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Research paper

Global well-posedness for a quasilinear combustion model in multilayer porous media

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ABSTRACT

We investigate a class of nonlinear reaction-diffusion-convection systems modeling combustion fronts in multilayer porous media. The model describes the coupled dynamics of temperature and fuel concentration, departing from previous approaches that assumed prescribed fuel profiles to simplify the analysis. By integrating the fuel concentration equations, we derive a non-autonomous quasilinear evolution problem formulated purely in terms of the temperature vector. To address the analytical difficulties arising from the absence of a linear part in the differential operator, we introduce a higher-order regularization strategy that ensures control over second derivatives in Sobolev spaces. Using Banach's fixed-point theorem and detailed Sobolev estimates, we establish global-in-time existence, uniqueness, and continuous dependence on both initial data and model parameters. By removing the assumption of prescribed fuel profiles and regularizing the fully nonlinear system, this work provides the first global well-posedness result for a quasilinear combustion model in multilayer porous media, with numerical simulations confirming the theoretical predictions and capturing physically consistent front propagation.

1. Introduction

Combustion in porous media has attracted significant attention due to its relevance in various industrial applications, including heat exchangers, waste-to-energy systems, and enhanced oil recovery. Notably, Trimis and Durst [1] developed a combustion-based heat exchanger that exploits the large surface area of porous structures to enhance heat transfer and allows precise control over combustion temperatures. Experimental studies, such as those by Chen et al. [2], revealed the propagation dynamics of combustion waves in foamed porous media, while Mujeebu et al. [3] provided a broad survey highlighting the diversity and practical scope of combustion systems in porous media. Additionally, applications to stratified geological formations, such as in-situ combustion, have been studied extensively [4,5]. Majdinasab et al. [6] provided a comprehensive review of landfill gas generation models, discussing their specific characteristics and categorizing them into different types, such as mathematical, numerical, and first-order decay models.

From a modeling standpoint, Da Mota and Schecter [7] proposed a two-layer model describing lateral combustion front propagation in porous media with distinct physical properties, the porous media where crude oil is found contain different layers that are characterized by differences in porosity, density, thermal conductivity, etc (see [8]). Their formulation admits traveling wave solutions connecting burned and unburned states, with the wave profiles depending on parameters such as interlayer heat transfer. Assuming that the fuel concentrations in both layers are prescribed functions, the existence and uniqueness of classical solutions to

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the associated Cauchy problem were established in [9] using the monotone iteraction method of upper and lower solutions. Assuming that the fuel concentrations are not prescribed, that is, they are also treated as dependent variables, the existence of a solution to the Cauchy problem was proved in [10] using the Arzelá-Ascoli's theorem and Zorn's Lemma.

This two-layer model was extended by Batista and Da Mota [11] to an arbitrary number of layers, leading to a coupled reaction-diffusion-convection system in which the fuel concentrations were assumed to be known functions. Existence and uniqueness of classical solutions were then obtained using monotone iteration techniques. Later, in [12], the assumption of prescribed fuel concentrations was removed, and the existence of local and global solutions was established using Schauder's fixed-point theorem, although uniqueness and continuous dependence were not addressed.

A new version of the layered combustion model introduced in [11], incorporating spatially varying parameters, was recently proposed in [8]. Analytical results for the corresponding Cauchy problem were obtained in this work and in [13], using semigroup theory and Kato's method. These analyses retained the simplifying assumption that the fuel concentrations were prescribed functions, thereby preserving the linearity of the evolution operator.

In the present work, we generalize the mathematical system modeling combustion in a porous medium by considering both temperature and fuel concentration as unknown variables. The problem consists of a nonlinear reaction-diffusion system of PDEs governing the temperature dynamics, derived from the energy balance, coupled with a system of ordinary differential equations (ODEs) modeling the fuel concentrations according to the Arrhenius law.

Following the integration of the ODEs, the problem reduces to a quasilinear PDE system depending solely on the temperature vector $u = (u_1, \dots, u_n)$, which takes the form:

$$\begin{cases} \partial_t u + L(x,t,u)u = f(x,t,u), & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \phi(x), \end{cases} \tag{1}$$

where $L(x,t,u)$ is a nonlinear differential operator reflecting diffusion, convection, and reactive terms, and $f(x,t,u)$ includes source terms arising from the fuel dynamics (see Section 2). This formulation leads to a non-autonomous quasilinear evolution equation outside the scope of classical semigroup theory.

Our main contributions are as follows:

- We address the full system where both temperatures and fuel concentrations are unknowns to be determined. This significantly generalizes previous models [7,10–12], removing a key simplifying assumption and allowing for more realistic simulations of combustion fronts in heterogeneous porous structures.
- We introduce a novel fourth-order parabolic regularization strategy for this quasilinear problem, inspired by ideas from [14–16], which provides H^s-norm control of second derivatives that would not be controlled by standard second-order smoothing.
- We prove the global-in-time existence and uniqueness of solutions in Sobolev spaces $H^s(\mathbb{R})^n$, for $s \geq 2$, even in the absence of a linear component in the evolution operator. This is achieved using energy estimates combined with a fixed-point argument.
- We establish uniform bounds via Banach's fixed-point theorem, in contrast with the more traditional approaches based on Kato-Ponce-type commutator estimates [17].
- We prove continuous dependence of solutions on both initial data and model parameters, a property not previously addressed in the more general model formulations.
- We present numerical simulations to support the analytical findings and show consistency with expected physical behavior, in the spirit of recent simulation-based studies such as [8,18].

These advances overcome limitations in earlier analyses, which typically assumed known fuel profiles or employed restrictive analytical frameworks, such as semigroup theory or monotone iterations under tight conditions.

We now state our two main theorems, which summarize the key mathematical results of this work.

Theorem 1.1 (Global solution). *Suppose $s \geq 2$, and let R be a constant defined in (13). If $\phi = (\phi_1, \dots, \phi_n) \in H^s(\mathbb{R})^n$, then for all $T > 0$, there exists a unique solution $u = (u_1, \dots, u_n) \in C\big([0,T], H^s(\mathbb{R})^n\big)$ to the initial-value problem (1), in the sense of the H^{s-2}-norm.*

Theorem 1.2 (Continuous dependence). *Under the same assumptions as in Theorem 1.1, the solution $u \in C\big([0,T], H^s(\mathbb{R})^n\big)$ depends continuously on the initial data ϕ and on the model parameters, for every $T > 0$.*

Since the IVP (1) has a quasilinear nature, one might consider applying Kato's quasilinear theory [19]. Nevertheless, we observe that this approach does not yield better results than the parabolic regularization technique. Indeed, by applying this theory, the hypothesis A3 of Theorem 6 (see [19]) imposes an additional regularity condition on the initial data, which requires us to have $s > \frac{5}{2}$, instead $s \geq 2$.

The remainder of this article is organized as follows. Section 2 provides a concise summary of the model rigorously derived in [8]. In Section 3, we introduce the notation and definitions used throughout this work. The proof of Theorem 1.1 is presented in Section 4 using a regularization approach. This proof relies on a sequence of estimates, established in the H^s, L^2, and L^∞ norms, which are proved by lemmas introduced earlier.

In Section 5, we establish the continuous dependence of solutions with respect to both the initial data and model parameters, based on uniform estimates and convergence arguments. Section 6 presents numerical simulations that validate the analytical results, demonstrating consistency between the integral solution and the profiles obtained via finite difference methods, as well as agreement with the expected physical behavior. Section 7 discusses the implications of our results and outlines directions for future research.

Appendix A includes background material on fractional derivatives, including the definition of the Stein derivative employed in Section 4. Finally, a table with some physical parameters used in the numerical simulations is presented in Appendix B.

2. Summary of the model

The model is based on the full formulation developed in [8], where a horizontal one-dimensional porous medium composed of n parallel layers was considered. Each layer initially contains a solid fuel, such as coke, with a prescribed concentration. Physical parameters within each layer are assumed to be functions of the spatial variable. The effects of radiation, viscous dissipation, and work due to pressure changes are neglected, and the hypothesis of local thermal equilibrium is adopted. However, the model includes reaction kinetics, longitudinal heat conduction, interlayer heat exchange, and heat loss to the surrounding rock formation. Assuming incompressibility, the Cauchy problem describing the temperature and fuel concentration dynamics in each layer is given by:

$$\begin{cases} \partial_t u_i - \alpha_i(y_i)\partial_x^2 u_i + \beta_i(y_i)\partial_x u_i = f_i(u, y_i), \quad x \in \mathbb{R},\ t > 0, \\ \partial_t y_i = -A_i y_i g(u_i), \quad x \in \mathbb{R},\ t > 0, \\ (u_i(x,0), y_i(x,0)) = (\phi_i(x), y_{i,0}(x)), \quad x \in \mathbb{R}, \end{cases} \tag{2}$$

where $u_i = u_i(x,t)$ and $y_i = y_i(x,t)$ denote the temperature and the fuel concentration in layer i, respectively. The partial differential equation (PDE) is derived from the energy balance, and the ordinary differential equation (ODE) arises from the Arrhenius law for the combustion reaction. Here, A_i is the pre-exponential Arrhenius factor for $i = 1, \dots, n$, and the initial data are given by ϕ_i and $y_{i,0}$.

In vector form, we write $u = (u_1, \dots, u_n)$, $y = (y_1, \dots, y_n)$, $f = (f_1, \dots, f_n)$, $\phi = (\phi_1, \dots, \phi_n)$, and $y_0 = (y_{1,0}, \dots, y_{n,0})$.

The coefficient functions α_i and β_i are defined by

$$\alpha_i(y_i) = \frac{\lambda_i}{a_i + b_i y_i}, \quad \beta_i(y_i) = \frac{c_i}{a_i + b_i y_i}, \tag{3}$$

and the source terms by

$$\begin{aligned} f_1(u, y_1) &= \frac{-(c_1)_x u_1}{a_1 + b_1 y_1} + \frac{(K_1 b_1 u_1 + d_1) y_1 g(u_1)}{a_1 + b_1 y_1} + \frac{q_1(u_2 - u_1)}{a_1 + b_1 y_1} - \frac{\bar{q}_1(u_1 - u_e)}{a_1 + b_1 y_1}, \\ f_i(u, y_i) &= \frac{-(c_i)_x u_i}{a_i + b_i y_i} + \frac{(K_i b_i u_i + d_i) y_i g(u_i)}{a_i + b_i y_i} - \frac{q_{i-1}(u_i - u_{i-1})}{a_i + b_i y_i} + \frac{q_i(u_{i+1} - u_i)}{a_i + b_i y_i}, \quad i = 2, \dots, n-1, \\ f_n(u, u_n) &= \frac{-(c_n)_x u_n}{a_n + b_n y_n} + \frac{(K_n b_n u_n + d_n) y_n g(u_n)}{a_n + b_n y_n} - \frac{q_{n-1}(u_n - u_{n-1})}{a_n + b_n y_n} - \frac{\bar{q}_2(u_n - u_e)}{a_n + b_n y_n}. \end{aligned} \tag{4}$$

Here, the parameters $a_i, b_i, c_i, d_i, K_i, \lambda_i$, and q_i, are non-negative functions of the spatial variable x, reflecting physical properties such as porosity, thermal conductivity, and initial fuel content. The other parameters $\bar{q}_1$, $\bar{q}_2$, A_i, and E are non-negative constants. Their detailed definitions are provided in [8]. The external ambient temperature is denoted by u_e and is treated as a constant as well. Notably, a_i and λ_i are strictly positive. The function g is given by:

$$g(\theta) = \begin{cases} e^{-\frac{E}{\theta}}, & \text{if } \theta > 0, \\ 0, & \text{if } \theta \le 0. \end{cases} \tag{5}$$

By integrating the second equation in (2) with the given initial condition, we obtain:

$$y_i(x, t, u_i) = y_{i,0}(x) \exp\left(-A_i \int_0^t g(u_i(x, \tau))\, d\tau \right). \tag{6}$$

Thus, once the temperature u_i is known, the fuel concentration y_i can be explicitly computed from (6).

Substituting (6) into the first equation of (2), as well as into (3) and (4), the problem (2) reduces to the quasilinear Cauchy problem for the temperature field, given in (1), where

$$L(x, t, u)u = \big(L_1(x, t, u_1)u_1, \dots, L_n(x, t, u_n)u_n \big), \tag{7}$$

with

$$L_i(x, t, u_i)u_i := -\alpha_i(x, t, u_i)\, \partial_x^2 u_i + \beta_i(x, t, u_i)\, \partial_x u_i, \quad i = 1, \dots, n. \tag{8}$$

3. Notations and general definitions

Throughout this study, the index i refers to layer i of the porous medium, where i takes values in the range $1, \dots, n$, unless specified otherwise.

When there is no risk of ambiguity, we will omit the variable x from notations such as $u_i(x,t)$, $u_{i,0}(x)$, $y_i(x,t,u)$, $f_i(x,t,u)$, $\alpha_i(x,t,u_i)$, and $\beta_i(x,t,u_i)$; we will instead utilize the simplified forms $u_i(t)$, $u_{i,0}$, $y_i(t,u)$, $f_i(t,u)$, $\alpha_i(t,u_i)$, and $\beta_i(t,u_i)$. Additionally, for convenience, we may also omit the variable t, resulting in the notations $y_i(u)$, $f_i(u)$, $\alpha_i(u_i)$, and $\beta_i(u_i)$.

The set of real numbers is denoted by $\mathbb{R}$, and an interval is represented by $I \subset \mathbb{R}$. The symbol T stands for a positive temperature. We designate X and Y as Banach spaces, where $Y \subset X$. The L^p spaces referenced in this study include L^1, L^2, and L^∞. The notations adopted are as follows:

$\|\cdot\|_X$ denotes the norm in the space X. The partial derivatives are represented by $\partial_t = \frac{\partial}{\partial t}$, $\partial_x = \frac{\partial}{\partial x}$, and $\partial_x^2 = \frac{\partial^2}{\partial x^2}$.

If there is no ambiguity, both norms $\|\cdot\|_{L^2(\mathbb{R})}$ and $\|\cdot\|_{L^2(\mathbb{R})^n}$ will be denoted as $\|\cdot\|$, where for $\psi = (\psi_1, \dots, \psi_n) \in L^2(\mathbb{R})^n$, we define $\|\psi\| = \|\psi\|_{L^2(\mathbb{R})^n} = \max_{1\le i\le n} \|\psi_i\|_{L^2(\mathbb{R})}$. For $\psi \in L^\infty(\mathbb{R})$, we denote $\|\psi\|_{L^\infty} = \|\psi\|_\infty$.

$C(I, X)$ represents the space of continuous functions defined from I into X. If I is compact, then it forms a Banach space with the supremum norm.

The Fourier transform of a function f is defined as:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x}\, dx, \qquad \xi \in \mathbb{R}.$$

For any $s \in \mathbb{R}$, The Bessel and the Riesz potentials of the function f are defined, respectively, via Fourier transforms as:

$$\widehat{J^s f}(\xi) = (1+\xi^2)^{s/2} \hat{f}(\xi) \quad \text{and} \quad \widehat{D^s f}(\xi) = |\xi|^s \hat{f}(\xi).$$

The Sobolev space, denoted by $H^s := H^s(\mathbb{R})$, is based on the L^2 norm and endowed with the norm $\|\cdot\|_s$, where $\|f\|_s := \|J^s f\|_{L^2(\mathbb{R})}$. The operator $D^s f$ may be referred to as the fractional derivative of order s of the function f.

For clarity, if there is no ambiguity, both norms $\|\cdot\|_{H^s(\mathbb{R})}$ and $\|\cdot\|_{H^s(\mathbb{R})^n}$ will be denoted by $\|\cdot\|_s$, where for $\psi = (\psi_1, \dots, \psi_n) \in H^s(\mathbb{R})^n$, we obtain $\|\psi\|_s := \|\psi\|_{H^s(\mathbb{R})^n} = \max_{1\le i\le n} \|\psi_i\|_{H^s(\mathbb{R})}$.

The inner product in $H^s(\mathbb{R})^n$ is defined as $(\psi, \varphi)_s = \max_{1\le i\le n} (\psi_i, \varphi_i)_s$, where ψ_i and φ_i are the ith components of the vectors ψ and φ, respectively, and

$$(\psi_i, \varphi_i)_s := \int_R (1+|\xi|^2)^s \hat{\psi}_i(\xi) \overline{\hat{\varphi}_i(\xi)} d\xi.$$

We denote by $\mathcal{Q}$, $\tilde{\mathcal{Q}}$, $\mathcal{Q}_1$, and $\mathcal{Q}_2$ continuous functions that are increasing in their arguments, as defined in Lemma 4.3. Additionally, we use $\mathcal{P}$, $\mathcal{P}_1$, and $\mathcal{P}_2$ to denote other continuous functions that are also increasing in their arguments, as outlined in Lemma 4.4.

The space of all absolutely continuous functions mapping from $[0,T]$ to H^s is denoted by $\mathcal{AC}([0,T]; H^s)$.

Throughout the text, generic constants are represented by C, while auxiliary constants are indexed (e.g., C_1, C_2, C_μ) upon their introduction.

For positive numbers a and b, we use the notation $a \lesssim b$ to indicate the existence of a constant C such that $a \le Cb$.

Several specific constants are defined in the text, including:

$r := \sup_{[0,T]} \|v\|_s$,

R, as defined in (13),

R^j and C^j, defined via (94) and (103), respectively.

4. Existence of solution

In this section we show the existence solution for the IVP (1). Initially, by setting

$$L(u)u = L(x,t,u)u = \left(L_1(x,t,u_1)u_1, \dots, L_n(x,t,u_n)u_n\right),$$

where L_i is given in (8), the initial value problem (1) assume the following form

$$\begin{cases} \partial_t u + L(u)u = f(u), & x \in \mathbb{R}, \\ u(0) = \phi, \end{cases} \tag{9}$$

where $0 \le t \le T$.

We then consider the next IVP

$$\begin{cases} \partial_t u + \mu \partial_x^4 u = h(u), & x \in \mathbb{R}, \\ u(0) = \phi, \end{cases} \tag{10}$$

where μ is the parabolic regularization parameter, $h(u) = f(u) - L(u)u$, and $0 \le t \le T$

Observe that we can write $h(u) = (h_1(u), \dots, h_n(u))$, for

$$h_i(u) = f_i(u) + \alpha_i(u_i)\partial_x^2 u_i - \beta_i(u_i)\partial_x u_i. \tag{11}$$

By setting

$$U_\mu(t)\phi = (e^{-\mu t\xi^4}\hat{\phi})^\vee := \left((e^{-\mu t\xi^4}\hat{\phi}_1)^\vee, \dots, (e^{-\mu t\xi^4}\hat{\phi}_n)^\vee\right),$$

it follows that the integral equation for (10) is given by

$$u(t) = U_\mu(t)\phi + \int_0^t U_\mu(t-\tau)h(u(\tau))d\tau. \tag{12}$$

To establish our main results, we require certain assumptions and definitions concerning the coefficients α_i, β_i, and the functions f_i. Thus, we assume $a_0 := \min_{1\le i\le n} \inf a_i > 0$. Also, for all $k \in \mathbb{N}$, by setting

$$\|y_0\|_s = \max_{1\le i\le n} \|y_{i,0}\|_s, \quad \|y_0''\|_\infty = \max_{1\le i\le n} \|y_{i,0}''\|_\infty,$$

$$\|q^{(k)}\|_\infty = \max\left\{\|q_1^{(k)}\|_\infty, \dots, \|q_{n-1}^{(k)}\|_\infty, |u_e|, \overline{q}_1, \overline{q}_2\right\},$$

and

$$\|a^{(k)}\|_\infty = \max_{1\le i\le n} \|a_i^{(k)}\|_\infty,$$

with similar definition for $\|b^{(k)}\|_\infty$, $\|c^{(k)}\|_\infty$, and $\|\lambda^{(k)}\|_\infty$, we define the following constant

$$R = \max_{0\le k\le [s]+1}\left\{\|a^{(k)}\|_\infty, \|b^{(k)}\|_\infty, \|c^{(k)}\|_\infty, \|\lambda^{(k)}\|_\infty, \|q^{(k)}\|_\infty, \|g^{(k)}\|_\infty, \|y_0\|_s, \|y_0''\|_\infty, a_0^{-1}, \max_{1\le i\le n}|A_i|\right\}, \tag{13}$$

where $[s]$ denotes the integer part function.

In our arguments, we also need the L^∞-norm of the coefficients $\alpha_i := \alpha_i(x,t,u_i)$, $\beta_i := \beta_i(x,t,u_i)$, and their derivatives. In what follows, due to the similarity of the expressions for α_i and β_i, it is enough to obtain the estimates for α_i, since the case for β_i is analogous. Here, we denote

$$\alpha_i = \alpha,\ \lambda_i = \lambda,\ a_i = a,\ b_i = b,\ u_i = v,\ A_i = A,\ y_{i,0} = y_0, \tag{14}$$

and

$$\psi = \psi(x,t,v) := e^{A\int_0^t g(v(x,\tau))d\tau}. \tag{15}$$

Hence, a straightforward computations reveals that

$$\partial_x\psi = A\,\psi \underbrace{\int_0^t g'\big(v(\tau)\big)\partial_x v(\tau)d\tau}_{\sigma} \tag{16}$$

and

$$\begin{aligned}\partial_x^2\psi &= \psi A^2\Big(\int_0^t g'\big(v(\tau)\big)\partial_x v(\tau)d\tau\Big)^2\\ &\quad + A\psi\int_0^t \Big(g''\big(v(\tau)\big)(\partial_x v(\tau))^2 + g'\big(v(\tau)\big)\partial_x^2 v(\tau)\Big)d\tau\\ &= A^2\psi\sigma^2 + A\psi\partial_x\sigma.\end{aligned} \tag{17}$$

The following lemma provides estimates for the coefficients $\alpha_i = \alpha_i(u_i) = \alpha_i(x,t,u_i)$ and $\beta_i = \beta_i(u_i) = \beta_i(x,t,u_i)$. Since the proof follows from standard calculations, we omit it here.

Lemma 4.1. *Suppose that $R < \infty$, then*

(i) $\|\alpha_i(u_i)\|_\infty, \|\beta_i(u_i)\|_\infty \le R^2$,

(ii) $\|\alpha_i(u_i) - \alpha_i(v_i)\|_\infty \le R^4\|u-v\|_\infty$, *and* $\|\beta_i(u_i) - \beta_i(v_i)\|_\infty \le R^4\|u-v\|_\infty$,

(iii) $\|\alpha_i(u_i) - \alpha_i(v_i)\| \le R^4\|u-v\|$.

For the next results, we denote $r := \sup_{[0,T]}\|v\|_s$.

Lemma 4.2. *Let $s \ge 2$. Assume that $v \in H^s(\mathbb{R})$, and let $\psi = \psi(x,t,v)$ be as defined in (15). Then for all $t \in [0,T]$*

$$\begin{cases}\|\partial_x\psi\| \le AtRr\,e^{At},\\ \|\partial_x^2\psi\| \le (AtRr)^2\big(r+1\big)\,e^{At},\\ \|\psi\|_\infty \le e^{At},\\ \|\partial_x\psi\|_\infty \le AtRr\,e^{At}.\end{cases} \tag{18}$$

Proof. Follows from an easy computation, using the fact that derivatives of g are bounded, (13) and (15)–(17), together with the inequalities

$$\|\partial_x^2 v\|, \|v\|_\infty, \|\partial_x v\|_\infty \lesssim \|v\|_s \le r.$$

□

Now, we proceed with the estimates for the coefficient α.

Substituting (6) in (3), and using (14) and (15), it follows that

$$\alpha = \frac{\lambda\psi}{a\psi + by_0}. \tag{19}$$

On the other hand, the Taylor expansion for the exponential implies that ψ does not belong to $L^2(\mathbb{R})$. Consequently, in our proofs, we need to use the $L^\infty(\mathbb{R})$-norm of ψ. Thus, putting $\nu := \frac{1}{a\psi+by_0}$, a straightforward calculation yields

$$\partial_x\alpha = \nu\Big[\big(\lambda'\psi + \lambda\partial_x\psi\big) - \alpha\big(a'\psi + a\partial_x\psi + b'y_0 + by_0'\big)\Big] = \nu(\lambda - \alpha a)\partial_x\psi + g_1, \tag{20}$$

where

$$g_1 := \nu\Big[(\lambda' - \alpha a')\psi - \alpha(b' y_0 + b y_0')\Big]. \tag{21}$$

Lemma 4.3. *Let $0 \le \delta < 2$, then*

$$\|\alpha(v)\partial_x^2 f\|_\delta \le \mathcal{Q}(r,t,R)\|f\|_{\delta+2}, \tag{22}$$

where $\mathcal{Q}$ is a continuous function and increasing in its arguments.

Proof. First, from (13), (18), and (20), we obtain

$$\begin{aligned}\|\partial_x\alpha\|_\infty &\le R^2\big[\big(1 + RArt\big)e^{At} + R\big(Re^{At}(1 + RArt) + 2R^2\big)\big] \\ &:= \tilde{\mathcal{Q}}(r,t,R).\end{aligned} \tag{23}$$

where $\tilde{\mathcal{Q}}$ is a continuous and increasing function in its arguments.

We divide the proof in two cases.

Case: $0 \le \delta < 1$. Using Holder's inequality, (A.2), and (A.3),

$$\begin{aligned}\|\alpha\partial_x^2 f\|_\delta &\lesssim \|\alpha\partial_x^2 f\| + \|\mathcal{D}^\delta\big(\alpha\partial_x^2 f\big)\| \\ &\lesssim \|\alpha\|_\infty\|\partial_x^2 f\| + \|\alpha\,\mathcal{D}^\delta\partial_x^2 f\| + \|\partial_x^2 f\,\mathcal{D}^\delta\alpha\| \\ &\lesssim R^2\|\partial_x^2 f\| + \|\alpha\|_\infty\|\mathcal{D}^\delta\partial_x^2 f\| + \|\mathcal{D}^\delta\alpha\|_\infty\|\partial_x^2 f\| \\ &\lesssim \big(3R^2 + \|\partial_x\alpha\|_\infty\big)\|f\|_{2+\delta}.\end{aligned} \tag{24}$$

Case: $1 \le \delta < 2$. Here, we can write $\delta = 1 + \delta_1$, for $\delta_1 \in [0,1)$.

Then

$$\begin{aligned}\|\alpha\partial_x^2 f\|_\delta &\lesssim \|\alpha\partial_x^2 f\| + \|\partial_x\mathcal{D}^{\delta_1}\big(\alpha\partial_x^2 f\big)\| \\ &\lesssim R^2\|\partial_x^2 f\| + \|\alpha\,\mathcal{D}^{\delta_1}\partial_x^3 f\| + \underbrace{\|\partial_x^2 f\,\mathcal{D}^{\delta_1}\partial_x\alpha\|}_{C_1}.\end{aligned} \tag{25}$$

Next, we deal with the C_1 term. First, observe that

$$\|\partial_x\psi\|_{\delta_1} \lesssim \|\partial_x\psi\|_1 \lesssim \|\partial_x\psi\| + \|\partial_x^2\psi\|. \tag{26}$$

Thus, from (18), (26), and (A.4),

$$\begin{aligned}\|\mathcal{D}^{\delta_1}\big(\nu(\lambda - \alpha a)\partial_x\psi\big)\| &\lesssim \big(\|\nu(\lambda - \alpha a)\|_\infty + \|\partial_x(\nu(\lambda - \alpha a))\|_\infty\big)\|\partial_x\psi\|_{\delta_1} \\ &\lesssim \mathcal{Q}_1(r,t,R),\end{aligned} \tag{27}$$

where $\mathcal{Q}_1$, calculated in Lemma A.3, is continuous and increasing in its arguments.

Recalling that g_1 is given in (21), it is easy to see that there exists $\mathcal{Q}_2$ (computed in Lemma A.3), continuous and increasing in its arguments, such that

$$\|g_1\|_\infty + \|\partial_x g_1\|_\infty \lesssim \mathcal{Q}_2(r,t,R). \tag{28}$$

Hence, from (25), (27), and (28),

$$\begin{aligned}C_1 &\lesssim \|\partial_x^2 f\mathcal{D}^{\delta_1}\big(\nu(\lambda - \alpha a)\partial_x\psi\big)\| + \|\partial_x^2 f\mathcal{D}^{\delta_1}g_1\| \\ &\lesssim \|\partial_x^2 f\|_\infty\|\mathcal{D}^{\delta_1}\big(\nu(\lambda - \alpha a)\partial_x\psi\big)\| + \|\partial_x^2 f\|\|\mathcal{D}^{\delta_1}g_1\|_\infty \\ &\lesssim \|\partial_x^2 f\|_\infty\|\mathcal{D}^{\delta_1}\big(\nu(\lambda - \alpha a)\partial_x\psi\big)\| + \|\partial_x^2 f\|(\|g_1\|_\infty + \|\partial_x g_1\|_\infty) \\ &\lesssim \|f\|_{2+\delta}\big(\mathcal{Q}_1(r,t,R) + \mathcal{Q}_2(r,t,R)\big),\end{aligned} \tag{29}$$

where we used Sobolev embedding to obtain $\|\partial_x^2 f\|_\infty \lesssim \|f\|_{2+\delta}$.

Therefore, in view of (25) and (29)

$$\|\alpha\partial_x^2 f\|_\delta \lesssim \big(3R^2 + \mathcal{Q}_1(r,t,R) + \mathcal{Q}_2(r,t,R)\big)\|f\|_{2+\delta}, \tag{30}$$

where Lemma 4.1 (i) is used to ensure that $\|\alpha\|_\infty \le R^2$. We also use the estimate

$$\|\mathcal{D}^{\delta_1}\partial_x^3 f\| \lesssim \|\partial_x^3 f\| + \|\mathcal{D}^{\delta_1}\partial_x^3 f\| \lesssim \|f\|_{2+\delta}.$$

Above we use $\mathcal{D}^0 := Id$. Finally, (24) and (30) imply the desired result.

This ends the proof. □

To obtain our next result we need the second derivative of α. From (20), we have

$$\partial_x^2\alpha = \nu(\lambda + \alpha a)\partial_x^2\psi + g_2, \tag{31}$$

where

$$g_2 := \nu\Big[\big(\psi\lambda'' + 2\lambda'\partial_x\psi\big) - 2\partial_x\alpha\big(a'\psi + a\partial_x\psi + b' y_0 + b y_0'\big) + \alpha\big(a''\psi + 2a'\partial_x\psi + b'' y_0 + 2b' y_0' + b y_0''\big)\Big]. \tag{32}$$

Lemma 4.4. *Let $v \in H^s(\mathbb{R})$, for $s \geq 2$. Then there exists a function $\mathcal{P}$, continuous and increasing in its arguments, such that*

$$\|\alpha(v)f\|_2 \leq \mathcal{P}(r,t,R)\|f\|_2, \tag{33}$$

where we recall that $r := \sup_{[0,T]}\|v\|_s$.

Proof. We observe that from the definition of H^2-norm it follows that

$$\begin{aligned}\|\alpha f\|_2 &\lesssim \|\alpha f\| + \|\partial_x^2(\alpha f)\|\\ &\lesssim \|\alpha\|_\infty\|f\| + \|\alpha\partial_x^2 f\| + 2\|\partial_x\alpha\,\partial_x f\| + \|\partial_x^2\alpha f\|\\ &\lesssim \|\alpha\|_\infty\|f\| + \|\alpha\|_\infty\|\partial_x^2 f\| + 2\|\partial_x\alpha\|_\infty\|\partial_x f\| + \|f\partial_x^2\alpha\|\\ &\lesssim 2R^2\|f\|_2 + 2\|\partial_x\alpha\|_\infty\|\partial_x f\| + \underbrace{\|f\partial_x^2\alpha\|}_{C_2}.\end{aligned} \tag{34}$$

Next, we deal with the term C_2.

First, note that by (13), (18), and (31) imply

$$\|v(\lambda+\alpha a)\|_\infty\|\partial_x^2\psi\| \lesssim \mathcal{P}_1(r,t,R) \tag{35}$$

and

$$\|g_2\|_\infty \lesssim \mathcal{P}_2(r,t,R), \tag{36}$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are continuous and increasing functions in their arguments.

Hence, by (31), (35), and (36) imply

$$\begin{aligned}C_2 &\lesssim \|f\,v(\lambda+\alpha a)\partial_x^2\psi\| + \|f\,g_2\|\\ &\lesssim \|f\,v(\lambda+\alpha a)\|_\infty\|\partial_x^2\psi\| + \|f\|\|g_2\|_\infty\\ &\lesssim \|f\|_\infty\|v(\lambda+\alpha a)\|_\infty\|\partial_x^2\psi\| + \|f\|P_2(r,t,R)\\ &\lesssim \big(\mathcal{P}_1(r,t,R)+\mathcal{P}_2(r,t,R)\big)\|f\|_2.\end{aligned} \tag{37}$$

Therefore, (23), (34), and (37) yields us the desired result.

This finishes the proof. □

Lemma 4.5. *Assume that $R < \infty$. Then the source function $f = (f_1,\dots,f_n) : [0,T]\times H^2(\mathbb{R})^n \to L^2(\mathbb{R})^n$, for any fixed $T > 0$, satisfies the following properties:*

i) There exist constants $\sigma_1,\sigma_2 > 0$, which depends on R, such that

$$\|f_i(t,w)\| \leq \|f(t,w)\| \leq \sigma_2\|w\| + \sigma_1$$

for all $t \in [0,T]$.

ii) For a fixed $w \in H^2(\mathbb{R})^n$, the function $t \in [0,T] \mapsto f(t,w) \in L^2(\mathbb{R})^n$ is continuous.

iii) For each $t \in [0,T]$, the function $w \in H^2(\mathbb{R})^n \mapsto f_i(w,y_i)$ is Lipschitz in $L^2(\mathbb{R})^n$, that is

$$\|f_i(u,y_i) - f_i(w,y_i)\| \leq \sigma_3\,\|u-w\|,$$

for all $u,w \in H^2(\mathbb{R})^n$, where $\sigma_3 = \sigma_3(\varrho,R)$ is increasing in its arguments and $\varrho = \max\{\sup\|u\|_2, \sup\|w\|_2\}$. In addition, σ_3 is independent of t.

Proof. First, we will proof (iii).

Initially, observe that, in view of (4), it is enough to deal with f_1, since the proof for $f_2,\dots,f_n$ is analogous. We define

$$\varphi(u_1) = y_1(x,t,u_1) = y_{1,0}(x)e^{-A_i\int_0^t g(u_1(x,\tau))d\tau},$$

with an analogous definition for $\varphi(w_1)$.

Thus, from (4)

$$\begin{aligned}f_1(u,\varphi(u_1)) - f_1(w,\varphi(w_1)) &= \frac{-(c_1)_x u_1}{a_1+b_1\varphi(u_1)} - \frac{(c_1)_x w_1}{a_1+b_1\varphi(w_1)}\\ &\quad+\underbrace{\frac{b_1K_1u_1\varphi(u_1)g(u_1)}{a_1+b_1\varphi(u_1)} - \frac{b_1K_1w_1\varphi(w_1)g(w_1)}{a_1+b_1\varphi(w_1)}}_{B}\\ &\quad+\frac{d_1u_1\varphi(u_1)g(u_1)}{a_1+b_1\varphi(u_1)} - \frac{d_1w_1\varphi(w_1)g(w_1)}{a_1+b_1\varphi(w_1)}\\ &\quad+\frac{q_1}{a_1+b_1\varphi(u_1)}(u_2-u_1) - \frac{q_1}{a_1+b_1\varphi(w_1)}(w_2-w_1)\\ &\quad-\frac{\overline{q}_1}{a_1+b_1\varphi(u_1)}(u_1-u_e) + \frac{\overline{q}_1}{a_1+b_1\varphi(w_1)}(w_1-u_e).\end{aligned} \tag{38}$$

Next, we deal with term $\mathcal{B}$. First, note that since the function

$$\theta \in \mathbb{R} \mapsto g_3(\theta) := \theta g(\theta)$$

has bounded derivative, using the Mean value theorem, it follows that

$$\begin{aligned} &\|u_1\varphi(u_1)g(u_1) - w_1\varphi(w_1)g(w_1)\| \le \|\big(u_1\, g(u_1) - w_1 g(w_1)\big)\varphi(u_1)\| \\ &+ \|w_1 g(w_1)(\varphi(u_1) - \varphi(w_1))\| \le \sup|g_3'|\|u_1 - w_1\| + \|w_1\|_\infty \|u_1 - w_1\|. \end{aligned} \tag{39}$$

We can write

$$\begin{aligned} \mathcal{B} &= b_1 K_1 \left[\frac{u_1\varphi(u_1)g(u_1)\big(a_1 + b_1\varphi(w_1)\big) - w_1\varphi(w_1)g(w_1)\big(a_1 + b_1\varphi(u_1)\big)}{\big(a_1 + b_1\varphi(u_1)\big)\big(a_1 + b_1\varphi(w_1)\big)} \right] \\ &= b_1 a_1 K_1 \Big[\frac{u_1\varphi(u_1)g(u_1) - w_1\varphi(w_1)g(w_1)}{\big(a_1 + b_1\varphi(u_1)\big)\big(a_1 + b_1\varphi(w_1)\big)} \Big] \\ &+ b_1^2 K_1 \Big[\frac{u_1\varphi(u_1)g(u_1)(\varphi(w_1) - \varphi(u_1)) + \varphi(u_1)\big((u_1\varphi(u_1)g(u_1)}{\big(a_1 + b_1\varphi(u_1)\big)\big(a_1 + b_1\varphi(w_1)\big)} \Big] \\ &- b_1^2 K_1 \Big[\frac{w_1\varphi(w_1)g(w_1)\big)}{\big(a_1 + b_1\varphi(u_1)\big)\big(a_1 + b_1\varphi(w_1)\big)} \Big]. \end{aligned} \tag{40}$$

Thus (39) and (40) give

$$\begin{aligned} &\|\mathcal{B}\| \lesssim R^3\big(\sup|g_3'|\|u_1 - w_1\| + \|w_1\|_\infty \|u_1 - w_1\|\big) \\ &\quad + R^5\big(\|u_1\|_\infty \sup|g_3| \sup|\varphi'|\|w_1 - u_1\| + \sup|\varphi'| \sup|g_3'|\|u_1 - w_1\|\big) \\ &\lesssim \Big[R^3\big(\sup|g_3'| + \|w\|_2\big) + R^5\big(\|u_1\|_2 \sup|g_3| + \sup|g_3'|\big)\Big]\|u - w\|, \end{aligned} \tag{41}$$

where we used that $\varphi, \varphi' \le 1$.

The other terms in (38) can be estimated in a manner similar to (41).

Thus we conclude the proof of (iii).

The argument for cases (i) and (ii) follows similarly.

This ends the proof. □

Lemma 4.6. *Let $\lambda \ge 0$, $s' \in \mathbb{R}$ and $t > 0$. Then,*

$$\|U_\mu(t)f\|_{s'+\lambda} \le \Big[1 + \Big(\frac{\lambda}{4\mu t}\Big)^{\frac{\lambda}{4}}\Big]\|f\|_{s'}.$$

Proof. It follows from similar ideas as in [20,21]. □

Proposition 4.7. *Let $\mu, \epsilon \in (0,1)$ and $\phi \in H^s(\mathbb{R})$, for $s \ge 2$. Then, there exists $T_\mu = T_\mu(\|\phi\|_s, \mu, \epsilon)$ and a unique solution $u_\mu \in C([0, T_\mu]; H^s)$ of (10). In addition, for all $T > 0$, u_μ can be extended to $C([0,T]; H^s)$ and*

$$\|u_\mu(t)\|_s \le \|\phi\|_s + \epsilon := \rho, \quad \textit{for all} \quad t \in [0,T]. \tag{42}$$

Proof.

To begin, we define an operator Φ by the right-hand side of (12), that is

$$\Phi u(t) = U_\mu(t)\phi + \int_0^t U_\mu(t-\tau)h(u(\tau))d\tau. \tag{43}$$

Next, we establish the following complete metric space

$$\chi_T = \big\{u \in C([0,T]; H^s) : \|u(t)\|_s \le \rho, \text{ for all } t \in [0,T]\big\}, \tag{44}$$

equipped with the metric $d(u,v) := \sup_{[0,T]} \|u(t) - v(t)\|_s$. By the Lebesgue dominated convergence theorem, it is easy to prove that if $u \in \chi_T$, then $\Phi u \in C([0,T]; H^s)$.

From (11), using Lemmas 4.1(i) and 4.5(i), for $u \in \chi_T$, we obtain

$$\begin{aligned} \|h_i(u(\tau))\| &\le \|\alpha_i\|_\infty \|\partial_x^2 u_i\| + \|\beta_i\|_\infty \|\partial_x u_i\| + \|f_i(u)\| \\ &\le R^2\|u\|_s + R^2\|u\|_s + \sigma_2\|u\| + \sigma_1 \\ &\le (2R^2 + \sigma_2)\rho + \sigma_1, \end{aligned} \tag{45}$$

where we have used the Sobolev embedding $H^s(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$, and the estimates

$$\|\partial_x u_i\|,\ \|\partial_x^2 u_i\| \le \|u\|_s.$$

Thus, by the definition of the norm $\|\cdot\|$ in $L^2(\mathbb{R})^n$, we have

$$\|h(u(\tau))\| \le (2R^2 + \sigma_2)\rho + \sigma_1. \tag{46}$$

Then, for $u \in \chi_T$, applying Lemma 4.6 with $\lambda = s$ and $s' = 0$, we obtain

$$\begin{aligned} \|\Phi u(t)\|_s \leq & \ \|U_\mu(t)\phi\|_s + \int_0^t \|U_\mu(t-\tau)h(u(\tau))\|_s d\tau \\ \leq & \ \|\phi\|_s + \int_0^t \Big[1 + \big(\frac{s}{4\mu(t-\tau)}\big)^{s/4}\Big] \|h(u(\tau))\| d\tau \\ \leq & \ \|\phi\|_s + \Big[t + \Big(\frac{s}{4\mu}\Big)^{\frac{s}{4}} \frac{4}{4-s} t^{\frac{4-s}{4}}\Big]\Big[(2R^2+\sigma_2)\rho + \sigma_1\Big]. \end{aligned} \tag{47}$$

Here, we assume $2 \leq s < 4$. The case $s > 4$ can be treated similarly, as demonstrated in [20]. A straightforward calculation shows that there exists a value $T'_\mu := T'_\mu(\|\phi\|_s, \mu, \epsilon) > 0$, sufficiently small, such that

$$\|\Phi u(t)\|_s \leq \rho, \ \text{ for all } \ t \in [0, T'_\mu],$$

which implies that

$$\Phi : \chi_{T'_\mu} \to \chi_{T'_\mu}.$$

On the other hand, for $\tau \in [0,t]$, $u = (u_1, \dots, u_n), v = (v_1, \dots, v_n) \in \chi_T$ and from Lemmas 4.1 and 4.5, it follows that

$$\begin{aligned} \|h_i(u) - h_i(v)\| \leq & \ \|\alpha_i(u_i) - \alpha_i(v_i)\|_\infty \|\partial_x^2 u_i\| + \|\alpha_i(v_i)\|_\infty \|\partial_x^2(u_i - v_i)\| \\ & + \|\beta_i(u_i) - \beta_i(v_i)\|_\infty \|\partial_x v_i\| + \|\beta_i(u_i)\|_\infty \|\partial_x(u_i - v_i)\| + \|f_i(u) - f_i(v)\| \\ \leq & \ R^4 \|u\|_s \|u-v\|_s + 2R^2 \|u-v\|_s + R^4 \|v\|_s \|u-v\|_s + \sigma_3 \|u-v\|_s \\ \leq & \ (2R^4 + 2R^2 + \sigma_3)\rho \|u-v\|_s, \end{aligned} \tag{48}$$

where we have also used the Sobolev embedding, $\|u-v\|_\infty \lesssim \|u-v\|_s$.

Consequently,

$$\|h(u) - h(v)\| \leq C_3 \rho \|u-v\|_s). \tag{49}$$

where $C_3 = 2R^4 + 2R^2 + \sigma_3$.

Thus, we obtain

$$\begin{aligned} \|\Phi u(t) - \Phi v(t)\|_s \leq & \int_0^t \Big[1 + \Big(\frac{s}{4\mu(t-\tau)}\Big)^{s/4}\Big] \|h(u) - h(v)\| d\tau \\ & \leq \Big[t + \Big(\frac{s}{4\mu}\Big)^{\frac{s}{4}} \frac{4}{4-s} t^{\frac{4-s}{4}}\Big] C_3 \, \rho \, d(u,v). \end{aligned} \tag{50}$$

Again, a straightforward calculation shows that there exists a value $T''_\mu > 0$, sufficiently small, such that

$$\Big[t + \Big(\frac{s}{4\mu}\Big)^{\frac{s}{4}} \frac{4}{4-s} t^{\frac{4-s}{4}}\Big] C_3 \, \rho < 1, \ \text{ for all } \ t \in [0, T''_\mu]. \tag{51}$$

Therefore, defining $T_\mu = \min\{T'_\mu, T''_\mu\}$, we conclude that $\Phi : \chi_{T_\mu} \to \chi_{T_\mu}$ is a contraction. Hence, using the Banach fixed-point theorem, there exists a unique solution u_μ of (10). The uniqueness of u_μ follows from an application of Gronwall's lemma.

Let $\mu \in (0,1)$. Then, its can possible to show that for all $T > 0$, the solution u_μ of (12) can be extend to $C([0,T]; H^s)$.

Indeed, let us define

$$\begin{aligned} T^*(\phi) = T^* = \sup \Big\{ & T_\mu > 0; \text{ there exists } u_\mu \in C([0,T_\mu]; H^s) \text{ solution of (12)}, \\ & \text{such that } \|u_\mu(t)\|_s \leq \rho, \text{ for all } t \in [0, T_\mu] \Big\}. \end{aligned} \tag{52}$$

Assume that $T^* < \infty$. Then, following the ideas in [22, Section 5.2] and using the results proved above, we can prove that there exists $\tilde{T}_\mu > 0$ such that u_μ can be extended to $C([0, T^* + \tilde{T}_\mu]; H^s(\mathbb{R})^n)$. This contradicts the definition of T^*. Therefore, $T^* = \infty$.

This ends the proof. □

Proposition 4.8. *Let u solution of IVP (10), then*

$$u \in C((0, T_\mu]; H^\infty). \tag{53}$$

Proof. It follows from the bootstrapping argument, see [20,21]. □

Lemma 4.9. *Let $\tau \geq 0$ and $C \geq 0$, then*

$$-\tau^2 + C\tau \lesssim C^2.$$

Proof. It follows from studying the maximum of the function $\varphi_1(x) = -x^2 + Cx$. □

Observe that from (2), we can assume that the coefficients α_i and β_i depend on u_i, meanwhile f_i depends on u. Using this information we can show the following result.

Proposition 4.10. *Let $s \geq 2$. Suppose $\mu, \nu \in (0,1)$ and u_μ, u_ν are solutions of (10), with $u_\mu(0) = u_\nu(0) = \phi$. Then*

$$\lim_{\mu,\nu\downarrow 0} \sup_{[0,T]} \|u_\mu(t) - u_\nu(t)\| = 0.$$

Proof. Let u_μ and u_ν such that $u_\mu(0) = v_\mu(0) = \phi$. By $u_{\mu,i}$ and $u_{\nu,i}$ we denote the ith coordinate of u_μ and u_ν, respectively. In our argumentation, we will omit the index i, then here we put $v := u_{\mu,i}$, $z := u_{\nu,i}$, $\alpha := \alpha_i$ and $\beta := \beta_i$. Thus, by setting $w = v - z$, the ith equation in (10) is given by

$$\partial_t w + (\alpha(z) - \alpha(v))\partial_x^2 z - \alpha(v)\partial_x^2 w + (\beta(v) - \beta(z))\partial_x v + \beta(z)\partial_x w = -\mu\partial_x^4 v + \nu\partial_x^4 z + f_i(u_\mu) - f_i(u_\nu). \tag{54}$$

To obtain our result, we use (53). Multiplying (54) by w, and integrate over $\mathbb{R}$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 + ((\alpha(z) - \alpha(v))\partial_x^2 z, w) - (\alpha(v)\partial_x^2 w, w) + ((\beta(v) - \beta(z))\partial_x v, w)\\ + (\beta(z)\partial_x w, w) = (-\mu\partial_x^4 v + \nu\partial_x^4 z, w) + (f_i(u_\mu) - f_i(v_\nu), w).\end{aligned} \tag{55}$$

Next, we estimate each term of (55).

First, by using Lemma 4.1 and Sobolev embedding

$$\|\alpha(z) - \alpha(v)\|_\infty \lesssim R^4\|w\|_1 \lesssim R^4(\|w\| + \|\partial_x w\|). \tag{56}$$

Hence, from (42) and (56), we have

$$\begin{aligned}\big((\alpha(z) - \alpha(v))\partial_x^2 z, w\big) \leq{}& \|\alpha(z) - \alpha(v)\|_\infty \|\partial_x^2 z\|\|w\|\\ \lesssim{}& R^4(\|w\| + \|\partial_x w\|)\rho\|w\|\\ \lesssim{}& R^4\rho\big(\|w\|^2 + \|w\|\|\partial_x w\|\big).\end{aligned} \tag{57}$$

Before dealing with the next term in (55), we need the following definition

$$\alpha_0 := \min_{1\leq i\leq n} \inf \alpha_i, \quad \text{where} \quad \inf \alpha_i = \frac{\inf \lambda_i}{\|a_i\|_\infty + \|b_i\|_\infty}. \tag{58}$$

From (23), we have that

$$\|\partial_x \alpha(v)\|_\infty \leq \tilde{Q}(r,t,R) \lesssim \tilde{Q}(\rho, T, R) := C_4, \tag{59}$$

where C_4 is a constant depending on $\|\phi\|$, R, and T. Hence, by using integration by parts, (58), and (59), we obtain

$$\begin{aligned}\big(\alpha(v)\partial_x^2 w, w\big) ={}& -\int \alpha(v)(\partial_x w)^2 - \int \partial_x\alpha(v) w\partial_x w\\ \leq{}& -\alpha_0\|\partial_x w\|^2 + \|\partial_x\alpha(v)\|_\infty\|w\|\|\partial_x w\|\\ \leq{}& -\alpha_0\|\partial_x w\|^2 + C_4\|w\|\|\partial_x w\|.\end{aligned} \tag{60}$$

By analogous to (57),

$$\big((\beta(v) - \beta(z))\partial_x v, w\big) \lesssim R^4\rho\big(\|w\|^2 + \|w\|\|\partial_x w\|\big), \tag{61}$$

and from Lemma 4.1,

$$\big(\beta(z)\partial_x w, w\big) \leq \|\beta(z)\|_\infty\|\partial_x w\|\|w\| \leq R^2\|\partial_x w\|\|w\|. \tag{62}$$

Integration by parts together with (42) yields us

$$\begin{aligned}\big(-\mu\partial_x^4 v + \nu\partial_x^4 z, w\big) \leq{}& \mu\|\partial_x^2 v\|\|\partial_x^2 w\| + \nu\|\partial_x^2 z\|\|\partial_x^2 w\|\\ \lesssim{}& (\mu + \nu)\rho^2(T).\end{aligned} \tag{63}$$

Finally, an application of Lemma 4.5 (iii) gives

$$\big(f_i(u_\mu) - f_i(v_\nu), w\big) \leq \|f_i(u_\mu) - f_i(v_\nu)\|\|w\| \leq \sigma_3\|u_\mu - u_\nu\|^2. \tag{64}$$

Therefore, from (54), (64), and Lemma 4.9, we have that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|w\|^2 \lesssim{}& -\alpha_0\|\partial_x w\|^2 + \underbrace{\big(2R^4\rho + C_4 + R^2\big)}_{C_5}\|w\|\|\partial_x w\|\\ &+ 2R^4\rho\|w\|^2 + \sigma_3\|u_\mu - u_\nu\|^2 + (\mu + \nu)\rho^2(T)\\ \lesssim{}& \big(\alpha_0^{-1}C_5^2 + 2R^4\rho\big)\|w\|^2 + \sigma_3\|u_\mu - u_\nu\|^2 + (\mu + \nu)\rho^2(T).\end{aligned} \tag{65}$$

Then, from the definition of norm in $H^s(\mathbb{R})^n$ together with Gronwall's lemma, we conclude that, for all $t \in [0,T]$,

$$\|u_\mu(t) - u_\nu(t)\|^2 \lesssim (\mu + \nu)\rho^2(T)\exp\Big(\big[\alpha_0^{-1}C_5^2 + 2R^4\rho + \sigma_1\big]\,T\Big), \tag{66}$$

which implies the desired result.

This finishes the proof. □

Proof of Theorem 1.1. We begin by considering the case $2 \le s < 4$. The case $s \ge 4$ follows analogously.

Propositions 4.10 and A.4 imply the existence of a weakly continuous function $u_0 : [0,T] \to H^s$ such that $u_\mu \rightharpoonup u_0$ in H^s, uniformly on $[0,T]$. Moreover, we have the uniform bound

$$\|u_0(t)\|_s \le \rho \quad \text{for all } t \in [0,T], \tag{67}$$

where $\rho = \|\phi\|_s + \epsilon$ by (42). For any $\kappa \in H^s$,

$$|(u_\mu(t), \kappa)_s| \le \|u_\mu(t)\|_s \|\kappa\|_s \le \rho \|\kappa\|_s. \tag{68}$$

Taking the limit as $\mu \downarrow 0$ yields

$$|(u_0(t), \kappa)_s| \le \rho \|\kappa\|_s,$$

and by taking the supremum over $\|\kappa\|_s = 1$, we recover (67).

From Proposition 4.10, we obtain the strong convergence

$$\|u_\mu - u_0\| \to 0 \quad \text{as } \mu \downarrow 0, \quad \text{uniformly on } [0,T]. \tag{69}$$

We now establish weak convergence results. The differential equation in (10) can be expressed as

$$\partial_t u_\mu = F_\mu(u_\mu), \tag{70}$$

where $F_\mu(u_\mu) = h(u_\mu) - \mu \partial_x^4 u_\mu$.

Let $u_{\mu,i}$, $u_{0,i}$, and $F_\mu(u_\mu)_i$ denote the ith components of u_μ, u_0, and $F_\mu(u_\mu)$, respectively. Setting $v := u_{\mu,i}$, $z := u_{0,i}$, $\alpha := \alpha_i$, and $\beta := \beta_i$, the ith component becomes

$$F_\mu(u_\mu)_i = -\mu \partial_x^4 v + \alpha(v)\partial_x^2 v - \beta(v)\partial_x v + f_i(u_\mu).$$

We will prove the weak convergence

$$F_\mu(u_\mu) \rightharpoonup F_0(u_0) \text{ in } H^{s-4}(\mathbb{R})^n, \text{ uniformly on } [0,T]. \tag{71}$$

To establish this, we analyze each term in the decomposition

$$\begin{aligned} F_\mu(u_\mu)_i - F_0(u_0)_i &= -\mu \partial_x^4 v + (\alpha(v) - \alpha(z))\partial_x^2 z + \alpha(v)\partial_x^2(v - z) \\ &\quad + (\beta(z) - \beta(v))\partial_x v - \beta(z)\partial_x(v - z) + f_i(u_\mu) - f_i(u_0). \end{aligned} \tag{72}$$

For any $\kappa \in H^{s-4}(\mathbb{R})$, we have

$$\mu(\partial_x^4 v, \kappa)_{s-4} \le \mu \|v\|_s \|\kappa\|_{s-4} \le \mu\rho \|\kappa\|_{s-4} \to 0 \text{ as } \mu \downarrow 0. \tag{73}$$

For $\epsilon_1 > 0$, choose $\kappa_1 \in H^\infty(\mathbb{R})$ satisfying

$$\|\kappa - \kappa_1\|_{s-4} < \epsilon_1 \left(2Q(\rho, T, R)\right)^{-1},$$

where Q is from Lemma 4.3. Using Lemma 4.5(iii), we estimate

$$\begin{aligned} |(f_i(u_\mu) - f_i(u_0), \kappa)_{s-4}| &\le \|f_i(u_\mu) - f_i(u_0)\|_{s-4} \|\kappa\|_{s-4} \\ &\le \|f_i(u_\mu) - f_i(u_0)\| \|\kappa\|_{s-4} \\ &\le \sigma_3 \|u_\mu - u_0\| \|\kappa\|_{s-4}. \end{aligned} \tag{74}$$

The convergence (69) implies that

$$(f_i(u_\mu) - f_i(u_0), \kappa)_{s-4} \to 0 \text{ as } \mu \downarrow 0, \text{ uniformly on } [0,T]. \tag{75}$$

Using Parseval's identity, we bound

$$\begin{aligned} \left|((\alpha(v) - \alpha(z))\, \partial_x^2 z, \kappa)_{s-4}\right| &= \left|\left(J^{s-4}\left[(\alpha(v) - \alpha(z))\, \partial_x^2 z\right],\, J^{s-4}\kappa\right)_{L^2}\right| \\ &\le \|(\alpha(v) - \alpha(z))\, \partial_x^2 z\|_{H^{s-4}}\, \|\kappa\|_{H^{s-4}} \\ &\le C\, \|\alpha(v) - \alpha(z)\|_{L^\infty}\, \|\partial_x^2 z\|_{H^{s-4}}\, \|\kappa\|_{H^{s-4}} \\ &\le C\, R^4 \|v - z\|\, \|z\|_{H^{s-2}}\, \|\kappa\|_{H^{s-4}}, \end{aligned} \tag{76}$$

where we applied Lemma 4.3 (with $\delta = s - 2$) and Lemma 4.1(iii). This yields

$$((\alpha(v) - \alpha(z))\partial_x^2 z, \kappa)_{s-4} \to 0 \text{ as } \mu \downarrow 0, \text{ uniformly on } [0,T]. \tag{77}$$

Similarly, we have

$$\begin{aligned} &|(\alpha(v)\partial_x^2(v - z), \kappa)_{s-4}| \\ &\quad \le \|\alpha(v)\partial_x^2(v - z)\|_{s-2} \|\kappa - \kappa_1\|_{s-4} + C_6 \\ &\quad \le 2Q(\rho, T, R)\|\kappa - \kappa_1\|_{s-4} + C_6 \lesssim \epsilon_1 + C_6, \end{aligned} \tag{78}$$

where C_6 satisfies

$$\begin{aligned} C_6 &= \left(\alpha(v)\,\partial_x^2(v-z),\,\kappa_1\right)_{s-4} = |(\partial_x^2(v-z),\alpha(v)J^{2s-8}\kappa_1)| \\ &\le \|J^{-2}\partial_x^2(v-z)\|\|J^2(\alpha(v)J^{2s-8}\kappa_1)\| \\ &\lesssim \mathcal{P}(\rho,T,R)\|J^{2s-8}\kappa_1\|\,\|u_\mu-u_0\| \to 0, \end{aligned} \tag{79}$$

uniformly on $[0,T]$ as $\mu \downarrow 0$. Thus,

$$(\alpha(u_\mu)\partial_x^2 z,\kappa)_{s-4} \to 0 \text{ uniformly on } [0,T] \text{ as } \mu\downarrow 0. \tag{80}$$

Analogous arguments show

$$((\beta(u_0)-\beta(u_\mu))\partial_x u_\mu,\kappa)_{s-4} \to 0 \text{ uniformly on } [0,T] \text{ as } \mu\downarrow 0, \tag{81}$$

and

$$(\beta(u_0)\partial_x z,\kappa)_{s-4} \to 0 \text{ uniformly on } [0,T] \text{ as } \mu\downarrow 0. \tag{82}$$

Combining these results establishes (71). Proposition A.5 then implies that $F_0(u_0)$ is weakly continuous in $H^{s-4}(\mathbb{R})^n$. For any $\tilde{\kappa}\in H^{s-4}(\mathbb{R})^n$ and $t\in[0,T]$, we have

$$\begin{aligned} (u_0(t)-\phi,\tilde{\kappa})_{s-4} &= \lim_{\mu\downarrow 0}(u_\mu(t)-\phi,\tilde{\kappa}) \\ &= \int_0^t (F_0(u_0(\tau)),\tilde{\kappa})_{s-4}\,d\tau. \end{aligned} \tag{83}$$

This implies $u_0(t)-\phi = \int_0^t F_0(u(\tau))\,d\tau$, showing that $u_0\in \mathcal{AC}([0,T];H^{s-4})\cap L^\infty([0,T];H^s)$ satisfies

$$\partial_t u_0 + L(u_0)u_0 = f(u_0) \text{ a.e. } t\in[0,T]. \tag{84}$$

To prove continuity at the origin, we use Proposition A.4 and (67):

$$\begin{aligned} \|u_0(t)-\phi\|_s^2 &= \|u_0(t)\|_s^2 + \|\phi\|_s^2 - 2\mathrm{Re}(u_0(t),\phi)_s \\ &\le \rho^2 + \|\phi\|_s^2 - 2\mathrm{Re}(u_0(t),\phi)_s \\ &\to \epsilon^2 + 2\|\phi\|_s^2 - 2\|\phi\|_s^2 = \epsilon^2, \text{ as } t\downarrow 0. \end{aligned} \tag{85}$$

Therefore, the result follows, since $\epsilon\in(0,1)$ is arbitrary.

Right continuity at $t_1\in(0,T)$ follows by observing that the differential equation in (9) is invariant under the transformation $\omega(t,x)=u_0(t_1+t,x)$, which is continuous at the origin. Left continuity at $t_1\in(0,T]$ is obtained via $\omega(t,x)=u_0(t_1-t,-x)$, leading to

$$\begin{cases} \partial_t\omega + \alpha\partial_x^2\omega + \beta\partial_x\omega = 0, \\ \omega(0) = \tilde{u}_0(t_1), \end{cases} \tag{86}$$

where $\tilde{u}_0(t_1)(x) := u_0(t_1)(-x)$. The solution's continuity at $t=0$ implies

$$\|\omega(t)-\omega(0)\|_s = \|u_0(t_1-t)-u_0(t_1)\|_s \to 0 \text{ as } t\downarrow 0.$$

We conclude that $u_0\in C([0,T];H^s)$. To verify (84) holds for all $t\in[0,T]$ (with left/right derivatives at endpoints), note that $F_0(u_0): H^{s-2}(\mathbb{R})^n \to H^{s-2}(\mathbb{R})^n$ is continuous. For $\epsilon_1>0$, there exists $\delta>0$ such that for $|h|<\delta$ and τ between t and $t+h$,

$$\|F_0(u_0(\tau)) - F_0(u_0(t))\|_{s-2} < \epsilon_1.$$

Thus, for $t\in[0,T]$,

$$\left\|\frac{u_0(t+h)-u_0(t)}{h} - F_0(u_0(t))\right\|_{s-2} \le \frac{1}{|h|}\int_t^{t+h} \|F_0(u_0(\tau))-F_0(u_0(t))\|_{s-2}d\tau < \epsilon_1. \tag{87}$$

Setting $u_0=u$ completes the proof. Next, we establish the uniqueness of solutions for the IVP (9). Suppose that u_0 and v_0 are solutions of the IVP (9) with the same initial data ϕ. Let u_μ and v_μ be solutions of the IVP (10) with initial data ϕ, that is, $u_\mu(0) = v_\mu(0) = \phi$. Then, by Proposition 4.7, it follows that $u_\mu \equiv v_\mu$. Hence, from (69), for all $t\in[0,T]$,

$$\begin{aligned} \|u_0(t)-v_0(t)\| &\le \|u_0(t)-u_\mu(t)\| + \|u_\mu(t)-v_\mu(t)\| + \|v_\mu(t)-v_0(t)\| \to 0, \\ \text{as} \quad & \mu\downarrow 0, \end{aligned} \tag{88}$$

which yields $u_0\equiv v_0$ and completes the proof of uniqueness. □

5. Continuous dependence

In this section, we deal with the continuous dependence on the initial data and parameters. Thus, we consider a sequence of problems similar to (9) as follows

$$\begin{cases} \partial_t u^j + L^j(u^j)u^j = f^j(u^j), \\ u^j(0) = \phi^j. \end{cases} \tag{89}$$

Here, $L^j(u^j)u^j := \left(L_1^j(u^j)u^j, \dots, L_n^j(u^j)u^j\right)$, where

$$L_i^j(u^j)u^j = -\alpha_i^j\, \partial_x^2 u^j + \beta_i^j\, \partial_x u^j, \tag{90}$$

with

$$\alpha_i^j = \frac{(\lambda_i)^j}{(a_i)^j + (b_i)^j (y_i)^j}, \text{ and } \beta_i^j = \frac{(c_i)^j}{(a_i)^j + (b_i)^j (y_i)^j}, \tag{91}$$

and

$$(y_i)^j(x) = (y_{i,0})^j(x) e^{-A_i \int_0^t g(u_i^j(x,\tau))d\tau}. \tag{92}$$

Finally, the source function in (89), is defined by $f^j = \left((f_1)^j, \dots, (f_n)^j\right)$, where

$$\begin{aligned}
(f_1)^j(z) &= \frac{-((c_1)^j)_x\, z_1}{(a_1)^j + (b_1)^j (y_1)^j} + \frac{\left((K_1)^j (b_1)^j z_1 + (d_1)^j\right)(y_1)^j\, g(z_1)}{(a_1)^j + (b_1)^j (y_1)^j} \\
&\quad + \frac{(q_1)^j (z_2 - z_1)}{(a_1)^j + (b_1)^j (y_1)^j} - \frac{\overline{q}_1 (z_1 - u_e)}{(a_1)^j + (b_1)^j (y_1)^j}, \\
(f_i)^j(z) &= \frac{-((c_i)_x)^j\, z_i}{(a_i)^j + (b_i)^j (y_i)^j} + \frac{\left((K_i)^j (b_i)^j z_i + (d_i)^j\right)(y_i)^j\, g(z_i)}{(a_i)^j + (b_i)^j (y_i)^j} \\
&\quad + \frac{(q_i)^j (z_{i+1} - z_i)}{(a_i)^j + (b_i)^j (y_i)^j} - \frac{(q_{i-1})^j (z_i - z_{i-1})}{(a_i)^j + (b_i)^j (y_i)^j}, \quad i = 2, \dots, n-1, \\
(f_n)^j(z) &= \frac{-((c_n)^j)_x\, z_n}{(a_n)^j + (b_n)^j (y_n)^j} + \frac{\left((K_n)^j (b_n)^j z_n + (d_n)^j\right)(y_n)^j\, g(z_n)}{(a_n)^j + (b_n)^j (y_n)^j} \\
&\quad - \frac{(q_{n-1})^j (z_n - z_{n-1})}{(a_n)^j + (b_n)^j (y_n)^j} - \frac{\overline{q}_2 (z_n - u_e)}{(a_n)^j + (b_n)^j (y_n)^j}.
\end{aligned} \tag{93}$$

To guarantee that, for each natural j, the IVP (89) has a solution u^j, Theorem 1.1 requires certain hypotheses on the coefficients of (89). To this end, we assume $a_0^j := \min_{1\le i\le n} \inf a_i^j > 0$. Also, for each $j, k \in \mathbb{N}$, by setting

$$\|y_0^j\|_s = \max_{1\le i\le n} \|y_{i,0}^j\|_s, \quad \|(y_0^j)''\|_\infty = \max_{1\le i\le n} \|(y_{i,0}^j)''\|_\infty,$$

$$\|(a^j)^{(k)}\|_\infty = \max_{1\le i\le n} \|(a_i^j)^{(k)}\|_\infty, \quad \|(q^j)^{(k)}\|_\infty = (\max_{1\le i\le n-1} \|(q_i^j)^{(k)}\|_\infty, \overline{q}_1, \overline{q}_2, u_e)$$

with similar definition for $\|b^{(k)}\|_\infty$, $\|c^{(k)}\|_\infty$, and $\|\lambda^{(k)}\|_\infty$, we define the following constant

$$\begin{aligned}
R^j = \max_{0\le k\le [s]+1} \Big\{ &\|(a^j)^{(k)}\|_\infty, \|(b^j)^{(k)}\|_\infty, \|(c^j)^{(k)}\|_\infty, (\lambda^j)^{(k)}\|_\infty, \\
&\|(q^j)^{(k)}\|_\infty, \|g^{(k)}\|_\infty, \|y_0^j\|_s, \|(y_0^j)''\|_\infty, (a_0^j)^{-1}, \max_{1\le i\le n} |A_i^j| \Big\},
\end{aligned} \tag{94}$$

where $[s]$ denotes the integer part function.

Suppose that there exists $\tilde{R} > 0$ such that, for all $j \in \mathbb{N}$

$$R^j \le \tilde{R}, \text{ and also suppose that } R < \infty, \tag{95}$$

where R is given by (13).

Let $s \ge 2$, $j \in \mathbb{N}$, $\phi = (\phi_1, \dots, \phi_n) \in H^s(\mathbb{R})^n$ and $\phi^j = (\phi_1^j, \dots, \phi_n^j) \in H^s(\mathbb{R})^n$ satisfying $\phi^j \to \phi$ in $H^s(\mathbb{R})^n$. Hence, there exists an open ball $\mathcal{B}(0, R_1)$ in $H^s(\mathbb{R})^n$, centered in 0 and radio R_1, such that

$$\phi, \phi^j \in \mathcal{B}(0, R_1), \tag{96}$$

where R_1 depends on $\|\phi\|_s$.

Suppose that (95)–(96) hold. Then, following the proof of Proposition 4.7, it follows that for all $T > 0$, if u_μ^j denotes the solution of the initial value problem (89) obtained by perturbing it with the term $-\mu\partial_x^4 u^j$, and u_μ is the solution of the Cauchy problem (10), we conclude that $u_\mu^j, u_\mu \in C([0,T]; H^s(\mathbb{R})^n)$, for all $j \in \mathbb{N}$. Then, by denoting $u := u_\mu$, $v := u_\mu^j$ and $w_\mu := u - v$, it follows that $w_\mu \in C([0,T]; H^s(\mathbb{R})^n)$.

Our strategy is to proceed as in Proposition 4.7 and show that it is possible to solve the IVP (89), where the solution w_μ is bounded by a suitable function (see Proposition 5.1 below).

Setting $w_\mu = w$ and $\psi := \phi^j$, we see that w satisfies the following problem

$$\begin{cases} \partial_t w + \mu\partial_x^4 w = \tilde{h}(w), \ t \in \mathbb{R}, \\ w(0) = \phi - \psi, \end{cases} \tag{97}$$

where $\tilde{h}(w) := f(u) - f^j(v) + L^j(v)v - L(u)u$ and

$$\begin{aligned} \tilde{h}_i(w) = (\alpha_i(u_i) - \alpha_i^j(v_i))\partial_x^2 u_i + \alpha_i^j(v_i)\partial_x^2 w_i + (\beta_i^j(v_i) - \beta_i(u_i))\partial_x v_i \\ - \beta_i(u_i)\partial_x w_i + f_i(u) - f_i^j(v). \end{aligned} \tag{98}$$

From the definition of U_μ, we can see that the integral equation for (97) is given by

$$w(t) = U_\mu(t)(\psi - \phi) + \int_0^t U_\mu(t-\tau)\tilde{h}(w(\tau))\,d\tau. \tag{99}$$

Next, we put $\bar{R} := \max\{\tilde{R}, R\}$. Then, from similar estimates to (49) and Gronwall's Lemma we conclude that for all $t \in [0, \tilde{T}]$

$$\|u(t)\|_s \le \|\phi\|_s \underbrace{\exp\left\{(2\bar{R}^2 + \sigma_2 + \sigma_1)\left[t + \left(\frac{s}{4\mu}\right)^{\frac{s}{4}} \frac{4}{4-s} t^{\frac{4-s}{4}}\right]\right\}}_{C_\mu(t)}, \tag{100}$$

and

$$\|v(t)\|_s \le C_\mu(t)\|\psi\|_s, \tag{101}$$

where σ_1 and σ_2 are given in Lemma 4.5, and here, they depend on $\bar{R}$.

In addition, we set

$$\begin{aligned} C_{i,j} = \max\Big\{ & \|(a_i)^j - a_i\|_\infty, \|(b_i)^j - b_i\|_\infty, \|(c_i)^j - c_i\|_\infty, \|(\lambda_i)^j - \lambda_i\|_\infty, \\ & \|(d_i)^j - d_i\|_\infty, \|(y_{i,0})^j - y_{i,0}\|_s, \|(c_i)_x^j - (c_i)_x\|_\infty, \\ & |K_i^j - K_i|, |A_i^j - A_i|, q^j \Big\}, \end{aligned} \tag{102}$$

where $q^j := \max_{1\le i\le n-1} \|(q_i)^j - q_i\|_\infty$. Moreover, we define

$$C^j = \max_{1\le i\le n} C_{i,j}. \tag{103}$$

Proposition 5.1. *Let $\mu \in (0,1)$, $\phi \in H^s(\mathbb{R})$, $s \ge 2$. Then, for each $T > 0$, there exists a unique solution $w_\mu \in C([0,T]; H^s)$ of (97), such that*

$$\|w_\mu(t)\|_s \le 2\|\psi - \phi\|_s + C^j, \quad \textit{for all } t \in [0,T]. \tag{104}$$

Proof. We denote by Ψw the right-hand side of (99). Let $T \le \tilde{T}$ for some fixed $\tilde{T} > 0$, and consider the following complete metric space:

$$\begin{aligned} \tilde{\chi}_T = \Big\{ w \in C([0,T]; H^s) : & \|w(t) - U_\mu(t)(\psi - \phi)\|_s \\ & \le \|\psi - \phi\|_s + C^j, \quad \text{for all } t \in [0,T] \Big\}, \end{aligned} \tag{105}$$

equipped with the metric $d(w^1, w^2) := \sup_{0\le t\le T} \|w^1(t) - w^2(t)\|_s$.

By the Lebesgue dominated convergence theorem, it is easy to see that if $w \in \tilde{\chi}_T$, then $\Psi w \in C([0,T]; H^s)$.

Let $w \in \tilde{\chi}_T$. Writing $f_i(u) - f_i^j(v) = f_i(u) - f_i(v) + f_i(v) - f_i^j(v)$ and proceeding analogously to (49), using (96), (100), and (101), we conclude that

$$\begin{aligned} \|\tilde{h}(w)\| &\le \underbrace{\left[(2\bar{R}^4 + C(\tilde{R}, R, T)(1 + C^j)\tilde{K}C_\mu(T) + 2\bar{R}^2 + \sigma_3\right]}_{\sigma_5} \|w\|_s \\ &\le \sigma_5\big(2\|\phi - \psi\|_s + C^j\big). \end{aligned} \tag{106}$$

Thus, from (106) and (100),

$$\begin{aligned} \|\Psi w(t) - U_\mu(t)(\psi - \phi)\|_s &\le \int_0^t \|U_\mu(t-\tau)\tilde{h}(w)\|_s\,d\tau \\ &\le \int_0^t \left[1 + \left(\frac{s}{4\mu(t-\tau)}\right)^{s/4}\right] \|\tilde{h}(w)\|\,d\tau \\ &\le \sigma_5 \left[t + \left(\frac{s}{4\mu}\right)^{s/4} \frac{4}{4-s} t^{\frac{4-s}{4}}\right] \big(2\|\phi - \psi\|_s + C^j\big). \end{aligned} \tag{107}$$

Moreover, if $w^1, w^2 \in \tilde{\chi}_T$, then by (98),

$$\tilde{h}_i(w^1) - \tilde{h}_i(w^2) = \alpha_i^j(v_i)\partial_x^2(w_i^1 - w_i^2) + \beta_i(u_i)\partial_x(w_i^1 - w_i^2), \tag{108}$$

hence

$$\|\tilde{h}(w^1) - \tilde{h}(w^2)\| \le 2\bar{R}^2\|w^1 - w^2\|_s. \tag{109}$$

Using (109), we obtain

$$\begin{aligned} \|\Psi w^1(t) - \Psi w^2(t)\|_s &\le \int_0^t \|U_\mu(t-\tau)\big(\tilde{h}(w^1) - \tilde{h}(w^2)\big)\|_s \, d\tau \\ &\le \int_0^t \left[1 + \left(\frac{s}{4\mu(t-\tau)}\right)^{s/4}\right] \|\tilde{h}(w^1) - \tilde{h}(w^2)\| \, d\tau \\ &\le 2\bar{R}^2\left[t + \left(\frac{s}{4\mu}\right)^{s/4} \frac{4}{4-s} t^{\frac{4-s}{4}}\right] d(w^1, w^2). \end{aligned} \tag{110}$$

Therefore, from (107) and (110), we conclude that there exists $T'_\mu > 0$ such that $\Psi : \tilde{\chi}_{T'_\mu} \to \tilde{\chi}_{T'_\mu}$ is a contraction. Also, for all $t \in [0, T'_\mu]$, note that

$$\|w_\mu(t)\|_s \le \|w(t) - U_\mu(t)(\psi - \phi)\|_s + \|U_\mu(t)(\psi - \phi)\|_s \le 2\|\psi - \phi\|_s + C^j.$$

Defining

$$\begin{aligned} T^* = \sup\Big\{ & T_\mu > 0 : \text{ there exists } w_\mu \in C([0, T'_\mu]; H^s) \text{ solving (99)}, \\ & \text{such that } \|w_\mu(t)\|_s \le 2\|\psi - \phi\|_s + C^j \text{ for all } t \in [0, T'_\mu] \Big\}, \end{aligned} \tag{111}$$

we conclude the desired result using the ideas from the proof of Proposition 4.7. □

Proof of Theorem 1.2. By the proof of Theorem 1.1, the solution w_0 of (99) (with $\mu = 0$) satisfies $w_0 = \lim_{\mu \downarrow 0} w_\mu \in C([0, T]; H^s)$.

Furthermore, by following the same approach as in the initial steps of the proof of Theorem 1.1, we conclude that

$$w_\mu \rightharpoonup w_0, \text{ in } H^s, \text{ uniformly in } [0, T], \text{ with } \mu \downarrow 0. \tag{112}$$

Consequently

$$\|w_0(t)\|_s \le 2\|\psi - \phi\| + C^j, \quad \text{for all } t \in [0, T]. \tag{113}$$

Indeed, for all $\varphi \in H^s$

$$|(w_\mu(t), \varphi)_s| \le \|w_\mu(t)\|_s\|\varphi\|_s \le \big(2\|\psi - \phi\| + C^j\big)\|\varphi\|_s. \tag{114}$$

Then, when $\mu \downarrow 0$ we obtain

$$|(w_0(t), \varphi)_s| \le \big(2\|\psi - \phi\| + C^j\big)\|\varphi\|_s.$$

Thus, taking the supremum over $\|\varphi\|_s = 1$, in the last inequality, we obtain inequality (113), which implies the desired result.

This concludes the proof. □

6. Numerical results

To illustrate the behavior of the solution to the initial value problem (1), we numerically implement the integral representation formula (12), which solution depends on the parabolic regularization parameter μ. The simulations are carried out in a domain composed of three horizontal layers, each representing a distinct porous region, over the spatial interval $0 \le x \le l$ and for $t \ge 0$. The theoretical solution for the temperature vector $u = (u_1, u_2, u_3)$ of (1), corresponds to the limit when μ goes to 0.

The initial conditions and physical parameters adopted in this work are exactly the same as those used in the previous study [8]. For clarity, we reproduce these values here. This consistency enables a direct and meaningful comparison of the results obtained using the new solution methodology. All physical parameters are assumed to be spatially uniform average values, rather than functions of x. The specific values and nomenclature are presented in Tables B.1.

Consequently, the parameters $a_i, b_i, c_i, d_i, A_i, \lambda_i, q_i, \bar{q}_1, \bar{q}_2$, and E, for $i = 1, 2, 3$, appearing in Section 2 are also treated as constants.

Using the dimensionless formulation and the reference values $x^\star = 3.0$ m, $t^\star = 1.0$ day, $\rho_g^\star = 0.717$ kg/m^3, and $\tilde{l} = l/x^\star$ (where the length l is still denoted simply by l), the parameters are defined as follows (see Appendix A in [8]):

$$a_i = \frac{\Phi_i\rho_g^*\rho_{g_i}c_{g_i} + (1-\Phi_i)\rho_{r_i}c_{r_i}}{\rho_{\bar{r}}c_r}, b_i = \frac{\eta_i^o c_{c_i}}{\rho_{\bar{r}}c_r}, c_i = \frac{\rho_g^*\rho_{g_i}c_{g_i}v_i}{\rho_{\bar{r}}c_r}, \tag{115}$$

$$d_i = \frac{A_i\eta_i^o Q_{h_i}}{T^*\rho_{\bar{r}}c_r}, A_i = t^*\hat{A}_i, E = \frac{\bar{E}}{\bar{R}T^*}, \lambda_i = \frac{t^*\lambda_{s_i}}{(x^*)^2\rho_{\bar{r}}c_r}, \tag{116}$$

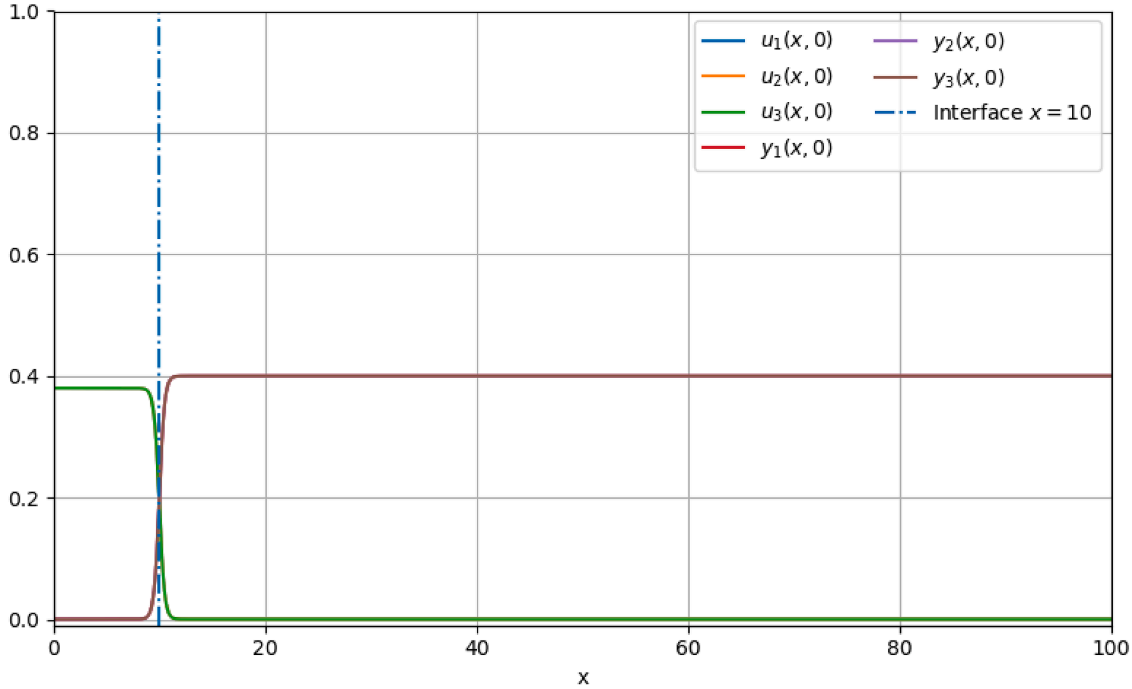

Fig. 1. Initial temperature and fuel concentration profiles in the three-layer medium.

$$q_i = \frac{t^* Q_i}{\bar{\rho_r c_r}}, \qquad \bar{q}_1 = \frac{t^* \bar{Q}_1}{\bar{\rho_r c_r}}, \qquad \bar{q}_2 = \frac{t^* \bar{Q}_2}{\bar{\rho_r c_r}}, \tag{117}$$

where $\bar{\rho_r c_r} = \frac{1}{3}(\rho_{r_1} c_{r_1} + \rho_{r_2} c_{r_2} + \rho_{r_3} c_{r_3})$.

Neumann boundary conditions with zero flux are imposed at $x = 0$ and $x = l$, and the initial temperature field is defined piecewise as:

$$\phi_i(x) = \begin{cases} u_0, & \text{if } 0 \le x \le l_1, \\ 0, & \text{if } l_1 < x \le l, \end{cases} \qquad i = 1, 2, 3. \tag{118}$$

This condition models the injection of a hot gas at the left boundary ($x = 0$) of each layer, initiating combustion and allowing the thermal front to propagate rightward.

The PDE system in (1) remains fully nonlinear, with its coefficients and reaction source terms evaluated iteratively using the updated fuel concentration at each time step. Unlike previous works, where $y(x,t)$ was prescribed as a fixed function, it is now dynamically computed by solving Eq. (6), subject to the following initial condition:

$$y_i(x, 0) = \begin{cases} 0.0, & \text{if } 0 \le x \le l_1, \\ y_0, & \text{if } l_1 < x \le l, \end{cases} \quad \text{for } i = 1, 2, 3. \tag{119}$$

Fig. 1 illustrates the initial states for $u_0 = 0.38$ and $y_0 = 0.40$. Initially, no fuel is present in the region $0 < x < l_1$, while in the interval $l_1 < x < l$, fuel is fully available but the temperature is zero.

In accordance with Theorems 1.1 and 1.2, the data used in the numerical simulation consist of smooth approximations of the initial profiles ϕ_i and y_i, for $i = 1, 2, 3$, chosen so that they belong to the space $H^2([0, l])$.

In order to focus on the integral Eq. (12), we assume average constant values for the physical parameters across the layers, except for porosity. The values used are $\Phi_1 = 0.50$, $\Phi_2 = 0.45$, and $\Phi_3 = 0.40$, consistent with typical physical values, as listed in Table B.1.

The temperature is also rescaled so that the reference value 0 corresponds to the initial (and ignition) temperature u_0, and 1 corresponds to $u_0 + 773.15$ K.

Under these simplifications, we adopt an iterative scheme based on a uniform discretization of the space-time domain to numerically solve Eq. (12). The spatial interval $[0, l]$ is divided into N subintervals with step size $h = \Delta x = l/N$, defining the spatial nodes $x_i = ih$, for $i = 0, 1, \dots, N$. The time domain is discretized using a time step $k = 0.25h$, so that the temporal nodes are given by $t_j = jk$, where $j = 0, 1, \dots, M$, and M is determined by the final simulation time.

The computation proceeds iteratively as follows. Suppose that the vector-valued function

$$u_i^j = \left((u_1)_i^j,\ (u_2)_i^j,\ (u_3)_i^j\right)$$

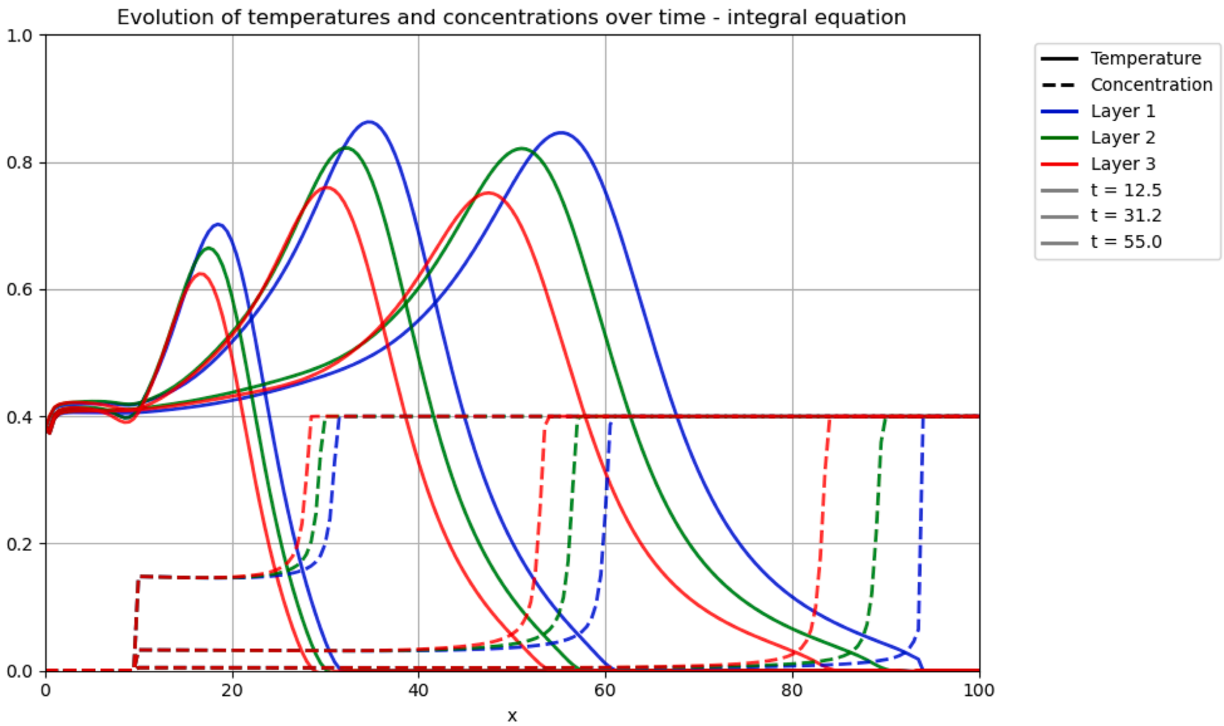

Fig. 2. Temperature and fuel concentration profiles obtained by solving the integral Eq. (12) with $\mu = 0.02$, for three different rescaled times, $t_1 = 12.5$, $t_2 = 31.2$, and $t_3 = 55$, based on the initial data shown in Fig. 1.

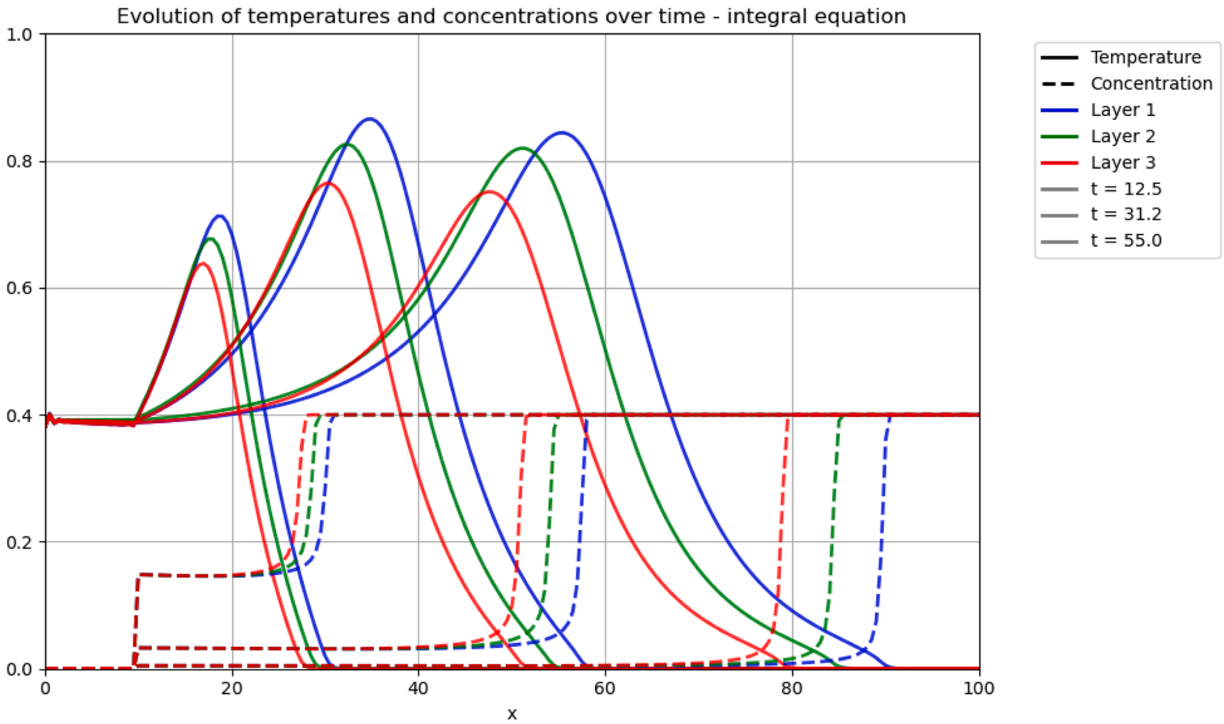

Fig. 3. Temperature and fuel concentration profiles obtained by solving the integral Eq. (12) with $\mu = 0.0002$, for three different rescaled times, $t_1 = 12.5$, $t_2 = 31.2$, and $t_3 = 55$, based on the initial data shown in Fig. 1.

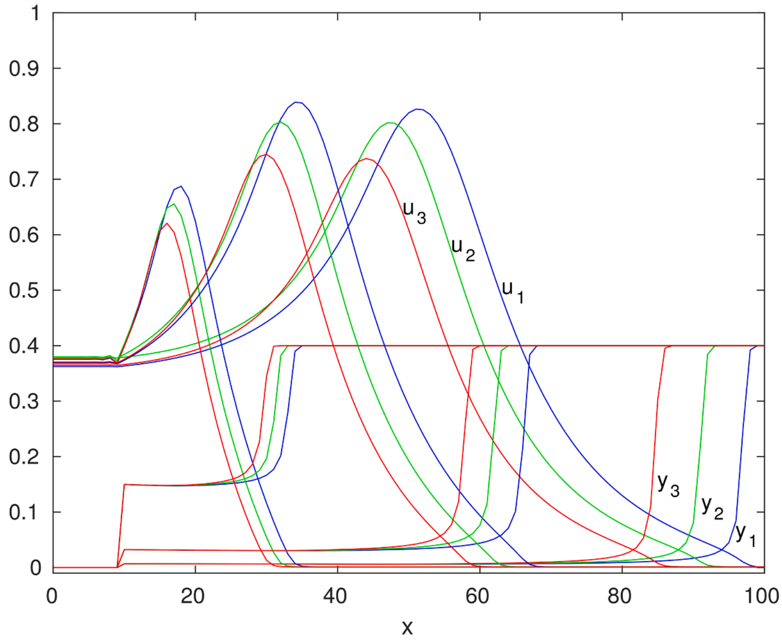

Fig. 4. Temperature and fuel concentration profiles obtained from the full model (2), using the Crank-Nicolson finite difference scheme, for three different rescaled times, $t_1 = 12.5$, $t_2 = 31.2$, and $t_3 = 55$, based on the initial data shown in Fig. 1.

is known at each grid point (x_i, t_j). Then, using Eq. (12), we compute the next time step

$$u_i^{j+1} = \left((u_1)_i^{j+1},\ (u_2)_i^{j+1},\ (u_3)_i^{j+1}\right)$$

iteratively, starting from the initial condition

$$u(x,0) = \phi(x) = \left(\phi_1(x),\ \phi_2(x),\ \phi_3(x)\right).$$

Figs. 2 and 3 present the graphs of the solutions to integral Eq. (12) for layers 1, 2, and 3 at three distinct time instances. The solutions were obtained using initial condition illustrated in Fig. 1 . Fig. 2 corresponds to the parabolic parameter $\mu = 0.02$, whereas Fig. 3 corresponds the case with $\mu = 0.0002$.

Fig. 4 shows the solution of Problem (2) obtained by directly solving this problem using the finite difference method (Crank-Nicolson scheme). We observe that as the parameter μ decreases, the graph produced by the integral equation becomes increasingly closer to the one obtained via finite differences, which confirms that our theoretical solution is consistent with the solution of the original problem.

Although a detailed discussion of the physical properties of the solution based on the variation of the parameters is beyond the scope of this work, we include here, for reference, Figs. 3, 5, and 6. These figures were generated using the same parameter values as in Table B.1, except for the interfacial transfer coefficients Q_1 and Q_2, which were set to $Q_1 = Q_2 = 0.0025$, $Q_1 = Q_2 = 0.02$, and $Q_1 = Q_2 = 1.0$, respectively.

We observe that as the values of Q_1 and Q_2 increase, the solutions for the three layers become increasingly similar. In the last case, with $Q_1 = Q_2 = 1.0$, the three solutions essentially coincide, behaving as if there were only a single homogeneous layer. This outcome is physically consistent, as higher transfer coefficients promote stronger coupling between layers, effectively eliminating thermal gradients at the interfaces.

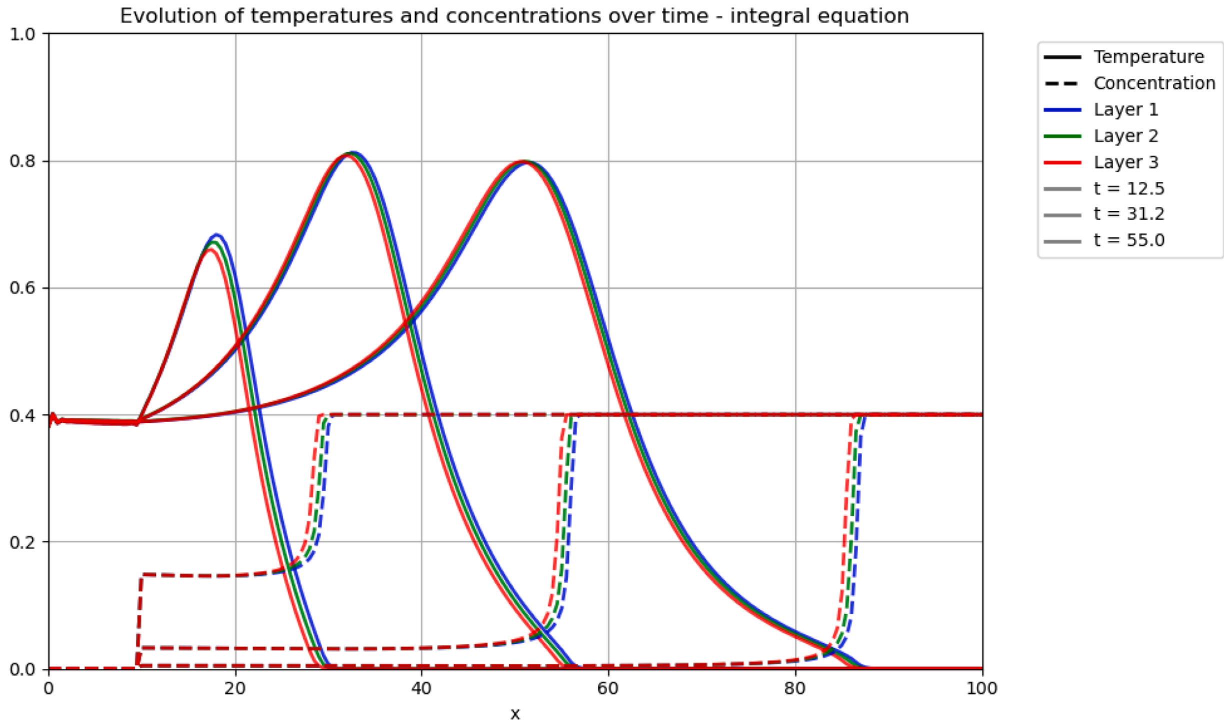

Fig. 5. Temperature and fuel concentration profiles obtained by solving the integral Eq. (12) with $Q_1 = Q_2 = 0.02$, for three different rescaled times, $t_1 = 12.5$, $t_2 = 31.2$, and $t_3 = 55$, based on the initial data shown in Fig. 1.

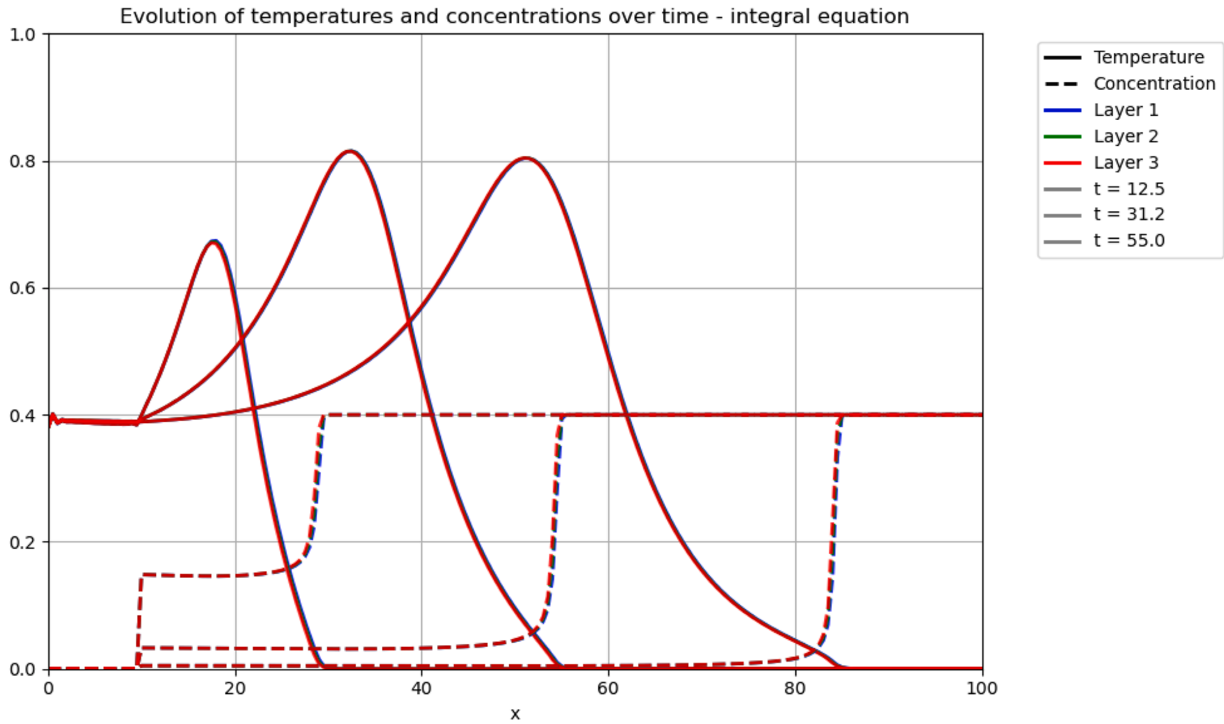

Fig. 6. Temperature and fuel concentration profiles obtained by solving the integral Eq. (12) with $Q_1 = Q_2 = 1.0$, for three different rescaled times, $t_1 = 12.5$, $t_2 = 31.2$, and $t_3 = 55$, based on the initial data shown in Fig. 1.

7. Conclusions

We have established a well-posedness theory for a class of quasilinear reaction-diffusion-convection systems modeling combustion in multilayer porous media, where both temperature and fuel concentrations evolve as coupled unknowns. By integrating the fuel equations and introducing a fourth-order parabolic regularization, we overcame the lack of a linear structure and proved the global existence, uniqueness, and continuous dependence of solutions in Sobolev spaces. This analysis significantly extends previous models by removing the assumption of prescribed fuel profiles and allowing spatially heterogeneous parameters. Numerical simulations further confirm the consistency of the solutions with expected combustion behavior. These results offer a more general and realistic framework for the mathematical modeling of reactive transport in complex porous structures, and open avenues for future work on stability analysis, asymptotic behavior, the extension to higher-dimensional geometries, and the inclusion of additional dependent variables such as gas-phase pressure and oxygen fraction.

CRediT authorship contribution statement

Marcos R. Batista: Writing – review & editing, Formal analysis; **Alysson Cunha:** Writing – original draft, Formal analysis, Conceptualization; **Jesus C. Da Mota:** Writing – review & editing, Validation, Formal analysis, Conceptualization; **Ronaldo A. Santos :** Writing – review & editing, Validation, Formal analysis, Conceptualization.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A. Stein derivative

In our work, we also use a notion of derivative equivalent to the fractional derivative, as presented in Section 3. Thus, let $L^p_b(\mathbb{R}^d)$ the fractional Sobolev space defined as $L^p_b(\mathbb{R}^d) := (1-\Delta)^{-\frac{b}{2}} L^p(\mathbb{R}^d)$. Such spaces can be characterized by the Stein derivative of order b.

Theorem A.1. *Let $b \in (0,1)$ and $\frac{2d}{d+2b} < p < \infty$. Then $f \in L^p_b(\mathbb{R}^d)$ if and only if*

a) $f \in L^p(\mathbb{R}^d)$,

b) $\mathcal{D}^b f(x) := \left(\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^2}{|x-y|^{d+2b}} dy \right)^{\frac{1}{2}} \in L^p(\mathbb{R}^d)$, *with*

$$\|f\|_{b,p} := \|J^b f\|_p \sim \|f\|_p + \|D^b f\|_p \sim \|f\|_p + \|\mathcal{D}^b f\|_p. \tag{A.1}$$

Proof. See [23]. □

From Fubini's theorem, can deduce the following product estimate

$$\|\mathcal{D}^b(fg)\|_{L^2(\mathbb{R}^d)} \le \|f\mathcal{D}^b g\|_{L^2(\mathbb{R}^d)} + \|g\mathcal{D}^b f\|_{L^2(\mathbb{R}^d)}. \tag{A.2}$$

As an application of the previous inequality, and the definition of the derivative $\mathcal{D}^b$, for the case $d=1$, we have:

Proposition A.2. *Let $b \in (0,1)$ and h be a measurable function on $\mathbb{R}$ such that $h, h' \in L^\infty(\mathbb{R})$. Then, for almost every $x \in \mathbb{R}$,*

$$\mathcal{D}^b h(x) \lesssim \|h\|_{L^\infty(\mathbb{R})} + \|h'\|_{L^\infty(\mathbb{R})}. \tag{A.3}$$

We will also need of the following inequality

$$\|\mathcal{D}^\delta(hf)\| \le (\|h\|_\infty + \|h'\|_\infty)\|f\| + \|h\|_\infty \|\mathcal{D}^\delta f\|, \tag{A.4}$$

which the proof can be deduced from (A.2) and (A.3).

Lemma A.3. *Let $Q_1 := Q_1(r,t,R)$ and $Q_2 := Q_2(r,t,R)$ as in (27) and (28), respectively. Then*

$$\begin{aligned} Q_1 = AtR^3re^{At}\big[R^2e^{At}(1+AtRr)(1+R^2) \\ +2+\tilde{Q}\big]\big[AtRr(r+1)+1\big] \end{aligned} \tag{A.5}$$

and

$$\begin{aligned} Q_2 =& R^2e^{At}\big[1+R^2(2+AtRr)\big((1+R^2)e^{At}+2R^3\big)\big]+2R^4+ \\ &+R^2\big[\big(1+\tilde{Q}+R^2+(1+R^2)AtRr\big)e^{At}+2R^2\tilde{Q}+4R^4\big], \end{aligned} \tag{A.6}$$

where $\tilde{Q}$ is given in (23).

Proof. The proof of (A.5) involves finding an upper bound for the right-hand side of (27). Thus, by using (13), (23) and (26) and the Sobolev embedding $\|y_0'\|_\infty \lesssim \|y_0\|_s \leq R$, we obtain the desired result. The proof of (A.6) corresponds to obtaining an upper bound for the left-hand side of (28) and follows a similar approach to the one above. □

Proposition A.4. *Let $f_\mu \in C([0,T]; H^s)$ such that*

$$\lim_{\mu,\nu\downarrow 0} \sup_{[0,T]} \|f_\mu - f_\nu\| = 0.$$

Then, there exists $f_0 : [0,T] \to H^s$ with

$$f_\mu \rightharpoonup f_0, \quad \text{as} \quad \mu \downarrow 0, \quad \text{uniformly in} \quad [0,T]. \tag{A.7}$$

Proof. See [15]. □

Proposition A.5. *Let $f_\mu \in C([0,T]; H^s)$ such that*

$$f_\mu \rightharpoonup f_0, \quad \text{as} \quad \mu \downarrow 0, \quad \text{uniformly in} \quad [0,T].$$

Then, $f_0 : [0,T] \to H^s$ is weakly continuous.

Proof. See [15]. □

Appendix B. Physical values

In this section, we present Tables B.1 with some average physical values used in the numerical simulations.

Table B.1
Nomenclature and typical physical values, for $i = 1, 2, 3$.

Notation	Meaning	International system
l	Reservoir length (rescaled)	100
u_e	Environment temperature (rescaled)	0.001
Φ_1, Φ_2, Φ_3	Porosity, layers 1, 2, 3 respectively	0.50, 0.45, 0.40
ρ_{gi}	Average density of gas	0.717 $\mathrm{kg/m^3}$
ρ_{ri}	Average density of rock	2400.0 $\mathrm{kg/m^3}$
c_{gi}	Thermal capacity of gas at constant pressure	0.0022 $\mathrm{kJ/(kg\,°C)}$
c_{ri}	Thermal capacity of rock at constant pressure	0.00084 $\mathrm{kJ/(kg\,°C)}$
c_{ci}	Thermal capacity of coke at constant pressure	0.00084 $\mathrm{kJ/(kg\,°C)}$
λ_i	Thermal conductivity of the system (rescaled)	0.00025 $\mathrm{kW/(m\,°C)}$
$\hat{A}_i$	Pre-exponential Arrhenius factor (rescaled)	1.0×10^{10}
$\bar{E}$	Activation energy	100 $\mathrm{kJ/kmol}$
$\bar{R}$	Gas constant	8.310 $\mathrm{J/(kmol\,K)}$
K_i	Absolute permeability (rescaled)	4.3867×10^{-10} $\mathrm{m^2}$
Q_{h_i}	Reaction heat per unit mass	352.80 $\mathrm{kJ/kg}$
Q_1, Q_2	Heat transfer between layers 1 and 2, and 2 and 3	0.0025 $\mathrm{KW/m°C\,m^2}$
$\bar{Q}_1, \bar{Q}_2$	Heat transfer between layers 1 and 3, and environment	0.001 $\mathrm{KW/m°C\,m^2}$
a_1, a_2, a_3	Constants defined by Eq. (115), layers 1, 2, 3	0.0010, 0.0011, 0.0012
b_i	Constants defined by Eq. (115)	0.90×10^{-3}
c_i	Constants defined by Eq. (115)	3.33×10^{-7}
d_i	Constants defined by Eq. (116)	1.22×10^{5}

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