -8a^7b^7$, $N(\theta,0) = 0$, and $\frac{\partial N}{\partial r}(\theta,0) = 0$.

The Taylor expansions of L, M and N are as follows:

$$\begin{aligned} L =& l_0 + l_1 r + h.o.t, \\ M =& m_0 + m_1 r + h.o.t, \\ N =& r^2(n_0/2 + n_1 r/6 + n_2 r^2/24 + h.o.t.) \end{aligned}$$

After a long calculation, corroborated by computer algebra, it follows that:

$$\begin{aligned} l_0 = &= 2a^5b^5[a_{30}b^3\cos^2\theta\sin\theta + a_{21}ab^2(2\cos\theta - 3\cos^3\theta) \\ &+ a_{12}a^2b(\sin\theta - 3\cos^2\theta\sin\theta) + a_{03}a^3(\cos^3\theta - \cos\theta)] \end{aligned} \tag{16}$$

$$\begin{aligned} n_0 =& 4b^5a^5[(-3a_{21}b^2a + a_{03}a^3)\cos^3\theta \\ &+ (a_{30}b^3 - 3a_{12}ba^2)\sin\theta\cos^2\theta \\ &+ (-a_{03}a^3 + 2a_{21}b^2a)\cos\theta + 4ba^2a_{12}\sin\theta] \end{aligned} \tag{17}$$

$$m_1 = -4b^5a^5[(a_{30}b^3 + a_{12}ba^2)\cos\theta + (a_{21}b^2a + a_{03}a^3)\sin\theta] \tag{18}$$

(19)

$$
\begin{aligned}
n_2 =& -2a^3b^3[(a^3b^4(24a_{30}a_{13}+6a_{03}a_{40}+108a_{21}a_{22}+72a_{12}a_{31})\\
&+6a^7a_{03}a_{04}-18b^2a^5(a_{21}a_{04}+4a_{12}a_{13}+2a_{03}a_22)-6ab^6(3a_{21}a_{40}\\
&+4a_{30}a_{31}))\cos^7\theta+(a^4b^3(6a_{04}a_{30}+72a_{21}a_{13}+108a_{12}a_{22}\\
&+24a_{03}a_{31})+6a_{40}b^7a_{30}-18b^5a^2(a_{40}a_{12}+4a_{21}a_{31}+2a_{30}a_{22})\\
&-6a^6b(4a_{03}a_{13}+3a_{12}a_{04}))\sin\theta\cos^6\theta+(24a^7b^2a_{03}+72b^6a^3a_{21}\\
&-b^4a^5(24a_{03}+72a_{21})+ab^6(50a_{30}a_{31}+36a_{21}a_{40})\\
&-a^3b^4(10a_{03}a_{40}+198a_{21}a_{22}+46a_{30}a_{13}+126a_{12}a_{31})\\
&+b^2a^3(18a_{12}^2a_{21}-6a_{21}^2a_{03}-12a_{30}a_{12}a_{03})+b^4a(3a_{30}^2a_{03}\\
&+6a_{30}a_{12}a_{21}-9a_{21}^3)+a^5b^2(54a_{03}a_22+114a_{12}a_{13}+30a_{21}a_{04})\\
&+a^5(3a_{03}^2a_{21}-3a_{12}^2a_{03})-8a_{03}a^7a_{04})\cos^5\theta\\
&+(a^6b(22a_{03}a_{13}+18a_{12}a_{04})+a^4b^5(72a_{12}+24a_{30})-10a_{40}b^7a_{30}\\
&-a^4b^3(90a_{21}a_{13}+126a_{12}a_{22}+8a_{04}a_{30}+26a_{03}a_{31})\\
&+a^2b^5(54a_{30}a_22+24a_{40}a_{12}+102a_{21}a_{31})+a^2b^3(12a_{30}a_{03}a_{21}\\
&+6a_{12}^2a_{30}-18a_{21}^2a_{12})-24b^7a^2a_{30}+3b^5(a_{21}^2a_{30}-a_{30}^2a_{12})\\
&-a^4b(3a_{03}^2a_{30}+6a_{21}a_{12}a_{03}-9a_{12}^3)-72a^6b^3a_{12})\sin\theta\cos^4\theta\\
&+(-2a_{03}a^7a_{04}-24a^7b^2a_{03}+6a^5(a_{12}^2a_{03}-a_{03}^2a_{21})-ab^6(16a_{21}a_{40}\\
&+24a_{30}a_{31})-a^5b^2(12a_{21}a_{04}+18a_{03}a_22+48a_{12}a_{13})\\
&+a^5b^4(24a_{03}+96a_{21})+a^3b^2(15a_{30}a_{12}a_{03}+12a_{21}^2a_{03}-27a_{12}^2a_{21})\\
&+a^3b^4(4a_{03}a_{40}+120a_{21}a_22+70a_{12}a_{31}+26a_{30}a_{13})\\
&+ab^4(6a_{21}^3-3a_{30}^2a_{03}-3a_{30}a_{12})a_{21}-96b^6a^3a_{21}))\cos^3\theta+(48a^6a_{21}b^3\\
&+(4a_{13}a_{03}+2a_{21}a_{04})a^6b+(9a_{21}a_{03}a_{21}+3a_{03}^2a_{30}-12a_{21}^3)a^4b\\
&-(48a_{21}+24a_{30})a^4b^5+(6a_{03}a_{31}+34a_{13}a_{21}+48a_{21}a_{22}\\
&+2a_{04}a_{30})a^4b^3+24b^7a^2a_{30}-(36a_{31}a_{21}+18a_{22}a_{30}+6a_{40}a_{21})a^2b^5\\
&+(9a_{21}^2a_{21}-9a_{30}a_{03}a_{21})a^2b^3)\sin\theta\cos^2\theta\\
&+(4a_{03}a^7a_{04}+(-3a_{12}^2a_{03}+3a_{03}^2a_{21})a^5\\
&+6a_{13}b^2a^5a_{12}-36a^5b^4a_{21}+3(3a_{12}^2a_{21}-2a_{21}^2a_{03}-a_{12}a_{03}a_{30})b^2a^3\\
&-4(6a_{21}a_{22}+a_{13}a_{30}+3a_{12}a_{31})b^4a^3+36b^6a^3a_{21})\cos\theta\\
&+(12b^5a^4a_{12}-12a^6b^3a_{12}-(2a_{12}a_{04}+2a_{13}a_{03})a^6b\\
&-(4a_{13}a_{21}-6a_{12}a_{22})a^4b^3+(3a_{12}^3-3a_{12}a_{03}a_{21})a^4b)\sin\theta].
\end{aligned}
$$

$$
\begin{aligned}
l_1 =& -\frac{1}{6}a^3b^3[(18b^5aa_{21}a_{30}+18ba^5a_{12}a_{03}\\
&-6b^3a^3(a_{30}a_{03}+9a_{21}a_{12}))\cos^6\theta\\
&+(3a^6a_{03}^2-9(2a_{21}a_{03}+3a_{12}^2)b^2a^4\\
&+9(2a_{30}a_{12}+3a_{21}^2)b^4a^2-3b^6a_{30}^2)\sin\theta\cos^5\theta\\
&+(-15b^5aa_{21}a_{30}+9b^3a^3(9a_{21}a_{12}+a_{30}a_{03})\\
&-39ba^5a_{12}a_{03}-32b^3a^5a_{13}+32b^5a^3a_{31})\cos^4\theta\\
&+(48b^4a^4a_{22}-8b^6a^2a_{40}-6b^4a^2(3a_{21}^2+2a_{12}a_{30})-8b^2a^6a_{04}\\
&-6a^6a_{03}^2+12b^2a^4(2a_{21}a_{03}+3a_{12}^2))\sin\theta\cos^3\theta\\
&+(-24b^5a^3a_{31}+40b^3a^5a_{13}+24ba^5a_{12}a_{03}\\
&-3b^3a^3(9a_{21}a_{12}+a_{30}a_{03}))\cos^2\theta\\
&+(-12b^6a^4-3b^2a^4(3a_{12}^2+2a_{21}a_{03})+8b^2a^6a_{04}\\
&+12b^4a^6+3a^6a_{03}^2-24b^4a^4a_{22})\sin\theta\cos\theta\\
&-(3ba^5a_{12}a_{03}+8b^3a^5a_{13})]
\end{aligned}
\tag{20}
$$

$$
\begin{aligned}
n_1 =& a^3b^3[(18b^5aa_{21}a_{30}-6b^3a^3(9a_{21}a_{12}+a_{30}a_{03})\\
&+18ba^5a_{12}a_{03})\cos^6\theta\\
&+(9b^4a^2(2a_{12}a_{30}+3a_{21}^2)-9b^2a^4(2a_{21}a_{03}+3a_{12}^2)\\
&-3b^6a_{30}^2+3a^6a_{03}^2)\sin\theta\cos^5\theta\\
&+(16b^3a^5a_{13}+9b^3a^3(a_{30}a_{03}+9a_{21}a_{12})\\
&-16b^5a^3a_{31}-39b^5aa_{21}a_{30}-15ba^5a_{12}a_{03})\cos^4\theta\\
&+(-24b^4a^4a_{22}+6b^6a_{30}^2-12b^4a^2(3a_{21}^2+2a_{12}a_{30})+4b^2a^6a_{04}\\
&+4b^6a^2a_{40}+6b^2a^4(2a_{21}a_{03}+3a_{12}^2))\sin\theta\cos^3\theta\\
&+(-20b^3a^5a_{13}-6ba^5a_{12}a_{03}+12b^5a^3a_{31}\\
&+18b^5aa_{21}a_{30}-3b^3a^3(13a_{21}a_{12}+a_{30}a_{03}))\cos^2\theta\\
&+(-12b^6a^4-3a_{12}^2b^2a^4+12b^4a^4a_{22}+12b^4a^6\\
&+6b^4a^2(2a_{21}^2+a_{12}a_{30})-3a^6a_{03}^2-4b^2a^6a_{04})\sin\theta\cos\theta\\
&+(4b^3a^5a_{13}+6b^3a^3a_{21}a_{12}+3ba^5a_{12}a_{03})].
\end{aligned}
\tag{21}
$$

Q.E.D.

3.2. Principal Nets around a Definite Critical End Points

Proposition 5. *Suppose that p is an end critical point and consider the chart defined in Lemma 3 such that h is given by equation (12), with* $a > 0$, $b > 0$ *(local minimum). Then the behavior of curvature lines near p is the following.*

i) *One principal foliation is radial.*

ii) *The other principal foliation surrounds p and the associated return map* Π *is such that* $\Pi(0) = 0$, $\Pi'(0) = 1$, $\Pi''(0) = 0$, $\Pi'''(0) = 0$ *and* $\Pi''''(0) = \dfrac{\pi}{2^{10}a^5b^5}\Delta$, *where*

$$\begin{aligned}\Delta =&12(a_{30}a_{21} + 3a_{03}a_{30} - 5a_{12}a_{21})b^6a^4\\ &+12(5a_{12}a_{21} - a_{12}a_{03} - 3a_{03}a_{30})a^6b^4\\ &+4(3a_{04}a_{30}a_{21} + a_{13}a_{21}^2 + 10a_{31}a_{03}a_{21})a^4b^4\\ &-4(10a_{13}a_{12}a_{30} + 3a_{40}a_{12}a_{03} + a_{31}a_{12}^2)a^4b^4\\ &+4(a_{13}a_{03}^2 - a_{04}a_{12}a_{03})a^8 + 4(a_{40}a_{30}a_{21} - a_{31}a_{30}^2)b^8\\ &+3(a_{30}^3a_{03} + 2a_{30}a_{21}^3 - 3a_{30}^2a_{21}a_{12})b^6\\ &+3(3a_{12}a_{03}^2a_{21} - 2a_{12}^3a_{03} - a_{30}a_{03}^3)a^6\\ &+4[a_{03}(2a_{13}a_{21} - 3a_{04}a_{30} - 3a_{31}a_{03} + 12a_{22}a_{12})\\ &+5a_{04}a_{12}a_{21} - 13a_{13}a_{12}^2]a^6b^2\\ &+4[a_{30}(+3a_{13}a_{30} - 2a_{31}a_{12} + 3a_{40}a_{03} - 12a_{22}a_{21})\\ &-5a_{40}a_{21}a_{12} + 13a_{31}a_{21}^2]a^2b^6\\ &+9(a_{30}a_{21}^2a_{03} - 2a_{30}a_{21}a_{12}^2 + a_{12}a_{21}^3)a^2b^4\\ &+9(-a_{12}^3a_{21} - a_{30}a_{03}a_{12}^2 + 2a_{12}a_{21}^2a_{03})a^4b^2\\ &+12b^2a^8a_{12}a_{03} - 12b^8a^2a_{30}a_{21}\end{aligned}$$

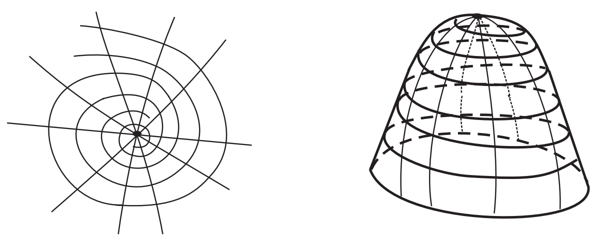

Fig. 6. Curvature lines near a definite focal critical end point, $\Delta > 0$.

Proof. Consider the implicit differential equation (15)

$$(l_0 + rl_1 + h.o.t)dr^2 + (m_0 + m_1 r + h.o.t)drd\theta$$
$$+r^2(n_0/2 + n_1 r/6 + n_2 r^2/24 + h.o.t.)d\theta^2 = 0.$$

As $m_0 = -8a^7b^7 \neq 0$ this equation factors in the product of two equations in the standard form, as follows.

$$\begin{aligned}\frac{dr}{d\theta} =& -\frac{1}{2}\frac{n_0}{m_0}r^2 + \frac{1}{6}\frac{3m_1 - n_0 + m_0 n_1}{m_0^2}r^3 \\ & -\frac{1}{24}\frac{(12m_1^2 n_0 + m_0^2 n_2 + 6l_0 n_0^2 - 6m_0 m_2 n_0 - 4m_0 m_1 n_1)}{m_0^3}r^4 \\ & +h.o.t. \\ =& \frac{1}{2}d_2(\theta)r^2 + \frac{1}{6}d_3(\theta)r^3 + \frac{1}{24}d_4(\theta)r^4 + h.o.t.\end{aligned} \tag{22}$$

$$\frac{d\theta}{dr} = -\frac{l_0}{m_0} + h.o.t. \tag{23}$$

The solutions of the nonsingular differential equation (23) defines the radial foliation.

Writing $r(\theta, h) := h + q_1(\theta)h + q_2(\theta)h^2/2 + q_3(\theta)h^3/6 + q_4(\theta)h^4/24 + h.o.t.$ as the solution of differential equation (22) it follows that:

$$\begin{aligned} q_1'(\theta) =& 0 \\ q_2'(\theta) =& d_2(\theta) = -\frac{n_0}{m_0} \\ q_3'(\theta) =& 3d_2(\theta)q_2(\theta) + d_3(\theta) \\ q_4'(\theta) =& 3d_2(\theta)q_2(\theta)^2 + 4d_2(\theta)q_3(\theta) + 6d_3(\theta)q_2(\theta) + d_4(\theta)\end{aligned} \tag{24}$$

As $q_1(0) = 0$ it follows that $q_1(\theta) = 0$. Also $q_i(0) = 0,\ i = 2,\ 3,\ 4$. So it follows that $q_2(\theta) = -\int_0^\theta \frac{n_0}{m_0}d\theta$. From the expression of n_0, an odd polynomial in the variables $c = \cos\theta$ and $s = \sin\theta$, it follows that $q_2(2\pi) = 0$ and therefore $\Pi'(0) = 1,\ \ \Pi''(0) = 1$.

Now, $q_3(\theta) = \int_0^\theta [q_2(\theta)q_2'(\theta) + d_3(\theta)]d\theta$.

Therefore $q_3(\theta) = \frac{1}{2}q_2^2(\theta) + \int_0^\theta d_3(\theta)d\theta$.

So,

$$\Pi'''(0) = q_3(2\pi) = \int_0^{2\pi} d_3(\theta)d\theta.$$

A long calculation, confirmed by algebraic computation, shows that $q_3(2\pi) = 0$.

Integrating the last linear equation in (24), it follows that:

$$q_4(\theta) = 3q_2(\theta)^3 + 4q_2(\theta)\int_0^\theta d_3(\theta)d\theta + 2\int_0^\theta q_2(\theta)d_3(\theta)d\theta + \int_0^\theta d_4(\theta)d\theta.$$

Therefore,

$$\Pi''''(0) = q_4(2\pi) = 2\int_0^{2\pi} q_2(\theta)d_3(\theta)d\theta + \int_0^{2\pi} d_4(\theta)d\theta.$$

Integration of the right hand member, corroborated by algebraic computation, gives $\Pi''''(0) = \frac{\pi}{2^{10}a^5b^5}\Delta$. This ends the proof. Q.E.D.

Remark 1. *When $\Delta \neq 0$ the foliation studied above spirals around p. The point is then called a focal definite critical end point.*

3.3. Principal Nets at Saddle Critical End Points

Let p be a saddle critical point of h as in equation (13) with the finite region defined by $h(u,v) > 0$.

Then the differential equation (14) is given by

(25)

$$\begin{aligned}
&Ldv^2 + Mdudv + Ndu^2 = 0,\\
L(u,v) =& -a^3u^2 + 2a^2uv - 3aa_{12}uv^2 + (2a^2a_{12} - 3aa_{21} - 2a_{30})u^2v\\
&+(aa_{30} + 2a^2a_{21})u^3 + (2a_{12} + aa_{03})v^3 + h.o.t.\\
M(u,v) =& -2a^2v^2 + (4a_{30} - a^2a_{12})uv^2 + (a^2a_{21} - 2aa_{30})u^2v\\
&+a^2a_{30}u^3 + (2aa_{12} - a^2a_{03} + 4a_{21})v^3 + h.o.t.\\
N(u,v) =& av^2[a^2 - 2(aa_{21} + a_{30})u - 2(aa_{12}u + a_{21})v] + h.o.t.
\end{aligned}$$

Proposition 6. *Suppose that p is a saddle critical point of the surface represented by $w = h(u,v)$ as in Lemma 3. Then the behavior of the extended principal foliations in the region ($h(u,v) \geq 0$), near p, is the following.*

- i) *If $aa_{30}(a_{03}a^3 + 3aa_{21} + 3a^2a_{12} + a_{30}) > 0$ then the curvatures of both branches of $h^{-1}(0)$ at p have the same sign and the behavior is as in Fig. 7, left -even case.*
- ii) *If $aa_{30}(a_{03}a^3 + 3aa_{21} + 3a^2a_{12} + a_{30}) < 0$ then the curvatures of both branches of $h^{-1}(0)$ at p have opposite signs and the behavior is as in the Fig. 7, right - odd case.*

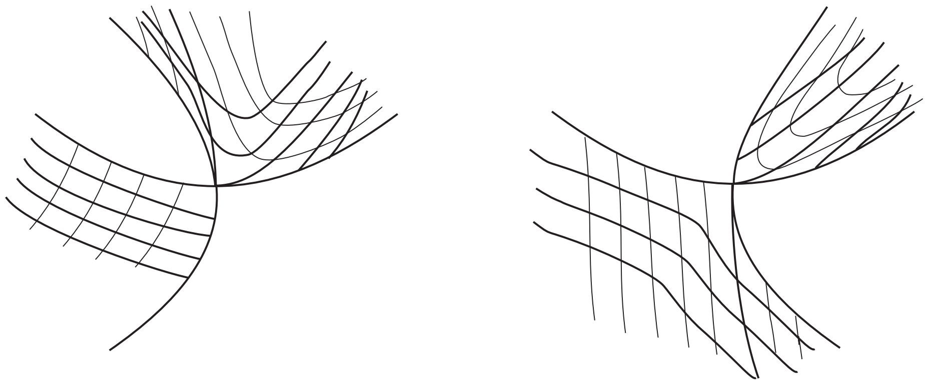

Fig. 7. Curvature lines near saddle critical end point: even case, left, and odd case, right.

Proof. In order to analyze the behavior of the principal lines near the branch of $h^{-1}(0)$ tangent to $v = 0$ consider the projective blowing-up $u = u, v = uw$.

The differential equation $Ldv^2 + Mdudv + Ndu^2 = 0$ defined by equation (25) is, after some simplification, given by:

$$(-\frac{1}{4}a_{30}^2 u + aa_{30}w + O(2))du^2 + (aa_{30}u - 2a^2 w + O(2))dwdu \\ -(a^2 u + O(2))dw^2 = 0. \tag{26}$$

To proceed consider the resolution of the singularity $(0, 0)$ of equation (26) by the Lie-Cartan line field $X = (q\mathcal{G}_q, \mathcal{G}_q, -(q\mathcal{G}_u + \mathcal{G}_w)), \quad q = \frac{du}{dw}$. Here $\mathcal{G}$ is

$$\mathcal{G} = (-\frac{1}{4}a_{30}^2 u + aa_{30}w + O(2))q^2 + (aa_{30}u - 2a^2 w + O(2))q - (a^2 u + O(2)).$$

The singularities of X, contained in axis q (projective line), are the solutions of the equation $q(2a - a_{30}q)(6a - a_{30}q) = 0$.

Also,

$$DX(0, 0, q) = \begin{pmatrix} \frac{1}{2}(qa_{30}(2a - a_{30}q)) & -2qa(a - a_{30}q) & 0 \\ \frac{1}{2}(a_{30}(2a - a_{30}q)) & -2a(a - a_{30}q) & 0 \\ 0 & 0 & A_{33} \end{pmatrix}$$

where $A_{33} = 3a^2 - 4aa_{30}q + \frac{3}{4}a_{30}^2 q^2$.

The eigenvalues of $DX(0, 0, q)$ are $\lambda_1(q) = -2a^2 + 3aa_{30}q - \frac{1}{2}a_{30}^2 q^2$, $\lambda_2(q) = 3a^2 - 4aa_{30}q + \frac{3}{4}a_{30}^2 q^2$ and $\lambda_3(q) = 0$.

Therefore the non zero eigenvalues of $DX(0)$ are $-2a^2$ and $3a^2$. At $q_1 = \frac{2a}{a_{30}}$ the eigenvalues of $DX(0,0,q_1)$ are $2a^2$ and $-2a^2$. Finally at $q_2 = \frac{6a}{a_{30}}$ the eigenvalues of $DX(0,0,q_2)$ are $6a^2$ and $-2a^2$.

As a conclusion of this analysis we assert that the net of integral curves of equation (26) near $(0,0)$ is the same as one of the generic singularities of quadratic differential equations, well known as the Darbouxian D_3 or a tripod, [14, 17]. See Fig. (5).

Now observe that the curvature at 0 of the branch of $h^{-1}(0)$ tangent to $v=0$ is precisely $k_1 = \dfrac{a_{30}}{a}$ and that $h^{-1}(0)\backslash\{0\}$ is solution of equation (25). So, after the blowing down, only one branch of the invariant curve $v = \dfrac{a_{30}}{6a}u^2 + O(3)$ is contained in the finite region $\{(u,v) : h(u,v) > 0\}$.

Analogously, the analysis of the behavior of the principal lines near the branch of $h^{-1}(0)$ tangent to $v = au$ can be reduced to the above case. To see this perform a rotation of angle $\tan\theta = a$ and take new orthogonal coordinates $\bar{u}$ and $\bar{v}$ such that the axis $\bar{u}$ coincides with the line $v = au$.

The curvature at 0 of the branch of $h^{-1}(0)$ tangent to $v = au$ is

$$k_2 = -\frac{a_{03}a^3 + 3aa_{21} + 3a^2a_{12} + a_{30}}{3a}.$$

Performing the blowing-up $v = v, \quad u = sv$ in the differential equation (25) we conclude that it factors in two transversal regular foliations.

Gluing the phase portraits studied so far and doing their blowing down, the net explained below is obtained.

The finite region $(h(u,v) > 0)$ is formed by two sectorial regions R_1, with $\partial R_1 = C_1 \cup C_2$ and R_2 with $\partial R_2 = L_1 \cup L_2$. The two regular branches of $h^{-1}(0)$ are given by $C_1 \cup L_1$ and $C_2 \cup L_2$.

If $k_1k_2 < 0$ – odd case – then one region, say R_1, is convex and ∂R_1 is invariant for one extended principal foliation and ∂R_2 is invariant for the other one. In each region, each foliation has an invariant separatrix tangent to the branches of $h^{-1}(0)$. See Fig. 7, right.

If $k_1k_2 > 0$ – even case – then in a region, say R_1, the extended principal foliations are equivalent to a trivial ones, i.e., to $dudv = 0$, with C_1 being a leaf of one foliation and C_2 a leaf of the other one. In the region R_2 each extended principal foliation has a hyperbolic sector, with separatrices tangent to the branches of $h^{-1}(0)$ as shown in Fig. 7, left. Here $C_2 \cup L_1$ are leaves of one principal foliation and $C_1 \cap L_2$ are leaves of the other one. Q.E.D.

§4. Concluding Comments and Related Problems

We have studied here the simplest patterns of principal curvature lines at end points, as the supporting smooth surfaces tend to infinity in $\mathbb{R}^3$, following the paradigm established in [6] to describe the structurally stable patterns for principal curvature lines escaping to infinity on algebraic surfaces.

We have recovered here –see Proposition 1 – the main results of the structurally stable inflexion ends established in [6] for algebraic surfaces: namely the hyperbolic and elliptic cases.

In the present context a surface $A(\alpha^c)$ with $\alpha^c \in \mathcal{A}_c^k$ is said to be *structurally stable* at a singular end point p if the C^s, topology with $s \leq k$ if the following holds. For any sequence of functions $\alpha_n^c \in \mathcal{A}_c^k$ converging to α^c in the C^s topology, there is a sequence p_n of end points of $A(\alpha_n^c)$ converging to p such that the extended principal nets of $\alpha_n = \alpha_n^c \circ \mathbb{P}$, at these points, are topologically equivalent to extended principal net of $\alpha = \alpha^c \circ \mathbb{P}$, at p.

Recall (see [6]) that two nets $N_i\; i = 1, 2$ at singular points $p_i\; i = 1, 2$ are topologically equivalent provided there is a homeomorphism of a neighborhood of p_1 to a neighborhood of p_2 mapping the respective points and leaves of the respective foliations to each other.

The analysis in Proposition 1 makes clear that the hyperbolic and elliptic inflexion end points are also structurally stable in the C^3 topology for defining α^c functions in the space $\mathcal{A}_c^k$, $k \geq 4$.

We have studied also six new cases –see Propositions 2 to 6 – which represent the simplest patterns where the structural stability conditions fail.

The lower smoothness class C^k for the validity of the analysis in the proofs of these propositions is as follows. In Propositions 2 and 6 we must assume $k \geq 4$. In Proposition 5, clearly $k \geq 5$ must hold.

In each of these cases it is not difficult to describe partial aspects of possible topological changes –bifurcation phenomena– under small perturbations of the defining functions α^c.

However it involves considerably technical work to provide the full analysis of bifurcation diagrams of singular end points and their global effects in the principal nets.

We recall here that the study of the bifurcations of principal nets away from end points, i.e., in compact regions was carried out in [13], focusing the umbilic singular points. There was also established the connection between umbilic codimension one singularities and their counterparts in critical points of functions and the singularities of vector fields, following the paradigm of first order structural stability in the sense of

Andronov and Leontovich [1], generalized and extended by Sotomayor [20]. Grosso modo this paradigm aims to characterize the structurally stable singularities under small perturbations inside the space of non-structurally stable ones.

To advance an idea of the bifurcations at end points, below we will suggest pictorially the local bifurcation diagrams in the three regular cases studied so far.

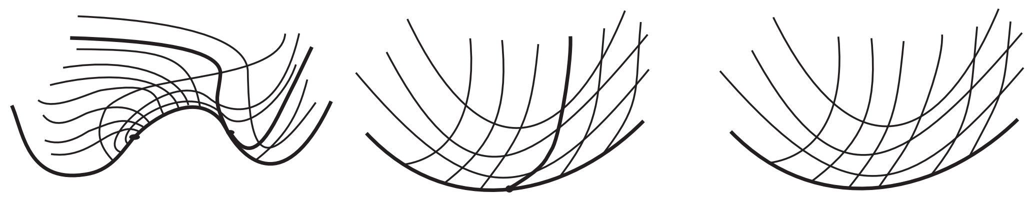

Fig. 8. Bifurcation Diagram of Curvature lines near regular end points: elimination of hyperbolic and elliptic inflexion points

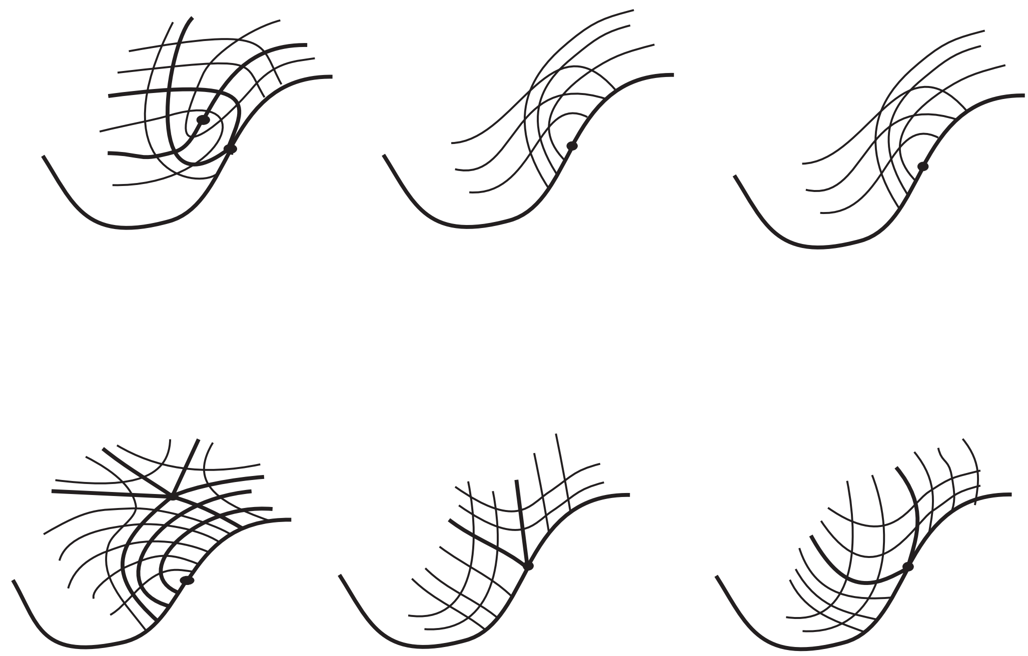

Fig. 9. Bifurcation Diagram of Curvature lines near umbilic-inflexion end points. Upper row: D_1 umbilic - hyperbolic inflexion. Lower row: D_3 umbilic - elliptic inflexion.

The description of the bifurcations in the critical cases, however, is much more intricate and will not be discussed here.

The full analysis of the non-compact bifurcations as well as their connection with first order structural stability will be postponed to a future paper.

Concerning the study of end points, see also [16], where Gutierrez and Sotomayor studied the behavior of principal nets on constant mean curvature surfaces, with special analysis of their periodic leaves, umbilic and end points. However, the patterns of behavior for this class of surfaces is non-generic in the sense of the present work.

We conclude proposing the following problem.

Problem 1. Concerning the case of the focal critical end point, we propose to the reader to provide a conceptual analysis and a proof of Proposition 5, avoiding long calculations and the use of Computer Algebra.

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