Eur. Phys. J. D (2013) 67: 156
DOI: 10.1140/epjd/e2013-40161-y

THE EUROPEAN
PHYSICAL JOURNAL D

Regular Article

Perfect transfer of quantum states in a network of harmonic oscillators

D. Portes Jr.[1], H. Rodrigues[1], S.B. Duarte[2,a], and B. Baseia[3]

[1] Centro Federal de Educação Tecnológica do Rio de Janeiro, Av. Maracanã 229, 20.271-110, Rio de Janeiro, RJ, Brazil
[2] Centro Brasileiro de Pesquisas Físicas, Rua Dr. Xavier Sigaud 150, 22.290-180, Rio de Janeiro, RJ, Brazil
[3] Instituto de Física, Universidade Federal de Goiás, P.O. Box-131, 74.001-970, Goiania, GO, Brazil

Received 22 March 2013 / Received in final form 14 May 2013
Published online 24 July 2013 –

Abstract. This work presents an exactly soluble scheme to address the problem of optimal transfer of quantum states through a set of s harmonic oscillators composing a network with connected ends as a closed quantum circuit. For this purpose we start from a general quadratic Hamiltonian form. The relationship between the parameters of the Hamiltonian, the network size, and the time interval required for such transfer are explicitly shown. Particular physical realizations of this Hamiltonian, transfer of entangled states, including transfer of states at the expense of a quantum entanglement, are also considered.

1 Introduction

Among the various interesting problems studied in quantum optics one may cite (and distinguish) the teleportation of states between two (non interacting) quantum systems [1,2], from one subsystem to the other, and alternatively, the transfer of states between (interacting) quantum systems [3,4]. In the first case the process occurs crucially due to the intervention of an entangled state that describes a combined bipartite system. In the second case the process is due to an appropriate type of interaction between the subsystems. This second scenario may also include the study of exchange of states between the subsystems [5–9]. In both scenarios the efficiency of the process deserves special attention, whose verification involves the calculation of the fidelity and the success probability of the operation. The case of state transfer is also interesting, e.g., in the realm of quantum spin networks [10–12]. Others have investigated the behavior of a quantum state propagating through a network of interacting oscillators [13,14], with few attention on how to find out the conditions that govern the connected oscillators to allow transmission of states closer to the ideal case. The issue concerns the perfect transfer of states in terms of fidelity and success probability, including the transfer of entangled states.

In this work we pursue the answer to the following query: What is, if any, the appropriated class of Hamiltonian that allows us to get such a state transfer through the coupled oscillators network? To this end, we developed a compact method to obtain a Hamiltonian form that produces a perfect state transfer through the mentioned network. In particular, we explicitly determine this Hamiltonian form for a time independent s-sized network with connected ends. Such network configuration simulates a closed quantum circuit of oscillators which is important to discuss possible occurrence of nonclassical effects, as state revival, squeezing, and others during the state propagation. We show that, for a perfect quantum state transfer along the system, the coupling can not be restricted to next neighbors, but it must embrace the whole network. Transfer of correlated quantum states in the network was also studied, including examples where these type of states play an auxiliary role for state transfers.

This paper is organized as follows. In Section 2 we develop the mathematical basis for our analysis to describe the time evolution of the characteristic function of the state of our s-sized HO-network. In Section 3 we use the result of Section 2 in order to obtain a suitable Hamiltonian form that yields perfect cyclic transfer of an arbitrary state. In Section 4 we discuss the time evolution of a quantum (number) state and also the transfer of entangled states. Section 5 treats the propagation of the (nearest classical) coherent state and discusses the physical realizations of our employed Hamiltonian form, including the transfer of states at the expense of a quantum entanglement. Section 6 contains the comments and conclusions.

2 The characteristic function of the system state

In the Schrödinger picture, the density operator $\rho(t)$ for the state of s identical coupled harmonic oscillator may

[a] e-mail: sbd@cbpf.br

be described by means of the characteristic function χ, in the form [15,16]

$$\rho(t) = \pi^{-s} \int \chi(\alpha_1, \ldots, \alpha_s; t)_{j=1}^{s} \left[\mathbf{D}_j^{-1}(\alpha_j)\, d^2\alpha_j\right], \quad (1)$$

with χ defined in terms of s complex quantities α_j as,

$$\chi(\alpha_1, \ldots, \alpha_s; t) \equiv \mathrm{Tr}\left[\rho(t)_{j=1}^{s} \mathbf{D}_j(\alpha_j)\right], \quad (2)$$

where

$$\mathbf{D}_j(\alpha) \equiv \exp\left(\alpha \mathbf{a}_j^\dagger - \alpha^* \mathbf{a}_j\right). \quad (3)$$

The integration in equation (1) is carried out in the whole α_j-plane, and the element $d^2\alpha_j$ is defined by

$$d^2\alpha_j = d(\mathrm{Re}(\alpha_j))\, d(\mathrm{Im}(\alpha_j)). \quad (4)$$

The reduced density matrix ρ_k for a kth subsystem is obtained from the partial trace of $\rho(t)$, taken over all other subsystems,

$$\rho_k(t) = \mathrm{Tr}_1\left[\ldots \mathrm{Tr}_{j\neq k}\left[\ldots \mathrm{Tr}_s[\rho(t)]\right]\right]. \quad (5)$$

By considering that

$$\rho_k(t) = \frac{1}{\pi}\int \chi_k(\alpha; t)\mathbf{D}_k^{-1}(\alpha)\, d^2\alpha, \quad (6)$$

and using equations (1) and (5) with the properties of the characteristic function from,

$$\mathrm{Tr}[\mathbf{D}(\alpha)] = \pi\delta^{(2)}(\alpha), \quad (7)$$

the reduced characteristic function can be written as,

$$\chi_k(\alpha; t) \equiv \chi(0, \ldots, \alpha_k, \ldots 0; t)|_{\alpha_k=\alpha}. \quad (8)$$

A complete transfer of the quantum states of the mth oscillator to the nth one is obtained by imposing the prescription,

$$\chi_n(\alpha; \tau) = \chi_m(\alpha; 0), \quad (9)$$

or

$$\chi(0, \ldots, \alpha_n, \ldots 0; \tau) = \chi(0, \ldots, \alpha_m, \ldots 0; 0). \quad (10)$$

The prescription (9) indicates that the state transfer is realized after the elapsed time τ.

Next, we use the well known functional identity for the characteristic function of a system composed by s quantum-mechanical oscillators [9],

$$\chi(\alpha_1, \ldots, \alpha_s; t) = \chi(\alpha_1(t), \ldots, \alpha_s(t); 0), \quad (11)$$

where the complex quantities $\alpha_j(t)$ are obtained from the inverse of the Bogoliubov transformation of the Heisenberg operators,

$$\mathbf{a}(t) = \mu(t)\mathbf{a}(0) + \nu(t)\mathbf{a}^\dagger(0), \quad (12)$$

namely,

$$\alpha(t) = \mu^\dagger(t)\alpha(0) - \nu^T(t)\alpha^*(0), \quad (13)$$

where the matrix representation was used for operators and coefficients in (12)–(13), $\mathbf{a} = [a_j]_{s\times 1}$, $\alpha = [\alpha_j]_{s\times 1}$, $\mu = [\mu_{jk}]_{s\times s}$ and $\nu = [\nu_{jk}]_{s\times s}$. It should be emphasized that the relation (13) is valid only for the Bogoliubov transformation, when the Hamiltonian is quadratic in the time-independent creation and annihilation operators, $\mathbf{a}$ and $\mathbf{a}^\dagger$. From equations (8), (11) and (13) the temporal evolution of the reduced characteristic function is given by

$$\chi_n(\alpha; t) = \chi(\mu_{n1}^*(t)\alpha - \nu_{n1}(t)\alpha^*, \ldots, \mu_{ns}^*(t)\alpha - \nu_{ns}(t)\alpha^*; 0). \quad (14)$$

From equation (9), we have the condition,

$$\begin{cases} \mu_{nj}^*(\tau)\alpha - \nu_{nj}(\tau)\alpha^* = \alpha & (j = m) \\ \mu_{nj}^*(\tau)\alpha - \nu_{nj}(\tau)\alpha^* = 0 & (j \neq m), \end{cases} \quad (15)$$

for arbitrary α. The solution of equation (15) is

$$\begin{cases} \mu_{nj}(\tau) = \delta_{jn}, & j = 1, \ldots, s \\ \nu_{nj}(\tau) = 0, & j = 1, \ldots, s. \end{cases} \quad (16)$$

Particularly, for a cyclic permutation representing the state transfer of an oscillator to one of its first neighbor, and considering the boundary condition of coincident network ends, plus the conditions

$$\chi_m(\alpha; \tau) = \chi_{m-1}(\alpha; 0) \text{ and } \chi_1(\alpha; \tau) = \chi_s(\alpha; 0), \quad (17)$$

we have

$$\mu(\tau) = C \equiv \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \quad (18)$$

and

$$\nu(\tau) = 0. \quad (19)$$

Substituting equations (18), (19) and (13) in equation (11), we find that

$$\chi(\alpha_1, \ldots, \alpha_s; \tau) = \chi(\alpha_2, \ldots, \alpha_s, \alpha_1; 0). \quad (20)$$

Thus, any state transfer between HO-oscillators in the network corresponds to a cyclic permutation of the arguments in the characteristic function of the whole system after a time interval τ. This result can be generalized to any type of permutation, hence not restricted to a cyclic one.

3 The Hamiltonian for perfect state transfer

To construct the time-independent Hamiltonian that describes the dynamics illustrated in the previous section, we consider a general quadratic form in the creation $\mathbf{a}^\dagger$ and annihilation $\mathbf{a}$ operators

$$\mathbf{H} = \hbar\sum_{j,k=1}^{s} \lambda_{jk}\mathbf{a}_j^\dagger\mathbf{a}_k + \hbar\sum_{j,k=1}^{s}\left(\gamma_{jk}\mathbf{a}_j^\dagger\mathbf{a}_k^\dagger + \gamma_{jk}^*\mathbf{a}_j\mathbf{a}_k\right). \quad (21)$$

This Hamiltonian can be diagonalized by a Bogoliubov transformation given by

$$\mathbf{a}'_k = \sum_{j=1}^{s} \left(W_{kj}\mathbf{a}_j + V_{kj}\mathbf{a}_j^\dagger \right), \tag{22}$$

which leads it to the Hamiltonian form

$$\mathbf{H} = \hbar \sum_{j}^{s} \omega_j \ \mathbf{a}_j'^\dagger \mathbf{a}'_j. \tag{23}$$

By taking the time evolution of the Heisenberg operators $\mathbf{a}(t)$ in equation (12) we obtain,

$$\mu(t) = W^\dagger e^{-i\Omega t} W - V^T e^{i\Omega t} V^*, \tag{24}$$
$$\nu(t) = W^\dagger e^{-i\Omega t} V - V^T e^{i\Omega t} W^*, \tag{25}$$

where

$$\Omega = \mathrm{diag}\,(\omega_1, \ldots, \omega_s). \tag{26}$$

To have the complete transfer of state, as mentioned in previous section (see Eqs. (18) and (19)), the matrices should satisfy $\mu(\tau) = C$ and $\nu(\tau) = 0$, thus we have

$$W^\dagger e^{-i\Omega\tau} W - V^T e^{i\Omega\tau} V^* = C, \tag{27}$$

and

$$W^\dagger e^{-i\Omega\tau} V - V^T e^{i\Omega\tau} W^* = 0. \tag{28}$$

A proof of uniqueness of Bogoliubov transformation satisfying the last two equations is shown in Appendix. This transformation is defined by $V = 0$ and the unitary matrix W that diagonalizes C, given explicitly by reference [17],

$$W_{jk} = \frac{1}{\sqrt{s}} e^{2\pi i jk/s}. \tag{29}$$

Thus, we have

$$WCW^\dagger = \mathrm{diag}(e^{-2\pi i(1/s)}, e^{-2\pi i(2/s)} \ldots, e^{-2\pi i}). \tag{30}$$

From equation (27), and taking $V = 0$, we obtain

$$WCW^\dagger = e^{-i\Omega t}. \tag{31}$$

Equations (30) and (31) lead to the relation

$$\omega_j = \frac{2\pi}{\tau} \left(\frac{j}{s} + m_j \right), \tag{32}$$

where m_j are arbitrary integers. We will impose that all $m_j \geq 0$ to ensure positive eigenvalues for the Hamiltonian. For $m_j = 0$ and any j we obtain the frequencies of the fundamental modes.

As $V = 0$ we see from equation (22) that the terms $\gamma_{jk}\mathbf{a}_j^\dagger\mathbf{a}_k^\dagger + \gamma_{jk}^*\mathbf{a}_j\mathbf{a}_k$ are excluded in the Hamiltonian (21). So the general form of our Hamiltonian is given by

$$\mathbf{H} = \hbar \ \mathbf{a}^\dagger W^\dagger \Omega W \mathbf{a}. \tag{33}$$

This Hamiltonian form propagates an arbitrary state of an oscillator in the network, allowing its complete transfer to another one after the time interval τ. More explicitly, substituting the form of Ω from equation (26) and that of W from equation (29) in equation (33) we obtain the coupling coefficients of the original Hamiltonian (21)

$$\lambda_{jk} = \frac{2\pi\hbar}{s\tau} \sum_{l=1}^{s} \left(\frac{l}{s} + m_l \right) \exp\left(2\pi i \,(j-k)\, \frac{l}{s} \right). \tag{34}$$

Since we have arbitrary integers m_l in the last equation, we indeed obtained an infinite and numerable family of Hamiltonian forms (labeled by the m_l-values), allowing the perfect state transfer. From equation (34) (with $k = j$) we can see that the diagonal elements,

$$\lambda_{jj} = \frac{2\pi\hbar}{s\tau} \sum_{l=1}^{s} \left(\frac{l}{s} + m_l \right), \tag{35}$$

are directly related with the trace of Ω matrix. Note that, in principle, none of the coupling λ_{jk} are null. So, during the elapsed time τ the perfect transfer of states between oscillators is an effect of collective coupling between oscillators in the network.

Next, substituting $V = 0$ into equations (24) and (25), one gets

$$\mu(t) = W^\dagger e^{-i\Omega t} W \tag{36}$$
$$v(t) = 0, \tag{37}$$

and then

$$\mu_{jk}(t) = \frac{1}{s} \sum_{l=1}^{s} \exp\left(2\pi i \left[j - k - \frac{t}{\tau} \right] \frac{l}{s} \right) \exp\left(-i \frac{2\pi}{\tau} m_l t \right). \tag{38}$$

The following properties are satisfied by the these matrices:

$$\mu_{jk}(t) = \mu_{sk}(t-\tau), \quad \text{for } j = 1, \tag{39}$$
$$\mu_{jk}(t) = \mu_{(j-1)k}(t-\tau), \quad \text{for } 2 \leq j \leq s, \tag{40}$$

and

$$\mu_{js}(t) = \mu_{jk}(t-\tau), \quad \text{for } k = 1, \tag{41}$$
$$\mu_{jk}(t) = \mu_{j(k+1)}(t-\tau), \ \text{for } 2 \leq k \leq s. \tag{42}$$

From these properties and knowing that $\mu(0) = 1$, we have $\mu(t)$ for all multiple of the period τ

$$\mu(n\tau) = \mu(\tau)^n. \tag{43}$$

In special, for $n = s$ we have $\mu(s\tau) = 1$, as expected, since we are using coincident ends for the network, as boundary condition.

4 Number state and entanglement transfer

Here we will firstly analyze the case where the initial configuration represents the first oscillator prepared in a number state $|n\rangle$, while the others in their fundamental modes, i.e.,

$$|\Psi(0)\rangle = |n, 0, \ldots, 0\rangle. \tag{44}$$

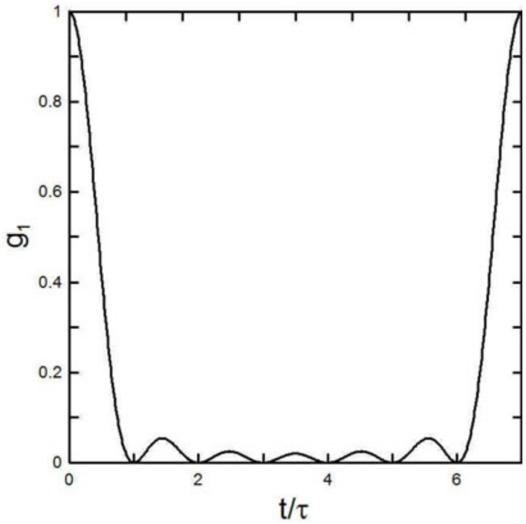

Fig. 1. Function $g_1(t)$, defined in the text, with no excitation ($m_j = 0$ for all oscillators) in a network with size $s = 7$.

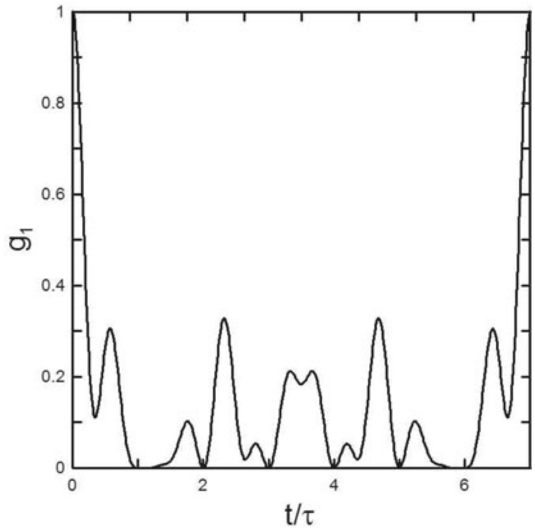

Fig. 2. Function $g_1(t)$, defined in the text, for $m_2 = 1$, $m_6 = 2$ and $m_j = 0$ for other oscillators in a network with $s = 7$.

Choosing the first oscillator instead of any other is irrelevant since the Hamiltonian is symmetrical by any cyclic permutation of the oscillator label, as seen in the previous section. The characteristic function associated with the state at $t = 0$ is

$$\chi(\alpha_1, \ldots, \alpha_s; 0) = f^{(n)}(\alpha_1) f^{(0)}(\alpha_2) \ldots . f^{(0)}(\alpha_s), \quad (45)$$

where [15,16]

$$f^{(n)}(\alpha) = e^{-\frac{1}{2}\alpha^*\alpha} L_n(\alpha^*\alpha), \quad (46)$$

and $L_n(x)$ stands for the Laguerre polynomial. Thus, taking into account that $L_0(x) = 1$, we get

$$\chi(\alpha_1, \ldots, \alpha_s; 0) = L_n(\alpha_1^*\alpha_1) \exp\left(-\frac{1}{2}\sum_{j=1}^{s} \alpha_j^*\alpha_j\right). \quad (47)$$

The time evolution of the reduced characteristic function can be easily obtained by using equation (14) with $\nu(t) = 0$. We thus find

$$\chi_j(\alpha; t) = e^{-\frac{1}{2}\alpha^*\alpha} L_n\left[g_j(t)\alpha^*\alpha\right], \quad (48)$$

where

$$g_j(t) = \mu_{j1}^*(t)\mu_{j1}(t). \quad (49)$$

Note that the sum $\sum \alpha_j^*\alpha_j$ is time invariant due to relation $\mu^\dagger\mu = 1$. According to equation (48) the jth oscillator is in its ground state when $g_j(t) = 0$ and in a number state when $g_j(t) = 1$. From the property given by equation (39) we have

$$g_j(t) = g_1\left(t - (j-1)\tau\right). \quad (50)$$

Therefore all oscillators in the network exhibit similar time evolution, except for a time delay.

Figure 1 shows the function $g_1(t)$ with $m_j = 0$ for all j, for $s = 7$. We can see that $g(t)$ has the expected behavior: for values of time multiples of τ ($t = k\tau$) we have either $g_1 = 0$ or $g_1 = 1$. We can also see that the transferred state runs cyclically through the network with the period $s\tau$. Figure 2 shows the function $g_1(t)$ for $m_2 = 1$, $m_6 = 2$ and $m_j = 0$ for other j. For $t = k\tau$ the behavior of $g_1(t)$, showed in Figure 2, remains the same; however for $t \neq k\tau$ this behavior is quite different with additional excitations, which reflects the higher energy level of the Hamiltonian.

It is also easy seeing that an initial entanglement, in the sense of EPR state [18] will propagate through the network. For example, if we take

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|n,0\rangle + |0,n\rangle\right) \times |0,\ldots,0\rangle, \quad (51)$$

due to the linear time evolution we have for $t = \tau$,

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}}|0,n,0\ldots,0\rangle + \frac{1}{\sqrt{2}}|0,0,n,0\ldots,0\rangle \quad (52)$$

$$= \frac{1}{\sqrt{2}}|0\rangle \times \left(|n,0\rangle + |0,n\rangle\right) \times |0,\ldots,0\rangle. \quad (53)$$

The same occurs for any time multiple of τ and oscillator interactions that occur during the state propagation restore the entanglement whenever $t = k\tau$. In addition, if any state correlations is produced by the oscillator interactions itself, they should vanish periodically at every time $t = k\tau$.

5 Coherent state propagation

Let us now suppose the first oscillator in a coherent state $|\beta\rangle$ at $t = 0$, again with the others in their fundamental mode, i.e.,

$$|\Psi(0)\rangle = |\beta, 0, \ldots, 0\rangle, \quad (54)$$

with $a_1|\Psi(0)\rangle = \beta|\Psi(0)\rangle$. The characteristic function associated with the coherent state $|\beta\rangle$ is

$$f^{(\beta)}(\alpha) = e^{-\alpha^*\alpha/2} e^{\alpha\beta^* - \alpha^*\beta}. \quad (55)$$

Thus,

$$\chi(\alpha_1,\ldots,\alpha_s;0) = \exp\left(\alpha_1\beta^* - \alpha_1^*\beta\right)\exp\left(\sum_{j=1}^{s}\alpha_j^*\alpha_j\right). \tag{56}$$

Analogously to the result obtained in equation (48), we have

$$\chi_j(\alpha;t) = \exp\left(-\alpha^*\alpha/2 + \mu_{j1}(t)\alpha\beta^* - \mu_{j1}^*(t)\alpha^*\beta\right), \tag{57}$$

and the propagation of coherent states is such that at $t = k\tau$ the coherence is restored.

As physical realizations of the Hamiltonian model in equation (21) we mention that: (i) it describes a set of s noninteracting HO (or equivalently, s field modes) when only λ_{jj} are the non null coefficients; (ii) when $\lambda_{jk} = \gamma_{jk} = 0$ for all $j \neq k$ it recovers the quadratic ("two-photon") Hamiltonian which also responds for the generation of squeezing effect (see Refs. [19,20]). One origin/realization of the "two-photon" Hamiltonian comes from the $\mathbf{xp} + \mathbf{px}$ interaction, where $\mathbf{x} \propto (\mathbf{a} + \mathbf{a}^\dagger)$ and $\mathbf{p} \propto (\mathbf{a} - \mathbf{a}^\dagger)$ stand for the position and momentum operators of the HO (or the quadratures of a field mode), respectively; (iii) when $\gamma_{jk} = 0$ for all j and k, it becomes bilinear in operators $\mathbf{a}$ and $\mathbf{a}^\dagger$ and concerns a set of s interacting HO (or field modes), constituting the so called "one-photon" interaction. In this case and in the interaction picture it recovers, for $s = 2$, the "beam-splitter" (BS) Hamiltonian for traveling fields, which furnishes the state of the output field from a given input field that impinges a BS as in reference [21]. Thus, with $\lambda_{12} = \lambda_{21} = \lambda$ and $|\Psi\rangle_{\text{in}} = |1_1, 0_2\rangle$, we get

$$\begin{aligned}|\Psi\rangle_{\text{out}} &= \exp\left(-i\tau\frac{\mathbf{H}}{\hbar}\right)|\Psi\rangle_{\text{in}} \\ &= \exp[-i\lambda(\mathbf{a}_1^\dagger\mathbf{a}_2 + \mathbf{a}_1\mathbf{a}_2^\dagger)]|1_1,0_2\rangle \\ &= \cos(\lambda\tau)|1_1,0_2\rangle + i\sin(\lambda\tau)|0_1,1_2\rangle,\end{aligned} \tag{58}$$

$\cos(\lambda\tau)$ and $\sin(\lambda\tau)$ being the transmission and reflection coefficients of the BS. Next, if we let the output (58) working as an input for a second BS (named BS_2 to differ it from the first BS, now named BS_1), similar calculations result in the new output ($T_i = \cos(\lambda\tau_i)$ and $R_i = \sin(\lambda\tau_i)$ being the transmission and reflection coefficients of the BS_i, respectively),

$$\begin{aligned}|\Psi_2\rangle_{\text{out}} &= (T_1T_2 - R_1R_2)|1_1,0_2\rangle \\ &\quad + i(R_1T_2 + R_2T_1)|0_1,1_2\rangle,\end{aligned} \tag{59}$$

that reduces to (for $R_1 = T_1$ and $R_2 = T_2$, or $R_1 = T_2$ and $R_2 = T_1$),

$$|\Psi_2\rangle_{\text{out}} = |0_1, 1_2\rangle, \tag{60}$$

which can also be viewed as a perfect transfer: of the one-photon state going from the field mode-1 of BS_1 to the field mode-2 of BS_2. Here, contrary to the classical case where transfer occurs between interacting subsystems, the two BS's do not interact and the event comes from intervention of the quantum entanglement in equation (59).

These examples and that of reference [4] constitute some applications of the Hamiltonian model (21). Besides the HOs and field modes, applications can be extended to nanotechnological devices as superconducting quantum circuits based on Cooper pair boxes, nanoresonators, etc. [22–24].

6 Final remarks and conclusions

In this work we have studied the state transfer between coupled HO in a s-sized network. The procedure is based on the dynamic evolution of the system described by the characteristic function introduced in Section 2. With this description we have shown that the complete exchange of state between oscillators can be accomplished in a characteristic elapse of time. We have determined the explicit Hamiltonian form that allows such state transfer. The procedure is somewhat similar to that of "engineering Hamiltonians". In this way, and exploring properties of the characteristic function of the system, we have also analyzed the transfer of a genuine quantum state and also the transfer of coherent states. For initial entangled states, it was shown that they are restored whenever the state transfer is accomplished.

Before finalizing, we call attention for the particular case $s = 2$, where one recovers the results found in reference [4] using the Wigner approach. In reference [4] cyclical transfer of states was studied via the Hamiltonian,

$$\begin{aligned}\mathbf{H} = \frac{\pi\hbar}{\tau}\ &\left[\left(\frac{3}{2} + m_1 + m_2\right)\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{a}_2^\dagger\mathbf{a}_2\right)\right. \\ &+ \left.\left(\frac{1}{2} + m_2 - m_1\right)\left(\mathbf{a}_2^\dagger\mathbf{a}_1 + \mathbf{a}_1^\dagger\mathbf{a}_2\right)\right],\end{aligned} \tag{61}$$

which can be put in the more compact form,

$$\mathbf{H} = \omega\hbar\left(\mathbf{a}_1^\dagger\mathbf{a}_1 + \mathbf{a}_2^\dagger\mathbf{a}_2\right) + c\hbar\left(\mathbf{a}_2^\dagger\mathbf{a}_1 + \mathbf{a}_1^\dagger\mathbf{a}_2\right). \tag{62}$$

Note that this Hamiltonian provides the relation between the coupling constant c and the characteristic field frequency ω

$$\frac{c}{\omega} = \frac{1 + 2m_2 - 2m_1}{3 + 2m_1 + 2m_2}, \tag{63}$$

and also the characteristic elapse of time τ value,

$$\tau = \left(\frac{1}{2} + m_2 - m_1\right)\frac{\pi}{c}. \tag{64}$$

Thus, for two oscillators the state transfer occurs when they have the same frequency, with the coupling parameter satisfying equation (A.8) for arbitrary non-negative integers m_1 and m_2. According to equation (63), we always have $c < \omega$. The case $c \ll \omega$ is usually assumed in quantum optics, and periodical transfer of states occurs at $t = (m + 1/2)\pi/c$, with $m = m_2 - m_1$ being an arbitrary integer, which recovers our result in reference [4].

The authors thank the Brazilian agency CNPq for the partial support.

Appendix

All Bogoliubov transformation, represented by the matrices W and V, must satisfy four conditions [25]:

$$WW^\dagger - VV^\dagger = 1, \tag{A.1}$$
$$W^\dagger W - V^T V^* = 1, \tag{A.2}$$
$$WV^T - VW^T = 0, \tag{A.3}$$

and

$$W^\dagger V - V^T W^* = 0. \tag{A.4}$$

Besides, we need to solve equations (27) and (28), namely

$$W^\dagger e^{-i\Omega\tau} W - V^T e^{i\Omega\tau} V^* = C, \tag{A.5}$$

and

$$W^\dagger e^{-i\Omega\tau} V - V^T e^{i\Omega\tau} W^* = 0, \tag{A.6}$$

where C is a unitary matrix. To this end we multiply from the left equation (A.6) by W and replace equations (A.1), (A.3) and (A.5) to get

$$e^{-i\Omega\tau} V = VC. \tag{A.7}$$

Next, multiplying equation (A.5) from the left by W, and replacing equations (A.1), (A.3) and (A.6), we obtain

$$e^{-i\Omega\tau} W = WC. \tag{A.8}$$

From equation (A.1) and $|\det(W)| \geq 1$, then W is invertible and

$$e^{-i\Omega\tau} = WCW^{-1}. \tag{A.9}$$

The matrix that diagonalizes a unitary matrix is also unitary, thus $WW^\dagger = 1$ and $V = 0$.

References

1. C.H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, W.K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993)
2. D. Bouwnmeeter, G.J. Pan, K. Mattle, M. Eigl, H. Weinfurter, A. Zeilinger, Nature **390**, 575 (1999)
3. K. Jahne, B. Yurke, U. Gavish, Phys. Rev. A **75**, 010301(R) (2007)
4. D. Portes Jr., H. Rodrigues, S.B. Duarte, B. Baseia, Eur. Phys. J. D **48**, 145 (2008)
5. A.S.M. de Castro, V.V. Dodonov, S.S. Mizrhai, J. Opt. B: Quantum Semiclass. Opt. **4**, 191 (2002)
6. D. Portes Jr., H. Rodrigues, B. Baseia, S.B. Duarte, Comput. Phys. Commun. **180**, 226 (2009)
7. W.P. Bastos, W.B. Cardoso, A.T. Avelar, B. Baseia, Quantum Inf. Process. **10**, 395 (2011)
8. W.P. Bastos, W.B. Cardoso, A.T. Avelar, N.G. de Almeida, B. Baseia, Quantum Inf. Process. **11**, 1867 (2012)
9. B.R. Mollow, Phys. Rev. **162**, 1256 (1967)
10. M. Christandl, N. Datta, Tony C. Dorlas, A. Ekert, A. Kay, A.J. Landahl, Phys. Rev. A **71**, 032312 (2005)
11. C. Di Franco, M. Paternostro, M.S. Kim, Phys. Rev. Lett. **101**, 230502 (2008)
12. M.A. Jafarizadeh, R. Sufiani, Phys. Rev. A **77**, 022315 (2008)
13. M.B. Plenio, J. Hartley, J. Eisert, New J. Phys. **6**, 36 (2004)
14. K. Audenaert, J. Eisert, M.B. Plenio, Phys. Rev. A **66**, 042327 (2002)
15. H.J. Carmichael, *Statistical Methods in Quantum Optics I: Master Equations and Fokker-Planck Equations* (Springer-Verlag, Berlin, 2002)
16. K.E. Cahill, R.J. Glauber, Phys. Rev. **177**, 1857 (1969)
17. F. Iachello, A. Del Sol Mesa, J. Math. Chem. **25**, 345 (1999)
18. A. Einstein, B. Podolsky, N. Rosen, Phys. Rev. **47**, 777 (1935)
19. D.F. Walls, Nature **306**, 141 (1983)
20. P. Maystre, M. Sargent III, *Elements of Quantum Optics* (Springer-Verlag, New York, 1990), Chap. 16
21. C.C. Gerry, P.L. Knight, *Introduction to Quantum Optics* (Cambridge University Press, Cambridge, 2005)
22. Q. You, F. Nori, Nature **474**, 592 (2011)
23. K. Jacobs, A.N. Jordan, E.K. Irish, Eur. Phys. Lett. **82**, 18003 (2008)
24. C. Valverde, B. Baseia, Quantum Inf. Process. **12**, 2019 (2013), and references therein
25. Y. Tikochinsky, J. Math. Phys. **19**, 270 (1978)