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PASCAL AND A TRIANGULAR BILLIARD

LILIANA GHEORGHE AND RONALDO GARCIA

Abstract. Is any triangle fit to be a billiard orbit in some ellipse? Is it unique? Can we draw it as a point-conic? We show that this is always the case and provide a synthetic proof leading to a new construction for both the billiard and its caustic. The link between the billiard, incircle, and Euler circle emerges almost naturally. Neat geometric ties between the caustic and exinscribed circles are also brought to light.

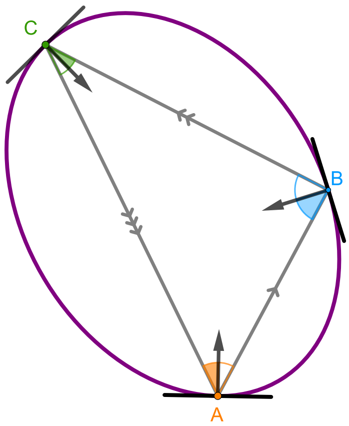

Figure 1. A triangular path obeying a billiard law: the incidence angle equals the reflection angle.

1. Introduction

An elliptic billiard is an ellipse in which a reflection law is in place: the incidence angle equal the reflection angle. Between collisions, the particle moves in straight line.

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If the polygonal orbit closes in three steps, we shall call both the orbit, as well as the ellipse itself a (triangular) billiard, or simply a billiard; see Figure 1.

Is any triangle is fit to be a billiard orbit?

This is a rhetorical question as it already has a well-known answer: any triangle is a (triangular) orbit inside the orthic conic [1] of its anti-pedal triangle. So, why bother?

Firstly, because the orthic conic is not a five-point conic, hence dynamical software, such as Geogebra, cannot perform it without human intervention. Secondly, because out proof shed new light on triangular billiards, placing it between the Euler circle and the incircle.

Main results. We initially regard the billiard as a six-point conic, later as a nine-point conic; see Theorem 3.1 and Corollary 4.1. In addition, we specify the caustic as a six-point conic; see Theorem 5.1.

Related work. Apparently, the first research on the general properties of systems formed by light rays concerns the caustics, dating back to 1682 and are attributed to Tschirnhausen. Decades later, Bernoulli, Carré, de l'Hôpital, just to mention a few, obtained the equation of plane caustics by reflection and refraction, regardless of the nature of the reflecting or refracting curve; see [11] for a summary of the state of the art in the mid-1800s.

Despite being the most Euclidean among conics —as the property of being a billiard is not preserved by projectivities— Euclidean phenomena in triangular (elliptic) billiards have not received explicit attention until recent years. A sample of articles are duly collected in [7]. An explicit parametrization of the triangular orbits of a billiard is found in [5] For a visual experience, refer to [5], [13] A comprehensive and detailed description in barycentric coordinates of the billiard as an isogonal conjugate can be found in [9]. Nine-point conics, which are natural extensions of the nine-point circle, are explored in the recent paper [12]. The finely-tuned paper [14] puts in perspective old and new facts on billiard poristic grids.

Notations. We shall denote by $\triangle = \triangle ABC$ the reference triangle, by $\triangle_m = \triangle A_mB_mC_m$ the midline triangle i.e. the triangle whose vertices are the midpoints of the sides of a $\triangle ABC$, and by $\triangle_a$ the anti-complementary triangle of $\triangle$, i.e. the triangle where the vertices of $\triangle$ are midpoints. By $\triangle_c = \triangle A_cB_cC_c$ we shall denote the triangle whose vertices are the contact points of the incircle of the anti-complementary triangle $\triangle_a$, with $\triangle$'s sides.

The triangular centers will be referred to as X, X_m, X_a, X_c respectively. At times, we shall use both classification of these centers, as in [8], as well as their classic names; for instance, $Na_m = X_{8m}$ denotes the Nagel point of the midline triangle. In some of the statements, we opt for a (deliberate) mixture of notations for triangular centers: some of them are foreseen as points of $\triangle$, other as points of $\triangle_m$, $\triangle_a$ or $\triangle_c$. We did so, as we tried to keep the triangular center number names as low as possible, in order to keep track of their geometrical properties.

[1] the conic that is tangent to the triangle's sides at the altitudes' feet

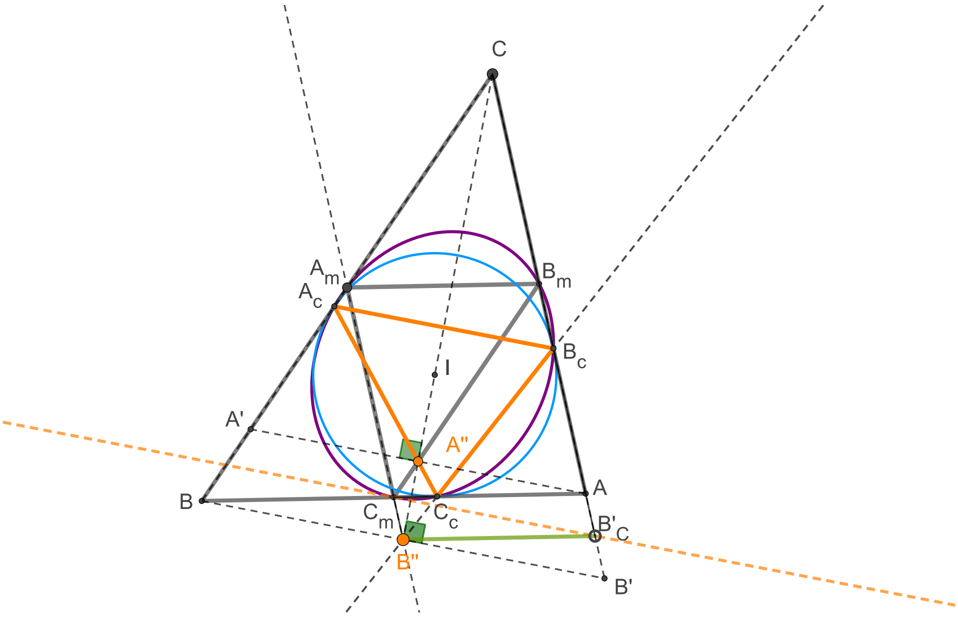

FIGURE 2. $\triangle_m$ is a billiard orbit in an ellipse $\mathcal{B}_m$, if and only if the tangents to $\mathcal{B}_m$ at the vertices of $\triangle_m$ are respectively parallel to the sides of $\triangle_c$.

On references. Since we deal with classic objects and facts, finding the exact source is often challenging. In most cases, we cite the most readily available one.

2. Some technical tips

2.1. On the tangents at the billiards' orbits.

Lemma 2.1. *Let $\triangle = \triangle ABC$ be a reference triangle and let $\triangle_m = \triangle A_m B_m C_m$ be the midbase triangle, whose vertices are the mid-points of the sides of $\triangle$. Let $\triangle_c = \triangle A_c B_c C_c$ be the contact triangle, whose vertices are the contact points of the inscribed circle, with the sides of $\triangle$. Finally, let $\mathcal{B}_m$ be any ellipse that circumscribes $\triangle_m$. Then $\triangle_m$ is a billiard orbit (in $\mathcal{B}_m$) if and only if the tangents to $\mathcal{B}_m$ in A_m, B_m, and C_m are respectively parallel to the sides of the contact-triangle $\triangle_c$.*

Proof. $\triangle_m$ is a billiard orbit within an ellipse $\mathcal{B}_m$, iff the tangents drawn at A_m, B_m, C_m to $\mathcal{B}_m$ are perpendicular to the internal bisectors of $\triangle_m$. These internal bisectors are, in turn, parallel to the internal bisectors of $\triangle$; hence, it suffice to prove that the internal bisectors of $\triangle$ are respectively perpendicular to the sides of the contact triangle $\triangle_c$, shown in Figure 2

To see this, note that the internal bisectors of $\angle BCA$, $\angle ABC$, $\angle BCA$ of $\triangle$ are also the internal bisectors of the angles $\angle A_c C B_c$, $\angle B_c A C_c$, and $\angle C_c B A_c$, viewed as angles in $\triangle A_c C B_c$, $\triangle B_c A C_c$, and $\triangle C_c B A_c$, respectively. Since the sides of ABC are tangent to the inscribed circle at A_c, B_c, C_c, these three triangles are obviously isosceles ($CA_c = CB_c$, $BA_c = BC_c$, $AB_c = AC_c$), hence the internal bisector of $\angle A_c C B_c$, $\angle B_c A C_c$, and $\angle C_c B A_c$, are perpendicular to the bases $A_c B_c$, $B_c C_c$, and $A_c C_c$. In other words, the

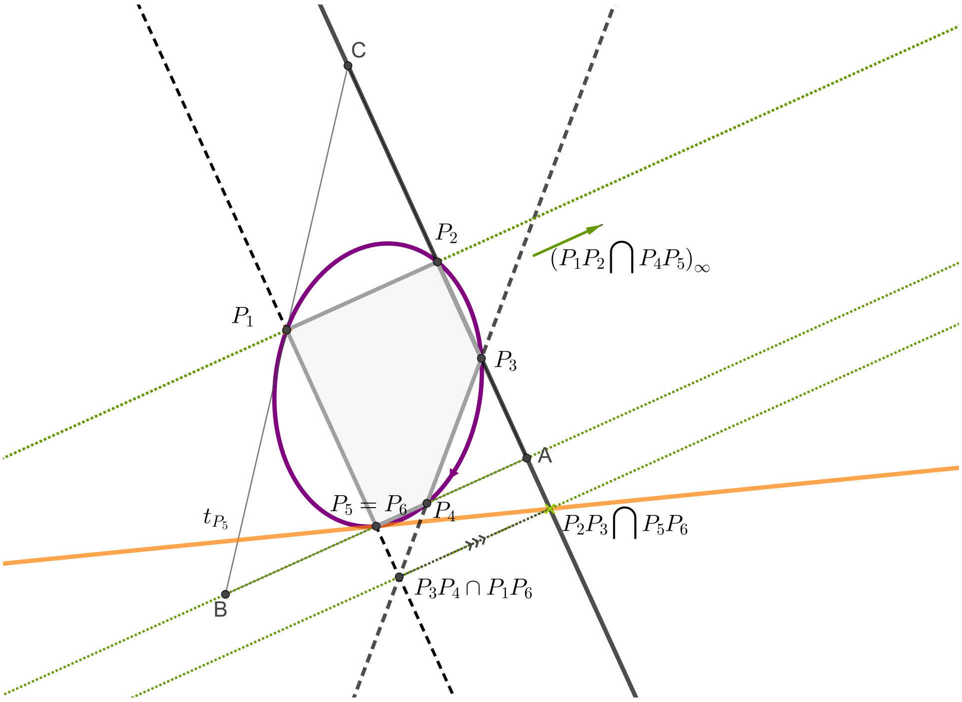

Figure 3. The construction of the tangent in $P_5 = P_6$ to the conic $[P_1P_2P_3P_4P_5]$, via Pascal theorem.

internal bisectors of $\angle A$, $\angle B$, and $\angle C$, are respectively perpendicular to the sides of $\triangle_c$, ending the proof.

2.2. **The tangent to a five-point conic, via Pascal theorem.** Another ingredient for our construction of a billiard is the geometric construction of a tangent to a five-point conic, utilizing Pascal's theorem. Although this is a classic fact, we include it here for the sake of readability.

Lemma 2.2. *Let $P_1P_2P_3P_4P_5$ be the vertices of a pentagon $\mathcal{P}$; see Figure 3. Let $P_1P_2 \bigcap P_4P_5 = Q_1$, $P_1P_5 \bigcap P_3P_4 = Q_3$, and let $P_5Q_1 \bigcap P_2P_3 = Q_2$. Then the tangent in P_5 to the five-point conic that circumscribes the pentagon $\mathcal{P}$ is the line P_5Q_2*

Proof. The tangent at P_5 to the five-point conic $P_1P_2P_3P_4P_5$ is the line passing through P_5 whose direction is the limit of (the directions of) the secants P_5P_6, where P_6 is a point on the conic approaching P_5.

The tangent at P_5 can be interpreted as a degenerate sixth side P_5P_6 of a hexagon inscribed into the conic, whose two vertices P_5 and P_6 coincides; as such, it can be obtained via Pascal theorem, as follows.

Let P_1P_2 intersect the line P_4P_5 at a point Q_1; then, let the lines P_3P_4 and $P_6P_1 = P_5P_1$ intersect at Q_3. According to Pascal theorem, the lines P_2P_3, P_5P_6 and Q_1Q_3 should intersect at a common point. In our case, the points P_5 and P_6 coincide, the direction of P_5P_6 (which represent the direction of the tangent to the five-point conic at the point P_5) is indeterminate. Nevertheless, the direction of the tangent at P_5 is the limit of the directions of the secants P_5P_6 passing thru P_5, as the point P_6 approaches P_5 along the

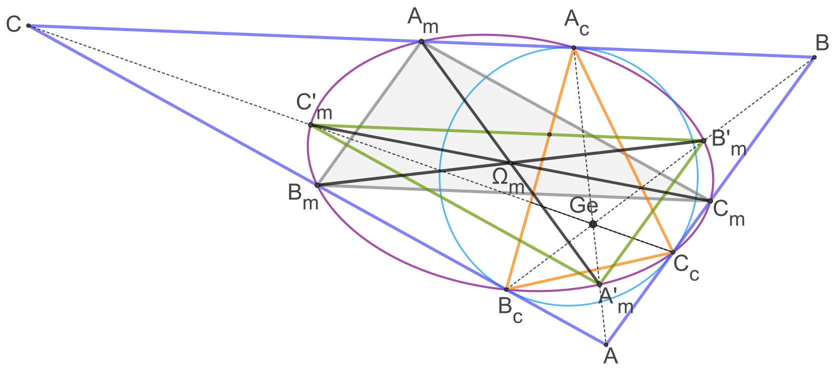

Figure 4. The midpoints of the sides A_m, B_m, C_m, the contact points of the incircle with the sides A_c, B_c, C_c, and the midpoints A'_m, B'_m, C'_m of the segments AGe, BGe, CGe, are on a conic, $\mathcal{B}_m$. This conic is centered at the Mittelpunkt Ω_m, which is the (common) midpoint of the segments $A_mA'_m$, $B_mB'_m$, $C_mC'_m$.

conic. Therefore, the tangent thru P_5 will necessarily intersect the Pascal line Q_1Q_3 at the same point where the line P_2P_3 does. Denoting this point by Q_2, we conclude that the tangent at P_5 to the conic is precisely the line P_5Q_2.

If $P_1P_2 \parallel P_4P_5$, then the point Q_1 does not exists; it is an improper point, called infinity point in the common direction of these lines. In this case, the line that join any (proper) point Q, with this improper point is nothing but the parallel through Q to (both) P_1P_2 and P_4P_5.

3. A six-point billiard

Now we are able to prove our main result.

Theorem 3.1. *In any triangle the mid-points of the sides and the contact-points of the incircle with the sides are on a conic, $\mathcal{B}_m$, in which $\triangle_m$ is a billiard orbit.*

Proof. $\triangle_m$ and $\triangle_c$ are both cevian triangles; the former is the cevian of the barycenter and the later is the cevian of Gergonne's point. By Carnot theorem [2],[4] these six points lie on a conic, denoted as $\mathcal{B}_m$. In order to prove that $\triangle_m$ is a billiard orbit (in $\mathcal{B}_m$), we need to show that the tangents at A_m, B_m, and C_m to $\mathcal{B}_m$ coincide with the external bisectors of the angles $\angle A_m$, $\angle B_m$, and $\angle C_m$, respectively. To establish this, according to Lemma 2.1, we need to show that the tangent at C_m to the six-point conic $\mathcal{B}_m$ is parallel to A_cB_c, and similarly for the other vertices. For clarity, we shall rename these points. Let the points

$$A_m, B_m, B_c, C_c, C_m \text{ be, respectively } P_1, P_2, P_3, P_4, P_5,$$

and let a sixth point P_6 coincide with P_5. The pentagon $P_1P_2P_3P_4P_5$, is inscribed into a conic: $\mathcal{B}_m$; hence, so is "the hexagon" $P_1P_2P_3P_4P_5P_6$. The tangent at C_m (alias P_5, or alias P_6) to $\mathcal{B}_m$, can be obtained via Pascal theorem, as explicitly described in Lemma 2.2.

So let P_1P_2 (alias A_mB_m) intersect the line P_4P_5 (alias C_cC_m) at a point Q_1; since these two parallel lines, then Q_1 is the point at infinity in the direction of the line P_1P_2 (alias A_mB_m).

Let B'' be the intersection point of P_3P_4 (B_cC_c) and P_6P_1 (C_mA_m); then the[2] Pascal line is the parallel through B'' to the line P_1P_2 (A_mB_m).

We need to intersect the lines P_2P_3 (or B_mB_c) and the line P_5P_6. As the points P_5 and P_6 coincide, the direction of this line (which is the direction of the tangent to the five-point conic at the point P_5) is not determinate. Nevertheless, as in Lemma 2.2, the tangent through P_5 to $\mathcal{B}_m$ necessarily meet the Pascal line of $P_1P_2P_3P_4P_5P_6$ at the same point where the line P_2P_3 does. Therefore, the tangent at P_5 (or C_m) to $\mathcal{B}_m$ is precisely the line $C_mB'_C$, where B'_C is the intersection point of the parallel through B'' to AB, with the line B_mB_c.

To finish the proof, we need to show that

$$C_mB'_C \parallel A_cB_c. \tag{1}$$

Let B' (resp. A') be the reflections of B (resp. A) about the internal bisector of $\angle C$. By Lemma A.1 the lines A_mC_m, CI, B_cC_c meet at a point B'', which is the projection of B onto the internal bisector CI. Thus, B'' is the midpoint of BB'. By hypothesis, $B''B'_C \parallel AB$, hence $B''B'_C$ is a midline in $\triangle ABB'$ and B'_C is the midpoint of AB'. Also by hypothesis, C_m is the midpoint of AB; hence the segment B'_CC_m is a midline, too, thus

$$C_mB'_C \parallel BB'. \tag{2}$$

Since CA_c and CB_c are the two tangents from C to the incircle, then

$$A_cB_c \perp CI,$$

hence $BB' \parallel AA' \parallel A_cB_c$ and therefore, $C_mB'_C \parallel A_cB_c$, ending the proof.

4. A nine-point billiard

The three lines AA_c, BB_c, CC_c, which join the vertices of a triangle, with the contact-points of the inscribed circle, are concurrent. Their common point is the Gergonne point, $Ge = X_7$.

Recall that any six-point conic passing through the midpoints A_m, B_m, C_m, as well as through the feet of the cevians through any point P, which does not lie on its sides, also passes through the midpoint of the segments AP, BP, CP, making it a nine-point conic; see for example [3] The results just proved can be restated as follows.

Corollary 4.1. *The six-point billiard $\mathcal{B}_m$ of $\triangle_m$ is the nine-point conic with the cevian point being the Gergonne point.*

In terms of the triangle, this can be restated as follows.

Corollary 4.2. *The conic in which ABC is a billiard orbit, is a six-point conic that passes through A, B, C and through the feet of the cevians through the Gergonne point of its anti-complementary triangle.*

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[2]the Pascal line associated with this order of the six vertices!

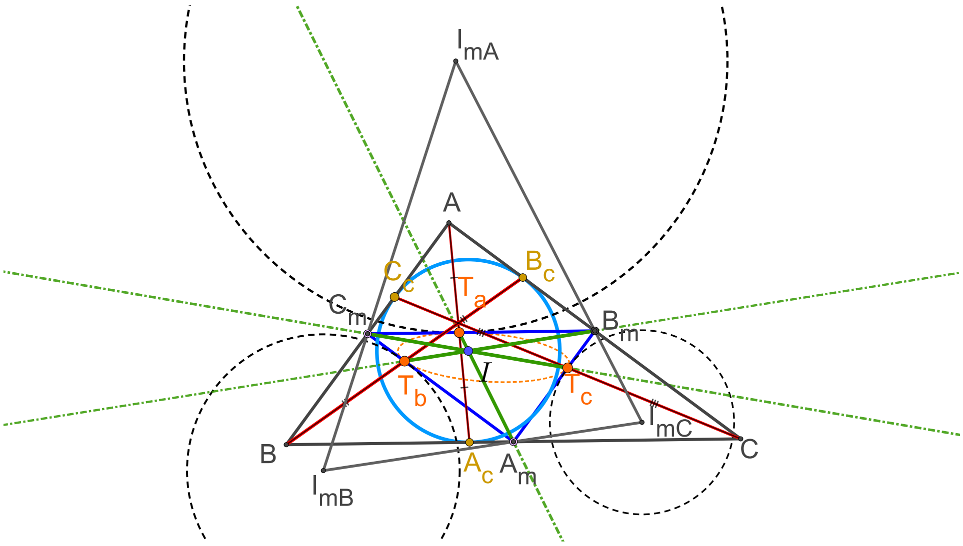

Figure 5. The lines AA_c, B_mC_m, and IA_m are concurrent at T_a, the contact point of the excircle of $\triangle_m$ with the side B_mC_m.

At this point, the simplest way to find the center of a billiard, is to foresee it as a nine-point conic.

Proposition 4.1. *With notations as in Figure 4, the center of a triangular billiard $\mathcal{B}_m$, circumscribed to $\triangle_m$ is the (common) midpoint of three segments: $A_mA'_m$, $B_mB'_m$, and $C_mC'_m$.*

Proof. If A'_m, B'_m, C'_m are the midpoints of AGe, BGe, CGe, then by [3] the lines $A_mA'_m$, $B_mB'_m$, $C_mC'_m$ meet at their common midpoint, which coincides with the billiards's center Ω_m.

While the sides of the medial triangle $\triangle_m$ obey a billiard law in the billiard $\mathcal{B}_m$, the sides of the contact triangle $\triangle_c$ do not; see Figure 4. What distinguishes the two is the fact that the sides of $\triangle_m$ are tangent to another special conic, confocal with $\mathcal{B}_m$ (the caustic), while the sides of $\triangle_c$ are not.

5. The caustic

In any elliptic billiard, regardless of whether the orbit closes or not, and regardless of the initial point, all the sides of any polygonal orbit are tangent to one and the same conic: the caustic. The caustic is always a conic that is confocal with the elliptic billiard $\mathcal{E}$. The nature of the caustic depends on the first strike. If this does not intersect the focal segment, the caustic will be a confocal ellipse; otherwise, is a hyperbola. If the first strike passes through the focus, the caustic does not exist. These are well-established facts; for concise proofs, refer to [10] These being recalled, we shall give here a construction of a caustic as a six-point conic. This method, which only works for caustics of triangular orbits, diverges from the standard approach used in constructing a caustic within a billiard grid, and uses the inscribed circle and the Euler circle.

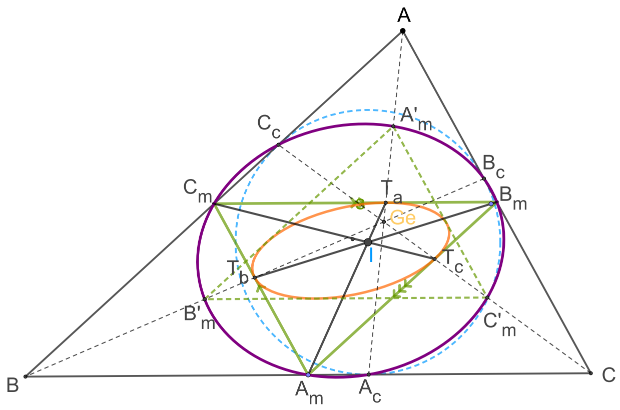

Figure 6. The caustic $\mathcal{K}_m$ is the conic inscribed in $\triangle_m$ and whose perspector is the Nagel point X_{8m}, which coincides with $I = X_1$.

Lemma 5.1. *Let $\triangle$, $\triangle_c$, and $\triangle_m$ be as in Figure 5. Then the lines AA_c, B_mC_m, and IA_m are concurrent at a point T_a, which is the contact point of the exscribed circle of $\triangle_m$ centered at I_{mA}.*

Proof. This is stated in [4.5.6, p.39 [2]]. A direct proof in barycentric coordinates is similar to the proof of Lemma A.1 and we omit.

Theorem 5.1. *The conic $\mathcal{K}_m$, centered at the Mittenpunkt Ω_m of $\triangle_m$ and passing through the mid-points of the segments AA_c, BB_c, CC_c, joining the vertices of A, B, C with the contact points with the incircle, is the caustic for the billiard of $\triangle_m$.*

Proof. Let T_a, T_b, T_c, be the midpoints of the segments AA_c, BB_c, CC_c; in barycentric coordinates
(3)
$T_a = [2a, a+b-c, a+c-b]$, $T_b = [a+b-c, 2b, b+c-a]$, $T_c = [a+c-b, b+c-a, 2c]$.

The conic $\mathcal{K}_m$ centered in

$$\Omega_m = [a(b+c) - (b-c)^2, b(a+c) - (c-a)^2, c(a+b) - (a-b)^2]$$

and passing through T_a, T_b, T_c is
(4)

$$f(a,b,c)x^2 + f(b,c,a)y^2 + f(c,a,b)z^2 + g(a,b,c)xy + g(b,c,a)yz + g(c,a,b)xz = 0$$

$$f(a,b,c) = -5c^4 + 4(3a+b)c^3 + (-6a^2 - 12ab + 2b^2)c^2 + (3a+5b-4c)(a-b)^3$$

$$g(a,b,c) = 6c^4 - 16(a+b)c^3 + (12a^2 + 8ab + 12b^2)c^2 - 2(a-b)^4$$

This result can be restated as follows.

Corollary 5.1. *The caustic (of a billiard in which a triangle is a billiard orbit) is the inscribed conic whose perspector is the triangle's Nagel point.*

Proof. For convenience, let's choose the caustic of $\mathcal{B}_m$ as the reference, as in Figure 6 According to [ex. 4.5.6, p.39, [2]] the three cevians A_mT_a, B_mT_b, and C_mT_c intersect at a common point I, which is the Nagel point of the midline triangle, as well. To conclude the proof, we recall from Theorem 5.1,

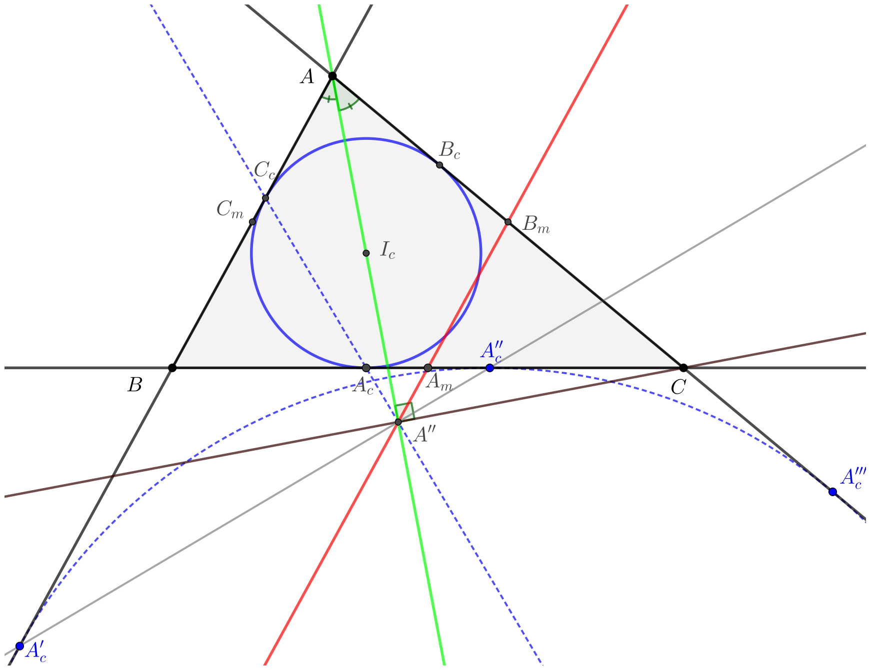

Figure 7. A five-line theorem: the internal bisector of $\angle A$, the midline A_mB_m, the internal contact-chord A_cC_c, the external contact-chord $A'_cA''_c$ and the perpendicular through C to the internal bisector of vertex A meet at the point $A'' = [c-b, b, c]$.

that T_a, T_b, and T_c are the contact points of the caustic with the sides of $\triangle_m$.

Appendix A. A technical lemma

The following fact, rooted in triangle geometry, is stated as an exercise in [4.5.4,p.29,[2]]. For convenience, we provide a direct proof based on straightforward verification; see Figure 7.

Lemma A.1. *In any triangle $\triangle = ABC$ the following five lines are concurrent:*

i) *AI_c, the internal bisector of $\angle A$;*
ii) *the midline A_mB_m;*
iii) *the internal contact-chord A_cC_c,*
iv) *the external contact-chord $A'_cA''_c$,*
v) *the perpendicular through C to AI_c.*

Proof. In barycentric coordinates the midpoints and the contact points of $\triangle$ are, respectively $A_m = [0,1,1], B_m = [1,0,1], C_m = [1,1,0]$, $A_c = [0, 1/(a-b+c), 1/(a+b-c)], B_c = [1/(-a+b+c), 0, 1/(a+b-c)]$ and $C_c = [1/(-a+b+c), 1/(a-b+c), 0]$. Also, the two external contact points are $A'_c = [-a-b+c, a+b+c, 0]$ and $A''_c = [0, a-b+c, a+b-c]$.

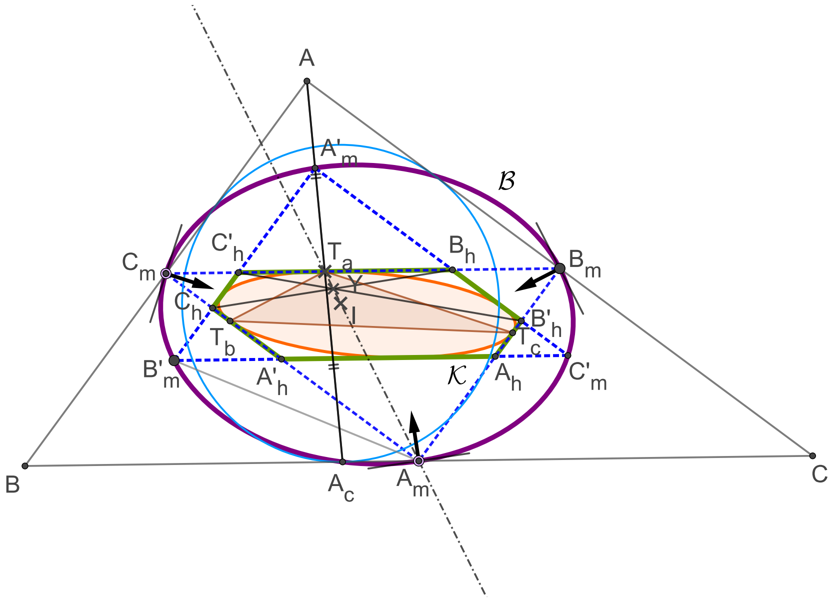

Figure 8. T_a, the contact point of the caustic with B_mC_m is the intersection of the line A_mY with B_mC_m.

The incenter is $I_c = [a, b, c]$, the lines A_mB_m (midline), A_cB_c (contact-chord), and AI_c (internal bisector) are given by:

$$\begin{aligned} A_mB_m &: x + y - z = 0 \\ A_cC_c &: (-a + b + c)x - (a - b + c)y + (a + b - c)z = 0 \\ AI_c &: cy - bz = 0 \\ C^{\perp} &: bx + (b - c)y = 0 \\ A'_cA''_c &: (a + b + c)x + y(a + b - c) + (-a + b - c)z = 0; \end{aligned}$$

their common point is clearly the point $A'' = [c - b, b, c]$.

Finally, to prove that the triangle $\triangle AA''C$ is rectangle, we check the Pythagoras's theorem. In fact, $|AA''|^2 = b(a + b + c)(b + c - a)/(4c)$ and $|CA''|^2 = b(a+b-c)(a-b+c)/(4c)$. Therefore, $|AA''|^2+|CA''|^2 = |AC|^2 = b^2$.

Appendix B. Caustic via Brianchon theorem

For the sake of completion, we present the classic construction of the caustic via Brianchon theorem.

Let $\triangle_m$ and $\triangle'_m$, its reflection about the Mittelpunkt (of $\triangle_m$ determine a hexagon $\mathcal{H} = A_hA'_hB_hB'_hC_hC'_h$, whose diagonals are concurrent; then, the conic inscribed in this hexagon is the caustic $\mathcal{K}$ of the billiard of $\triangle_m$. In order to obtain the contact points between $\mathcal{K}$ and the sides of $\triangle_m$ and $\triangle'_m$ let $Y = B_hC_h \bigcap B'_hC''_h$; by Brianchon theorem, the contact point T_a of $\mathcal{K}$ with the line B_mC_m is the intersection of the line A_mY, with B_mC_m; similarly we obtain other five contact points.

Remarkably, I, the i-center of ABC is on the line A_mY, too. Direct, yet long calculations in barycentric coordinates, that we skip, confirm these facts.

Since I coincide with the Nagel point of $\triangle_m$ the later fact also tells us that T_a is the contact point of the excircle of $A_mB_mC_m$, with the side B_mC_m.

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DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE FEDERAL DE PERNAMBUCO
RECIFE, (PE) BRASIL
E-mail address: `liliana@dmat.ufpe.br`

ISTITUTO DE MATEMÁTICA E ESTATÍSTICA
UNIVERSIDADE FEDERAL DE GOIÁS
GOIÂNIA, (GO) BRASIL
E-mail address: `ragarcia@ufg.br`