STRUCTURAL STABILITY OF PARABOLIC POINTS AND PERIODIC ASYMPTOTIC LINES

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Abstract

In this paper are studied the simplest qualitative properties of asymptotic lines of a surface immersed in Euclidean space. These lines are the integral curves of the null directions of the second fundamental form (normal curvature), on the closure of the hyperbolic region of the immersion, where the Gaussian curvature is negative. Conditions for local structural stability of asymptotic lines around parabolic points and periodic asymptotic lines are established.

Resumo

Neste trabalho estudamos as propriedades qualitativas mais simples das linhas assintóticas numa superfície imersa no espaço Euclideano. Estas linhas são as curvas integrais das direções nulas da segunda forma fundamental (curvatura normal), no fecho da região hiperbólica da imersão na qual a curvatura Gaussiana é negativa. São estabelecidas condições para estabilidade estrutural local das linhas assintóticas em torno dos pontos parabólicos e linhas assintóticas fechadas.

1. Introduction

Consider a C^∞, i.e. smooth, immersion α of a smooth, oriented, two-dimensional manifold $\mathbb{M}$ into Euclidean space $\mathbb{E}^3$.

The *Fundamental Forms* of α at a point p of $\mathbb{M}$ are the symmetric bilinear forms on $\mathbb{T}_p\mathbb{M}$ defined as follows [St, Sp]:

The *First Fundamental Form:* $I_\alpha(p; v, w) = \langle D\alpha(p; v), D\alpha(p; w)\rangle$.

The *Second Fundamental Form*: $II_\alpha(p; v, w) = -\langle DN_\alpha(p; v), D\alpha(p; w)\rangle$.

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Here, $< .,. >$ is the Euclidean inner product on $\mathbb{E}^3$ and N_α is the positive normal of the immersion:

$$N_\alpha = \frac{\alpha_u \wedge \alpha_v}{\mid \alpha_u \wedge \alpha_v \mid},$$

where (u, v) is a positive chart on $\mathbb{M}$ and $\wedge$ is the vector (wedge) product associated to a once for all fixed orientation on $\mathbb{E}^3$, $\alpha_u = \dfrac{\partial \alpha}{\partial u}$ and $\alpha_v = \dfrac{\partial \alpha}{\partial v}$:

A line $\ell = \mathbb{R}.v$, tangent at a point p of $\mathbb{M}$ (i.e. $v \in \mathbb{T}_p\mathbb{M}\backslash 0$), along which the *normal curvature*

$$k_n(p;\ell) = \frac{II_\alpha(p; v, v)}{I_\alpha(p; v, v)}$$

vanishes, is called an *asymptotic direction of* α at p.

A maximal, regular curve $c : (a, b) \to \mathbb{M}$, parametrized by arc length s, whose tangent line is an asymptotic direction is called an *asymptotic line* of α. That is, for every s in (a, b), it holds that $II_\alpha(c(s); c'(s), c'(s)) = 0$. In a local chart this equation writes as:

$$e(u, v)du^2 + 2f(u, v)dudv + g(u, v)dv^2 = 0,$$

where $e =< \alpha_{uu}, \alpha_u \wedge \alpha_v >$, $f =< \alpha_{uv}, \alpha_u \wedge \alpha_v >$ and $g =< \alpha_{vv}, \alpha_u \wedge \alpha_v >$

Through every point p of the *hyperbolic region* $\mathbb{H}_\alpha$ of the immersion α, characterized by the condition that the Gaussian Curvature $\mathcal{K} = \det(DN_\alpha)$ is negative, pass two transverse asymptotic lines of α, tangent to the two asymptotic directions through p. This follows from the usual existence and uniqueness theorems on Ordinary Differential Equations. In fact, on $\mathbb{H}_\alpha$ the local line fields are defined by the kernels $\mathcal{L}_{\alpha,1}$, $\mathcal{L}_{\alpha,2}$ of the smooth one-forms $\omega_{\alpha,1}$, $\omega_{\alpha,2}$ which locally split $II_\alpha = \omega_{\alpha,1} \otimes \omega_{\alpha,2}$.

The forms $\omega_{\alpha,i}$ are locally defined up to a non vanishing factor and a permutation of their indices. Therefore, their kernels and integral foliations are locally well defined only up to a permutation of their indices.

Under the orientability hypothesis imposed on $\mathbb{M}$, it is possible to globalize, to the whole $\mathbb{H}_\alpha$, the definition of the line fields $\mathcal{L}_{\alpha,1}$, $\mathcal{L}_{\alpha,2}$ and of the choice of an ordering between them, as follows:

Consider the field $\mathcal{C}_\alpha$ of tangent cones on $\mathbb{H}_\alpha$, defined by the non-negative part of the second fundamental form, i.e. $I_\alpha(p; v, v) = 1 \quad II_\alpha(p; v, v) \geq 0$, oriented compatibly with $\mathbb{M}$. Call $\{e_1(p), e_2(p)\}$ a positive basis for $\mathbb{T}_p\mathbb{M}$ consisting of unit asymptotic vectors, positive also for $\mathcal{C}_\alpha(p)$.

This choice of a basis can also be defined as follows:

$D\alpha(p, e_1(p)) \wedge D\alpha(p, e_2(p)) = N_\alpha(p)$ and $II_\alpha(p; v, v) > 0$, for $v = e_1(p) + e_2(p)$.

There is only one other different choice, $\{e'(p), e'_2(p)\}$, for such a basis; both choices define the same *asymptotic line fields* of α:

$\mathcal{L}_{\alpha,1}(p) = \mathbb{R}.e_1(p)) = \mathbb{R}.e'_1(p)$ and $\mathcal{L}_{\alpha,2}(p) = \mathbb{R}.e_2(p) = \mathbb{R}.e'_2(p)$.

These two line fields, called the *asymptotic line fields* of α, are smooth on $\mathbb{H}_\alpha$; they are distinctly defined together with the ordering between them given by the subscripts $\{1, 2\}$ which define their *orientation ordering*: *"1 "* for the *first asymptotic line field* $\mathcal{L}_{\alpha,1}$, *"2"* for the *second asymptotic line field* $\mathcal{L}_{\alpha,2}$. They will be presented as an ordered pair $\mathcal{L}_\alpha = \{\mathcal{L}_{\alpha,1}, \mathcal{L}_{\alpha,2}\}$.

The *asymptotic foliations* of α are the integral foliations $\mathcal{A}_{\alpha,1}$ of $\mathcal{L}_{\alpha,1}$ and $\mathcal{A}_{\alpha,2}$ of $\mathcal{L}_{\alpha,2}$; they fill out the hyperbolic region $\mathbb{H}_\alpha$. The *ordered asymptotic net* of the immersion α is the ordered pair $\mathcal{A}_\alpha = \{\mathcal{A}_{\alpha,1}, \mathcal{A}_{\alpha,2}\}$, the index $i = \{1, 2\}$ will be called the *orientation ordering* of the *asymptotic foliation.*

Clearly, an exchange in the orientations either of $\mathbb{M}$ or of $\mathbb{E}^3$ produces an inversion in the orientation ordering of the asymptotic line fields.

When non-empty, the region $\mathbb{H}_\alpha$ is bounded by the set (generically, i.e. for most $\alpha's$, a regular curve [Ke-Th, Ba - Th ,Bl-W]) $\mathbb{P}_\alpha$ of *parabolic* points of α, on which $\mathcal{K}_\alpha$ vanishes. On $\mathbb{P}_\alpha$, the pair of asymptotic directions degenerate into a single one or into the whole tangent plane at points where $II_\alpha = 0$, called flat umbilic points.

The parabolic points will be regarded here as the singularities of the asymptotic net. In fact, in the context of Singularity Theory, $\mathbb{P}_\alpha$ is the singular set of the the Normal Map N_α from $\mathbb{M}$ to the unit sphere $\mathbb{S}^2$. On the *Elliptic Region* $\mathbb{E}_\alpha$, defined by $\mathcal{K}_\alpha > 0$, the asymptotic directions are imaginary and will not be studied here. Thus the domain for real asymptotic directions and their integral curves in the present work will be the set $\{\mathcal{K}_\alpha \leq 0\}$ of non elliptic points, which generically is either the empty set or a manifold with boundary coincident with $\mathrm{Clos}\mathbb{H}_\alpha$.

An immersion α is said to be *C^s -local asymptotic structurally stable at a compact set S* in $\mathrm{Clos}\mathbb{H}_\alpha$ if for any sequence α_n converging to α together with its first s derivatives in a compact neighborhood V_S of S there is a sequence of compact subsets S_n and a sequence of homeomorphisms h_n mapping S to S_n, converging to the identity of $\mathbb{M}$ such that on V_S it maps arcs of the asymptotic foliations $\mathcal{A}_{\alpha,i}$ to arcs of that of $\mathcal{A}_{\alpha_n,i}$ for $i = 1, 2$.

Asymptotic lines, together with geodesics and principal curvature lines are studied in Classical Differential Geometry [Da, H-CV, An, B-F, G-S,1,2,3, Ke-Th, Ba-Th, Ba-Ga-McC, Sp, St].

This paper is devoted to an initial study of the simplest qualitative aspects of asymptotic lines on surfaces immersed into Euclidean space, focusing on their local structural stability and genericity properties near the parabolic curve and periodic asymptotic lines. The results establish necessary and sufficient conditions for an immersion α to be *C^s-local asymptotic stable*, $s \geq 5$, at parabolic points and periodic asymptotic lines.

The precise formulation of the results and their proofs are given in Theorem 1, section 2 and Theorem 2, section 3. In section 4 two examples are given to illustrate the conditions expressed in these theorems.

2. Asymptotic Lines near Parabolic points

In this section it will be established the behavior of the asymptotic foliations near parabolic points, in terms of differential geometric invariants of the immersion α.

Let $c : [0, L] \to \mathbb{M}^2$ be a regular arc of parabolic points, parametrized by arc length u. To fix the notation, suppose that $k_{2|c} = 0$ and $k_{1|c} < 0$, where $k_1$and k_2 are the principal curvatures of the immersion α. Let $\varphi(u)$ the angle between $c'(u) = t(u)$ and the principal direction $L_2(\alpha)$, corresponding to k_2, at the point $c(u)$. Denote by $k_g(u)$ the geodesic curvature of c at the point $c(u)$.

The main result of this section is formulated now.

Theorem 1. *Let $c : [0, L] \to \mathbb{M}$ be a regular curve of parabolic points as above. Then the following holds:*

1) *If $\varphi(u) \neq 0$, the asymptotic foliation, near $c(u)$, is as shown in Fig. 2.1.a (cuspidal type).*

2) *If $\varphi(u) = 0$ and $\varphi'(u) \neq 0$ there are three cases:*

a) $k_g(u)/\varphi'(u) < 1$,

b) $1 < k_g(u)/\varphi'(u) < 9$

c) $9 < k_g(u)/\varphi'(u)$

In cases a), b) and c) above the asymptotic foliation is as shown in the figures 2.1.b, 2.1.c and 2.1.d respectively; and correspond respectively to the folded saddle, focus and node types parabolic points.

3) *The set of immersions whose parabolic points satisfy conditions 1) and 2) is open and dense in C^5-topology.*

4) *The points described in 1) and 2) are the only stable locally asymptotic structurally stable parabolic points.*

Asymptotic foliations near parabolic points

Remark. The pictures above are well known in the literature. Appear for example in the book of Banchoff-Gafney-McCroy, [Ba-Ga-McC] and in the work of Thom and Banchoff,[Ba-Th]. The classification given above, in terms of geometric invariants, has not been found elsewhere.

The following lemma and calculations will be useful in the proof.

Lemma 2.1. *Let* $c : [0, L] \to \mathbb{M}^2$ *be a regular arc of parabolic points, parametrized by arc length* u. *Then the expression:*

$$\alpha(u,v) = (\alpha \circ c)(u) + v(N \wedge t)(u) + [k_n^{\perp}(u)\frac{v^2}{2} + v^2 A(u,v)]N(c(u)) \qquad (1)$$

where, $A(u,0) = 0$ *and* $k_n^{\perp}(u) = k_n(c(u), N \wedge t(u))$ *defines a local chart of class* C^{∞} *around* c.

Proof. The map $\alpha(u,v,w) = (\alpha \circ c)(u) + v(N \wedge t)(u) + wN(u)$ is a local diffeomorphism. Therefore, solving the equation $< \alpha(u,v,w(u,v)), N(u) >= 0$ and using Hadamard's lemma follows the result asserted. □

Calculations in the chart (u,v)

The Darboux equations for the positive frame $\{t, N\wedge t, N\}$ are:

$$\left\{\begin{aligned} t'(u) &= k_g(u)(N\wedge t)(u) + k_n(u)N(u) \\ (N\wedge t)'(u) &= -k_g(u)t(u) + \tau_g(u)N(u) \\ (N)'(u) &= -\tau_g(u)(N\wedge t)(u) - k_n(u)t(u) \end{aligned}\right. \tag{2}$$

with $\tau_g^2(u) = k_n^{\perp}(u)k_n(u)$. This is because c is a parabolic curve.

Also, using Euler's formula, $k_n = k_1\sin^2\varphi + k_2\cos^2\varphi$, [Sp, St], follows that,

$k_n^{\perp} = k_1\cos^2\varphi$,

$k_n = k_1\sin^2\varphi$,

$k_n^{\perp} + k_n = 2\mathcal{H}$,

$\tau_g = k_1\sin\varphi\cos\varphi$.

Computation of the Second Fundamental Form of α

In what follows it will be calculated the coefficients and the derivatives of the second fundamental form of α in the chart introduced in lemma 3.1 For the sake of simplicity in the expressions that follow, write

$A = A(u,v)$, $N = (N\circ c)(u)$, $k_n = k_n(u)$, $k_n^{\perp} = k_n^{\perp}(u)$ and $k_g = k_g(u)$.

Moreover the following notation will be used:

$$\begin{array}{llll} E & =<\alpha_u, \alpha_u> & \quad e & =<\alpha_u\wedge\alpha_v, \alpha_{uu}> \\ F & =<\alpha_u, \alpha_v> & \quad f & =<\alpha_u\wedge\alpha_v, \alpha_{uv}> \\ G & =<\alpha_v, \alpha_v> & \quad g & =<\alpha_u\wedge\alpha_v, \alpha_{vv}> \end{array}$$

Here E, F, G and $e/\mid\alpha_u\wedge\alpha_v>\mid$, $\quad f/\mid\alpha_u\wedge\alpha_v>\mid$ and $\quad g/\mid\alpha_u\wedge\alpha_v>\mid$ are respectively the coefficients of the first and second fundamental forms of α, expressed in the chart (u,v).

Differentiating (1), using (2), obtain:

$$\begin{aligned} \alpha_u &= [1 - k_g v - k_n(k_n^{\perp}\frac{v^2}{2} + v^2A)]t - \tau_g(k_n^{\perp}\frac{v^2}{2} + v^2A)N\wedge t \\ &+[\tau_g v + (k_n^{\perp})'\frac{v^2}{2} + v^2A_u]N \end{aligned} \tag{3}$$

$$\alpha_v = N \wedge t + (k_n^{\perp} v + 2vA + v^2 A_v)N \tag{4}$$

$$\begin{aligned}
\alpha_u \wedge \alpha_v \quad &= -[\tau_g v + (k_n^{\perp})' v^2 + v^2 A_u + \tau_g (k_n^{\perp} \frac{v^2}{2} + v^2 A) \\
&\quad (k_n^{\perp} v + 2vA + v^2 A_v)] t - [(1 - k_g v - k_n (k_n^{\perp} \frac{v^2}{2} + v^2 A)) \\
&\quad (k_n^{\perp} v + 2vA + v^2 A_v)] N \wedge t \; [1 - k_g v - k_n (k_n^{\perp} \frac{v^2}{2} + v^2 A)] N
\end{aligned} \tag{5}$$

$$\begin{aligned}
\alpha_{uu} \quad &= [-k_g' v - k_n ((k_n^{\perp})' \frac{v^2}{2} + v^2 A_u) - k_n' (k_n^{\perp} \frac{v^2}{2} + v^2 A) \\
&\quad - k_n (\tau_g v + (k_n^{\perp})' \frac{v^2}{2} + v^2 A_u) + k_g \tau_g (k_n^{\perp} \frac{v^2}{2} + v^2 A)] t \\
&\quad + [k_g (1 - k_g v - k_n (k_n^{\perp} \frac{v^2}{2} + v^2 A)) - \tau_g (\tau_g v + (k_n^{\perp})' \frac{v^2}{2} + v^2 A_u) \\
&\quad - \tau_g' ((k_n^{\perp}) \frac{v^2}{2} + v^2 A) - \tau_g ((k_n^{\perp})' \frac{v^2}{2} + v^2 A_u)] N \wedge t \\
&\quad + [k_n (1 - k_g v - k_n (k_n^{\perp} \frac{v^2}{2} + v^2 A)) - \tau_g^2 (k_n^{\perp} \frac{v^2}{2} + v^2 A) \\
&\quad + \tau_g' v + (k_n^{\perp})'' \frac{v^2}{2} + v^2 A_{uu}] N
\end{aligned} \tag{6}$$

$$\begin{aligned}
\alpha_{uv} \quad &= -[k_g + k_n (k_n^{\perp} v + 2vA + v^2 A_v)] t \\
&\quad + [\tau_g + (k_n^{\perp})' v + 2v A_u + v^2 A_{uv}] N \\
&\quad - \tau_g (k_n^{\perp} v + 2vA + v^2 A_v) N \wedge t
\end{aligned} \tag{7}$$

$$\alpha_{vv} = [k_n^{\perp} + 2A + 4vA + v^2 A_{vv}] N \tag{8}$$

From (3) to (8), it results that

$$\begin{aligned}
e(u,0) &= k_n(u) = k_1 \sin^2 \varphi \\
f(u,0) &= \tau_g(u) = k_1 \sin \varphi \cos \varphi \\
g(u,0) &= k_n^{\perp}(u) = k_1 \cos^2 \varphi \\
e_v(u,0) &= -k_g (2k_n + k_n^{\perp}) + \tau_g' \\
f_v(u,0) &= (k_n^{\perp})' \\
g_v(u,0) &= -k_g k_n^{\perp} + 6A_v \\
E_v(u,0) &= -2k_g \\
F_v(u,0) &= 0 \\
G_v(u,0) &= 0
\end{aligned} \tag{9}$$

From the relation,

$$2\mathcal{H}(EG-F^2)^{\frac{3}{2}} = eG - 2fF + gE$$

and (9) follows that,

$$6A_v(u,0) = 2\mathcal{H}_v - 6k_g\mathcal{H} + k_g(2k_n + 4k_n^{\perp}) - \tau_g' \tag{10}$$

Also, from

$$\mathcal{K}(EG-F^2) = eg - f^2,$$

and equations (9) and (10), it is obtained that,

$$\mathcal{K}_v(u,0) = k_g[k_1^2\cos 2\varphi - 2\tau_g^2] + k_1\tau_g' - 2\tau_g(k_n^{\perp})' - 2\mathcal{H}_v k_n \neq 0 \tag{11}$$

which expresses condition of regularity of the parabolic set.

Proof of Theorem 1.

1) *The cuspidal case: transversal crossing*

Suppose that the principal foliation $F_2(\alpha)$ is transversal to the parabolic line at the point u_0. This means that $\varphi(u_0) \neq 0$.

Using Hadamard lemma and equations (9) and (10), write:

$e(u,v) = k_n(u) + v[-k_g(2k_n + k_n^{\perp}) + \tau_g'] + vA_1(u,v)$

$f(u,v) = \tau_g(u) + v(k_n^{\perp})' + v^2A_2(u,v)$

$g(u,v) = k_n^{\perp}(u) + v[2\mathcal{H}_v - 6k_g\mathcal{H} + k_g(2k_n + 3k_n^{\perp}) - \tau_g'] + v^2A_3(u,v)$,

with, $k_n(u) = k_1\sin^2\varphi, \quad k_n^{\perp}(u) = k_1\cos^2\varphi, \quad \tau_g(u) = k_1\sin\varphi\cos\varphi.$

The differential equation of the asymptotic lines are given by:

$$edu^2 + 2fdudv + gdv^2 = 0$$

Then,

$$du/dv = \frac{-f \pm [f^2 - eg]^{\frac{1}{2}}}{e}$$

Let $v = w^2$. So, it follows that,

$$\begin{cases} \dfrac{du}{dw} & = -2w\dfrac{f}{e} \pm \dfrac{2w^2W(u,w^2)}{e} \\ u(u_0,0) & = u_0 \end{cases}$$

where $W(u,0) = [\mathcal{K}_v(u,0)]^{\frac{1}{2}} > 0$ by transversality conditions. The expression of $\mathcal{K}_v(u,0)$ is given by equation (11).

Solving the Cauchy problem above it results that:

$$u(u_0,v) = u_0 - cotg\varphi(u_0)v \pm W(u_0,0)v^{\frac{3}{2}} + \dots$$

Therefore near a cuspidal parabolic point the asymptotic foliation $\mathcal{A}_{\alpha,1}$ and $\mathcal{A}_{\alpha,2}$ are as shown in the Fig. 2.1.a.

Remark . It follows from [Ar] that there exist a system of coordinates (U,V) near a cuspidal parabolic point such that the differential equation of the asymptotic lines is given by

$$(dV/dU)^2 = U.$$

2) *The singular case: point of quadratic tangency*

Now suppose that $\varphi(u_0) = 0$, $u_0 = 0$, or equivalently $\tau_g(u_0) = 0$. This means that the parabolic line is tangent to the principal foliation $F_2(\alpha)$ at u_0. In fact, at a parabolic point the principal direction corresponding to the zero principal curvature is an asymptotic direction. Suppose also that at the point of tangency u_0 the contact above is quadratic, which is expressed by the conditions $\varphi(0) = 0$ and $\varphi'(0) \neq 0$.

Consider the implicit differential equation,

$$F(u,v,p) = e + 2fp + gp^2 = 0, \quad p = dv/du$$

and the line field given locally by the vector field X,

$$X : \begin{cases} u' & = F_p \\ v' & = pF_p \\ p' & = -(F_u + pF_v) \end{cases} \tag{12}$$

The projections of the integral curves of X by $\Pi(u,v,p) = (u,v)$ are the asymptotic lines of α. On the surface $F^{-1}(0)$ there exist a canonical involution σ that exchanges the families of asymptotic lines. This procedure is used in a more general setting in [Ar, Dv] for the study of Implicit Differential Equations. The singularities of X in $F^{-1}(0)$ are given by: $(u_0, 0, 0)$, where $\tau_g(u\ ) = 0$. Suppose $u_0 = 0$.

It results that the Jacobian matrix of $DX(0)$ is given by:

$$DX(0) = \begin{pmatrix} 2f_u & 2f_v & 2g \\ 0 & 0 & 0 \\ -e_{uu} & -e_{uv} & -(2f_u + e_v) \end{pmatrix} \tag{13}$$

Using (9) and (13) it results that the eigenvalues of $DX(0)$ are given by:

$$\lambda_1, \quad \lambda_2 = (\frac{k_n^\perp}{2})\{(k_g - \varphi') \pm [(k_g - \varphi')(k_g - 9\varphi')]^{\frac{1}{2}}\}$$

The eigenspace associated to λ_i is given by:

$$E_i = (1, 0, \frac{\varphi'}{4}[(a-1) \pm \sqrt{(a-1)(a-9)} - 4])$$

where $a = \dfrac{k_g}{\varphi'}$.

The tangent space of $\Pi^{-1}(\{v = 0\}) \cap F^{-1}(0)$ at the point $(u_0, 0, 0)$ is generated by $(1, 0, 0)$.

Therefore E_i is transversal to the singular set $\Pi^{-1}(\{v = 0\}) \cap \{F = 0\}$.

In the case of the saddle point $(\lambda_1\lambda_2 < 0)$, although the eigenspaces have inclinations of same sign, that is, $(\lambda_1 - 2k_n^\perp\varphi')(\lambda_2 - 2k_n^\perp\varphi') > 0$, the vector $(1, 0, 0)$ bisects the acute angle formed by E_1 and E_2. This implies that the asymptotic foliations near a saddle folded parabolic point are as shown in Fig. 2.1.b.

In the case of a focus singularity $(\lambda_1 = \bar{\lambda_2}\ ,\ Re(\lambda_1) \neq 0)$ the asymptotic foliations are as shown in Fig. 2.1.c.

In the case of a nodal singularity $(\lambda_1\lambda_2 > 0)$ the two eigenspaces also have inclinations of the same sign, but here $(1, 0, 0)$ bisects the obtuse angle formed by E_1 and E_2. Also E_2 bisects the angle formed by $(1,0,0)$ and E_1 (the tangent space to the strong separatrix). Therefore near a nodal folded parabolic point the asymptotic foliations are as shown in Fig. 2.1.d.

Remark. Related results has been obtained in the smooth category by Davydov, [Dv], where normal forms are obtained under generic non resonance conditions. Topological normal forms for singularities of binary implicit differential equations appear in the works of Bruce and Tari, [B-T,1,2] and Palmeira [Pa].

3) *Openness and Density.*

Openness is obvious from the transversality conditions involved in the expressions that define 1) and 2). Notice here that the class C^5 is essential for the analysis in the nodal point, where a blowing up must be performed on the field X in (12) in order to conclude that its nearby orbits behave as illustrated in Fig 2.1.c. That is, the strong separatrix splits a neighborhood into a parabolic and parallel sectors.

Density follows from the approximation results of Bleeker and Wilson [Be-W] which establishes that generically the Gaussian Map of an immersion has only Whitney folds and cusps singularities [Wh]. The folds singularities correspond to case 1) $(\varphi \neq 0)$. The Whitney cusps correspond to case 2) $(\varphi = 0, \ \ \varphi' \neq 0)$. At this point, using the third expression in 9), approximate the immersion so that only one of the conditions a), b) and c) hold at each Whitney cusp.

4) *Local Stability.*

The construction of the local topological equivalence, using the method of canonical regions, can be done in the same way as in Gutierrez and Sotomayor [G-S, 1, 3]. A detailed construction will appear in the global version of asymptotic stability [Ga-G-S]. □

3. Periodic Asymptotic Lines and their First Return Maps

For the purpose of this section a periodic asymptotic line is a compact leave of one of the asymptotic foliations $\mathcal{A}_{\alpha,i}$ in particular it is disjoint from the parabolic points.

Here will be established an integral expression for the derivative of the first return map of a periodic asymptotic line in terms of curvature functions of the immersion α. Also, it will be shown how to deform the immersion in order to make hyperbolic a periodic asymptotic line. This means that the derivative of the return map is different from one.

Lemma 3.1. *Let $c : [0, L] \to \mathbb{M}^2$ be a periodic asymptotic line, positively oriented, parametrized by arc length u. Then the expression:*

$$\alpha(u,v) = (\alpha \circ c)(u) + v(N \wedge t)(u) + [\mathcal{H}(u)v^2 + A(u,v)v^2]N(c(u)) \qquad (1)$$

where $A(u,0) = 0$ and $\mathcal{H}$ is the Mean Curvature of α, defines a local chart of class C^∞ around c.

Proof. Similar to that of lemma 2.1; the coefficient of v^2 stated there is given by $k_n^\perp$.

Using that $k_n(u) = k_n(c(u), t(u)) = 0$ for an asymptotic line and applying Euler's formula it follows that, $k_n^\perp + k_n = 2\mathcal{H}$. □

Proposition 3.2. *Let $c : [0, L] \to \mathbb{M}^2$ be a periodic asymptotic line, positively oriented, parametrized by arc length u. Then the derivative of the Poincaré map Π, associated to it is given by:*

$$\Pi'(0) = \exp \int_0^L \frac{k_g \mathcal{H}}{\sqrt{-\mathcal{K}}} du$$

where k_g is the geodesic curvature of c and $\sqrt{-\mathcal{K}} = \tau_g$ is the geodesic torsion of c.

Proof. The Darboux equations for the positive frame $\{t, N\wedge t, N\}$ are:

$$\begin{aligned} t'(u) &= k_g(u)(N\wedge t)(u) \\ (N\wedge t)'(u) &= -k_g(u)t(u) + \tau_g(u)N(u) \\ N'(u) &= -\tau_g(u)(N\wedge t)(u) \end{aligned} \tag{2}$$

The same calculation procedure used in lemma 2.1 gives that:

$$\begin{aligned} e(u,0) &= 0, & e_v(u,0) &= \tau_g' - 2\mathcal{H}(u)k_g(u) \\ f(u,0) &= \tau_g(u) & g(u,0) &= 2\mathcal{H}(u) \end{aligned} \tag{3}$$

The differential equation of asymptotic lines in a neighborhood of the line $\{v=0\}$ is given by:

$$e + 2f\frac{dv}{du} + g(\frac{dv}{du})^2 = 0 \tag{4}$$

Denote by $v(u,r)$ the solution of the (4) with initial condition $v(0,r)=r$. Therefore the return map Π is clearly given by $\Pi(r) = v(L,r)$.

Differentiating (4) with respect to r, it results that:

$$g_r v_r (dv/du)^2 + (2g v_{ur} + 2f_v v_r)(dv/du) + e_v v_r = 0$$

Evaluating at $v=o$, it follows that:

$$2f(u,0)v_{ur}(u,0) + e_v(u,0)v_r(u,0) = 0 \tag{5}$$

Therefore, using the expressions for f an e_v found in (3), integration of (5) it is obtained:

$$\ln \Pi'(0) = \int_0^L \frac{-\tau_g' + 2\mathcal{H}k_g}{2\tau_g} du$$

Performing the integration in the equation above we obtain the result stated. This ends the proof. □

Proposition 3.3 *Let $c : [0,L] \to \mathbb{M}^2$ be a regular periodic asymptotic line, positively oriented, parametrized by arc length u. Then there exist a deformation α_ϵ which for $\epsilon \neq 0$ small has c as a hyperbolic periodic asymptotic line.*

Proof. Consider the following one parameter deformation of α:

$$\alpha_\epsilon(u,v) = \alpha(u,v) + \epsilon w(u)\delta(v)v^2 N(u)$$

where $\delta \mid_c = 1$ and has small support.

Performing the calculation as in lemma 2.1 it follows that:

$e(\epsilon, u, 0) = 0$

$f(\epsilon, u, 0) = \tau_g(u)$

$g(\epsilon, u, 0) = 2(\mathcal{H}(u) + \epsilon w(u))$

$e_v(\epsilon, u, 0) = -2[\mathcal{H}(u) + \epsilon w(u)]k_g(u) + \tau_g'(u)$

Therefore $\{v = 0\}$ is a closed asymptotic line for α_ϵ and the derivative of the Poincaré map Π_{α_ϵ}, associated to it is given by:

$$\Pi'_{\alpha_\epsilon} = exp\left[\int_0^L \frac{k_g(u)[\mathcal{H}(u) + \epsilon w(u)]}{\tau_g(u)} du\right]$$

Take $w(u) = k_g(u)$, it holds that:

$$\frac{d}{d\epsilon}\left(\Pi'_{\alpha_\epsilon}(0)\right) \mid_{\epsilon=0} = \int_0^L \frac{k_g(u)^2}{\tau_g(u)} du$$

This integral expression never vanishes. In fact on asymptotic lines the the geodesic curvature coincides with the ordinary curvature. Therefore, if $k_g \mid_c = 0$ identically, contradicts the periodicity of c. □

The main result of this section is given by the theorem below, whose proof is immediate after the propositions above.

Theorem 2. *Let $c : [0, L] \to \mathbb{M}$ be a closed asymptotic line, parametrized by arc length u, of an immersion α.*

Then α is C^s- local asymptotic structurally stable at $c[0, L]$, $s \geq 4$, if only if, it is hyperbolic i.e. $\displaystyle\int_0^L \frac{k_g \mathcal{H}}{\sqrt{-\mathcal{K}}} du \neq 0$

4. Examples

In this section are given two examples that illustrate the two results of this work. Proposition 4.1 focuses on three types of parabolic singularities. Proposition 4.2 exhibits an immersed surface with a hyperbolic asymptotic line. The authors have not found in the classical literature examples of asymptotic generic immersions exhibiting periodic asymptotic lines.

Proposition 4.1. *Let $c : [0, L] \to \mathbb{E}^3$ be closed curve of class C^∞, parametrized by arc length u, with positive curvature k and torsion τ. Consider the mapping:*

$$\alpha(u, v) = c(u) + r \cos v N(u) + r \sin v B(u)$$

where $\{T, N, B\}$ is the orthonormal Frenet frame of c.

It holds that:

i) *For r small α is an immersion of the Torus whose hyperbolic region $\mathbb{H}_\alpha$ is the annulus given by $\{(u, v) : -\frac{\pi}{2} < v < \frac{\pi}{2}\}$.*

ii) *If $\tau(u) \neq 0$ then $(u, \epsilon\frac{\pi}{2})$, $\epsilon = \pm$, are cuspidal parabolic points.*

iii) *If $\tau(u_0) = 0$ and $\tau'(u_0)\epsilon < 0$ both points are folded saddles*

iv) *If $\tau(u_0) = 0$ and $\tau'(u_0)\epsilon > 0$ then $(u, \epsilon\frac{\pi}{2})$ is a folded node (resp. folded focus) if $k(u_0) - 8r\epsilon\tau'(u_0) > 0$(resp. < 0).*

Proof. The map α is an immersion when $1 - kr \cos v \neq 0$ Direct calculation gives that the coefficients of the second fundamental form are proportional to :

$$\begin{aligned} e(u, v) &= -r\tau^2 + k \cos v(1 - kr \cos v) \\ f(u, v) &= r^2\tau \\ g(u, v) &= -r^2 \end{aligned} \tag{1}$$

Therefore the parabolic set is given by $eg - f^2 = -kr \cos v(1 - kr \cos v) = 0$, which holds if and only if $v = \pm\frac{\pi}{2}$. The hyperbolic region is $\{(u, v) : -\frac{\pi}{2} < v < \frac{\pi}{2}\}$.

Consider the implicit differential equation of asymptotic lines,

$$F(u, v, p) = e + 2fp + gp^2 = 0, \quad p = dv/du$$

where the coefficients are given by (1).

The suspended line field is given by X,

$$X : \begin{cases} u' = F_p \\ v' = pF_p \\ p' = -(F_u + pF_v) \end{cases} \tag{2}$$

The projections of the integral curves of X by $\Pi(u,v,p) = (u,v)$ are the asymptotic lines of α.

The singularities of X in $F^{-1}(0)$ are given by: $(u_0, \pm\frac{\pi}{2}, 0)$, where $\tau(u_0) = 0$.

The linear part of X at the singular points is given by:

$$DX(u_0, \epsilon\frac{\pi}{2}, 0) = \begin{pmatrix} -2r\tau' & 0 & 2r \\ 0 & 0 & 0 \\ -2r(\tau')^2 & k'\sin(\frac{\epsilon\pi}{2}) & 2r\tau' - k\sin(\frac{\epsilon\pi}{2}) \end{pmatrix}$$

where $\epsilon = \pm 1$.

Then the eigenvalues of $DX(u_0, \epsilon\frac{\pi}{2}, 0)$ are

$$\lambda_1, \lambda_2 = \frac{-k\epsilon \pm \sqrt{k(k - 8r\epsilon\tau')}}{2}$$

Therefore for $\tau'(0)\epsilon < 0$ the point is a folded saddle.

When $\tau'(0)\epsilon > 0$ the point is a node or a focus provided $k - 8r\tau' > 0$ or < 0.

When $\tau(u) \neq 0$ the field (2) is transversal to the suspended parabolic line $\{v = 0\}$. □

Proposition 4.2. *Let $c : [0, L] \to \mathbb{E}^3$ be a closed curve, parametrized by arc length s, such that the curvature $k(s)$ and the torsion $\tau(s)$ of c are different from zero for all $s \in [0, L]$. The ruled surface*

$$\alpha(s, v) = c(s) + vn(s).$$

has c as a hyperbolic periodic asymptotic line.

Proof. Direct calculation gives that:

$e(s,v) = (-k\tau + \tau')v + (\frac{k}{\tau})'\tau^2 v^2$

$f(s,v) = \tau$

$g(s,v)=0$

Therefore the every point is hyperbolic and differential equations of the asymptotic lines are given by:

$$\begin{cases} ds = 0 \\ \dfrac{dv}{ds} = (-k+\dfrac{\tau'}{\tau})v + (\dfrac{k}{\tau})'\tau v^2 \end{cases} \tag{3}$$

This is a Riccati differential equation. The Poincaré map is given by $\pi(v_0) = v(L,v_0)$, where v is the solution of (3) with $v(0,v_0) = v_0$. Clearly $\pi'(0) = \exp\int_0^L -k(s)ds \neq 0$. This ends the proof. □

Remark 1. Curves with the properties in 4.2 above are provided by the toroidal helices in [Be-Go].

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