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Nonlinear Differential Equations and Applications NoDEA

Limit cycles of discontinuous piecewise differential systems formed by two rigid systems separated by a straight line

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Abstract. In this work we study the maximum number of limit cycles in some classes of discontinuous piecewise differential systems formed by two rigid systems separated by a straight line. These rigid systems are formed by a linear center plus a homogeneous polynomial of degree $2, 3, 4, 5, 6$. For these classes of piecewise differential systems we solve the extended 16th Hilbert problem.

Mathematics Subject Classification. Primary: 34C05.

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1. Introduction and statements of the main results

Following [4,12] a *center* is a singularity $p \in \mathbb{R}$ of a planar analytical differential system such that there exists an open neighborhood U of p such that all solutions in $U \backslash \{p\}$ are periodic. Denote by T_q the period of the periodic orbit through the point $q \in U \backslash \{p\}$. We say that p is an *isochronous center* if T_q is constant for $q \in U \backslash \{p\}$. Furthermore, an isochronous center is *uniform* or *rigid* if the angular velocity of the vector field is the same for all periodic orbits in $U \backslash \{p\}$. In other words, an isochronous center is rigid if in polar coordinates (r, θ) defined by $x = r\cos\theta$, $y = r\sin\theta$, it can be written as $dr/dt = G(r,\theta), d\theta/dt = k$, where $k \neq 0$ is constant. Besides, according to [3] a rigid differential system in the plane $\mathbb{R}^2$ having a focus or a center at the origin of coordinates can be written in the form

$$\dot{x} = \frac{dx}{dt} = -y + xF(x,y), \qquad \dot{y} = \frac{dx}{dt} = x + yF(x,y), \tag{1}$$

where $F(x,y)$ is a smooth real function. This system has only one singular point at the origin of coordinates.

For system (1), the center-focus problem, which consists of determining conditions that differentiate the cases in which the origin is a focus from the cases in which it is a center, is equivalent to the isochronicity problem [5]. The isochronicity problem has been studied by several authors due to its applications in physics, see [10,15]. Futhermore, nowadays, isochronicity has also been explored for discontinuous piecewise differential systems, considering the coupling of two or more rigid systems and investigating their dynamics, see [8,9].

In this paper we study the limit cycles of the piecewise differential systems separated by a straight line, that without loss of generality we take $y = 0$, and formed by the two rigid systems

$$\begin{aligned} \dot{x} &= -y + x(\lambda + P(x,y)) = X_m^1(x,y), \\ \dot{y} &= x + y(\lambda + P(x,y)) = X_m^2(x,y), \end{aligned} \tag{2}$$

where $P(x,y)$ is a homogeneous polynomial of degree $m-1$, and

$$\begin{aligned} \dot{x} &= -y + x(\lambda + Q(x,y)) = Y_n^1(x,y), \\ \dot{y} &= x + y(\lambda + Q(x,y)) = Y_n^2(x,y), \end{aligned} \tag{3}$$

where $Q(x,y)$ is a homogeneous polynomial of degree $n-1$. As usual the dot denotes derivative with respect to the time t. The interest in studying the limit cycles of discontinous piecewise differential systems has increased in the last years due to a big number of applications of these systems, see for instance the recent papers [2,6,11,13,14].

The famous 16th Hilbert problem asks for an upper bound for the maximum number of limit cycles that the polynomial vector fields in the plane of a given degree can exhibit, see [7]. For the moment this problem remains open. Here we solve the extension of the 16th Hilbert problem for some classes of piecewise differential systems separated by the straight line $y = 0$ and formed by the two rigid polynomial differential systems (2) and (3) of degree m and n, respectively.

Our main result is the following one.

Theorem 1. *Consider the piecewise differential systems separated by the straight line $y = 0$ and formed by the two rigid polynomial differential systems (2) and (3) of degree m and n respectively, with $m \le n$, where P and Q are homogeneous polynomials of degree $m-1$ and $n-1$, respectively. Then the piecewise differential system has*

(a) at most 1 limit cycle if $m = n = 2$ and the bound is reached;
(b) at most 2 limit cycles if $m = 2$ and $n = 3$ and the bound is reached;
(c) at most 3 limit cycles if $m = 2$ and $n = 4$
(d) at most 4 limit cycles if $m = 2$ and $n = 5$;
(e) at most 5 limit cycles if $m = 2$, and $n = 6$;
(f) at most 1 limit cycles if $m = n = 3$ and the bound is reached;
(g) at most 4 limit cycles if $m = 3$ and $n = 4$;
(h) at most 2 limit cycles if $m = 3$ and $n = 5$ and the bound is reached;
(i) at most 5 limit cycles if $m = 3$ and $n = 6$;

(j) *at most* 1 *limit cycles if* $m = 4$ *and* $n = 4$ *and the bound is reached;*
(k) *at most* 6 *limit cycles if* $m = 4$ *and* $n = 5$*;*
(l) *at most* 6 *limit cycles if* $m = 4$ *and* $n = 6$*;*
(m) *at most* 1 *limit cycles if* $m = n = 5$ *and the bound is reached;*
(n) *at most* 8 *limit cycles if* $m = 5$ *and* $n = 6$*.*

The rest of the paper is dedicated to prove Theorem 1.

2. Proof of theorem 1

If the degrees of the polynomials $P(x, y)$ and $Q(x, y)$ are ≥ 2, then the eigenvalues of the linear part of the differential systems (2) and (3) at the origin of coordinates are $\lambda \pm i$. So the origin is a hyperbolic focus if $\lambda \neq 0$, and it is a weak focus or a center if $\lambda = 0$.

Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{2}, \mathbf{2})$. Then the homogeneous polynomials $P(x, y)$ and $Q(x, y)$ have degree one, and we have the piecewise differential system separated by $y = 0$ and formed by the two rigid systems defined by the vector fields

$$X_2(x, y) = \begin{cases} X_2^1(x, y) = -y + x(\lambda + a_1 x + a_2 y), \\ X_2^2(x, y) = x + y(\lambda + a_1 x + a_2 y), \end{cases} \tag{4}$$

and

$$Y_2(x, y) = \begin{cases} Y_2^1(x, y) = -y + x(\lambda + b_1 x + b_2 y), \\ Y_2^2(x, y) = x + y(\lambda + b_1 x + b_2 y). \end{cases} \tag{5}$$

We write these two differential systems in polar coordinates (r, θ), where $x = r \cos \theta$ and $y = r \sin \theta$. Then system (4) becomes

$$\dot{r} = r\lambda + r^2(a_1 \cos \theta + a_2 \sin \theta), \qquad \dot{\theta} = 1.$$

or equivalently

$$r' = \frac{dr}{d\theta} = r\lambda + r^2(a_1 \cos \theta + a_2 \sin \theta).$$

Thus, the solution $r_1(\theta)$ of the differential equation $r'(\theta)$ such $r_1(0) = r_0$ is

$$r_1(\theta) = \frac{e^{\theta\lambda} r_0 (1 + \lambda^2)}{1 - a_2 r_0 + a_1 r_0 \lambda + \lambda^2 + e^{(\theta\lambda)} r_0 (a_2 - a_1 \lambda) \cos \theta - e^{(\theta\lambda)} r_0 (a_1 + a_2 \lambda) \sin \theta}. \tag{6}$$

Similarly system (5) in polar coordinates taken θ as the new independent variable writes

$$r' = \frac{dr}{d\theta} = r\lambda + r^2(b_1 \cos \theta + b_2 \sin \theta).$$

and its solution $r_2(\theta)$ such $r_2(\pi) = r_1$ is

$$r_2(\theta) = \frac{e^{\theta\lambda} r_1 (1 + \lambda^2)}{e^{\pi\lambda}(1 + b_2 r_1 - b_1 r_1 \lambda + \lambda^2) - e^{(\theta\lambda)} r_1 ((-b_2 + b_1 \lambda) \cos \theta + (b_1 + b_2 \lambda) \sin \theta)}. \tag{7}$$

The trajectory $r_1(\theta)$ of the vector field X_2 such that $r_1(0) = r_0$ intersects the straight line $y = 0$ at the point $(r_1(\pi), 0)$. Then since we want to concatenate the solution $r_1(\theta)$ with a trajectory of vector field Y_2, we consider the solution $r_2(\theta)$ such that $r_2(\pi) = r_1(\pi)$. Now we want that after a time π the solution $r_2(\theta)$ intersects the straight line $y = 0$ at r_0, in this way the piecewise differential system will have a periodic orbit. If this periodic orbit is isolated in the set of all periodic orbits of the piecewise differential system it will be a limit cycle.

In summary, for studying the limit cycles of the piecewise differential system formed by the vector fields X_2 and Y_2 we must solve the equation $r_2(2\pi) = r_0$ for r_0, i.e. the equation

$$\frac{e^{2\pi\lambda}\left(\lambda^2+1\right) r_0}{1+\lambda^2+r_0(1+e^{\pi\lambda})(a_1\lambda-a_2+e^{\pi\lambda}(b_2-b_1\lambda))} = r_0. \tag{8}$$

The solutions of this equation are

$$r_{0,1} = 0 \quad \text{and} \quad r_{0,2} = \frac{\left(1-e^{\pi\lambda}\right)\left(\lambda^2+1\right)}{a_2-a_1\lambda+e^{\pi\lambda}\left(b_1\lambda-b_2\right)}. \tag{9}$$

Since the origin is always an equilibrium point of the system, we only have the solution r_0^2 providing a periodic orbit. More precisely, if $\lambda = 0$, then $r_0^2 = 0$ and if $(a_2 - a_1\lambda + e^{\pi\lambda}(b_1\lambda - b_2)) = 0$, then r_0^2 does not exist. So, in order that r_0^2 be positive and provides a periodic orbit we must have $\left(1-e^{\pi\lambda}\right)(a_2 - a_1\lambda + e^{\pi\lambda}(b_1\lambda - b_2)) > 0$, and then since the piecewise differential system has only one periodic orbit, this is a limit cycle. Statement (a) of Theorem 1 is proved.
Case(m,n)=(2,3). Now the piecewise differential system is formed by the vector field X_2 defined in the previous case and the vector field (3) with $n = 3$, i.e.

$$Y_3(x,y) = \begin{cases} Y_3^1(x,y) = -\,y + x(\lambda + b_1x^2 + b_2xy + b_3y^2), \\ Y_3^2(x,y) = x + y(\lambda + b_1x^2 + b_2xy + b_3y^2). \end{cases} \tag{10}$$

This vector field in polar coordinates (10) provides the differential system

$$\dot{r} = r\lambda + \frac{1}{2}r^3(b_1 + b_3 + (b_1 - b_3)\cos(2\theta) + b_2\sin(2\theta)), \quad \dot{\theta} = 1, \tag{11}$$

or the equivalent differential equation

$$r' = \frac{dr}{d\theta} = r\lambda + \frac{1}{2}r^3(b_1 + b_3 + (b_1 - b_3)\cos(2\theta) + b_2\sin(2\theta)).$$

The solution $r_2(\theta)$ of this differential equation such that $r_2(\pi) = r_1$ is

$$r_2(\theta) = \frac{e^{\theta\lambda}\sqrt{2\lambda(1+\lambda^2)}}{\sqrt{E_{11}/r_1^2 + E_{12}}} \tag{12}$$

where

$$\begin{aligned} E_{11} &= e^{2\pi\lambda}((b_1+b_3)r_1^2 + (2 - b_2r_1^2)\lambda + 2b_1r_1^2\lambda^2 + 2\lambda^3), \\ E_{12} &= -e^{2\theta\lambda}((b_1+b_3)(1+\lambda^2) + \lambda(-b_2 + (b_1 - b_3)\lambda)\cos(2\theta) \\ &\qquad + \lambda(b_1 - b_3 + b_2\lambda)\sin(2\theta)). \end{aligned}$$

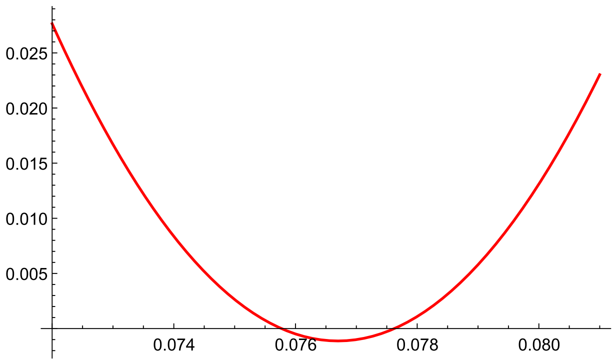

FIGURE 1. Real positive roots for case $(\mathbf{m},\mathbf{n})=(\mathbf{2},\mathbf{3})$

In this way, the limit cycles satisfy the equation $(r_2(2\pi))^2-r_0^2=0$, then we consider the numerator of the equation $(r_2(2\pi))^2-r_0^2=0$, and we obtain $\left(1+e^{\pi\lambda}\right)r_0^2p_1(r_0)$, being $p_1(r_0)$ the following polynomial of degree 2 in r_0

$$p_1(r_0)=(C_{0,1}+C_{1,1}r_0+C_{2,1}r_0^2), \tag{13}$$

where

$$\begin{aligned}
C_{0,1}=&2\left(e^{\pi\lambda}-e^{2\pi\lambda}+e^{3\pi\lambda}-1\right)\lambda\left(\lambda^2+1\right)^2,\\
C_{1,1}=&4\lambda\left(\lambda^2+1\right)(a_2-a_1\lambda),\\
C_{2,1}=&-\lambda\left(2a_1^2\left(e^{\pi\lambda}+1\right)\lambda^2-4a_1a_2\left(e^{\pi\lambda}+1\right)\lambda+2a_2^2\left(e^{\pi\lambda}+1\right)\right.\\
&\left.+b_2e^{2\pi\lambda}\left(e^{\pi\lambda}-1\right)\left(\lambda^2+1\right)\right)\\
&+b_1e^{2\pi\lambda}\left(e^{\pi\lambda}-1\right)\left(2\lambda^4+3\lambda^2+1\right)+b_3e^{2\pi\lambda}\left(e^{\pi\lambda}-1\right)\left(\lambda^2+1\right).
\end{aligned}$$

The polynomial $p_1(r_0)$ can have two real positive roots. For instance, for the values $\lambda=-1$, $a_1=10$, $a_2=15$, $b_1=14$, $b_2=12$ and $b_3=-2$ it has two real positive roots given by $r_0^1=0.0757703$ and $r_0^2=0.0776237$, see Figure 1. For these values of the variables and of the roots the denominator of the equation $(r_2(2\pi))^2-r_0^2$ is different from zero. So the piecewise differential system separated by the straight line $y=0$ with the vector fields X_2 and Y_3 can have two limit cycles. Statement (b) of Theorem 1 is proved.

Case $(\mathbf{m},\mathbf{n})=(\mathbf{2},\mathbf{4})$. Consider the piecewise differential system separated by the straight line $y=0$ and formed by vector field X_2 defined in (4) and the vector field (3) with $n=4$, i.e.

$$Y_4(x,y)=\begin{cases}Y_4^1(x,y)=&-y+x(\lambda+b_1x^3+b_2x^2y+b_3xy^2+b_4y^3),\\ Y_4^2(x,y)=&x+y(\lambda+b_1x^3+b_2x^2y+b_3xy^2+b_4y^3).\end{cases} \tag{14}$$

The vector field Y_4 provides in polar coordinates the differential system

$$\dot{r}_2=r\lambda+r^4(b_1\cos^3\theta+b_2\cos^2\theta\sin\theta+b_3\cos\theta\sin^2\theta+b_4\sin^3\theta),\quad \dot{\theta}=1,$$

or equivalently the differential equation

$$r' = \frac{dr}{d\theta} = r\lambda + r^4(b_1\cos^3\theta + b_2\cos^2\theta\sin\theta + b_3\cos\theta\sin^2\theta + b_4\sin^3\theta).$$

Then the solution $r_2(\theta)$ of this differential equation such that $r_2(\pi) = r_1$ is

$$r_2(\theta) = -\frac{2^{\frac{2}{3}}e^{\theta\lambda}r_1((1+\lambda^2)(1+9\lambda^2))^{\frac{1}{3}}}{E_2^{\frac{1}{3}}} \tag{15}$$

with

$$\begin{aligned}E_2 =& -4e^{3\pi\lambda}(1+\lambda^2)(1+9\lambda^2) - 4e^{3\pi\lambda}r_1^3(b_2+2b_4+3b_2\lambda^2-\lambda(7b_1+2b_3+9b_1\lambda^2))\\ &+ e^{3\theta\lambda}r_1^3\big(-3(b_2+3b_4-3(3b_1+b_3)\lambda)(1+\lambda^2)\cos\theta + (-b_2+b_4+b_1\lambda-b_3\lambda)\\ &(1+9\lambda^2)\cos 3\theta + 3(3b_1+b_3+3b_2\lambda+9b_4\lambda)(1+\lambda^2)\sin\theta\\ &+(b_1-b_3+b_2\lambda-b_4\lambda)(1+9\lambda^2)\sin 3\theta\big).\end{aligned}$$

Thus, the limit cycles satisfy the equation $(r_2(2\pi))^3 - r_0^3 = 0$. Then we consider the numerator of equation $(r_2(2\pi))^3 - r_0^3 = 0$ and we obtain the following polynomial of degree 3 in r_0

$$p_2(r_0) = r_0^3(C_{0,2} + C_{1,2}r_0 + C_{2,2}r_0^2 + C_{3,2}r_0^3), \tag{16}$$

where

$$\begin{aligned}C_{0,2} =& -(e^{6\pi\lambda}-1)(1+\lambda^2)^3(1+9\lambda^2),\\ C_{1,2} =& 3(1+e^{\pi\lambda})(-a_2+a_1\lambda)(1+\lambda^2)^2(1+9\lambda^2),\\ C_{2,2} =& 3(1+e^{\pi\lambda})^2(a_2-a_1\lambda)^2(1+\lambda^2)^2(1+9\lambda^2),\\ C_{3,2} =& e^{3\pi\lambda}\Big(\lambda\Big(\lambda^3\left(9a_1^3\lambda + b_2\left(3\lambda^2+7\right)+2b_4\right)+\lambda\left(a_1^3\lambda+5b_2+4b_4\right)\\ &-b_1\left(\lambda^2+1\right)^2\left(9\lambda^2+7\right)\\ &-2b_3\left(\lambda^2+1\right)^2\Big) - 3a_1^2a_2\lambda^2\left(9\lambda^2+1\right)+3a_1a_2^2\lambda\left(9\lambda^2+1\right)\\ &-\left(a_2^3\left(9\lambda^2+1\right)\right)+b_2+2b_4\Big)\\ &+3e^{\pi\lambda}\left(9\lambda^2+1\right)(a_1\lambda-a_2)^3+3e^{2\pi\lambda}\left(9\lambda^2+1\right)(a_1\lambda-a_2)^3\\ &+\left(9\lambda^2+1\right)(a_1\lambda-a_2)^3\\ &+e^{6\pi\lambda}\left(\lambda^2+1\right)^2\left(-\lambda\left(9b_1\lambda^2+7b_1+2b_3\right)+3b_2\lambda^2+b_2+2b_4\right).\end{aligned}$$

The polynomial $p_2(r_0)$ can have three real positive roots. For instance, for the values $\lambda = -1$, $a_1 = -512$, $a_2 = 2048$, $b_1 = b_2 = b_3 = 0$ and $b_4 = 414992$ it has three real positive roots given by $r_0^1 = 0.00124798$, $r_0^2 = 0.00124814$ and $r_0^3 = 0.00124832$, see Figure 2. With these values the denominator of the equation $(r_2(2\pi))^3 - r_0^3$ is different from zero. So the piecewise differential system separated by the straight line $y = 0$ with the vector fields X_2 and Y_4 can have three limit cycles. Statement (c) of Theorem 1 is proved.

Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{2}, \mathbf{5})$. We consider the piecewise differential system separated by straight line $y = 0$ and formed by two vector fields X_2 defined in (4) and

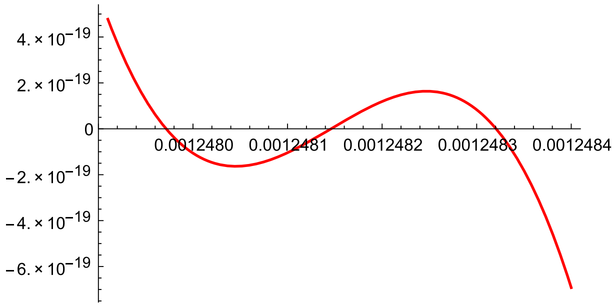

FIGURE 2. Real positive roots for case $(\mathbf{m},\mathbf{n}) = (\mathbf{2},\mathbf{4})$

Y_5

$$Y_5(x,y) = \begin{cases} Y_5^1(x,y) = -y + x(\lambda + b_1x^4 + b_2x^3y + b_3x^2y^2 + b_4xy^3 + b_5y^4),\\ Y_5^2(x,y) = x + y(\lambda + b_1x^4 + b_2x^3y + b_3x^2y^2 + b_4xy^3 + b_5y^4). \end{cases} \tag{17}$$

The vector field (17) in polar coordinates becomes

$$\dot r_2 = r\lambda + r^5\left(b_1\cos^4\theta + b_2\cos^3\theta\sin\theta + b_3\cos^2\theta\sin^2\theta + b_4\cos\theta\sin^3 + b_5\sin^4\theta\right)$$
$$\dot\theta = 1$$

or equivalently

$$r' = \frac{dr}{d\theta} = r\lambda + r^5\left(b_1\cos^4\theta + b_2\cos^3\theta\sin\theta + b_3\cos^2\theta\sin^2\theta + b_4\cos\theta\sin^3 + b_5\sin^4\theta\right).$$

Thus, the solution $r_2(\theta)$ of equation $r'(\theta)$ such that $r_2(\pi) = r_1$ is

$$r_2(\theta) = \frac{2^{3/4}e^{\theta\lambda}\lambda^{1/4}(1+5\lambda^2+4\lambda^4)^{1/4}r_1}{E_3^{1/4}},$$

where

$$\begin{aligned} E_3 =& 8e^{4\pi\lambda}(\lambda+5\lambda^3+4\lambda^5) - r_1^4\left(-e^{4\pi\lambda}\left(b_3+3b_5-(5b_2+3b_4)\lambda + 4(b_3-2b_2\lambda)\lambda^2\right.\right.\\ &\left.\left.+b_1(3+32\lambda^2+32\lambda^4)\right) + e^{4\theta\lambda}\left((3b_1+b_3+3b_5)(1+5\lambda^2+4\lambda^4)\right.\right.\\ &-4\lambda(b_2+b_4+4(-b_1+b_5)\lambda)(1+\lambda^2)\cos(2\theta) + \lambda(-b_2+b_4\\ &+(b_1-b_3+b_5)\lambda)(1+4\lambda^2)\cos(4\theta) + 8\lambda(b_1-b_5+(b_2+b_4)\lambda)(1+\lambda^2)\\ &\left.\left.\times\sin(2\theta) + \lambda\left(b_1-b_3+b_5+b_2\lambda-b_4\lambda\right)(1+4\lambda^2)\sin(4\theta)\right)\right) \end{aligned}$$

Then assuming $r_1 = r_1(\pi) = r_2(\pi)$, where $r_1(\theta)$ is the solution of (4) defined in (6). Now consider the numerator of the equation $(r(2\pi))^4 - r_0^4 = 0$, and we obtain $(1+e^{\pi\lambda})r_0^4p_3(r_0)$, being $p_3(r_0)$ the following polynomial of degree 4 in r_0

$$p_3(r_0) = C_{0,3} + C_{1,3}r_0 + C_{2,3}r_0^2 + C_{3,3}r_0^3 + C_{4,3}r_0^4,$$

where

$$\begin{aligned} C_{0,3} =& 8(-1+e^{\pi\lambda})(1+e^{2\pi\lambda})(1+e^{4\pi\lambda})\lambda(1+\lambda^2)^4(1+4\lambda^2),\\ C_{1,3} =& -32\lambda(a_1\lambda-a_2)(1+\lambda^2)^3(1+4\lambda^2);\\ C_{2,3} =& -48(1+e^{\pi\lambda})\lambda(a_2-a_1\lambda)^2(1+\lambda^2)^2(1+4\lambda^2),\\ C_{3,3} =& 32(1+e^{\pi\lambda})^2\lambda(a_2-a_1\lambda)^3(1+5\lambda^2+4\lambda^4),\\ C_{4,3} =& -8(1+e^{\pi\lambda})^3\lambda(a_2-a_1\lambda)^4(1+4\lambda^2)+e^{4\pi\lambda}(-1+e^{\pi\lambda})(1+e^{2\pi\lambda})(1+\lambda^2)^3\\ & \times(3b_1+b_3+3b_5-(5b_2+3b_4)\lambda+(32b_1+4b_3)\lambda^2-8b_2\lambda^3+32b_1\lambda^4). \end{aligned}$$

Now we shall use the Descartes'theorem (see [1]).

Theorem 2. (Descartes Theorem) *Consider the real polynomial $p(x)=a_{i_1}x^{i_1}+a_{i_2}x^{i_2}+\cdots+a_{i_r}x^{i_r}$ with $0\le i_1<i_2<\cdots<i_r$ and $a_{i_j}\neq 0$ real constants for $j\in\{1,2,\cdots,r\}$. When $a_{i_j}a_{i_{j+1}}<0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m, then $p(x)$ has at most m positive real roots. Moreover, if the coefficients of the polynomial $p(x)$ are independent, then it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has exactly m positive real roots.*

By Descartes Theorem the polyomial $p_3(r_0)$ can have at most four real positive roots. For instance, with the conditions, $a_1=-0.1$, $a_2=1$, $b_1=-2$, $b_2=0$, $b3=10$, $b_4=-1$, $b_5=-\frac{4}{3}$ and $\lambda=-1$ we have four variations of sign. Therefore the piecewise differential system separated by the straight line $y=0$ with the vector fields X_2 and Y_5 can have at most four limit cycles. Statament (d) of Theorem 1 is proved.

Case $(\mathbf{m},\mathbf{n})=(\mathbf{2},\mathbf{6})$. We consider the piecewise differential system separated by the straight line $y=0$ and formed by the vector field X_2 defined in (4) and the vector field (3) for $n=6$

$$Y_6(x,y)=\begin{cases} Y_6^1(x,y)=-y+x(\lambda+b_1x^5+b_2x^4y+b_3x^3y^2+b_4x^2y^3+b_5xy^4+b_6y^5),\\ Y_6^2(x,y)=x+y(\lambda+b_1x^5+b_2x^4y+b_3x^3y^2+b_4x^2y^3+b_5xy^4+b_6y^5). \end{cases} \tag{18}$$

The vector field (18) in polar coordinates becomes

$$\begin{aligned} \dot r_2 =& r\lambda+r^6\big(b_1\cos^5\theta+b_2\cos^4\theta\sin\theta+b_3\cos^3\theta\sin^2\theta+b_4\cos^2\theta\sin^3\theta\\ & +b_5\cos\theta\sin^4\theta+b_6\sin^5\theta\big)\\ \dot\theta =& 1 \end{aligned}$$

or equivalently

$$\begin{aligned} r'=\frac{dr}{d\theta} =& r\lambda+r^6\big(b_1\cos^5\theta+b_2\cos^4\theta\sin\theta+b_3\cos^3\theta\sin^2\theta+b_4\cos^2\theta\sin^3\theta\\ & +b_5\cos\theta\sin^4\theta+b_6\sin^5\theta\big) \end{aligned}$$

Thus, the solution $r_2(\theta)$ of equation $r'(\theta)$ such that $r_2(\pi)=r_1$ is

$$r_2(\theta)=-\frac{2^{\frac{4}{5}}e^{\theta\lambda}((1+\lambda^2)(1+25\lambda^2)(9+25\lambda^2))^{\frac{1}{5}}r_1}{E_4^{\frac{1}{5}}}, \tag{19}$$

where

$$\begin{aligned}E_4 =& -16e^{5\pi\lambda}\left(9+3(3b_2+2b_4+8b_6)r_1^5-(149b_1+26b_3+24b_5)r_1^5\lambda\right.\\&+(259+10(11b_2+3b_4)r_1^5)\lambda^2-50(15b_1+b_3)r_1^5\lambda^3+125(7+b_2r_1^5)\lambda^4\\&\left.-625b_1r_1^5\lambda^5+625\lambda^6\right)+e^{5\theta\lambda}r_1^5\left(-10(b_2+b_4+5b_6-5(5b_1+b_3+b_5)\lambda)\right.\\&\times(1+\lambda^2)(9+25\lambda^2)\cos\theta-5(1+\lambda^2)(1+25\lambda^2)\,(9b_2+3b_4\\&+5((-5b_1+b_3+3b_5)\lambda-3b_6))\cos 3\theta+(b_4-b_2-b_6+(b_1-b_3+b_5)\lambda)\\&\times(9+250\lambda^2+625\lambda^4)\cos 5\theta+10(5b_1+b_3+b_5+5(b_2+b_4+5b_6)\lambda)\\&\times(1+\lambda^2)(9+25\lambda^2)\sin\theta+5(15b_1-3b_3-9b_5+5(3b_2+b_4-5b_6)\lambda)\\&\times(1+\lambda^2)(1+25\lambda^2)\sin 3\theta+(b_1-b_3+b_5+(b_2-b_4+b_6)\lambda)\\&\left.\times\ (9+250\lambda^2+625\lambda^4)\sin 5\theta\right).\end{aligned}$$

Now consider $r_1 = r_1(\pi) = r_2(\pi)$ with $r_1(\theta)$ defined in (6).Then consider the numerator the equation $r_2(2\pi)^5 - r_0^5 = 0$ and we obtain $(1 + e^{\pi\lambda})r_0^5 p_4(r_0)$, being $p_4(r_0)$ the following polynomial of degree 5 in r_0

$$p_4(r_0) = C_{0,4} + C_{1,4}r_0 + C_{2,4}r_0^2 + C_{3,4}r_0^3 + C_{4,4}r_0^4 + C_{5,4}r_0^5,$$

where

$$\begin{aligned}C_{0,4} =&(1-e^{\pi\lambda}+e^{2\pi\lambda}-e^{3\pi\lambda}+e^{4\pi\lambda})(1-e^{5\pi\lambda})(1+\lambda^2)^5(9+250\lambda^2+625\lambda^4),\\C_{1,4} =&5(a_1\lambda-a_2)(1+\lambda^4)(9+250\lambda^2+625\lambda^4),\\C_{2,4} =&10(1+e^{\pi\lambda})(a_2-a_1\lambda)^2(1+\lambda^2)^3(9+250\lambda^2+625\lambda^4),\\C_{3,4} =&10(1+e^{\pi\lambda})^2(a_1\lambda-a_2)^3(1+\lambda^2)^2(9+250\lambda^2+625\lambda^4),\\C_{4,4} =&5(1+e^{\pi\lambda})^3(a_2-a_1\lambda)^4(1+\lambda^2)(9+250\lambda^2+625\lambda^4),\\C_{5,4} =&(1+e^{\pi\lambda})^4(a_1\lambda-a_2)^5(9+250\lambda^2+625\lambda^4)\\&+e^{5\pi\lambda}(1-e^{\pi\lambda}+e^{2\pi\lambda}-e^{3\pi\lambda}+e^{4\pi\lambda})\\&\times(1+\lambda^2)^4\left(24b_6+6b_4(1+5\lambda^2)+b2(9+110\lambda^2+125\lambda^4)\right.\\&\left.-\lambda\left(149b_1+26b_3+24b_5+50(15b_1+b_3)\lambda^2+625b_1\lambda^4\right)\right)).\end{aligned}$$

The polynomial $p_4(r_0)$ can have at most 5 positive real roots. For example, consider $\lambda = -1$, $a_1 = 10$, $a_2 = 15$, $b_1 = b_2 = b_3 = b_4 = 0$, $b_5 = -14$ and $b_6 = -12$, with these values there are 5 variations of sign. So By Descartes Theorem it can have at most 5 positive real roots, is that, the piecewise differential system formed by the two vector fields X_2 and Y_6 and separated by the straight line $y = 0$ can have at most 5 limit cycles. Statement (e) of Theorem 1 is proved.

Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{3})$. We consider the piecewise differential system separated by the straight line $y = 0$ and formed by the vector field Y_3 defined in (10) and the vector field

$$X_3(x,y) = \begin{cases} x_3^1(x,y) = -y + x(\lambda + a_1x^2 + a_2xy + a_3y^2), \\ x_4^2(x,y) = x + y(\lambda + a_1x^2 + a_2xy + a_3y^2). \end{cases} \tag{20}$$

Thus the solution $r_1(\theta)$ of the vector field X_3 in polar coordinates such that $r_1(0) = r_0$ is

$$r_1(\theta) = \frac{e^{\theta\lambda}\sqrt{2\lambda(1+\lambda^2)}}{\sqrt{2(\lambda+\lambda^3)/r_0^2 + E_5}} \tag{21}$$

where

$$E_5 = e^{2\theta\lambda}(-(a_1+a_3)(1+\lambda^2) + \lambda(a_2 + (-a_1+a_3)\lambda)\cos(2\theta) \\ - \lambda(a_1 - a_3 + a_2\lambda)\sin(2\theta)) + \lambda(-a_2 + 2a_1\lambda) + a_1 + a_3.$$

The solution $r_2(\theta)$ of the vector field Y_3 such that $r_2(\pi) = r_1$ is given in (12). Then solving the equation $r_2(2\pi) = r_0$ for r_0 we have the following solution

$$r_0 = \sqrt{\frac{-2\lambda(1+e^{2\pi\lambda})(1+\lambda^2)}{a_1 + a_3 - a_2\lambda + 2a_1\lambda^2 + e^{2\pi\lambda}(b_1 + b_3 - b_2\lambda + 2b_1\lambda^2)}}.$$

When $r_0 > 0$ the piecewise differential system separated by the straight line $y = 0$ and formed by the vector fields X_3 and Y_3 has at most one limit cycle. Statement (f) of Theorem 1 is proved.

Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{4})$. We consider the piecewise differential systems separated by the straight line $y = 0$ and formed by the two vector fields X_3 defined in (20) and Y_4 defined in (14). We know their solutions $r_1(\theta)$ such that $r_1(0) = r_0$ and $r_2(\theta)$ such that $r_2(\pi) = r_1(\pi)$. Then the numerator of the equation $r_2(2\pi)^3 - r_0^3 = 0$ is $A + B = 0$ where

$$A = 2\sqrt{2}e^{3\pi\lambda}\lambda^{3/2}\left(\lambda^2+1\right)^{3/2} r_0^3\left(9b_1\lambda^3 + 7b_1\lambda - b_2\left(3\lambda^2+1\right) + 2b_3\lambda - 2b_4\right) \\ + 2\sqrt{2}e^{6\pi\lambda}\lambda^{3/2}\left(\lambda^2+1\right)^{3/2}\left(\lambda r_0^3(7b_1 + 2b_3) + 9b_1\lambda^3 r_0^3 \right. \\ \left. - r_0^3(b_2 + 2b_4) + \lambda^2\left(10 - 3b_2r_0^3\right) + 9\lambda^4 + 1\right),$$

$$B = \left(\lambda^2+1\right)\left(9\lambda^2+1\right) r_0\left(\left(e^{2\pi\lambda}-1\right) r_0^2\left(2a_1\lambda^2 + a_1 - a_2\lambda + a_3\right) - 2\left(\lambda^3+\lambda\right)\right) \\ \sqrt{a_1 + a_3 + \lambda(-a_2 + 2a_1\lambda) - e^{2\pi\lambda}(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2) + 2\left(\lambda^3+\lambda\right)/r_0^2}.$$

Now we consider the equation $A^2 - B^2 = 0$ and we obtain the polynomial

$$p_5(r_0) = C_{0,5} + C_{25}r_0^2 + C_{3,5}r_0^3 + C_{4,5}r_0^4 + C_{6,5}r_0^6,$$

where

$$C_{0,5} = 8\left(e^{12\pi\lambda}-1\right)\lambda^3\left(\lambda^2+1\right)^5\left(9\lambda^2+1\right)^2,$$

$$C_{2,5} = 12\left(e^{2\pi\lambda}-1\right)\left(\lambda^2+1\right)^4\left(9\lambda^3+\lambda\right)^2\left(2a_1\lambda^2 + a_1 - a_2\lambda + a_3\right),$$

$$C_{3,5} = 16e^{9\pi\lambda}\left(e^{3\pi\lambda}+1\right)\lambda^3\left(\lambda^2+1\right)^4\left(9\lambda^2+1\right)\left(9b_1\lambda^3 + 7b_1\lambda - b_2\left(3\lambda^2+1\right) \right. \\ \left. + 2b_3\lambda - 2b_4\right),$$

$$C_{4,5} = 6\left(1-e^{2\pi\lambda}\right)^2\lambda\left(\lambda^2+1\right)^3\left(9\lambda^2+1\right)^2\left(2a_1\lambda^2 + a_1 - a_2\lambda + a_3\right)^2,$$

$$C_{6,5} = a_1^3\left(2\lambda^2+1\right)^3\left(9\lambda^4 + 10\lambda^2 + 1\right)^2\left(e^{2\pi\lambda}-1\right)^3 \\ + 3\left(18\lambda^6 + 29\lambda^4 + 12\lambda^2 + 1\right)^2\left(e^{2\pi\lambda}-1\right)^3$$

$$
\begin{aligned}
&(a_3-a_2\lambda)a_1^2+3a_1\left(2\lambda^2+1\right)\left(9\lambda^4+10\lambda^2+1\right)^2\left(e^{2\pi\lambda}-1\right)^3(a_3-a_2\lambda)^2\\
&+\left(\lambda^2+1\right)^2\Big(\lambda^3\Big(8e^{6\pi\lambda}\left(e^{3\pi\lambda}+1\right)^2\left(\lambda^2+1\right)\\
&\left(-9b_1\lambda^3-7b_1\lambda+3b_2\lambda^2+b_2-2b_3\lambda+2b_4\right)^2\\
&-a_2^3\left(e^{2\pi\lambda}-1\right)^3\left(9\lambda^2+1\right)^2\Big)+3a_2^2a_3\lambda^2\left(9\lambda^2+1\right)^2\\
&\left(e^{2\pi\lambda}-1\right)^3-3a_2\lambda\left(e^{2\pi\lambda}-1\right)^3\\
&\left(9a_3\lambda^2+a_3\right)^2+a_3^3\left(9\lambda^2+1\right)^2\left(e^{2\pi\lambda}-1\right)^3\Big).
\end{aligned}
$$

By Descartes Theorem the polynomial $p_5(r_0)$ can have at most four positive real roots. So the piecewise differential system separated by the straight line $y=0$ and formed by the two vector fields X_3 and Y_4 can have at most four limit cycles. Statement (g) of Theorem 1 is proved.

Case $(\mathbf{m},\mathbf{n})=(\mathbf{3},\mathbf{5})$. We consider the piecewise differential systems separated by the straight line $y=0$ and formed by the two vector fields X_3 defined in (20) and Y_5 defined in (17). We know their solutions $r_1(\theta)$ such that $r_1(0)=r_0$ and $r_2(\theta)$ such that $r_2(\pi)=r_1(\pi)$. Then we consider the numerator of the equation $(r_2(2\pi))^4-r_0^4=0$ and we get $(-1+e^{2\pi\lambda})r_0^4p_6(r_0)$, where $p_6(r_0)$ is the following polynomial of degree 4 in r_0

$$p_6(r_0)=C_{0,6}+C_{2,6}r_0^2+C_{4,6}r_0^4,$$

with

$$
\begin{aligned}
C_{0,6}=&8(1+e^{2\pi\lambda})(1+e^{4\pi\lambda})\lambda^2(1+\lambda^2)^2(1+4\lambda^2),\\
C_{2,6}=&8\lambda(a_1+a_3-a_2\lambda+2a_1\lambda^2)(1+5\lambda^2+4\lambda^4),\\
C_{4,6}=&2(1-e^{2\pi\lambda})(1+4\lambda^2)(a_1+a_3-a_2\lambda+2a_1\lambda^2)^2+e^{4\pi\lambda}(1+e^{2\pi\lambda})(\lambda+\lambda^3)\\
&\times(b_3+3b_5-5b_2\lambda-3b_4\lambda+4\lambda^2(b_3-2b_2\lambda)+b_1(3+32(\lambda^2+\lambda^4))).
\end{aligned}
$$

By Descartes Theorem the polynomial $p_6(r_0)$ can have at most two positive real roots. For exemple, for the values $\lambda=0.01$, $a_1=-4.95409$, $a_2=10$, $a_3=5$, $b_1=b_2=b_3=0$ $b4=250000$ and $b_5=2500.02$ it has two real positive roots, $r_0^1=1$ and $r_0^2=2$. See Figure 3. For theses values the denominator of the equation $(r_2(2\pi))^4-r_0^4$ is different from zero. So the piecewise differential system separated by the straight line $y=0$ with the vector fields X_3 and Y_5 can have two limit cycles. Statement (h) of Theorem 1 is proved.

Case $(\mathbf{m},\mathbf{n})=(\mathbf{3},\mathbf{6})$. We consider the piecewise differential systems separated by the straight line $y=0$ and formed by the two vector fields X_3 defined in (20) and Y_6 defined in (18). We know their solutions $r_1(\theta)$ such that $r_1(0)=r_0$ and $r_2(\theta)$ such that $r_2(\pi)=r_1(\pi)$. Then the numerator the equation $r_2(2\pi)^5-r_0^5=$

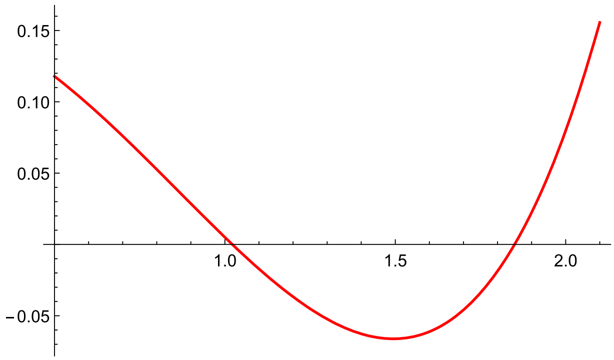

FIGURE 3. Real positive roots for case $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{5})$

0 is $A + B = 0$ where

$$\begin{aligned} A =& - r_0(1+\lambda^2)(1+25\lambda^2)(9+25\lambda^2) \\ & \sqrt{(1-e^{2\pi\lambda})(a_1+a_3-a_2\lambda+2a_1\lambda^2)+2(\lambda+\lambda^3)/r_0^2}; \\ B =& 4\sqrt{2}e^{5\pi\lambda}\lambda^{\frac{5}{2}}(1+\lambda^2)^{\frac{5}{2}} \left(e^{5\pi\lambda}(9+259\lambda^2+875\lambda^4+625\lambda^6)-(1+e^{5\pi\lambda})r_0^5\right. \\ & \times \left(24b_6+6b_4(1+5\lambda^2)+b_2(9+110\lambda^2+125\lambda^4)-\lambda\left(149b_1+26b_3+24b_5\right.\right. \\ & \left.\left.\left.+50(15b_1+b_3)\lambda^2+625b_1\lambda^4\right)\right)\right). \end{aligned}$$

Now consider the equation $A^2 - B^2 = 0$ and we obtain $(1+e^{\pi\lambda})(1+\lambda^2)p_7(r_0)$, where $p_7(r_0)$ is a polynomial of degree 10 in r_0

$$p_7(r_0) = C_{0,7} + C_{2,7}r_0^2 + C_{4,7}r_0^4 + C_{5,7}r_0^5 + C_{6,7}r_0^6 + C_{8,7}r_0^8 + C_{10,7}r_0^{10},$$

where

$$\begin{aligned} C_{0,7} =& 32(1-e^{\pi\lambda})(1+e^{2\pi\lambda})(1+e^{4\pi\lambda}+e^{8\pi\lambda}+e^{12\pi\lambda}+e^{16\pi\lambda})\lambda^5 \\ & \times(9+250\lambda^2+625\lambda^4)^2(1+\lambda^2)^5, \\ C_{2,7} =& 80(1-e^{\pi\lambda})(a_1+a_3-a_2\lambda+2a_1\lambda^2)(\lambda+\lambda^3)^4(9+250\lambda^2+625\lambda^4)^2, \\ C_{4,7} =& 80(-1+e^{\pi\lambda})^2(1+e^{\pi\lambda})\lambda^3(1+\lambda^2)^3(9+250\lambda^2+625\lambda^4)^2 \\ & (a_1+a_3-a_2\lambda+2a_1\lambda^2)^2, \\ C_{5,7} =& 64e^{15\pi\lambda}(1-e^{\pi\lambda}+e^{2\pi\lambda}-e^{3\pi\lambda}+e^{4\pi\lambda})\lambda^5(1+\lambda^2)^4(9+250\lambda^2+625\lambda^4) \\ & \times\left(24b_6+6b_4(1+5\lambda^2)+b_2(9+110\lambda^2+125\lambda^4)-\lambda\left(149b_1\right.\right. \\ & \left.\left.+26b_3+24b_5+50(15b_1+b_3)\lambda^2+625b_1\lambda^4\right)\right), \\ C_{6,7} =& 40(1-e^{2\pi\lambda})^3(1+e^{\pi\lambda})^2\lambda^2(a_1+a_3-a_2\lambda+2a_1\lambda^2)^3(1+\lambda^2)^2 \\ & \times(9+250\lambda^2+625\lambda^4)^2, \\ C_{8,7} =& 10(-1+e^{\pi\lambda})^4(1+e^{\pi\lambda})^3\lambda(1+\lambda^2)(a_1+a_3-a_2\lambda+2a_1\lambda^2)^4 \\ & \times(9+250\lambda^2+625\lambda^4)^2, \\ C_{10,7} =& (-1-e^{\pi\lambda})\left((-1+e^{\pi\lambda})^5(1+e^{\pi\lambda})^3(a_1+a_3-a_2\lambda+2a_1\lambda^2)^5\right. \end{aligned}$$

$$\times (9 + 250\lambda^2 + 625\lambda^4)^2$$
$$+32e^{10\pi\lambda}(1 - e^{\pi\lambda} + e^{2\pi\lambda} - e^{3\pi\lambda} + e^{4\pi\lambda})^2\lambda^5(1 + \lambda^2)^3$$
$$\times \big(24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4)$$
$$-\lambda \left(149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4\right)\big)^2\Big).$$

Note that the signs of the coefficients C_0, C_4 and C_8 are determined only by the sign of λ. Assume $\lambda < 0$, then $C_0 < 0$, $C_4 < 0$ and $C_8 < 0$. In this way, between C_0, C_2 and C_4 there are at most two variations of sign, between C_4, C_5, C_6 and C_8 there are at most two variations of sign and between C_8 and C_{10} there is at most one variation of sign. Therefore, there are at most five variations of sign between the coefficient of $p_7(r_0)$. More specifically, the sign of the coefficient C_2 is determinated by the sign of $A_1 = (a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)$. We assume $A_1 > 0$ in order to $C_2 > 0$. The sign of C_5 is given by

$$A_2 = \big(24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda \left(149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4\right)\big),$$

then assuming $A_2 < 0$ we have $C_5 > 0$. Now as $A_1 > 0$ then $C_6 > 0$. Finally as $A_1 > 0$ we have $C_{10} > 0$. Thus, we obtain five variations of sign, i.e $C_0 < 0$, $C_2 > 0$, $C_4 < 0$, $C_5 > 0$, $C_6 > 0$, $C_8 < 0$ and $C_{10} > 0$ for $\lambda < 0$. On the other hand, assuming $\lambda > 0$ we have $C_0 < 0$, $C_4 > 0$ and $C_8 > 0$. Thus between C_0, C_2 and C_4 there is at most one variation of sign and between C_4, C_5, C_6 and C_8 there are at most two variations of sign and between C_8 and C_{10} there is one variation. In this way, for $\lambda > 0$ there are at most four variations of sign. For instance, consider $\lambda < 0$, $a_1 > 0$, $a_2 \geq 0$, $a_3 \geq -a_1$, $b_1 = b_2 = 0$, $b_3 < 0$, $b_4 < 0$, $b_5 = -\frac{13b_3}{12}$ and $b_6 < 0$, wit this conditions we have five variations of sign. In conclusion by Descartes Theorem the polynomial $p_7(r_0)$ can have at most five positive real roots. In this way, it follows that the piecewise differential system separated by the straight line $y = 0$ and formed by the two vector fields X_3 and Y_6 can have at most five limit cycles. Statement (i) of Theorem 1 is proved.

Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{4}, \mathbf{4})$. Now consider the piecewise differential system separated by the straight line $y = 0$ and formed by vector field Y_4 defined in (14) and the vector field (2) for $m = 4$, i.e

$$X_4(x, y) = \begin{cases} X_4^1(x, y) = -y + x(\lambda + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3), \\ X_4^2(x, y) = x + y(\lambda + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3). \end{cases} \qquad (22)$$

The vector field (22) in polar coordinates provides the differential system

$$\dot{r}_1(\theta) = r\lambda + r^4(a_1 \cos^3\theta + a_2 \cos^2\theta \sin\theta + a_3 \cos\theta \sin^2\theta + a_4 \sin^3\theta), \quad \dot{\theta} = 1$$

or equivalently the differential equation

$$r' = \frac{dr}{d\theta} = r\lambda + r^4(a_1 \cos^3\theta + a_2 \cos^2\theta \sin\theta + a_3 \cos\theta \sin^2\theta + a_4 \sin^3\theta).$$

Then the solution $r_1(\theta)$ of the equation $r'(\theta)$ such that $r_1(0) = r_0$ is

$$r_1(\theta) = -\frac{2^{\frac{2}{3}} e^{\theta\lambda} r_0((1+\lambda^2)(1+9\lambda^2))^{\frac{1}{3}}}{E_6^{\frac{1}{3}}}. \tag{23}$$

Where

$$\begin{aligned} E_6 =& e^{3\theta\lambda} r_0^3 \big(-3(a_2+3a_4-3(3a_1+a_3)\lambda)(1+\lambda^2)\cos\theta + (-a_2+a_4+a_1\lambda-a_3\lambda) \\ & \times (1+9\lambda^2)\cos 3\theta + 3(3a_1+a_3+3a_2\lambda+9a_4\lambda)(1+\lambda^2)\sin\theta \\ & +(a_1-a_3+a_2\lambda-a_4\lambda)(1+9\lambda^2)\sin 3\theta\big) - 4(1+\lambda^2)(1+9\lambda^2) \\ & + 4r_0^3(a_2+2a_4+3a_2\lambda^2-\lambda(7a_1+2a_3+9a_1\lambda^2)). \end{aligned}$$

Then assuming $r_1 = r_1(\pi) = r_2(\pi)$, where $r_2(\theta)$ is the solution of (14) defined in (15), we solve $r_2(2\pi) = r_0$ for r_0. More specifically we consider the numerator of equation $(r_2(2\pi))^3 - r_0^3 = 0$ and we get $(1+e^{3\pi\lambda})r_0^3 p_8(r_0)$, with $p_8(r_0)$ being the following polynomial of degree 3 in r_0

$$p_8(r_0) = C_{0,8} + C_{3,8} r_0^3,$$

where

$$\begin{aligned} C_{0,8} =& (1-e^{3\pi\lambda})(1+10\lambda^2+9\lambda^4), \\ C_{3,8} =& (7a_1+2a_3)\lambda + 9a_1\lambda^3 - a_2(1+3\lambda^2) + e^{3\pi\lambda}(b_2+2b_4 \\ & -(7b_1+2b_3)\lambda + 3b_2\lambda^2 - 9b_1\lambda^3) - 2a_4. \end{aligned}$$

The polynomial $p_8(r_0)$ can have at most one real positive root $r_0 = \left(\frac{-C_0}{C_3}\right)^{1/3}$. Therefore the piecewise differential system separated by the straight line $y = 0$ formed by the vector fields X_4 and Y_4 can have at most one limit cycle. Statement (j) of Theorem 1 is proved.

Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{4}, \mathbf{5})$. Now consider the piecewise differential systems separated by the straight line $y = 0$ and formed by the two vector fields X_4 defined in (22) and Y_5 defined in (17). We consider their solutions $r_1(\theta)$ such that $r_1(0) = r_0$ and $r_2(\theta)$ such that $r_2(\pi) = r_1(\pi)$. Then the numerator of the equation $(r_2(2\pi))^4 - r_0^4 = 0$ is $A - B = 0$ where

$$\begin{aligned} A =& -8\lambda(1+5\lambda^2+4\lambda^4)\big((1+\lambda^2)(1+9\lambda^2) - (1+e^{\pi\lambda})(1-e^{\pi\lambda}+e^{2\pi\lambda})r_0^3 \\ & \times (a_2+2a_4-7a_1\lambda-2a_3\lambda+3a_2\lambda^2-9a_1\lambda^3)\big) \\ & \times \big((-1-\lambda^2)(1+9\lambda^2) + (e^{\pi\lambda}+1) \\ & \times (1-e^{\pi\lambda}+e^{2\pi\lambda})r_0^3(a_2+2a_4-7a_1\lambda-2a_3\lambda+3a_2\lambda^2-9a_1\lambda^3)\big)^{1/3} \\ B =& e^{4\pi\lambda}((1+\lambda^2)(1+9\lambda^2))^{4/3}\big(8e^{4\pi\lambda}(\lambda+5\lambda^3+4\lambda^5) \\ & + (-1+e^{\pi\lambda})(1+e^{\pi\lambda})(1+e^{2\pi\lambda})r_0^4 \\ & \times (b_3+3b_5-(5b_2+3b_4)\lambda+4b_3\lambda^2-8b_2\lambda^3+b_1(3+32\lambda^2+32\lambda^4))\big). \end{aligned}$$

Now we consider the equation $A^3 - B^3$ and we obtain $(1+e^{\pi\lambda})(1+\lambda^2)^3 p_9(r_0)$, being $p_9(r_0)$ the following polynomial of degree 12 in r_0

$$p_9(r_0) = C_{0,9} + C_{3,9}r_0^3 + C_{4,9}r_0^4 + C_{6,9}r_0^6 + C_{8,9}r_0^8 + C_{9,9}r_0^9 + C_{12,9}r_0^{12}.$$

Where

$$\begin{aligned}
C_{0,9} =& -512(1-e^{\pi\lambda}+e^{2\pi\lambda})(-1+e^{3\pi\lambda})(1+e^{6\pi\lambda})(1+e^{12\pi\lambda})\lambda^3 \\
& \times(1+\lambda^2)^4(1+4\lambda^2)^3(1+9\lambda^2)^4, \\
C_{3,9} =& -2048(1-e^{\pi\lambda}+e^{2\pi\lambda})(\lambda+14\lambda^3+49\lambda^5+36\lambda^7)^3 \\
& \times\left(a_2+2a_4+3a_2\lambda^2-\lambda\left(7a_1+2a_3+9a_1\lambda^2\right)\right), \\
C_{4,9} =& -192e^{20\pi\lambda}(-1+e^{\pi\lambda})(1+e^{2\pi\lambda})\lambda^2(1+\lambda^2)^3(1+9\lambda^2)^4 \\
& \times(b_3+3b_5-(5b_2+3b_4)\lambda \\
& +4(b_3-2b_2\lambda)\lambda^2+b_1(3+32\lambda^2+32\lambda^4))\,(1+4\lambda^2)^2, \\
C_{6,9} =& 3072(1+e^{\pi\lambda})(1-e^{\pi\lambda}+e^{2\pi\lambda})^2(\lambda+4\lambda^3)^3(1+10\lambda^2+9\lambda^4)^2 \\
& \times\left(a_2+2a_4+3a_2\lambda^2-\lambda\left(7a_1+2a_3+9a_1\lambda^2\right)\right)^2, \\
C_{8,9} =& -24e^{16\pi\lambda}(1+e^{\pi\lambda})(-1+e^{\pi\lambda}-e^{2\pi\lambda}+e^{3\pi\lambda})^2\lambda(1+\lambda^2)^2(1+4\lambda^2)(1+9\lambda^2)^4 \\
& \times\left(b_3+3b_5-(5b_2+3b_4)\lambda+4(b_3-2b_2\lambda)\lambda^2+b_1(3+32\lambda^2+32\lambda^4)\right)^2, \\
C_{9,9} =& -2048(1+e^{\pi\lambda})^2(1-e^{\pi\lambda}+e^{2\pi\lambda})^3\lambda^3(1+4\lambda^2)^3(1+10\lambda^2+9\lambda^4) \\
& \left(a_2+2a_4+3a_2\lambda^2-\lambda\left(7a_1+2a_3+9a_1\lambda^2\right)\right)^3, \\
C_{12,9} =& -(1+e^{\pi\lambda})^2\left(-512(1+e^{\pi\lambda})(1-e^{\pi}\lambda+e^{2\pi\lambda})^4(\lambda+4\lambda^3)^3\,(a_2+2a_4\right. \\
& \left.+3a_2\lambda^2-\lambda\left(7a_1+2a_3+9a_1\lambda^2\right)\right)^4+e^{12\pi\lambda}(-1+e^{\pi\lambda})^3(1+e^{2\pi\lambda})^3 \\
& \times(1+\lambda^2)(1+9\lambda^2)^4\left(b_3+3b_5-(5b_2+3b_4)\lambda+4(b_3-2b_2\lambda)\lambda^2\right. \\
& \left.+b_1(3+32\lambda^2+32\lambda^4)\right)^3.
\end{aligned}$$

By Descartes Theorem the polynomial $p_9(r_0)$ can have at most six positive real roots. For exemple, for the values $\lambda = 0.1$, $a_1 = 1$, $a_2 = 0$, $a_3 = -2$, $a_4 = -1$, $b_1 = 1$, $b_2 = b_3 = 0$, $b_4 = -1$, $b_5 = 1$ it has six variations of sign, that is, $p_9(r_0)$ can have at most six positive real roots. So the piecewise differential system separated by the straight line $y = 0$ with the vector fields X_4 and Y_5 can have at most six limit cycles. Statament (k) of Theorem (1) is proved.

Case $(\mathbf{m},\mathbf{n}) = (\mathbf{4},\mathbf{6})$. We consider the piecewise differential systems separated by the straight line $y = 0$ and formed by the two vector fields X_4 defined in (22) and Y_6 defined in (18). We know their solutions $r_1(\theta)$ such that $r_1(0) = r_0$ and $r_2(\theta)$ such that $r_2(\pi) = r_1(\pi)$. Then the numerator of the equation

$r_2(2\pi)^5 - r_0^5 = 0$ is $r_0^5(A - B) = 0$ where

$$
\begin{aligned}
A =&(1+10\lambda^2 + 9\lambda^4 - (1+e^{3\pi\lambda})r_0^3(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))) \\
&\times (9 + 259\lambda^2 + 875\lambda^4 + 625\lambda^6)\left(-1 - 10\lambda^2 - 9\lambda^4 + (1 + e^{3\pi\lambda})r0^3\right. \\
&\left.\times\ (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)))\right)^{2/3};\\
B =&e^{5\pi\lambda}(1 + \lambda^2)^{2/3}(1 + 9\lambda^2)^{2/3}(1 + 10\lambda^2 + 9\lambda^4) \\
&\times \left(e^{5\pi\lambda}(1 + \lambda^2)(1 + 25\lambda^2)(9 + 25\lambda^2)\right. \\
&+ (r_0^5 + e^{5\pi\lambda}r_0^5)\left(-6(b_4 + 4b_6) + (149b_1 + 26b_3 + 24b_5)\lambda\right. \\
&\left.\left.-b_2(9 + 110\lambda^2 + 125\lambda^4) + 5\lambda^2(-6b_4 + 5\lambda(2b_3 + 5b_1(6 + 5\lambda^2))))\right)\right).
\end{aligned}
$$

Then we consider the equation $A^3 - B^3 = 0$ and we get the polynomial

$$
\begin{aligned}
p_{10}(r_0) = &C_{0,10} + C_{3,10}r_0^3 + C_{5,10}r_0^5 + C_{6,10}r_0^6 \\
&+ C_{9,10}r_0^9 + C_{10,10}r_0^{10} + C_{12,10}r_0^{12} + C_{15,10}r_0^{15},
\end{aligned}
$$

where

$$
\begin{aligned}
C_{0,10} =&(1 - e^{30\pi\lambda})(1 + \lambda^2)^8(1 + 9\lambda^2)^5(9 + 250\lambda^2 + 625\lambda^4)^3,\\
C_{3,10} =& - 5(1 + e^{3\pi\lambda})(1 + \lambda^2)^7(1 + 9\lambda^2)^4(9 + 250\lambda^2 + 625\lambda^4)^3 \\
&\times \left(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)\right),\\
C_{5,10} =&3e^{25\pi\lambda}(1 + e^{5\pi\lambda})(1 + \lambda^2)^7(1 + 9\lambda^2)^5(9 + 250\lambda^2 + 625\lambda^4)^2 \\
&\times \left(24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda\right. \\
&\left.\left(149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4)\right)\right),\\
C_{6,10} =&10(1 + e^{3\pi\lambda})^2(1 + \lambda^2)^6(1 + 9\lambda^2)^3(9 + 250\lambda^2 + 625\lambda^4)^3 \\
&\times \left(a_2 + 2a_4 + 3a_2\lambda^2\lambda(7a_1 + 2a_3 + 9a_1\lambda^2)\right)^2,\\
C_{9,10} =& - 10(1 + e^{3\pi\lambda})^3(1 + \lambda^2)^5(1 + 9\lambda^2)^2(9 + 250\lambda^2 + 625\lambda^4)^3 \\
&\times \left(a_2 + 2a_4 + 3a_2\lambda^2\lambda(7a_1 + 2a_3 + 9a_1\lambda^2)\right)^3;\\
C_{10,10} =& - 3e^{20\pi\lambda}(1 + e^{5\pi\lambda})^2(1 + \lambda^2)^6(1 + 9\lambda^2)^5(9 + 250\lambda^2 + 625\lambda^4) \\
&\times \left(24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4)\right. \\
&\left.-\lambda\left(149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4)\right)\right)^2,\\
C_{12,10} =&5(1 + e^{3\pi\lambda})^4(1 + \lambda^2)^4(1 + 9\lambda^2)(9 + 250\lambda^2 + 625\lambda^4)^3 \\
&\times \left(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)\right)^4,\\
C_{15,10} =& - (1+e^{\pi\lambda})(1+\lambda^2)^3\left((1+e^{3\pi\lambda})^2(1-e^{\pi\lambda}+e^{2\pi\lambda})^5(9 + 250\lambda^2 + 625\lambda^4)^3\right. \\
&\times \left(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)\right)^5 \\
&+ (1 - e^{\pi\lambda} + e^{2\pi\lambda} - e^{3\pi\lambda} + e^{4\pi\lambda})^3 \\
&\times \left(-\left(24b_6 + 6b_4(1 + \lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda\left(149b_1 + 26b_3\right.\right.\right. \\
&\left.\left.\left.+24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4)\right)\right)^3 e^{15\pi\lambda}(1 + \lambda^2)^2(1 + 9\lambda^2)^5\right).
\end{aligned}
$$

Since we are assuming that C_i is not null for all i, then we have that $C_6 > 0$, $C_{10} < 0$ and $C_{12} > 0$. In this way, note that between C_0, C_3, C_5 and C_6 there are at most three variations of sign. Now between C_6, C_9, C_{10} and C_{12} there are at most two variations of sign and between C_{12} and C_{15} there is at most one variation. Therefore there are at most six variations of sign. For example, consider $\lambda = 0.1$, $a_1 = 0$, $a_2 = -2$, $a_3 = 1$, $a_4 = 1$, $b_1 = b_2 = 0$, $b_3 = 1$, $b_4 = -0.5$, $b_5 = -13/12$ and $b_6 = -1$, with this conditions there are six variations of sign. So by the Descartes Theorem the polynomial $p_{10}(r_0)$ can have at most six positive real roots. Thus, the piecewise differential system separated by the straight line $y = 0$ and formed by the two vector fields X_4 and Y_6 can have at most 6 limit cycles. Statement (l) of Theorem 1 is proved.

Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{5}, \mathbf{5})$. Now consider the piecewise differential systems separated by the straight line $y = 0$ and formed by the two vector fields Y_5 defined in (17) and

$$X_5(x,y) = \begin{cases} X_5^1(x,y) = -\,y + x(\lambda + a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4), \\ X_5^2(x,y) = x + y(\lambda + a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4). \end{cases} \tag{24}$$

Thus the solution $r_1(\theta)$ of the vector field X_5 in polar coordinates such that $r_1(0) = r_0$ is

$$r_1(\theta) = \frac{2^{3/4}e^{\theta\lambda}\lambda^{1/4}(1 + 5\lambda^2 + 4\lambda^4)^{1/4}r_0}{E_7^{1/4}} \tag{25}$$

where

$$\begin{aligned} E_7 =& 8(\lambda + 5\lambda^3 + 4\lambda^5) - r_0^4\Big(e^{4\pi\lambda}\big(-3a_5 - a_3(1 + 4\lambda^2) - a_1(3 + 32\lambda^2 + 32\lambda^4) \\ &+\lambda(3a_4 + a_2(5 + 8\lambda^2))\big) + e^{4\theta\lambda}\big(3a_1 + a_3 + 3a_5 + (3a_1 + a_3 + 3a_5)\lambda^2(5 + 4\lambda^2) \\ &- 4\lambda(a_2 + a_4 + 4(-a_1 + a_5)\lambda)(1 + \lambda^2)\cos 2\theta + \lambda(-a_2 + a_4 + (a_1 - a_3 + a_5)\lambda) \\ &\times (1 + 4\lambda^2)\cos 4\theta + 8\lambda(a_1 - a_5 + (a_2 + a_4)\lambda)(1 + \lambda^2)\sin 2\theta + \lambda\,(a_1 - a_3 \\ &+ a_5 + a_2\lambda - a_4\lambda)\,(1 + 4\lambda^2)\sin 4\theta\big)\Big). \end{aligned}$$

We consider the solution $r_2(\theta)$ of Y_5 such that $r_2(\pi) = r_1$ and (25), then we assume that $r_1 = r_1(\pi) = r_2(\pi)$. Thus we consider the numerator of the equation $(r_2(2\pi))^4 - r_0^4 = 0$ and we obtain $(1 - e^{4\pi\lambda})r_0^4p_{11}(r_0)$, being $p_{11}(r_0)$ the following polynomial of degree 4 in r_0

$$p_{11}(r_0) = C_{0,11} + C_{4,11}r_0^4,$$

where

$$\begin{aligned} C_{0,11} =& 8(1 + e^{4\pi\lambda})(\lambda + 5\lambda^3 + 4\lambda^5); \\ C_{4,11} =& 3a_1 + a_3 + 3a_5 - (5a_2 + 3a_4)\lambda \\ &+ (32a_1 + 4a_3)\lambda^2 - 8a_2\lambda^3 + 32a_1\lambda^4 + e^{4\pi\lambda}\big(3b_1 + b_3 \\ &+ 3b_5 - (5b_2 + 3b_4)\lambda + (32b_1 + 4b_3)\lambda^2 - 8b_2\lambda^3 + 32b_1\lambda^4\big). \end{aligned}$$

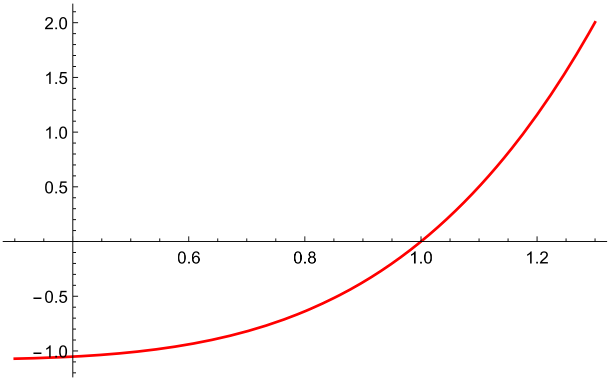

FIGURE 4. Real positive root for case $(\mathbf{m},\mathbf{n})=(\mathbf{5},\mathbf{5})$

The polynomial $p_{11}(r_0)$ can have one real positive root. For instance, for the values $\lambda=-0.1$, $a_1=2.83501$, $a_2=9$, $a_3=-9$, $a_4=1$, $a_5=-1$, $b_1=b_2=b_3=b_4=0$ and $b_5=-1$ it has one real positive root $r_0=1$, see Figure 4. With theses values the denominator of the equation $(r_2(2\pi))^4-r_0^4$ is different from zero. So the piecewise differential system by the straight line $y=0$ with the vector fields X_5 and Y_5 can have one limit cycle. Statement (m) of Theorem 1 is proved.

Case $(\mathbf{m},\mathbf{n})=(\mathbf{5},\mathbf{6})$. Now consider the piecewise differential systems separated by the straight line $y=0$ and formed by the two vector fields X_5 defined in (24) and Y_6 defined in (18). We consider their solutions $r_1(\theta)$ such that $r_1(0)=r_0$ and $r_2(\theta)$ such that $r_2(\pi)=r_1$. Then the numerator of the equation $(r_2(2\pi))^5-r_0^5=0$ is $r_0^5(A-B)=0$ where

$$\begin{aligned}A =&8\times 2^{3/4}e^{10\pi\lambda}\lambda^{5/4}(1+\lambda^2)^2(1+5\lambda^2+4\lambda^4)^{5/4}(9+286\lambda^2+1625\lambda^4+2500\lambda^6)\\&+8\times 2^{3/4}e^{5\pi\lambda}(1+e^{5\pi\lambda})r_0^5\lambda^{5/4}(1+5\lambda^2+4\lambda^4)^{5/4}\left((149b_1+26b_3+24b_5)\lambda\right.\\&\left.-6(b_4+4b_6)-b_2(9+110\lambda^2+125\lambda^4)+5\lambda^2(-6b_4+5\lambda(2b_3+5b_1(6+5\lambda^2)))\right),\\ B =&(9+259\lambda^2+875\lambda^4+625\lambda^6)\Big(8\lambda+40\lambda^3+32\lambda^5+(1-e^{4\pi\lambda})r_0^4\\&(a_3+3a_5-(5a_2+3a4)\lambda\\&+4a_3\lambda^2-8a_2\lambda^3+a_1(3+32\lambda^2+32\lambda^4))\Big)^{5/4}.\end{aligned}$$

Now consider the equation $A^4-B^4=0$ and we obtain the polynomial

$$\begin{aligned}p_{12}(r_0)=&C_{0,12}+C_{4,12}r_0^4+C_{5,12}r_0^5+C_{8,12}r_0^8+C_{10,12}r^{10}\\&+C_{12,12}r_0^{12}+C_{15,12}r_0^{15}+C_{16,12}r_0^{16}+C_{20,12}r^{20},\end{aligned}$$

where

$$\begin{aligned}C_{0,12}=&32768(-1+e^{40\pi\lambda})(1+\lambda^2)^9(\lambda+4\lambda^3)^5(9+250\lambda^2+625\lambda^4)^4,\\ C_{4,12}=&20480(-1+e^{4\pi\lambda})\lambda^4(1+\lambda^2)^8(1+4\lambda^2)^4(9+250\lambda^2+625\lambda^4)^4\\&(a_3+3a_5-(5a_2+3a_4)\\&\lambda+4a_3\lambda^2-8a_2\lambda^3+a_1(3+32\lambda^2+32\lambda^4)),\end{aligned}$$

$$
\begin{aligned}
C_{5,12} =& -131072e^{35\pi\lambda}(1+e^{5\pi\lambda})(1+\lambda^2)^8(9+250\lambda^2+625\lambda^4)^3\big(24b_6+6b_4(1+5\lambda^2)\\
&+b_2(9+110\lambda^2+125\lambda^4)-\lambda(149b_1+26b_3+24b_5\\
&+50(15b_1+b_3)\lambda^2+625b_1\lambda^4)\big)(\lambda+4\lambda^3)^5,\\
C_{8,12} =& -5120(-1+e^{4\pi\lambda})^2(1+\lambda^2)^7(\lambda+4\lambda^3)^3(9+250\lambda^2+625\lambda^4)^4\\
&(a_3+3a_5-(5a_2+3a_4)\\
&\lambda+4a_3\lambda^2-8a_2\lambda^3+a_1(3+32\lambda^2+32\lambda^4)\big)^2,\\
C_{10,12} =& 196608e^{30\pi\lambda}(1+e^{5\pi\lambda})^2(1+\lambda^2)^7(\lambda+4\lambda^3)^5(9+250\lambda^2+625\lambda^4)^2\\
&\big(24b_6+6b_4(1+5\lambda^2)\\
&+b_2(9+110\lambda^2+125\lambda^4)-\lambda(149b_1+26b_3+24b_5\\
&+50(15b_1+b_3)\lambda^2+625b_1\lambda^4)\big)^2,\\
C_{12,12} =& 640(-1+e^{4\pi\lambda})^3(1+\lambda^2)^6(\lambda+4\lambda^3)^2(9+250\lambda^2+625\lambda^4)^4\\
&\times\big(a_3+3a_5-(5a_2+3a_4)\lambda+4a_3\lambda^2-8a_2\lambda^3+a_1(3+32\lambda^2+32\lambda^4)\big)^3,\\
C_{15,12} =& -131072e^{25\pi\lambda}(1+e^{5\pi\lambda})^3(1+\lambda^2)^6(9+250\lambda^2+625\lambda^4)\big(24b_6+6b_4(1+5\lambda^2)\\
&+b_2(9+110\lambda^2+125\lambda^4)-\lambda(149b_1+26b_3+24b_5+50(15b_1+b_3)\lambda^2\\
&+625b_1\lambda^4)\big)^3(\lambda+4\lambda^3)^5,\\
C_{16,12} =& -40(-1+e^{4\pi\lambda})^4\lambda(1+\lambda^2)^5(1+4\lambda^2)(9+250\lambda^2+625\lambda^4)^4\\
&\big(a_3+3a_5-(5a_2+3a_4)\lambda+4a_3\lambda^2-8a_2\lambda^3+a_1(3+32\lambda^2+32\lambda^4)\big)^4,\\
C_{20,12} =& (-1+e^{4\pi\lambda})^5(1+\lambda^2)^4(9+250\lambda^2+625\lambda^4)^4\big(a_3+3a_5-(5a_2+3a_4)\lambda+4a_3\lambda^2\\
&-8a_2\lambda^3+a_1(3+32\lambda^2+32\lambda^4)\big)^5+32768e^{20\pi\lambda}\lambda^5\,(1+e^{5\pi\lambda})^4\\
&\big(24b_6+6b_4(1+5\lambda^2)+b_2(9+110\lambda^2+125\lambda^4)-\lambda(149b_1+26b_3+24b_5\\
&+50(15b_1+b_3)\lambda^2+625b_1\lambda^4)\big)^4(1+5\lambda^2+4\lambda^4)^5.
\end{aligned}
$$

By Descartes Theorem the polynomial $p_{12}(r_0)$ can have at most eight positive real roots. For instance, for the values $\lambda = 1$, $a_1 = -1$, $a_2 = 0$, $a_3 = 0$, $a_4 = 1$, $a_5 = -1$, $b_1 = b_3 = b_4 = 0$, $b_2 = b_6 = -1$ and $b_5 = 1$ there are eight variations of sign between the coefficients, that is, $p_{12}(r_0)$ can have at most eight positive real roots. Therefore the piecewise differential system separated by the straight line $y = 0$ with the vector fields X_5 and Y_6 can have at most eight limit cycles. Statement (n) of Theorem 1 is proved.

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