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## A NOTE ON THE FUNDAMENTAL GROUP OF A ONE-POINT EXTENSION

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### **Abstract**

*In this note, we consider an algebra  $A$  which is a one-point extension of another algebra  $B$  and we study the morphism of fundamental groups induced by the inclusion of (the bound quiver of)  $B$  into (that of)  $A$ . Our main result says that the cokernel of this morphism is a free group and we prove some consequences from this fact.*

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## 1. Introduction

In the representation theory of finite dimensional algebras over algebraically closed fields, the use of covering techniques, initiated by Gabriel, Riedtmann and others (see, for instance [3]) has shown the importance of the fundamental groups of bound quivers (see [7] for the definition). In this note, we consider an algebra  $A$  which is a one-point extension of another algebra  $B$  and the morphism of fundamental groups induced by the inclusion of (the bound quiver of)  $B$  into (that of)  $A$ . Our main result says that the cokernel of this morphism is a free group. We then deduce various consequences from this fact.

## 2. Preliminaries

### 2.1. Algebras and quivers.

By an algebra is always meant a basic and connected finite dimensional algebra over an algebraically closed field  $k$ , and by module is meant a finitely generated right module. For every algebra  $A$  there exists a (unique) quiver  $Q_A$  and (at least) a surjective morphism of algebras  $\nu : kQ_A \rightarrow A$  so that, setting  $I_\nu = \text{Ker}\nu$ , we have  $A \simeq kQ_A/I_\nu$  (see [3]). The bound quiver  $(Q_A, I_\nu)$  is then called a *presentation* of  $A$ . An algebra  $A$  is said to be *triangular* whenever  $Q_A$  has no oriented cycles. If  $x$  is a point in  $Q_A$ , we denote by  $e_x$  the corresponding idempotent and by  $P_x = e_x A$  the corresponding indecomposable projective module.

### 2.2. Fundamental groups.

Let  $(Q, I)$  be a connected bound quiver. For the definitions and properties of the fundamental group  $\pi_1(Q, I)$ , we refer to [7, 2]. A triangular algebra is called *simply connected* if, for every presentation  $(Q, I)$  of  $A$ , we have  $\pi_1(Q, I) = 1$ . On the other hand, to a given connected bound quiver  $(Q, I)$ , one can associate a CW complex  $\mathcal{B} = \mathcal{B}(Q, I)$  called the *classifying space* of  $(Q, I)$ , and this construction behaves well with respect to homotopy, in particular one has  $\pi_1(Q, I) \simeq \pi_1(\mathcal{B})$  (see [6]). The fundamental group of a bound quiver  $(Q, I)$  affords the following description: let  $T$  be a maximal tree in  $Q$ ,  $F$  be the free group with basis the set of all arrows of  $Q$ , and  $K$  be the normal subgroup of  $F$  generated by

1. all the arrows in  $T$ ; and

2. all expressions of the form  $(\beta_1\beta_2\cdots\beta_p)(\gamma_1\gamma_2\cdots\gamma_q)^{-1}$  where  $\beta_1\beta_2\cdots\beta_p$  and  $\gamma_1\gamma_2\cdots\gamma_q$  are two paths appearing in a minimal relation (in the sense of [7]).

Then,  $\pi_1(Q, I) \simeq F/K$ , see [6] (3.2).

### 2.3. One-point extensions.

Let  $A$  be an algebra,  $x$  be a source in  $Q_A$ , and  $B = A/Ae_xA$ . Letting  $M = \text{rad}_x$ , the algebra  $A$  can be written in matrix form

$$A = \begin{bmatrix} B & 0 \\ M & k \end{bmatrix}$$

with the usual matrix addition, and the multiplication induced from the  $B$ -module structure of  $M$ . We say that  $A$  is a *one-point extension* of  $B$ , and write  $A = B[M]$ . Let  $(Q_A, I_{vA})$  be a presentation of  $A$ , and  $(Q_B, I_{vB})$  be the induced presentation of  $B$ . Let  $Q^{(1)}, \dots, Q^{(c)}$  be the connected components of  $Q_B$ , and  $I_v^{(j)} = I_{vB} \cap kQ^{(j)}$ , for  $1 \leq j \leq c$ . For each  $j$ , the embedding of  $Q^{(j)}$  in  $Q_A$  induces a group morphism  $\eta_j : \pi_1(Q^{(j)}, I_v^{(j)}) \rightarrow \pi_1(Q_A, I_{vA})$ . Our aim is to compute the cokernel of the induced morphism

$$\eta = (\eta_j)_j : \prod_{j=1}^c \pi_1(Q^{(j)}, I_v^{(j)}) \rightarrow \pi_1(Q_A, I_{vA})$$

Following [2] (2.1), we denote by  $x^\rightarrow$  the set of all the arrows of  $Q_A$  starting in  $x$ . Let  $\approx$  be the least equivalence on  $x^\rightarrow$  such that  $\alpha \approx \beta$  whenever there exists a minimal relation  $\sum_{i=1}^r \lambda_i w_i$  such that  $w_1 = \alpha w'_1$ ,  $w_2 = \beta w'_2$ . We denote by  $[\alpha]_{\nu_A}$  the equivalence class of  $\alpha$ , and by  $t(\nu_A)$  the number of equivalence classes  $[\alpha]_{\nu_A}$  in  $x^\rightarrow$ . Finally, we denote by  $s = s(x)$  the number of indecomposable direct summands of  $M$ . It is shown in [2] (2.2) that  $c \leq t(\nu_A) \leq s$ .

## 3. The results

### 3.1. Theorem

Let  $A = B[M]$  be given an arbitrary presentation  $(Q_A, I_{vA})$ . Then the cokernel of the morphism  $\eta : \prod_{j=1}^c \pi_1(Q^{(j)}, I_v^{(j)}) \rightarrow \pi_1(Q_A, I_{vA})$  induced by the inclusion is the free group  $L_{t(\nu_A)-c}$  in  $t(\nu_A) - c$  generators.

**Proof :**

Clearly, if  $\delta_1$  is an arrow from  $x$  to a point in  $Q^{(1)}$ , and  $\delta_2$  is an arrow from  $x$  to a point in  $Q^{(2)}$ , then  $\delta_1 \not\approx \delta_2$ . We may then assume, without loss of generality that  $B$  is connected, that is,  $c = 1$ . We set  $t = t(\nu_A)$ , and let  $\{\alpha_1, \alpha_2, \dots, \alpha_t\}$  be a complete set of representatives of the classes  $[\alpha]_\nu$ . For  $i \in \{1, 2, \dots, t\}$ , we set  $[\alpha_i]_\nu = a_i$ . Our aim is to show that the cokernel of  $\eta$  is the free group  $L_{t-1}$  having  $\{a_2, \dots, a_t\}$  as set of (free) generators.

Let  $T_B$  be a maximal tree in  $Q_B$ , and  $F_B, K_B$  be as in (2, 2). Denote by  $T_A$  the maximal tree in  $Q_A$  obtained from  $T_B$  by adding the arrow  $\alpha_1$  and the vertex  $x$ . Denote again by  $F_A, K_A$  the groups as in (2.2). We have  $\pi_1(Q_B, I_{vB}) \simeq F_B/K_B$ ,  $\pi_1(Q_A, I_{vA}) \simeq F_A/K_A$ , and the morphism  $\eta : F_B/K_B \rightarrow F_A/K_A$  is given by  $\alpha K_B \mapsto \alpha K_A$  (where  $\alpha$  is an arrow in  $Q_B$ ).

We define a group morphism  $\tilde{\varphi} : F_A \rightarrow L_{t-1}$  as follows. For an arrow  $\alpha$  in  $Q_A$ , we set

$$\tilde{\varphi}(\alpha) = \begin{cases} a_i & \text{if } \alpha \approx \alpha_i \text{ for some } i \in \{2, \dots, t\}, \\ 1 & \text{otherwise.} \end{cases}$$

We claim that  $\tilde{\varphi}(K_A) = 1$ . Clearly, if  $\alpha$  is an arrow in  $T_A$ , then  $\tilde{\varphi}(\alpha) = 1$ . Let  $\beta_1\beta_2 \cdots \beta_p$  and  $\gamma_1\gamma_2 \cdots \gamma_q$  be two paths in  $(Q_A, I_{\nu_A})$  appearing in some minimal relation. If the source of this relation is not  $x$ , then it lies in  $Q_B$  so that

$$\tilde{\varphi}(\beta_1\beta_2 \cdots \beta_p) = 1 = \tilde{\varphi}(\gamma_1\gamma_2 \cdots \gamma_q)$$

If, on the other hand, the source is  $x$ , then  $\beta_1, \gamma_1 \in x^\rightarrow$ . Moreover, there exists  $i_0 \in \{1, 2, \dots, t\}$  such that  $[\beta_1]_\nu = [\gamma_1]_\nu = a_{i_0}$ . Therefore

$$\begin{aligned} \tilde{\varphi}(\beta_1\beta_2 \cdots \beta_p) &= \tilde{\varphi}(\beta_1)\tilde{\varphi}(\beta_2 \cdots \beta_p) \\ &= \tilde{\varphi}(\gamma_1)\tilde{\varphi}(\gamma_2 \cdots \gamma_q) \\ &= \tilde{\varphi}(\gamma_1\gamma_2 \cdots \gamma_q) \end{aligned}$$

This establishes our claim. We infer the existence of a group morphism (which is in fact an epimorphism)  $\varphi : F_A/K_A \rightarrow L_{t-1}$  defined by

$$\varphi(\alpha K_A) = \begin{cases} a_i & \text{if } \alpha \approx \alpha_i \text{ for some } i \in \{2, \dots, t\} \\ 1 & \text{otherwise.} \end{cases}$$

(where  $\alpha$  denotes an arrow in  $Q_A$ ).

Clearly, we have  $\varphi\eta = 1$ . Now, assume there exist a group morphism  $\varphi' : F_A/K_A \rightarrow G$  such that  $\varphi'\eta = 1$ . We define  $\overline{\varphi} : L_{t-1} \rightarrow G$  by  $a_i \mapsto \varphi'(\alpha K_A)$ , where  $[\alpha]v = a_i$ . We must verify that  $\varphi'(\alpha K_A)$  does not depend on the choice of the representative  $\alpha$ . Indeed, if  $[\beta]_v = [\gamma]_v$ , we may assume without loss of generality that there exist two paths  $\beta\beta_2 \cdots \beta_p$ , and  $\gamma\gamma_2 \cdots \gamma_q$  in  $Q_A$  appearing in the same minimal relation. But then

$$\begin{aligned} \varphi'(\beta K_A) &= \varphi'(\beta K_A) \varphi'\eta(\beta_2 \cdots \beta_p K_B) \\ &= \varphi'(\beta K_A) \varphi'(\beta_2 \cdots \beta_p K_A) \\ &= \varphi'(\beta\beta_2 \cdots \beta_p K_A) \\ &= \varphi'(\gamma\gamma_2 \cdots \gamma_q K_A) \\ &= \varphi'(\gamma K_A) \varphi'(\gamma_2 \cdots \gamma_q K_A) \\ &= \varphi'(\gamma K_A) \varphi'\eta(\gamma_2 \cdots \gamma_q K_B) \\ &= \varphi'(\gamma K_A) \end{aligned}$$

Thus,  $\overline{\varphi}$  is well-defined. Clearly  $\overline{\varphi}\varphi = \varphi'$ , and  $\overline{\varphi}$  is uniquely determined (because  $\varphi$  is an epimorphism). This completes the proof.

**3.2. Corollary [2] (2.3).** Let  $A = B[M]$  be a given by an arbitrary presentation  $(Q_A, I_{vA})$ , and  $Z$  be any abelian group. There exists an exact sequence of abelian groups

$$\begin{aligned} 0 \rightarrow z^{t(vA)-c} \rightarrow \text{Hom}_Z(\pi_1(Q_A, I_{vA}), Z) \\ \xrightarrow{\text{Hom}_z(\eta, Z)} \prod_{j=1}^c \text{Hom}_z\left(\pi_1\left(Q^{(j)}, I_v^{(j)}\right), Z\right) \end{aligned}$$

□

**3.3. Corollary.** Let  $A = B[M]$ . There exists a presentation  $(Q_A, I_{vA})$  such that the cokernel of the morphism  $\eta : \prod_{j=1}^c \pi_1(Q^{(j)}, I_\mu^{(j)}) \rightarrow \pi_1(Q_A, I_{\mu A})$  is the free group  $L_{s-c}$  in  $s - c$  generators.

**Proof:**

By [2] (2.2), there exists a presentation  $(Q_A, I_{\mu A})$  such that  $t(\mu) = s$ .

□

**3.4 Corollary [2] (2.6).** Let  $A$  be simply connected. Then all sources of  $Q_A$  are separating (see for instance [2]).

**Proof :**

Since  $A$  is simply connected, it follows from the previous result that  $s = c$ . □

**3.5. Corollary.** Let  $A = B[M]$  and  $(Q_A, I_{\mu_A})$  be a presentation of  $A$  as in corollary 3.3. If  $\pi_1(Q_A, I_{\mu_A})$  is finite, then  $M$  is a separated module.

**Proof:**

If this is not the case, then  $L_{s-c} \neq 1$  and, since the short exact sequence  $1 \rightarrow \text{Ker}\varphi \rightarrow \pi_1(Q_A, I_{\mu_A})^\varphi \rightarrow L_{s-c} \rightarrow 1$  splits (because  $L_{s-c}$  is a free group), we get  $\pi_1(Q_A, I_{\mu_A}) \simeq L_{s-c} \rtimes \text{Ker}\varphi$ , which is infinite.

**3.6. Corollary.** Let  $A = B[M]$  be given an arbitrary presentation  $(Q_A, I_{\nu_A})$  and assume  $B$  to be a direct product of simply connected algebras. Then

- a)  $\pi_1(Q_A, I_{\nu_A}) \simeq L_{t(\nu_A)-c}$ , and
- b)  $A$  is simply connected if and only if  $M$  is separated.

**Proof:**

Since a) is clear we only prove b): By definition,  $M$  is separated if and only if  $s = c$ . Moreover, it follows from [2] (2.2) that  $s = c$  if and only if  $t(\nu_A) = c$  for every presentation  $(Q_A, I_{\nu_A})$ . Finally, using a) one can see that this holds if and only if  $A$  is simply connected. □

Note that the sufficiency of b) was proven in [2] (2.5), [8] (2.3).

We now suppose  $A$  to be triangular and schurian (that is, for every two vertices  $x, y$  of  $Q_A$ , we have  $\dim_k e_x A e_y \leq 1$ ). In this case, the fundamental group does not depend on the presentation (see [4]), so it can unambiguously be denoted by  $\pi_1(A)$ . Moreover, the classifying space  $\mathcal{B}$  of  $(Q_A, I_{\nu_A})$  is a simplicial complex [5]. We denote by  $\text{SH}_1(A)$  the first simplicial homology group of  $A$ , and recall that  $\text{SH}_1(A)$  is the abelianisation of  $\pi_1(A)$ .

**3.7. Corollary.** Let  $A = B[M]$  be a triangular schurian algebra, and  $B_1, \dots, B_c$  be the connected components of  $B$ . Then, there exists an exact sequence of abelian groups

$$\prod_{j=1}^c SH_1(B_j) \xrightarrow{\eta_*} SH_1(A) \rightarrow \mathbf{Z}^{s-c} \rightarrow 0$$

Furthermore, if  $A$  contains no quasi-crowns (in the sense of [1]), then  $\eta_*$  is a monomorphism.

**Proof:** By 3.2 there exists, for each abelian group  $Z$ , an exact sequence of abelian groups.

$$0 \rightarrow \mathbf{Z}^{s-c} \rightarrow Hom_{\mathbf{Z}}(\pi_1(A), Z) \xrightarrow{Hom_{\mathbf{Z}}(\eta, Z)} \prod_{j=1}^c Hom_{\mathbf{Z}}(\pi_1(B_j), Z)$$

By [1] (3.8), the absence of quasi-crowns forces  $Hom_{\mathbf{Z}}(\eta, Z)$  to be an epimorphism. Since the above sequence may be rewritten as

$$0 \rightarrow Hom_{\mathbf{Z}}(\pi_1(A), Z) \rightarrow Hom_{\mathbf{Z}}(SH_1(A), Z) \xrightarrow{Hom_{\mathbf{Z}}(\eta, Z)} \prod_{j=1}^c Hom_{\mathbf{Z}}(\pi_1(B_j), Z)$$

where  $\eta_*$  is induced from  $\eta$  by abelianising, replacing  $Z$  by the injective cogenerator  $Q/Z$  yields the result.

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