Homogeneous cosmologies and the Maupertuis-Jacobi principle

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A recent work showing that homogeneous and isotropic cosmologies involving scalar fields are equivalent to the geodesics of certain effective manifolds is generalized to the nonminimally coupled and anisotropic cases. As the Maupertuis-Jacobi principle in classical mechanics, such a result permits us to infer some dynamical properties of cosmological models from the geometry of the associated effective manifolds, allowing us to go a step further in the study of cosmological dynamics. By means of some explicit examples, we show how the geometrical analysis can simplify considerably the dynamical analysis of cosmological models.

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I. INTRODUCTION

The dynamical system approach has a prominent role in modern cosmology. Any candidate to a realistic cosmological model must exhibit certain qualitative dynamical behaviors as, among others, robustness against small perturbations (to avoid fine-tuning problems, see [1] for a recent review), specific kinds of phase-space attractors (to describe, for instance, the recent accelerated expansion phase of the Universe [2,3]), and some classes of solutions with determined intermediate time behavior (for instance, tracker solutions [3,4], candidates to explain the "cosmic coincidence" or "Why Now?" problem). In a recent work [5], Townsend and Wohlfarth presented an interesting and promising tool for the dynamical analysis of cosmological models. Essentially, they show that the equations of motion of homogeneous and isotropic cosmological models with multiple minimally coupled scalar fields, self-interacting through an arbitrary potential, do indeed correspond to the geodesic equations of a certain effective Lorentzian manifold. Such correspondence, closely related to the Maupertuis-Jacobi principle of classical mechanics [6], permits us to infer some of the dynamical properties of a given cosmological model from the underlying geometry of the associated effective manifold, allowing one to go a step further in the dynamical analysis of realistic cosmological models. Townsend and Wohlfarth, indeed, apply their own formalism to identify asymptotic accelerated expansion phases in models with exponential potentials [5]. As we will see, such geometrical considerations can simplify considerably the dynamical study of cosmological models.

In this paper, we show that the work of Townsend and Wohlfarth can be largely extended, allowing the inclusion of nonminimally coupled scalar fields and also the anisotropic case, opening many new possibilities for the dynamical characterization of such cosmological models. Our main results and some applications are presented in the next sections, after the following brief introduction to the Maupertuis-Jacobi principle.

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The Maupertuis-Jacobi principle

The Maupertuis-Jacobi principle in classical mechanics [6] establishes that the dynamics of a given system can be viewed as geodesic motions in an associated Riemannian manifold. In order to recall it briefly, let us consider a classical mechanical system with N degrees of freedom described by the Lagrangian

$$L(q,\dot{q}) = \tfrac{1}{2}g_{ij}(q)\dot{q}^i\dot{q}^j - V(q), \qquad (1)$$

where $i, j = 1, 2, \ldots, N$, the dot stands for differentiation with respect to the time t, and g_{ij} is the Riemannian metric on the N-dimensional configuration space $\mathcal{M}$. All the quantities here are assumed to be smooth. The Euler-Lagrange equations of (1) can be written as

$$\ddot{q}^{\,i} + \Gamma^i_{jk}\dot{q}^j\dot{q}^k = -g^{ij}\partial_j V(q), \qquad (2)$$

where Γ^i_{jk} is the Levi-Civita connection for the metric g_{ij}. The Hamiltonian of the system described by (1)

$$\mathcal{H}(q,p) = \tfrac{1}{2}g^{ij}(q)p_ip_j + V(q), \qquad (3)$$

with $p_i = g_{ij}\dot{q}^j$, is obviously a constant of motion, namely, the total energy. For a fixed energy E, the trajectories in the $2N$-dimensional phase-space $(q^i; p_j)$ are confined to the hypersurface $E = \frac{1}{2}g^{ij}p_ip_j + V(q)$. The admissible region for the trajectories in the configuration space is given by

$$\mathcal{D}_E = \{q \in \mathcal{M}: V(q) \le E\}, \qquad (4)$$

which can be either bounded or unbounded, connected or not. If the potential V has no critical points on the boundary $\partial\mathcal{D}_E$, then $\partial\mathcal{D}_E$ is a $N-1$-dimensional submanifold of $\mathcal{M}$. We can see easily that if a trajectory reaches the boundary at a point q_0, its velocity at this point vanishes and the trajectory approach or depart from q_0 perpendicularly to $\partial\mathcal{D}_E$. In particular, there is no allowed trajectory along the boundary.

One can show that the equations of motion (2) are, in the interior of $\mathcal{D}_E$, fully equivalent to the geodesic equation of the "effective" Riemannian geometry on $\mathcal{M}$ defined from the Jacobi metric [6]

$$\hat{g}_{ij}(q) = 2(E - V(q))g_{ij}(q). \tag{5}$$

The geodesic equation in question is given by

$$\hat{\nabla}_u u = \frac{d^2q^i}{ds^2} + \hat{\Gamma}^i_{jk}\frac{dq^j}{ds}\frac{dq^k}{ds} = 0, \tag{6}$$

where $u = dq^i/ds$ is the tangent vector along the geodesic, $\hat{\nabla}$ and $\hat{\Gamma}^i_{jk}$ are, respectively, the covariant derivative and the Levi-Civita connection for the Jacobi metric $\hat{g}_{ij}$, and s is a parameter along the geodesic obeying

$$\frac{ds}{dt} = 2(E - V(q)). \tag{7}$$

As for any classical topic, there is a vast literature on the Maupertuis-Jacobi principle. We notice only that, motivated by the celebrated result due to Anosov stating that the geodesic flow in a compact manifold with all sectional curvatures negative at every point is chaotic [6], the Maupertuis-Jacobi principle has been recently invoked for the study of chaotic dynamics. (See, for instance, [7] and the references therein).

Townsend and Wohlfarth consider homogeneous and isotropic cosmological models with N self-interacting minimally coupled scalar fields ϕ^α taking their values in a Riemannian target space endowed with a metric $G_{\alpha\beta}$. The corresponding action is

$$S = \int d^Dx\sqrt{-g}(R - g^{ij}G_{\alpha\beta}(\phi)\partial_i\phi^\alpha\partial_j\phi^\beta - 2V(\phi)), \tag{8}$$

where R stands for the scalar curvature of the D-dimensional spacetime metric g_{ij}. By considering the Friedman-Robertson-Walker homogeneous and isotropic metric

$$ds^2 = -dt^2 + a^2(t)d\Sigma^2_\kappa, \tag{9}$$

where Σ_κ represents the $(D-1)$-dimensional spatial sections of constant curvature κ, they showed that the equations of motion associated to the action (8) do indeed correspond to the geodesics of a certain effective Jacobi (pseudo)metric on a Lorentzian manifold. For the spatially flat case ($\kappa = 0$), for instance, the geodesics corresponding to the equations of motion derived from (8) are timelike, null, or spacelike according, respectively, if $V > 0$, $V = 0$, or $V < 0$. Such results have been applied already to the dynamical study of the models governed by actions of the type (8), see [8]. For $\kappa \neq 0$, Townsend and Wohlfarth had to introduce some higher dimensional effective manifolds to accomplish their analysis. As we will see, such higher dimensional manifolds are unnecessary; all values of κ eventually can be treated in the same framework.

Applications of the Maupertuis-Jacobi principle to the field equations obtained from Hilbert-Einstein-like actions have also a long history. Nonhomogeneous and anisotropic cases were considered in [9]. Applications involving distinct differential spaces instead of differential manifolds were discussed in [10]. Nonminimally coupled fields, however, have not been considered so far.

II. NONMINIMALLY COUPLED HOMOGENEOUS AND ISOTROPIC COSMOLOGIES

Nonminimally coupled scalar fields are quite common in cosmology. In particular, they have been invoked recently to describe dark energy (see [11,12], and the references therein, for, respectively, models using conformal coupling and more general ones; see also [1,3]). The nonminimal coupling generalization of (8) we consider here is

$$S = \int d^4x\sqrt{-g}(F(\phi)R - g^{ij}G_{\alpha\beta}(\phi)\partial_i\phi^\alpha\partial_j\phi^\beta - 2V(\phi)). \tag{10}$$

Integrating by parts the action (10), taking into account that $R = 6\dot{H} + 12H^2 + 6\kappa/a^2$ for the metric (9), with $H = \dot{a}/a$, we obtain the following Lagrangian:

$$L(a, \dot{a}, \phi_\alpha, \dot{\phi}^\alpha) = a^3\Big(-6H^2F(\phi) - 6H\dot{\phi}^\alpha\partial_\alpha F(\phi) + G_{\alpha\beta}(\phi)\dot{\phi}^\alpha\dot{\phi}^\beta + 6\kappa\frac{F(\phi)}{a^2} - 2V(\phi)\Big) \tag{11}$$

on the $(N+1)$-dimensional configuration space spanned by (a, ϕ^α). Although we will discuss here only the $D = 4$ case, the D-dimensional one follows straightforwardly by using that $R = 2(D-1)\dot{H} + D(D-1)H^2 + (D-1)\times(D-2)\kappa/a^2$ in order to derive (11). By introducing the following Lorentzian metric

$$G_{AB}(a, \phi^\alpha) = \begin{pmatrix} -6aF & -3a^2\partial_\beta F \\ -3a^2\partial_\alpha F & a^3G_{\alpha\beta}\end{pmatrix} \tag{12}$$

on the configuration space (upper case roman indices run over $1\ldots N+1$), the Lagrangian (11) can be cast in the form

$$L(\phi^A, \dot{\phi}^A) = G_{AB}(\phi)\dot{\phi}^A\dot{\phi}^B - 2V_{\rm eff}(\phi^A), \tag{13}$$

where $\phi^A = (a, \phi^\alpha)$ and

$$V_{\rm eff}(\phi^A) = a^3V(\phi) - 3\kappa aF(\phi). \tag{14}$$

It is evident the similarity between (1) and (13), provided that $\det G_{AB} \neq 0$, which we discuss below. Before, let us recall that our manipulations imply that all solutions of the Euler-Lagrange of (10) are also solutions of the Euler-Lagrange equations of (11), but not the converse. Einstein equations form a constrained system, the solutions of (10) correspond, indeed, to a subset of the solutions of (11), as one can realize by considering the Hamiltonian associated to (11)

$$\mathcal{H}(\phi^A, \pi_A) = G^{AB}\pi_A\pi_B + 2V_{\rm eff}(\phi^A), \tag{15}$$

which is a constant of motion, where $\pi_A = G_{AB}\dot{\phi}^B$. The Euler-Lagrange equations of (10), on the other hand, implies that $\mathcal{H} = 0$ (the so-called energy constraint). Hence, we must bear in mind that the relevant solutions of our original problem correspond, in fact, to the $\mathcal{H} = 0$ subset of the dynamics governed by (11).

A proper interpretation of (12) as a metric tensor does require the essential assumption $\det G_{AB} \neq 0$. In order to grasp its actual meaning and implications, let us restrict ourselves to the $N = 1$ and $G = 1$ case, for which

$$\det G_{AB} = -6a^4(F(\phi) + \tfrac{3}{2}(F'(\phi))^2). \quad (16)$$

The vanishing of the quantity between parenthesis is known for a long time to be associated with the existence of some severe and unavoidable dynamical singularities [13] (see also [14]), which render the associated cosmological model unphysical. Here, the vanishing of (16) also leads generically to an actual geometrical singularity. Thus, the assumption of $\det G_{AB} \neq 0$ assures that the model in question is free of these singularities, geometrical and dynamical, a basic requirement for any realistic model. This can be viewed as the first application of the formalism; we will return to this issue in the last section. Note that one of the most interesting peculiarities of the conformal coupling ($N = 1$, $F(\phi) = 1 - \phi^2/6$) is that it can always evade the $\det G_{AB} = 0$ singularity, since $F(\phi) + \frac{3}{2} \times (F'(\phi))^2 = 1$ for such case.

The Lagrangian (13) is ready to be considered under the Maupertuis-Jacobi principle. For $V_{\rm eff} = 0$, the Euler-Lagrange equations of (13) already correspond to geodesics of (12). Moreover, from the energy constraint $\mathcal{H} = 0$, one has that they in fact correspond to timelike geodesics. For $V_{\rm eff} \neq 0$, let us introduce the Jacobi (pseudo)metric

$$\hat{G}_{AB} = 2|V_{\rm eff}|G_{AB}. \quad (17)$$

One can see that, in accordance with the Maupertuis-Jacobi principle, the Euler-Lagrange equations of (13) correspond to geodesics of (17), parameterized by s, see Eq. (7). From the energy constraint, one gets

$$\hat{G}_{AB}\frac{d\phi^A}{ds}\frac{d\phi^B}{ds} = -\frac{V_{\rm eff}}{|V_{\rm eff}|}, \quad (18)$$

implying that the geodesics are timelike and spacelike for, respectively, $V_{\rm eff} > 0$ and $V_{\rm eff} < 0$. All the results [5] that have motivated the present work can be obtained in a simpler way by setting $F = 1$. The analysis corresponding to the $\kappa = 0$ case has appeared already in [15].

III. THE ANISOTROPIC CASE

Our analysis can be extended in order to include also anisotropic models. Let us consider a model with action (10) and a Bianchi I type D-dimensional metric

$$ds^2 = -dt^2 + \sum_{i=1}^{D-1} a_i^2(t)dx^i, \quad (19)$$

for which the scalar curvature is given by

$$R = 2\bigg(\sum_{i=1}^{D-1}(\dot{H}_i + H_i^2) + \sum_{i=1,j>i}^{D-1} H_iH_j\bigg), \quad (20)$$

where $H_i = \dot{a}_i/a_i$. Again, by integrating the action (10) by parts, we obtain the following Lagrangian:

$$L = \bigg(\prod_{i=1}^{D-1} a_i\bigg)\bigg(-2F(\phi)\sum_{i=1,j>i}^{D-1} H_iH_j - 2\bigg(\sum_{i=1}^{D-1} H_i\bigg)\dot{\phi}^\alpha\partial_\alpha F(\phi) + G_{\alpha\beta}(\phi)\dot{\phi}^\alpha\dot{\phi}^\beta - 2V(\phi)\bigg) \quad (21)$$

on the $(N + D - 1)$-dimensional configuration space spanned by (a_i, ϕ^α). As in the previous section, we will restrict ourselves here to the $D = 4$ case; the D-dimensional one follows straightforwardly and without surprises. By introducing the following metric:

$$G_{AB} = \begin{pmatrix} 0 & -a_3F & -a_2F & -a_2a_3\partial_\beta F \\ -a_3F & 0 & -a_1F & -a_1a_3\partial_\beta F \\ -a_2F & -a_1F & 0 & -a_1a_2\partial_\beta F \\ -a_2a_3\partial_\alpha F & -a_1a_3\partial_\alpha F & -a_1a_2\partial_\alpha F & a_1a_2a_3G_{\alpha\beta} \end{pmatrix} \quad (22)$$

on the configuration space (upper case roman indices run now over $1\ldots N+3$), the Lagrangian (21) can be cast in the form given by (13), with $\phi^A = (a_1, a_2, a_3, \phi^\alpha)$ and

$$V_{\rm eff}(\phi^A) = a_1a_2a_3V(\phi^\alpha). \quad (23)$$

Note that the (pseudo) metric (22) has signature $(3, N)$. As in the isotropic case, the determinant $\det G_{AB}$ plays a central role in the geometrical and dynamical analyses. For $N = 1$ and $G = 1$, it reads

$$\det G_{AB} = -2(a_1a_2a_3F(\phi))^2\bigg(F(\phi) + \frac{3}{2}(F'(\phi))^2\bigg). \quad (24)$$

In this case, in addition to the singularity present for the isotropic models, we have also a singularity for $F(\phi) = 0$. Such anisotropic singularity has been described already in [14] in a dynamical analysis, where it is shown that it can be indeed used to rule out large classes of models. We notice, nevertheless, that the dynamical analysis of [14] is considerably more involved than the geometrical analysis presented here.

As in the isotropic case, with the introduction of the metric (22) and the effective potential (23), the system is ready to be considered under the Maupertuis-Jacobi principle.

IV. DISCUSSION

The great value of the dynamical analysis comes from the possibility of ruling out large classes of models that are not viable from the theoretical point of view. As we have already said, any candidate to a realistic cosmological model must exhibit certain qualitative dynamical behaviors. For instance, the singularities corresponding to the condition $\det G_{AB} = 0$ render the associated cosmological model inviable. Geometrically, such singularities imply that no geodesic can be extended beyond the singular points, what would imply a kind of future singularity in the associated cosmological model. This is precisely the case of the isotropic singularity of Sec. II and the anisotropic one of Sec. III. However, we stress that the identification of such singularities in the dynamical analyses [13,14] is considerably more involved than in the present geometrical approach. For multiple scalar fields, the dynamical analysis is much harder (see, for instance, [16]), and the present geometrical approach can be even more useful.

The far most common dynamical analysis in cosmology is the stability classification of certain solutions. De Sitter asymptotically stable fixed points are particularly relevant to the description of the recent accelerated expansion phase of the Universe [3]. Such fixed points correspond to isotropic limits such that ϕ^α and H are constants, $\phi^A(s) = (a_0 e^{Ht(s)}, \phi_0^\alpha)$. The geometrical analysis presented here can help the identification of such points. For this purpose, let us consider the geodesic deviation equation, which governs the local tendency of nearby geodesics to converge or to diverge from each order

$$\dot{\phi}^A \nabla_A \dot{\phi}^B \nabla_B n^D = R_{ABC}{}^D n^A \dot{\phi}^B \dot{\phi}^C, \tag{25}$$

where $R_{ABC}{}^D$ is the curvature tensor of the Jacobi metric, and n^A is a vector orthogonal to $\dot{\phi}^A$, pointing to the direction of the deviation. Let us take, for instance, the same case of Sec. II, $N = 1$ and $G = 1$, with $V_{\rm eff}(\phi) > 0$ now. For the geodesic corresponding to a de Sitter fixed point, one has $\dot{\phi}^A = ((a_0 H/2V_{\rm eff})e^{Ht}, 0)$. The only nonvanishing component of (25) reads

$$\ddot{n}(s) = -R_{121}{}^2 n(s)(\dot{\phi}^{(1)})^2, \tag{26}$$

The stability of the de Sitter solution requires a bounded $n(s)$ for $s \to \infty$ and, hence, $R_{121}{}^2$ cannot be negative, eliminating, in this way, the possibility of finding out viable de Sitter fixed points inside regions where $R_{121}{}^2 < 0$. In the present case, one has

$$R_{121}{}^2 = \frac{3aF}{2V_{\rm eff}}\left(\frac{\Box V_{\rm eff}}{V_{\rm eff}} - \frac{\partial_a V_{\rm eff}\partial^a V_{\rm eff}}{V_{\rm eff}^2} - \frac{FF'' - \frac{1}{2}(F')^2}{a^3(F(\phi) + \frac{3}{2}(F'(\phi))^2)^2}\right). \tag{27}$$

For $F = 1$, we have $R_{121}{}^2 = 3V''(\phi_0)/(2a^5 V^2(\phi_0))$ on the fixed point ϕ_0, and the usual requirement of stability $V''(\phi_0) > 0$ is recovered. For more general cases, the criteria $R_{121}{}^2 > 0$ and <0 can be used to locate regions where de Sitter fixed points could and could not appear, respectively. We finish by noticing that our analysis is in perfect agreement with the comprehensive study of [17].

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