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Rule-Following and Functions[1]

Resumo

Este artigo apresenta uma nova reconstituição dos famosos (e controversos) argumentos sobre seguir regras de Wittgenstein. Duas são as novidades dessa reconstituição. Em primeiro lugar, propomos uma mudança no foco central daquela discussão, da semântica geral e filosofia da mente, para a filosofia da matemática e a rejeição da noção de função. A segunda novidade é positiva: argumentamos que Wittgenstein oferece uma noção alternativa nova para a noção de regra (destinada a tomar o lugar das funções), uma noção que lembra a idéia de morfismo da Teoria das Categorias.

Palavras-chave: Seguir regras . regras . funções extensionais . operações intensionais

Abstract

This paper presents a new reconstruction of Wittgenstein's famous (and controversial) rule-following arguments. Two are the novel features offered by our reconstruction. In the first place, we propose a shift of the central focus of the discussion, from the general semantics and the philosophy of mind to the philosophy of mathematics and the rejection of the notion of a function. The second new feature is positive: we argue that Wittgenstein offers us a new alternative notion of a rule (to replace the rejected functions), a notion reminiscent of Category Theory's notion of a morphism.

Keywords: Rule-following . rules . extensional functions . intensional operations

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1 I dedicate this paper to Prof. Sören Stenlund. Our discussions on Wittgenstein's Philosophy of Mathematics prompted me to write the present work.

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> *The style of my sentences is extraordinarily strongly influenced by Frege. And if I wanted to, I could establish this influence where at first sight no one would see it.*
> Wittgenstein 1970, §712

Introduction

What are the *rule-following* considerations about? The argument which (supposedly) begins on §185, with the famous story about a pupil learning the progression 2, 4, 6, 8, …, and ends on §242, just before the private language argument; what is the main topic of that long section of Wittgenstein's *Philosophical Investigations*? Possibly no other segment of that famous work has received so much attention and generated such polemic as the rule-following considerations, but there seems to be a consensus as to the main subject discussed in those paragraphs: the notions of meaning and understanding:

> *The rule-following passages in the Investigations and Remarks on the Foundations of Mathematics in fact raise a number of distinct (though connected) issues about rules, meaning, objectivity and reasons…* (Wright 2007, 481)

> *… the passages on rule-following are concerned with some of the weightiest questions in the theory of meaning, questions – involving the reality, reducibility and privacy of meaning.* (Boghossian 2002, 141)

It is not a purpose of this paper to argue that the semantical notion of meaning and the psychological notion of understanding are *not* important elements in that discussion. But I will argue that they are not *the* central notion dealt with along those paragraphs. Instead, I will claim that the mathematical notion of a *function* is the key concept discussed there. So I am proposing a shift of focus from general semantics and philosophy of mind to philosophy of mathematics. I should iterate what I said before: I am not saying that the notions of meaning and understanding bear no relation to that discussion. On the contrary, I will argue that there is an intimate connection between those two topics thanks to Frege's grandiose proposal of a functional compositionalism in semantics. But a key element was missing if we leave out the notion of function.

My argument will be entirely based on what I believe to be a crucial distinction between *rule-as-a-function* (or simply a function) and *rule-as-a-relation* (the logical conception of a relation, as opposed to the set theoretical one). The first notion of a *rule-as-a-function* is rashly dismissed by Wittgenstein as a "*mythical idea of a rule*". His argument is completely *negative*: both the *classical concept* of a *function* as a *set of ordered* pairs and the *intuitionistic version as a method of obtainment* are fundamentally misguided and should be rejected. But there is a positive side to his proposal: the new notion of a rule as *relative definition*. We will argue that up to a certain point this positive alternative conception is surprisely close to the notion of morphism from *Category Theory*.

One last methodological remark, before we move directly to our reconstruction below. In this paper we try to offer a new vantage point from which to evaluate the entire rule-following argument. Our ambition is *panoramic*. Thus, as a methodological precaution we will repeatedly avoid getting tangled in various (important!) side discussions in order not to lose sight of the main picture we are trying to sketch.

Wittgenstein's Opponent

Let us go directly to the beginning of the *rule-following argument*, on §185 of the *Philosophical Investigations*. There, as we know, we find the story of a pupil learning simple mathematical progressions of the form:

$$0, n, 2n, 3n, \ldots$$

The child is drilled in many such progressions, always up to the number 1000. Then, he is invited to pass beyond that limit for the progression +2

$$0, 2, 4, 6, \ldots$$

and he writes his famous non-standard continuation

$$1000, 1004, 1008, 1012, \ldots$$

The suggestion I would like to make is strategical: let us avoid for a moment the direct exegetical task. Instead, let us ask the question: *why* is that tale there? What is the connection between the story of the boy learning mathematical progressions and the rest of the argument in the *Philosophical Investigations*? Above all, why the mathematical problem of an iterated application of the function n.n+2 beyond the bound 1000? We know that Wittgenstein's original plan was to include a section dealing with the foundations of mathematics (Baker and Hacker 2000, pg 3). Should we take §185 on as a kind of *pentimento* of those original plans? Of course, not. But then, how exactly should we construe the relationship between that material and the rest of the initial section of the book?

To understand (at least part of) the connection between §185 and the rest of the argument, I should like to step back a little and to take a very broad view of the entire beginning of the *Philosophical Investigations*. There is a widespread consensus that this first section is devoted to a criticism of a particular view of language (and of logic). There appears to be less agreement as to exactly which conception of language should we take as Wittgenstein's main opponent all through those pages. A pretty standard account of what that opponent might be is offered by Baker and Hacker and their *Augustinian Conception of Language* (Baker and Hacker 2005, Chap I). But there is a problem there: on their account the whole emphasis falls on *word-meaning* and *denotation*.

> *There can be no doubt that as far as Wittgenstein is concerned, the importance of Augustine's picture lay in the conception of word-meaning which it presupposes.*
> *(...) the essential function of words is to stand for things, that the things words stand for are what they mean and that words are correlated with their meanings by ostension, which connects language to reality.* (Baker and Hacker 2005, pg 3)

Even the notion of sentence appears to be somewhat secondary on their account; it merely *"invites incorporation into the Augustinian conception"*. (Baker and Hacker 2005, pg 11) Still according to them in the end all semantical properties are supposed to be miraculously derived from denotation

> *[The] Ostensive definition must be complete, i.e., fully determine the use of the word it links with the world.* (Baker and Hacker 2005, pg 9, cf also 5)

Frege's crucial *context-principle*, the primacy of the *sentence-sense* over *word-meaning* is left out as an optional, non essential ingredient of the conception. (Baker and Hacker 2005, pg 5) Yet, is it this rather crude conception of language that these authors invite us to take as the root of a "widely ramifying *Weltanschauung* endemic to modern philosophy" (BAKER and HACKER 2000, pg 13). Wittgenstein's main opponent both in his philosophy of mind and in his philosophy of mathematics:

> *...the Augustinian picture [is] the trunk from which his critical investigations of mathematical and psychological concepts spring.* (Baker and Hacker 2000, pg. 15)

This is not the place to review the multiple problems the election of such primitive view as Wittgenstein's main theoretical target produces in the exegesis of several specific paragraphs within the first sections of the Philosophical Investigations (say, from §1 to § 87). Our problem with such selection is that in our view it completely blocks out all our chances to connect §185 with the first part of that work, all the way up to rule-following. If we focus our discussion on *word-meaning*, downplaying even the notion of *sentence*, then the notion of *grammar* rule is also somewhat moved to the background, and the task of connecting that initial discussion to the general problem of rule-following is obscured. But I think that there is an even more problematic feature with our election of the crude Augustinian Conception.

Let me reiterate my problem: exactly *what is* the connection between the repeated application of the function [$n.n+2$] and the general discussion on (grammatical) rules? Why the choice of that *mathematical* example? And why the sudden issue about novel applications of that function beyond 1000? My proposal is this: we should *not* construe Wittgenstein's opponent as Baker and Hacker's crude Augustinian Conception of Language and its emphasis on words. We should chose instead a much more sophisticated view of semantics. I am thinking about Frege's sophisticated *Functional Compositionalism* as it is incarnated both in his *Grundgesetze der Arithmetik* and in Wittgenstein's own *Tractatus Logico-Philosophicus*.

Frege's Functional Compositionalism

In the introduction of Richard Mendelsohn's recent new book on Frege we read:

> *What makes Frege's distinction [sense/reference] so noteworthy? The answer lies with his compositionality principles, one for reference and the other for sense. These represent a genuine advance. Frege conceived of the semantic value of a complex construction in language as being determined by the simpler ones from which it is built in a mathematically rule-governed manner. These rules provided him with a framework within which rationally to connect and unify the semantic story posited for various linguistic entities.* (Mendelsohn 2005, pg XV)

This is Frege's famous *Functional Compositionalism*. A complex inductive structure of functions fixing, from base level up, the *truth conditions* – i.e., the meaning – of each new level of semantical complexity. To be sure, it is true that the entire bottom of Frege's structure is made up of names that extract their meaning largely from their denotation (as Baker and Hacker would emphasize).[2] But on our new account, this is just the beginning of a much more complex story.

Frege's main insight was based on a rather uniform way of dealing with semantical complexity. His answer involved traditional compositionality initially: *the sense of a complex* is to be construed as depending on the *meaning of its constituents*. But there was a further new ingredient, characteristically fregean. This new ingredient began by an audacious broadening of the concept of function:

> Now how has the reference of the word 'function' been extended by the progress of science? We can distinguish two directions in which this has happened.
> In the first place, the field of mathematical operations that serve for constructing functions has been extended. Besides addition, multiplication, exponentiation… transition to the limit have been introduced …. People have gone further still, and have actually been obliged to resort to ordinary language....

2 In the Tractatus case, it really is pure denotation.

> Secondly, the field of possible arguments and values for functions has been extended by the admission of complex numbers. In conjunction with this, the sense of the expressions 'sum,' 'product,' etc., had to be defined more widely.
> In both directions I go still further.

This was the key to the grandiose unification of the *"semantic story posited for various linguistic entities"* extolled above by Mendelsohn. For Frege, as we know, there are just two basic kinds of semantical entities, the saturated names and the unsaturated *functions-expressions*. No further third semantical category was needed.

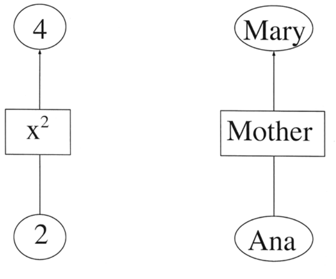
Once the broadening of the notion of function was in place, striking approximations between ordinary sentences and mathematical formulas became possible. Common sentences such as

The mother of Ana is Mary.

and mathematical equations such as

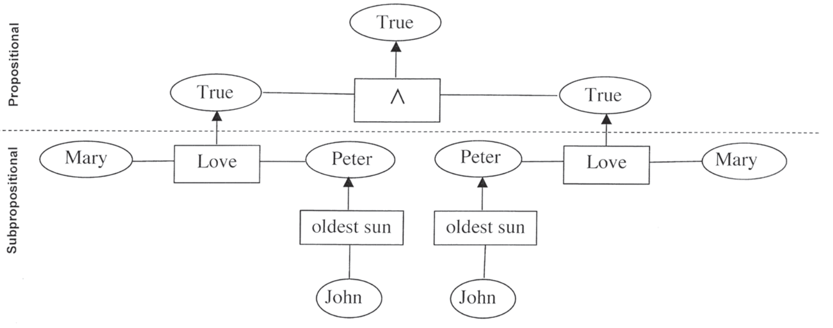
$$2^2 = 4$$

suddenly appeared to be strangely similar to each other. According to Frege, they shared a common logical structure:



Hidden logical complexities could be systematically uncovered, as for example, in sentences such as:

Mary loves John's oldest son and is reciprocated for which we get a "deep structure" somewhat like:



Following Frege, we now have a uniform way of representing both *subpropositional and propositional complexities.*[3] And, once we saturate all *bottom leafs* of the inverted tree, all other semantic determinations – i.e., the *truth* conditions and *relative denotations* – are functionally fixed, all the way up to the top. This is the magnificent unification that Mendelsohn was praising.

The Macrostructure of *Philosophical Investigations* Beginning Section

Our main interest in the *Philosophical Investigations* is, of course, the so-called *rule-following considerations*. According to the standard literature, this argument begins on § 185 and extends all the way to § 242, just before the equally famous *private language argument*. This demarcation of the rule-following considerations seems to have determined the very subdivision of the first two volumes of Baker and Hacker's monumental *Analytical Commentary* on the *Philosophical Investigations*. Their first volume, called *Understanding and Meaning*, begins with their discussion of the *Augustinian Conception of Language* of §1 and continues all the way up to § 184, just before what they take to be the beginning of the *rule-following argument*. Their second, smaller, volume called *Rules*, *Grammar and Necessity* focus on the *rule-following argument*

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3 In the Tractatus case, to be sure, subpropositional structure is just one layer and it is not functionally construed. But the rest of propositional complexity, all the way up from the mysterious elementary propositions, is (truth) functional.

itself, and extends from §185 to §242, before the beginning of the *private language argument*.

Now, how reasonable is this way to subdivide the initial section of that book? The very first sentence of § 185 is only an invitation for us to return to a previous paragraph, §143. If we go to that paragraph, what we discover is the initial segment of the famous tale of the pupil learning number-sequences. So §185 only resumes the previous story. It is true that between that initial segment and the famous continuation of the story in §185, Wittgenstein has inserted a long discussion on *reading*. But this topic, which ends on §178, is followed by a discussion on the problem of number-sequences' continuations. In fact, the whole stretch, from §143 onwards (maybe with the exception of the section on reading) is centrally dedicated to the problem of the connection between simple *arithmetical and algebraic formulas* and the implementations of the *operations they represent*.[4] In short, what we have here is a long discussion on arithmetical and algebraic *functions* and the problem of their (potential) implementations. And this, of course, is obviously connected to the argument from §185 onwards. We should look at that segment, §143 up, as the initial part of the rule-following considerations.

It is not our intention to embark on a widescale analysis of the structure of the entire first portion of the *Philosophical Investigations*. This would take us too far afield, away from our main topic, *rule-following*. Our point here is simply this: if we keep in mind Frege's grandiose connection of the semantical notions of *grammar-rules*, *sentence-compositionality*, etc. and the mathematical notion of *function*, then the role within the *Philosophical Investigations* of Wittgenstein's strange discussion on arithmetical and algebraic examples becomes comprehensible. In very rough terms, we could visualize the initial structure of that book as follows:[5] an initial part (say, from §1 to §88) on the notions of *name* and *denotation* (Frege's first key semantical entity, the *saturated* components), and a further stretch (say, from §133 all the way to §242) on rules and *mathematical functions* (Frege's second semantical entity). In between these two sections, we have, of course, a more general discussion on philosophy and ideality.[6]

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4 Check, for example, § 151 on the formula $a_n = n^2 + n - 1$.

5 Of course, I am not making here the absurd claim that this is the ***only*** way to view that book.

6 As such, the structure is strangely reminiscent of the ***Tractatus***: paragraphs 2 and 3 on sub-propositional structure, denotation, etc., paragraphs 5 and 6 on propositional, (truth) functional compositionality, and a more general section, paragraph 4, in between.

The *Novel Sentence's* Argument

As we have said, it is outside the scope of the present paper a detailed exegetical reconstruction of the *Philosophical Investigations*. We would like to focus instead on another crucial element of Frege's Proposal, his argument of the *Novel Sentence*. In his later paper *Compound Thoughts*, Frege writes:

> *It is astonishing what language can do. With a few syllables it can express an incalculable number of thoughts, so that even a thought grasped by a human being for the very first time can be put into a form of words which will be understood by someone to whom the thought is entirely new. This would be impossible, were we not able to distinguish parts in the thought corresponding to the parts of a sentence, so that the structure of the sentence serves as an image of the structure of the thought.* (Frege 1977, pg 55)

In this excerpt, Frege has concocted an argument in favor of compositionality that was later to become (sometimes annoyingly) standard in contemporary linguistic literature.[7] It involves a challenge: how else could one account for the *creativity of language*[8] – our capacity to understand a potentially *unlimited* (or *infinite*) number of novel, never encountered before, sentences – if not by appealing to some kind of combinatorial strategy by which, out of an *initially finite* stock of words we can produce a potentially *infinite* number word-combinations, i.e., sentences. Compositionality would be the only path which could lead us from *finite* lexicons to the potential infinity of our *statements*.

The same idea is also explicitly presented in Wittgenstein's *Tractatus*:[9]

> 4.02 We can see this from the fact that we understand the sense of a propositional sign without its having been explained to us.
> 4.024 To understand a proposition means to know what is the case if it is true. (One can understand it, therefore, without knowing whether it is true.) It is understood by anyone who understands its constituents.

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7 O the side of philosophy, a very short list could include (Chomsky 2002, pg 15), (Fodor and Katz 1971, pg 475) and (Davidson 1991, pg 3).

8 Also known as *productivity of language.* Cf. (Fodor and Lepore 2002, pg 1).

9 We could read that entire work as extended development of Frege's Functional Compositionalism. Cf. (Porto 2005)

> 4.025 When translating one language into another, we do not proceed by translating each proposition of the one into a proposition of the other, but merely by translating the constituents of propositions. (And the dictionary translates not only substantives, but also verbs, adjectives, and conjunctions, etc.; and it treats them all in the same way.).
> 4.026 The meanings of simple signs (words) must be explained to us if we are to understand them. With propositions, however, we make ourselves understood.
> 4.027 It belongs to the essence of a proposition that it should be able to communicate a new sense to us.

Creativity is supposed to be an essential property of a language. But, if this were so, by a sort of transcendental argument, compositionality would also be mandatory, for it would be our only hope of explaining creativity. And so, in a further development of the same argumentative line, we would have no choice but to posit, as a *condition for the very possibility of any language*, the Tractatus extravagant conceptions of (deep) *logical form*, that "enormously complicated" understructure of "tacit conventions", all hidden from our ordinary view. (Wittgenstein 1971, § 4.002). As we will see, this is Plato's ghost of ideal grammatical rules "flying ahead of you", "determining long before you get there" the meaning and the very possibility (i.e., grammaticality) of all sentences. (Wittgenstein and Diamond (ed.) 1976, pg 124). This, the idea of a magical potentiality "running ahead of us" is the real prima donna of §143-242 of the Philosophical Investigations. But let us not move ahead of time. Let us go to §185.

Philosophical Investigations §185: The Famous Tale

We now have all the elements we need to finally plunge into the famous paragraph 185 of the *Philosophical Investigations* in which we supposedly find the beginning of the *rule-following argument*. Here, as we know, we find a student learning what one could probably call "the simplest possible examples of infinite mathematical recursions":

0, 1, 2, 3,…[10]

and

0, 2, 4, 6,…

So we have here a situation involving simple *infinite mathematical progressions*, i.e., infinite iterations of equally modest *functions* (+1, +2).[11] We did offer above a possible connection of the notion of *function* and the rest of the discussion in that work. But why *infinity*? Why the infinite progressions 0, 1, 2, 3,… and 0, 2, 4, 6,…?

To understand that, we have to include another parallel thought-experiment which was created by Kripke precisely in connection to Wittgenstein's passage. We are talking about the situation involving the skeptic and the sum:

68 + 57

In his explanation of this strange choice of example Kripke writes:

> *Let me suppose, for example, that '68 + 57' is a computation that I have never performed before. Since I have performed – even silently to myself, let alone in my publicly observable behavior – only finitely many computations in the past, such an example surely exists. In fact, the same finitude guarantees that there is an example exceeding, in both its arguments, all previous computations.* (Kripke 1982, pg 8)

Kripke's is a make-belief example. It simply isn't true that no one has ever executed before the quite ordinary sum "68+57". Let us be more careful and try to improve Kripke's example. Let us try to find a better, more realistic example. In a recent posting in the Internet we read:

> *A pair of Japanese and US computer whizzes claim to have calculated Pi to five trillion decimal places – a number which if verified eclipses the previous record set by a French software engineer. (…) It took 90 days to*

10 "…so at the order of '+1' he writes down the series of natural numbers..". (Wittgenstein 2001, § 185)

11 This is typical of Wittgenstein's methodological strategy: to choose the simplest, most transparent examples to serve as his testing case for his philosophical discussions, examples which are free from all useless technical distractions.

> *calculate pi at Kondo's home using a desktop computer with 20 external hard disks. Verification took 64 hours.* (American Physical Society 2010)

So, according to this post, some folks have calculated Pi all the way up to five trillion decimal places. This is an amazing number of decimal places indeed. But, as we all know, they could have gone further. And since this was a new world record, Kripke's idea, of a completely virgin calculation, is exactly the case: no one has ever calculated beyond those decimal places (at least up to 2009).

What about Pi's 5,000,000,000,000+1st decimal? If we represent Pi's decimal expansion by the function:

$$[\lambda n.\ \mathit{Pi\_decimal\_place}_{(n)}]$$

then our question would be:

$$[\lambda n.\ \mathit{Pi\_decimal\_place}_{(n)}]\ 5{,}000{,}000{,}000{,}00 + 1 = \ ?$$

We can clearly understand the role of infinity in Wittgenstein's (and in Kripke's) example: to ensure the very possibility of talking about novel, uncalculated values for their functions. This is why Kripke writes above:

> *…the same finitude guarantees that there is an example exceeding, in both its arguments, all previous computations.* (Kripke 1982, pg 8)

So the target here is really a novel *function's value*. *Infinity* is just the means to ensure its availability. And we can now understand also Wittgenstein choice of the simple recursions "+1" and "+2": they are clearly unbounded. No matter how far we could end up calculating (as in Pi's colossal example above), we could always extend our computations a little further. This is precisely what goes on in Wittgenstein's tale in §185. He writes:

> *Now we get the pupil to continue a series (say + 2) beyond 1000. – and he writes 1000, 1004, 1008, 1012.* (Wittgenstein 2001, § 185)

In the *rather restricted context of that pupil's mathematical life*, the teacher's request involved the calculus of *four new values* of the recursion "+2". We could break down the teacher's request as the 4 new computations:

$Sup\ \{f_{+2}(n) < 1000 : n \in N\} = ?$

$(Sup\ \{f_{+2}(n) < 1000 : n \in N\} + 2) = ?$

$((Sup\ \{f_{+2}(n) < 1000 : n \in N\} + 2) + 2) = ?$

$(((Sup\ \{f_{+2}(n) < 1000 : n \in N\} + 2) + 2) + 2) = ?$

Just as in the case of Pi's 5,000,000,000,00 + 1st decimal, we have novel, never before calculated values of certain functions. The only difference between the two is that in the pupil's case, novelty is just contextual, restricted to that student's experiences. Pi's case is an example of absolute novelty.

Function and Modality: The Ordinary Conception

Lets us summarize what we've been trying to do so far. We've tried to approximate Wittgenstein's tale in §185 and Frege's famous argument in favor of compositionality, the *novel sentence's argument*. There is a lot to be said about this approximation. One detail we will dwell on later is the difference between *novel function values* and *novel sentences*. This distinction is important because, as we will see, it is connected to the difference between *rules-as-functions* and *rules-as-relations* mentioned in the beginning of this paper. But before we go into that, let us go back once more to the notion of *function* and the idea of *potency*.

From an ordinary, intuitive point of view, there seems to be an inherently modal component which plays a central role in the notion of function: the idea of *potentiality*. When dealing with functions, we ordinarily tend to employ a *temporal*, *dynamical* terminology. Let us take a very simple example, say, the addition:

$$2 + 3 = 5$$

We tend to say things like "one *obtains* a five when one adds two and three" or "two plus three produces a five". In constructivist's texts, we even get the more colorful terminology of "*generation*": "the sum of a two and a three *generates a five*".

In all such cases, of course, we have the notion of *potency* operating. First, we have the operation, in our example, the binary of Addition:

$$f_{+}(\ \ ,\ \ )$$

Following Frege, we could say that once we *saturate* its two argument-places:

$$f_{+}(2\ ,\ \ )$$
$$\uparrow$$

and

$$f_{+}(2\ ,3)$$
$$\uparrow$$

we *produce*, *generate*, the result:

$$f_{+}(2\ ,3) \rightarrow 5$$

The image can be likened to that of a *machine*. We *input* the numbers 2, and 3, and then the "apparatus" *spits* out the number 5. It is natural to think in terms of a *potential resulting behavior* here: once the two numbers, 2, and 3, were fed to the machine, the number 5 was somehow "*already around*": its *production* was just a question of time. This is precisely Wittgenstein's point in §193

> *The machine as symbolizing its action: the action of a machine – I might say at first – seems to be there in it from the start. What does that mean? – If we know the machine and…its movement, seems to be already completely determined.* (Wittgenstein 2001, § 193)

This is the notion of *potency* right at the very core of the idea of a *function*. Thus, a function is normally understood as a kind of *potency* to produce a *certain result* (once certain supplies, certain arguments, are offered). That was exactly the image suggested by our analysis above of the proposition "*Mary loves John's oldest son and is reciprocated*" along the lines of Frege's *Compositionalism*. Just as in the case of 2+3, once we saturated the bottom leafs of that complex functional structure with the names "*Mary*" and "*John*", the rest was all "predetermined", "automatic", all the way to the top truth-value for the entire complex, the value: "*True*!". This was the Tractatus "enormously complicated hidden logical machinery" at work, all there, waiting to be saturated, the *prima donna* we were talking about.

Function and *Modality*: The Classical Version

We have argued that the modal notion of *potency* is right at the core of our ordinary, intuitive understanding of the notion of *function*. But, what about technical, more sophisticated construals? In that case it is important to realize that one has to distinguish *two quite distinct alternatives* within contemporary foundational studies. First, there is the classic, *platonic* rendering in which all *modality* is reduced to abstract *existence*. This is incarnated into the set theoretical construal of a *function* we will briefly discuss bellow. We leave the alternative, intuitionist version of a function as a *method of obtainment* for our next section.

Let us begin with the classical, platonic notion of *function*. There are two main components in that construal: the idea that *functions* are a special kind of *relations* and that *relations* are further construed as sets of ordered *pairs*. Let us quickly comment each of these aspects, commencing with the idea that functions are relations. Let us take a common example of *relation*, say, love:

John loves Mary

It is natural here to think of a logical priority of the *identification of two objects involved*, over the *establishment of the relation between* them. In quite traditional terms, first we have to know *what objects are we talking about*. Following Frege's famous suggestion of not distinguishing the *direct object* from the *grammatical subject*, we could say: we are talking about John and Mary (in that order). Only then we could move on to establish whether the relation of love obtains between the first of those objects with respect to the second one. The point we wished to make is that it seems natural here to think of the establishment of the relation as *logically* posterior to the identification of both objects involved.

Let us now compare this with the case of a commonplace *function statement*, say:

$$9^2 = 81$$

Would it be natural to attribute the same logical priority we had in the *relational case* here too? Should we say that we first have 9 and 81 and only then we move on to establish the relation of *Squarehood* between them? Wouldn't it be more natural to say that we first have only the *argument-value*:

$$9$$

and then we apply the *squaring-function* to it:

$$9^2$$

finally obtaining:

$$81$$

the result?

True: maybe it isn't natural to use the intuitionist's terminology here, saying that the squaring function has "*generated*" the number 81. The number 81 "existed already". But what about the *process of obtainment*, so graphically represented in the contemporary notation:

$$9 \xmapsto{\textit{squaring}} 81$$

The relational picture of us having first *both objects*, the argument 9 *and the result* 81, and then moving on to establish if the first maintains *Squarehood Relation* to the second:

$$\begin{array}{c} \text{Squarehood ?} \\ \uparrow \\ (9, 81) \end{array}$$

seems to completely alter the natural epistemological order of: *"object, operation, result"*.

The second main component of the classical, set theoretical construal of the notion of function is further reduction of ***relations*** to (infinite) ***lists***. According to that classical approach, the notation

$$R\ (a, b)$$

should be understood as a mere abbreviation of:

$$(a, b) \in R$$

This is the famous set theoretical interpretation of *relations* as *sets* (of ordered pairs).

Let us see what this reduction amounts to, but this time, following Frege, let us pick as our example an ordinary relation: *love*. Just as we have said that

$$Squarehood\ (9, 81)$$

could be reduced to:

$$(9, 81) \in \{ (0, 0), (1, 1), (2, 4), (9, 81), \ldots\}$$

we could now say that the relation of love between *Romeo* and *Juliet* could be reduced to the fact that they belong to the list:

(*Juliet, Romeo*) ∈ {(*Heloise, Abelard*), (*Eve, Adam*), (*Juliet, Romeo*), …}

There are a few problems here. Clearly for reduction of our ordinary example to work, it might be advisable to include *all future couples* in our list above. Otherwise what would we do about future couples? Should they be left out of our list? But then, should we say that no one else should ever be allowed to fall in love, from now on? So, our best option is to include all future couples, from here all the way to eternity. But, apparently, even *that* won't do. Our proposal would still be open to questioning: if we really were to identify ***love*** with that list (future couples included), couldn't we end up confusing our relation with some other *co-extensional relation*? Just as in Quine's famous example of the "*creatures with a heart*" and "*creatures with a kidney*" (Quine 1971, pg 31), we could end up confusing love with some other connection which happened to unite the same couples. In order to avoid this further difficulty, perhaps the best option would be to include, not only all past, present and *future* couples, but also all purely *potential* ones. That is, we would then extend the domain of our quantifiers to include no only our actual world, but also all *possibilia* as well.[12]

Combined with the reduction of functions to relations, the further reduction of relations to sets erases all the traces of the *time*, occurrence and all *epistemological* process of *obtainment*. Everything is reduced to an extravagant notion of *purely abstract existence* (as our strange list of couples above). This is the platonic idea of reducing all *modality* into *abstract existence*. In one of his lectures on the philosophy of mathematics, in 1939, Wittgenstein comments:

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12 ven then we might worry about necessary co-extensionality such as the pair "Triangles" and "Trilaterals", but let us not go into that here.

> *Frege, who was a great thinker, said that although it is said in Euclid that a straight line can be drawn between any two points, in fact the line already exists even if no one has drawn it. The idea is that there is a realm of geometry in which the geometrical entities exist. What in the ordinary world we call a possibility is in the geometrical world a reality. In Euclidean heaven two points are already connected. This is a most important idea: the idea of possibility as a different kind of reality; and we might call it a shadow of reality.* (Wittgenstein and Diamond (ed.) 1976, pg 144)

Function and Modality: The Intuitionist Version

It is important to point out here that this is not the only approach to modal notions available within current contemporary logic-mathematical literature. There is a notorious alternative proposal put forward (mainly) by the intuitionists. As usual, we are not going to be able to go down to details here.[13] But the general features of the approach are quite straightforward. Instead of reducing *possibility* to *existence*, we proceed the opposite way: we reduce (objectual) existence to *possibility of concrete instantiation*. An object exists if we *could construct it*, i.e., if we have a *method* of obtaining it.

There is one important proviso we should keep in mind about the intuitionist reduction of *existence* to *possibility of construction*. The modal notion of *possibility* evoked by the intuitionists is not the one usually referred to as "*real possibility*", i.e., the possibility of *actual obtainment*. As we will see, Intuitionism does preserve a certain amount of *abstractness* in its key modal category. Instead of real *possibility*, the modal notion employed is that of *in principle*, or *theoretical possibility*. Dummett refers to that as a "*minimal undeniable concession to realism*" (Dummett 1991, pg. 267). He writes:

> *…[for intuitionists] it is not normally considered legitimate to assert a disjunction… only when we actually have a proof of one or other disjunct. For instance, it would be quite in order to assert that*
>
> $10^{10^{10}}+1$ *is either prime or composite*

13 Well leave that for a future paper.

> *without being able to say which alternative held good... What makes this legitimate, on the standard intuitionist view, is that we have a method which is in principle effective for deciding which of the two alternatives is correct.* (Dummett 1978, pg 239)

Existence is thus explained, not as *possibility of actualization*, but as *theoretical possibility of obtainment*. This more elaborate possibility is assured, not by any *real* possibility of instantiation, but by our possession of a *method* which could *in principle* ("theoretically") lead us to that instantiation. In the end, existence and *possibility* are both reduced by the intuitionists to the *possession of a method*. It is thus not entirely correct to say that within the intuitionistic approach all *existence* is reduced to *possibility*. We have to distinguish two different notions here: *existence of an object* (which is truly reduced to modality) and *existence of a method*. The second notion is not further reducible. It should simply be taken as primitive.[14]

It is perhaps important to point out here a common misidentification of the classical notion of a *mechanical procedure* and the intuitionistic notion of *method of obtainment*. The classical counterpart is usually further reduced to one of various (equivalent) alternative characterizations, such as *Turing computability*, for example. But, as Dag Prawitz emphasizes:

> *It seems that the notion of constructive procedure used here must be taken as a primitive notion. For instance, as perhaps first pointed out by [Rozsa] Peter, it is not possible to define it as a Turing Machine that always yields a value when applied to an argument; the quantifier in this definition must then be understood intuitionistically and this means that to understand the definition we must already know what such as constructive procedure is.* (Prawitz 1977, pg 27)

The presence of classical quantifiers completely distorts the original constructive intentions. And, as Prawitz himself warns us, there is always the threat of "… *an infinite regress* [which] *would defeat the entire project* ***of a*** [intuitionistic] *theory of meaning*". (Prawitz 1977, pg 27)

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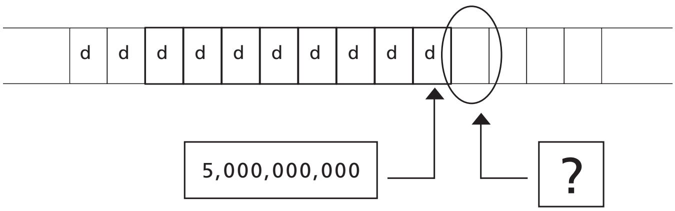
14 Cf. (P. MARTIN-LÖF 1990, pg 141)

Pi's Uncalculated Decimals

We have quickly presented the classical and the intuitionistic approaches to modality and the notion of function because, as we will see, they will provide quite helpful contrasts to Wittgenstein's own views. Once more the key comparison will be offered by the traditional problem of *Pi's new decimal* and the general idea of *novel*, *never* before *calculated values of functions*. We have "discovered" all the first 5,000,000,000,000 places of Pi's decimal expansion. But that is an infinite expansion. What should we say about the 5,000,000,000,001st decimal?



What should we say about the digit that one day may occupy Pi 5,000,000,000,001*st* decimal place? Should we say it *exists*? Certainly not in the sense in which we say that the previous, calculated ones exist. But, then, should we say it doesn't *exist at all*? If so, how are we able to discover it (later)? This challenge about Pi's uncalculated decimal is going to be our leading guide all through the rest of this paper. It will provide us with a nice contrasting element between Wittgenstein's and the Intuitionist and Classical proposals.

Let us start with the classicist's answer to this question. The Platonist reaction to our question should be quite obvious: based on his luxuriant notion of *abstract existence*, he unrepentantly says that that digit *does exist*, albeit "*abstractly*". And, it is quite true what Bernays once declared: "*Platonism reigns today in mathematics*". (Bernays 1983, pg 261, 258) But ever since set theory introduced its extravagant *transfinite hierarchy*, populated in its upper sections by such enormous multiplicities as to defy any instantiation in reality[15], there has been some doubt as to the reasonableness of the classical proposal.

15 Charles Parsons writes: "If the physically possible is what can in some sense be realized in space and time, then structures of sufficiently high cardinality whose acceptance is uncontroversial among set theorists (…) are not physically possible." (PARSONS 1983, pg 191)

In particular the constructivists and the intuitionists were always very skeptical of the classicist's over-permissive notion of existence:[16]

> *All objects that we shall consider are to be constructive objects.... Constructive objects are to be considered as concrete objects, that is, in the very end as existing in time and space.* (Martin-Löf 1970, pg 9)

So, what do intuitionists say about the challenges presented by Pi's uncalculated decimals? Contrary to the classicists, do they claim they *don't* exist? To understand the intuitionist's reaction to this question we have to recall the intuitionist's crucial tenet: the *propositions-as-domains idea*. In one single stroke this precept offers them a *theory of propositions* (parallel to Tarski's famous classical proposal) and a *theory of logical expressions*. Every *proposition* is likened to a *domain*: for it to be *true* is just an *elliptical statement* for the idea that that domain is *populated*, i.e., it isn't *empty*.[17]

In the specific case statements *existential statements*, $\exists x(Px)$, the domain in question is a derived one: the *disjoint union* $[\Sigma\, x : A\ (\ B\ (x))]$. The *quantifier notion* of *existence*[18] is thus reduced to:

$$\exists\ x : A(P(x)) \equiv [\Sigma\ x : A(B(x))]$$

As Martin-Löf himself explains:

> *In accordance with the intuitionistic interpretation of the existential quantifier, the rule of $\sum$ -introduction may be interpreted as saying that a (canonical) proof of $(\exists x \in A)B(x)$ is a pair (a, b), where b is a proof of the fact that a satisfies B.* (P. Martin-Löf 1984, pg 42)

So, for an intuitionist, the very *meaning* of an existence claim such as $\exists x(Px)$ is identified with the possession of a method for (potentially) exhibiting an *a* such that one *could construct* (*a canonical*) that *a* satisfies *P*. And, as we've stressed above, this *potentiality* is not bound by any concrete realizability constraints.

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16 The paradoxes and Cohen's results about the underdetermination of Set Theory's models didn't help the situation, of course.

17 *Apud* (Sommaruga 2000, pg 219)

18 Cf. note 13 above.

The possession of a method of obtainment determines the (abstract) existence of the *output* of a method: a proof of $P(a)$.

What about our question above, regarding Pi? Intuitionistically speaking, should we say that the 5,000,000,000,00+1st decimal of Pi *doesn't exists*? The crucial element for answering this question is the fact that we do have a general method for constructing Pi decimals for any given argument (Leibniz expansion, for instance). So, intuitionistically, we cannot answer this question but by accepting that, not only Pi's 5,000,000,000,00+1th decimal *have to exist*, but also the *whole infinity of decimals beyond that point*. They will all have to be "*abstractly there*". This is part of "*minimal undeniable concession to realism*" Dummett was talking about. Such concessions do not appear to be very congenial to a constructivist, though. In a famous passage by Michael Dummett we read:

> *It seems that we ought to interpose between the platonist and the constructivist picture [a reference to Wittgenstein] an intermediate picture, say of objects springing into being in response to our probing. We do not make the objects but must accept them as we find them (this corresponds to the proof imposing itself on us); but they were not already there for our statements to be true of false of before we carried out the investigations which brought them into being* (Dummett 1978, pg 185)

The Rejection of All *Abstract Potentiality:* the "Shadows of Reality"

Let us go back once more to Wittgenstein. We can now present rather sharply a central feature of his proposal: in contrast to both the *classicist* and the *intuitionist*, Wittgenstein bluntly rejects *all forms of abstract potency*. He writes:

> *There is a feeling: "There can't be actuality and possibility in mathematics. Everything is on one level. And in fact, is in a certain sense actual". – And that's correct.* (Wittgenstein 2005, pg 495)

The point is a recurrent theme through out his later philosophy, one which is already very much present all the way back from his *Big Typescript*.

In a characteristically Wittgensteinian manner, he even coins a special *motto* for the philosophical point. He is always warning us about the danger of accepting any form of abstract potentiality, what he derogatorily refers to as "*shadows of reality*".

> *The difficulty here is to defend oneself against the thought that possibility is a kind of shadowy reality.*
> *It is one of the most deep-rooted mistakes of philosophy to see possibility as a shadow of reality.* (Wittgenstein 2005, pg 258, 259)

In sharp contrast to his other later books, his middle masterpiece is neatly divided into special philosophical topics.[19] There, the very title of one of these sections (§80) reads:

> 80 - *"The Proposition Determines which Reality Makes it True." It Seems to Provide a Shadow of this Reality. A Command Seems to Anticipate its Execution in a Shadowy Way.* (Wittgenstein 2005, pg 277)

And in a striking anticipation of his famous remarks on rule-following in §188 of the *Investigations*, about a function "*flying ahead*" and "*taking all the steps*" in advance of us, he writes:

> *I could also say: It seems to us that by understanding the command we add something to it that fills the gap between the command and its execution. And surely that means, something that executes the command in a shadowy way. It is as if in the command there were already a shadow of its execution.*
> *Being able to do something has a shadowy quality, i.e. it seems like a shadow of actually doing it, just as the sense of a proposition seems like the shadow of its verification; or the understanding of a command the shadow of its being carried out. The command "casts its shadow ahead of itself, as it were", or "The act casts its shadow ahead of itself" in the command.* (Wittgenstein 2005, pgs 14, 226, 112)

Just as we anticipated before, it is important not to loose sight of one of Wittgenstein's key opponents here: Frege's Functional Compositionalism (and his own Tractatus) and their postulation of a *hidden abstract grammar* which would *run ahead of us* and would *fix the sense of all potential* new combinations *of old words*.[20] The very core of compositionality is unequivocally discarded:

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19 This maybe the appropriate opportunity to oppose the old fashioned view of Wittgenstein's work as divided in two opposing philosophies and strongly advocate the middle period, and specially the Big Typescript as a key element in his development.

20 An ontological counterpart of this can be read in Tractatus 2.0124: If all objects are given, then at the same time all possible states of affairs are also given.

> *Once again, the difficulty is that it can look as if a sentence containing the word "square", for example, already contained the shadows of other sentences that are formed with this word. – That is to say, the possibility of forming sentences, which, as I said, is contained in the sense of the word "square".* (Wittgenstein 2005, pgs 125)

Wittgenstein's Intermediary Period: the Rejection of the Extensional Notion of *Function*

So Wittgenstein rejects such "shadows of reality", the idea of an "abstract potency". But, what about the all important mathematical notion of a *function*? If we completely reject the idea of an *abstract potency*, what is left of that central mathematical notion? What is the fate of the entire family of kindred abstract notions such as *operation*, *mapping*, *algorithm*, etc.? As we have anticipated before, it is a notorious trace, distinctive of Wittgenstein's entire later philosophy (not only of his philosophy of mathematics), the parallel *complete rejection of all such notions!*

It is important to distinguish two important different phases in the long process of rejecting such central notion: first the rejection of the extensional construal, the idea of a function as an *infinite* object, and then the rejection of the intensional version, the constructive idea of an *operation as a method of obtainment*. The first step was rather easy for Wittgenstein: the rejection of the classical, extensional idea of a *function as infinite object*, a list of all ordered pairs (*input*, *output*) had been already insinuated in the *Tractatus* handling of the quantifiers an his notion of *number* as the "*exponent of an operation*". But the complete dismissal of the classical, extensional notion of a *function* was only fully achieved in the intermediary period. About this point, and his initial, tractarian period, Wittgenstein writes:

> *My understanding of the general proposition [in the Tractatus] was that (∃x). fx is a logical sum, and that although its terms weren't enumerated there, they could be enumerated."* (Wittgenstein 2005, pg 249)

Independent of how much credit we might give to Wittgenstein's autobiographical assessment, it is beyond dispute that his intermediary period includes a harsh critical rejection of the extensional viewpoint.[21] There are innumerous passages which one could use to attest this dismissal. Perhaps two of the most direct ones are:

> *We should distinguish between the "and so on" which is, and the "and so on" which is not, an abbreviated notation. "And so on ad inf." is not such an abbreviation. The fact that we cannot write down all the digits of* π *is not a human shortcoming, as mathematicians sometimes think.* (Wittgenstein 2001, § 208)
> *...the infinite has nothing to do with size at all. There is a constant temptation to picture an enormous extension (...) [it's] as though the whole extension has been given. We tend to think of the development as an actual enumeration.* (Wittgenstein and Ambrose (ed.) 1979, pg 180)

In sharp contrast to his rejection of the extensional picture, his dismissal of the intensional notion of an *operation* as a *method of obtainment* (just like the intuitionists) was a very painful and arduous process, only fully achieved well in his mature years.[22] As Pasquale Frascolla has argutely identified, all through his intermediary period the notion of *constructive operation*, of *"sign transformation rules"* was still very much in operation in Wittgenstein thought:

> *…in his intermediate phase, Wittgenstein exploits a not completely problematic notion of sign transformation rule… a notion still partially safe from the attacks he launches a short while afterwards.* (Frascolla 1994, pg 55)

For the middle-Wittgenstein, just as for the intuitionists above, the key idea was still the intensional notion of a *method of obtainment*:

> *What one calls mathematical problems may be utterly different. There are the problems one gives a child, e.g., for which it gets an answer according to the rules it has been taught. But there are also those to which the mathematician tries to find an answer which are stated without a method of solution.* (Wittgenstein and Ambrose (ed.) 1979, pg 185)

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21 Cf. the entire last part of his Big Typescript.

22 1935 was probably a crucial year, considering the lectures preserved in (Wittgenstein and AMBROSE (ed.) 1979)

In cases for which one has an algorithmic method of obtainment, Wittgenstein accepted even an approximation which he will later abhor: the matching of a *mathematical rule* and an *empirical proposition*:

> *In this respect the questions as to π and as to whether every equation has roots are alike, and they are unlike such questions as "What is the result of 26 × 13?", (…) These latter belong to a whole system of questions. We have a method of answering them, and the answers within the system of answers are like ordinary empirical propositions in the respect that one could give a method for deciding them.* (Wittgenstein and Ambrose (ed.) 1979, pg 198)

Again, in a striking parallel to the intuitionist's *"in principle existence"*, in his intermediary period Wittgenstein accepted that for algorithmic operations such as the multiplication of 61 × 175 there is a sense in which the entire procedure (*including its result*), is somehow "already given to us" even before we implement it:

> *We may not have been taught to do 61 × 175, but we do it according to the rule which we have been taught. Once the rule is known, a new instance is worked out easily. We are not given all the multiplications in the enumerative sense, but we are given all in one sense: any multiplication can be worked out according to rule. Given the law for multiplying, any multiplication can be done.* (Lect 32-35, pg. 8)

For Wittgenstein, just like the intuitionists, the availability of a method guaranteed that any such operation *"could be worked out"*, an appeal to a "theoretical modal" obviously resembling the intuitionist's notion of in principle possibility. Frascolla goes as far as asserting that, just as in the case of Dummett's large prime example above ($10^{10^{10}}+1$):

> *…according to Wittgenstein's strong Verificationism, a statement such as "11,003 is prime" can be understood without knowing its eventual proof. … Knowing the existence of this relation means knowing the assertibility-conditions for the proposition "11,003 is prime": i.e. which algorithm (defined in general terms) must be applied, and what results (again described in general terms such as "remainder of a division", "different from 0" etc.) such eventual application should have, in order that the conclusion that 11,003 is prime can be inferred.* (Frascolla 1994, 122-3)

In striking contrast to what will happen in his final period, Frascolla claims that for middle-Wittgenstein:

> *The meaning of a general mathematical term of such a kind [algorithmic cases]… transcends the set of the sign figures acknowledged – in any given moment – as resulting from correct applications of the method.* (Frascolla 1994, pg 56)

For Wittgenstein in this intermediary period, just as for the intuitionists, Pi's uncalculated decimals did exist, if only in an "abstract, non-enumerative" sense.

Wittgenstein's Final Stance: the Rejection of the Intensional Notion of *Function*

As many authors have correctly identified before, the rule-following considerations involve a rejection of something variously referred to as the *"pattern idea"* (Mcdowell 2002, pg 1, note 4), the *"metaphysical grounds for correct continuation"* (Goldfarb 2012, pg. 6), *"the meaning-fact"* (Kripke 1982, pg 11), or the *"normative force"* (Frascolla 1994, pg 116). I submit: in all such cases most of what is being rejected is a central notion in both classical and intuitionistic mathematics, the idea of a *function*, in both its extensional and its *intensional versions*. The rule-following considerations could also have been named, perhaps more aptly, as *Wittgenstein's rejection of the rule-as-a-function* idea.[23]

Much of the textual evidence for the rejection of the intensional picture, of an *"in principle potency"* sponsored by the availability of a method of obtainment, is pretty well known. In an image reminiscent of Dummett's objects "springing into being", in the famous §188 of the *Philosophical Investigations* Wittgenstein talks about an abstract functional potency "flying ahead of us" and fixing the offshoots of infinite expansions (like the sequence 0, 2, 4, 6, …), prior to all actual implementations:

23 Wittgenstein consistently avoided technical notions, perhaps as a strategy for escaping from old habits of thought. It is high time for us though to start connecting his jargon back to the more technical terminology, and thus help breaking the regrettable state of isolation of his philosophy of mathematics. (BANGU 2012, pg 2, FRASCOLLA 1994, pg vii)

> *Here I should first of all like to say: your idea was that that act of meaning the order had in its own way already traversed all those steps: that when you meant it your mind as it were flew ahead and took all the steps before you physically arrived at this or that one. Thus you were inclined to use such expressions as: "The steps are really already taken, even before I take them in writing or orally or in thought." And it seemed as if they were in some unique way predetermined, anticipated--as only the act of meaning can anticipate reality.* (Wittgenstein 2001, § 188)

Nothing, not even general formulations such as

$$\begin{cases} f(0) = 0 \\ f(sn) = f(n) + 2 \end{cases}$$

or algorithms such as:

```
value, upper bound;
value = 0;
upper bound = n;
while (value < upper bound)
        {
        print (value);
        value = value + 2;
        }
end;
```

can be said to "(potentially) fix ahead of us" the terms of some "infinite" sequence intimated by the first, initial exemplifications: 0, 2, 4, 6, …:

> *We have then a rule for dividing, expressed in algebraic or general terms,- and we have also examples. One feels inclined to say, "But surely the rule points into infinity – flies ahead of you – determines long before you get there what you ought to do." "Determines" – in that it leads you to do so-and-so. But this is a mythical idea of a rule – flying through the whole arithmetical series.* (Wittgenstein and Diamond (ed.) 1976, pg 124)

In a direct reference to *both the extensional* ("the infinite *thing*") and the *intensional* construals ("a potency beyond actual realizability") regarding the same "infinite sequence" 0, 2, 4, 6, …, Wittgenstein writes:

> *The generality of m = 2n is an arrow that points along the series generated by the operation. And in fact you can say that the arrow points into the infinite; but does that mean that there is a something, the infinite, that it points to – as to a thing? – The arrow designates the possibility of things lying in its direction, as it were. But the word "possibility" is misleading for, as someone will say, what is possible might now become actual. Furthermore, we always think of temporal processes in this context, and infer from the fact that mathematics has nothing to do with time, that in it possibility is already actuality.* (Wittgenstein 2005, pg 493)

Even the computer-like algorithm above seems to have been somehow prefigured in the *Philosophical Investigations.*[24] In another famous sequence of paragraphs regarding a *machine as symbolizing its action*, Wittgenstein writes

> *…the way it moves must be contained in the machine-as-symbol far more determinately than in the actual machine. As if it were not enough for the movements in question to be empirically determined in advance, but they had to be really – in a mysterious sense – already present.* (PI, § 193, pg. 66)

Let us be precise about what exactly is being denied here. The problem of what the potential output of a *particular machine* (say, this desk computer) in a specific implementation (tomorrow) is a perfectly legitimate one, even for Wittgenstein. But this is an *empirical* problem about the specific behavior of that machine, and it includes all kinds of possible malfunctions as well:

> *… we forget the possibility of their [parts] bending, breaking off, melting, and so on.* (Wittgenstein 2001, § 193)[25]

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24 The connections between Wittgenstein and contemporary Computability Theory would be an extremely challenging and fruitful topic for further exploration. But, as in many other points in this paper, that exploration will have to be postponed to a future article.

25 This is reminiscent of course of Kripke's well know observation that the dispositions of any specific agent, despite how trustworthy (it, she or he) might appear, do involve dispositions for making mistakes. (Kripke 1982, pg 28-9)

In the same manner, we can talk about the potential behavior of any specific calculating agent:

> *If we watch a man dividing 1 by 3, then the question whether he will always write 3´s is like a question of physics – like asking whether a comet will describe a parabola.* (Wittgenstein 1979, 212)

Nothing of that kind is problematic for Wittgenstein. What is really problematic, according to him, is to enquire about the *potentiality*, not of the behavior of an empirically given calculating agent (such as a specific machine or calculating person), and a pure abstract potentiality of an *operation* (as determined by a *definition* or an *algorithm* such as the one above), understood as being completely independent of the behavior of all agents:

> *A definition as a part of the calculus cannot act at a distance. It acts only by being applied.* (Wittgenstein 1974, pg 81)

This is the *abstract* functional potency we were talking about before. And *that* is what is rejected by the later Wittgenstein.

The Positive Side of Wittgenstein's Proposal

So, according to us, Wittgenstein has refused both the classical construal of a function as a set of ordered pairs, and the constructivist picture as an *abstract method* of *obtainment*, an "*algorithm*". But, if this is true, what exactly is left of mathematics? In Tractatus 6.2 Wittgenstein had already declared that all that science was constituted by "pseudo-propositions" (possibly the *Unsinn* of that work's terminology). Is he back again conspiring against mathematics, now in a more mature form? At this point in our argument, it is perhaps advisable at least to register that Wittgenstein himself was very much aware of the (apparently) devastating character of his proposals:

> *Where does our investigation get its importance from, since it seems only to destroy everything interesting, that is, all that is great and important? (All the buildings, as it were, leaving behind only bits of stone and rubble.)* (Wittgenstein 2005, pg 304)

> … *I was thinking about my work in philosophy and said to myself: "I destroy, I destroy, I destroy –"* (Wittgenstein 1977, pg 21)

How soothing can that be, though, the mere assurance that Wittgenstein himself was aware that he might be destroying all mathematics, constructivist and classical? Was that all he had to offer us, his awareness of the destruction? What about mathematics, should we simply accept letting it go down the drain? Is that a credible proposal?

There is something much more important for us to point out here. Wittgenstein was confident of a *positive aspect* hidden somewhere within his utterly iconoclastic suggestions in the philosophy of mathematics. He writes:

> *This seems to abolish logic, but does not do so.* (Wittgenstein 2001, § 242)
> *What we are destroying is nothing but houses of cards and we are clearing up the ground of language on which they stand.* (Wittgenstein 2005, pg 304)

This is the second fundamental point we wish to suggest in this article: Wittgenstein's position is *not entirely destructive*. There is a *positive* side to it: the idea of a *rule* as *relative definitional constraints*, what we've called above a *rule-as-a-relation*. But before we can move to these more substantial issues, there is an important ambiguity in Wittgenstein's usage of the term "*rule*" which we will have to try to clear up. As we will point out, this ambiguity produces a related obscurity in his usage of another key term in the philosophy of mathematics: the notion of *proof*. We will try to show that in many passages Wittgenstein's employment of the word "*proof*" is simply *quite misleading*, and that all these semantical imprecisions have conspired to block what we think is an adequate construal of many key concepts in his philosophy of mathematics.

What Exactly is a *"Rule"*?

Let us go back to our discussion of simple arithmetical operations, like the multiplication 61 × 175 above. In this example, it is very natural to construe the term *rule* as connected to the notion of *algorithm*. An *algorithm* is a set of *rules* one sequentially applies to the arguments (61 and 175) to finally obtain the result (10,675). This is the sense employed by Wittgenstein above when he talks about doing 61 × 175 "according to the *rule* which we have been taught".[26] And this is exactly the sense in which he writes, also in his middle period, that:

> *If the equation* $x^2 + 2x + 2 = 0$ *yields, by applying the algebraic* rules, = *- 1* ± $\sqrt{-1}$,… (Wittgenstein 1975, § 176, pg 214)

and, in general, that:

> *'The equation yields a' means: if I transform the equation in accordance with certain rules I get a.* (Wittgenstein 1975, § 150, pg 29)27

It is important to realize here that in these passages the word *rule* is being employed in an acceptation which connects it to the family of notions (of varying degree of abstractness) at the core of which we find the notion of a *function*. Thus the notion of *algorithm* is normally further analyzed as something like, say, "one (of the possibly many) sets of rules which one could employ to implement a *function*" (in our case, the binary operation of *multiplication*). In Frege's terminology, there is a crucial *unsaturation* in all these notions. Or, in the modal terms we have emphasized above, the notion of *potency* runs through all of them. The arguments are the *input*, the result, the *output*, and the *rules* establish the *potential links* connecting each step of the calculation to the next. The rules are "in the back", all through the calculation, determining each correct new step. And we tend to say: they did that, fixed the correct continuation, even before any of us ever begins calculating. This seems unavoidable, after all we didn't invent these rules exclusively to regulate that implementation.

My point should be quite clear by now: this is precisely the later Wittgenstein's dreaded idea of rules "flying ahead of you – determining long before you get there what you ought to do", what Wittgenstein, in his last, most mature period, called a "mythical idea of a rule". But the important point is: is that the only sense in which Wittgenstein, in his later period, employs the word rule? The answer is: no! In his final works we find the term rule employed in a slight, but significantly, different way.

26 My emphasis.

27 My emphases.

Consider these passages extracted from his Remarks on the Foundations of Mathematics:

> *For I want to say: "One can only see that 13 × 13 = 169, … one can – more or less blindly – accept a* rule".
> *In so far as 8 × 9 = 72 is a rule, of course it means nothing to say that that shows me how 8 × 9 = 72;* (Wittgenstein 1998, pgs 77, 306)[28]

A striking difference between these later usages and the ones extracted from the *Philosophical Remarks* (his first book after his return to philosophical activity in 1929) is of course the presence of the *result of the calculation* here, side by side with its "functional part". This is not a trifle difference for Wittgenstein:

> *"One can't believe that the multiplication 13 × 13 yields 169, because the result is part of the calculation.*
> *For example it is the property of '5' to be the subject of the rule '3 + 2 = 5'. For only as the subject of the rule is this number the result of the addition of the other numbers.* (Wittgenstein 1998, pgs 79, 69)

He is quite insistent about the role of the result within his (new) notion of *rule*:

> *For example it is the property of '5' to be the subject of the rule '3 + 2 = 5'. For only as the subject of the rule is this number the result of the addition of the other numbers.*
> *What does it mean for me to say e.g.: this number can be got by multiplying these two numbers? This is a rule telling us that we must get this number if we multiply correctly;* (Wittgenstein 1998, pgs 69, 40)

He is even willing to formulate, in quite general terms, his new role for the result:

> *The reason why "If you follow the rule, this is where you'll get to" is not a prediction is that this proposition simply says: "The result of this calculation is..."*
> *What does it mean for me to say e.g.: this number can be got by multiplying these two numbers? This is a rule telling us that we must get this number if we multiply correctly;* (Wittgenstein 1998, pgs 318, 50)

28 My emphases.

The very notion of *inference* is treated in his final, mature phase, not as an "*unsaturated*", *functional notion*, but as a (explicit) *relation* between the arguments *and the result*:

> *When I say "This proposition follows from that one", that is to accept a rule.* (Wittgenstein 1998, pg 50)

Summing up in even more general terms, Wittgenstein is willing to then affirm:

> *Mathematics – I want to say – teaches you, not just the answer to a question, but a whole language-game with questions and answers.* (Wittgenstein 1998, pg 381)
> *But you can't give an internal relation except by giving the two things between which it holds.* (Wittgenstein and Diamond (ed.) 1976, pg 85)

About the notion of *proof*, he writes:

> *A proof – I might say – is a single pattern, at one end of which are written certain sentences and at the other end a sentence (which we call the 'proved proposition'.)* (Wittgenstein 1998, pg 48)

Let us focus for a while on this last quote above, about the notion of *proof*. Wittgenstein's employment of the term *proof* can be very strange, quite far from the ordinary usage of that word. I think it could be readily agreed that the notion of proof is usually conceived somehow related to the notion of *theorem*. In quite ordinary terms, we say that we start out with an unproved proposition, a *mathematical conjecture*:

Propostion

and latter we may find a proof for it, i.e., we may now have "*grounds for accepting it*":

Propostion
↗
Proof

Only then our original proposition can now be called a theorem: i.e., a proved proposition. As I said before, here the notion of proof is presented in opposition to the notion of *theorem*: the *proof* is not the *entire sequence*, (theorem included), it is just the sequence that *leads to our acceptance of the theorem*. It would be strange to include the original proposition as part of its demonstration. The *proof* is just the first, usually longer part. The final part is the *theorem*.

Let us now go over Wittgenstein's proposed employment of the term. In a striking contrast to the ordinary usage, his suggestion is that we should use the term *proof* to cover the *whole sequence* "*ordinary proof* + *theorem*":

$$\overbrace{\textit{Ordinary proof} + \textit{Theorem}}^{\textit{Wittgenstein's ``proof''}}$$

Going back to his own words above, one does not ordinarily think of a *proof* as "*single pattern*, *at one end of which are written certain sentences* [the ordinary proof] *and at the other end a sentence* [the theorem]".

The same thing that happens, I submit, to the pair *calculation/result* above. The normal asymmetrical picture is this. First, we have the calculation:

$$13 \times 13$$

and then the calculation *produces*, *generates* the *result*:[29]

$$\begin{array}{c} \quad 169 \\ \nearrow \\ 13 \times 13 \end{array}$$

This usage matches Wittgenstein's own employment of the term *rule* in his intermediary period. But what about his later habit of calling "rules" complete equational statements such as $13 \times 13 = 169$ (the *result* included)? Just as in the case of the term *proof*, here we have an inclusive, *relational* construal: the notion of *rule* is used for the *entire sequence: "calculation + result":*

29 The difference here is that in the calculus we start with the "lower" element and in the proof we start with the "upper" element, the proposition in need of a proof.

$$\overbrace{\textit{Calculation + Result obtained}}^{\textit{Wittgenstein's new "rule"}}$$

In a quite unexpected approximation to the classical, platonist interpretation, instead of the *unleveled* functional treatment of the intuitionists, we have purely *flat* reading of the rule.[30]

The New Relational Rules

In a book written many years ago on Wittgenstein whole philosophical development Robert Fogelin proposed the idea of "rules for the identity of descriptions".[31] He writes:

> *…the rules for the identity of descriptions…. relate two ways of describing a collection of things. (…) The identity statement lays down the principle that where one mode of description is correct, so, too, is the other.* (Fogelin 1987, 214-5)

This is the same idea I referred to as "*back and forth correction*" in (Porto 2012): the idea of a *rule* as a *paradigm* providing *relative definitional constraints*.[32] This conception of a rule is extremely prominent in Wittgenstein's mature texts on the philosophy of mathematics.

> *The proposition proved by means of the proof serves as a rule – and so as a paradigm. For we go by the rule.* (Wittgenstein 1998, pg 163)
> *Accepting a proof: one may accept it as the paradigm of the pattern that arises when these rules are correctly applied to certain patterns.* (Wittgenstein 1998, pgs 163, 168)
> *We deposit the picture in the archives, and say, "This is now regarded as a standard of comparison by means of which we describe future experiments."* (Wittgenstein and Diamond (ed.) 1976, pg 104)

30 As we will see, Wittgenstein's notion is "metalinguistic" though.

31 Cf. (Fogelin 1987), also (BANGU 2012, pg 8)

32 This relational view of rules strangely approximates Wittgenstein's proposal and the classical construal of a function as a kind of relation.

The kernel of this new relational conception of a rule is the establishment of a *definitional link* between two *concepts*.[33] Adapting one of Wittgenstein's many examples, let us consider the concept "*counting* 625 *things*". And let us also consider the concept "*organizing* 25 *rows* of 25 *columns of things*". His proposal is that the "*rule* 25 × 25 = 625" establishes a necessary connection between the two concepts:

> *The fact that I have 25 x 25 nuts can be verified by my counting 625 nuts, but it can also be discovered in another way which is closer to the form of expression "25 x 25". And of course it is in the linking of these two ways of determining a number that one point of multiplying lies.* (Wittgenstein 1998, pg 357)

Whenever we have *this* we must also have *this*. A rule establishes a connection constraining empirical applications of two concepts:

> *The proof (the pattern of the proof) shows us the result of a procedure (the construction); and we are convinced that a procedure regulated in this way always leads to this configuration.* (Wittgenstein 1998, pg 159)

In his Lectures on the Foundations of Mathematics of 1939 Wittgenstein is very emphatic about the definitional aspect of his construal:

> "The number of so-and-so's is equal to the number of so and-so's": experiential or mathematical. One can affix to the mathematical proposition "by definition".
>
> In a most crude way – the crudest way possible – if I wanted to give the roughest hint to someone of the difference between an experiential proposition and a mathematical proposition … I'd say that we can always affix to the mathematical proposition a formula like "*by definition*".
>
> Mathematical and logical propositions are still preparations for a use of language-almost as definitions are. (Wittgenstein and Diamond (ed.) 1976, pgs 111, 112, 249)

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33 his new relational conception is derived from an older notion of substitution or replacement rule (WITTGENSTEIN 2005, pg 383) and even earlier from his tractarian notion identity as synonymy. (WITTGENSTEIN 1971, 6.23)

> *In a clear effort to stress the non-assertoric,* purely normative *character of his "definitions", Wittgenstein goes as far as proposing a* purely deontic *reading of arithmetical equational rules:*
> *Suppose we look at mathematical propositions as commandments, and even utter them as such? "Let 25^2 be 625."*
> *Can we imagine all mathematical propositions expressed in the imperative? For example: "Let 10 × 10 be 100".* (Wittgenstein 1998, pgs 271, 276)

The same construal strategy is applied to a large number of different arithmetical examples of varying kinds of complexity:

> *The proof is now our model of correctly counting 200 apples and 200 apples together: that is to say, it defines a new concept: "the counting of 200 and 200 objects together". Or, as we could also say: "a new criterion for nothing's been lost or added.*
> *The equating of 252 and 625 could be said to give me a new concept. And the proof shows what the position is regarding this equality. – "To give a new concept" can only mean to introduce a new employment of a concept, a new practice.* (Wittgenstein 1998, pgs 161, 432)

We are not going to explore here the multiple challenges on the way to presenting Wittgenstein's notion of *rule* as a viable alternative to the extensional set of ordered pairs and to the intuitionist method of obtainment. In a previous paper we have tried to approximate Wittgenstein's notion of rule to Category Theory's notion of *morphism*. In fregean terms, in both construals we reject talking directly about objects.[34] Instead, *rules* (and *morphisms*) establish definitional relations between concepts, which can latter be used to assert (empirical) properties and relations about objects.[35]

> *Such definitions [category-theoretical definitions] may be said to be abstract, structural, operational, relational, or external …. The idea is that objects and arrows are determined by the role they play in the category via their relations to other objects and arrows, … and not by what they "are" or "are made of" in some absolute sense."* (Awodey 2010, pg 25)

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34 Cf. (Frege 1978, § 47)

35 Differently from category theory, the constrained interpretations are not conceived as abstract structures, but ordinary empirical assertions.

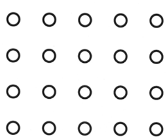
Just as in Category Theory, Wittgenstein sees his "definitional construal" as opposed to the traditional "one layer conception" in which mathematical statements assert directly about objects:

> *The "by definition" always refers to a picture lying in the archives there. – If we forget this, we get into one queer trouble: one asks such a thing as what mathematics is about-and someone replies that it is about numbers.* (Wittgenstein and Diamond (ed.) 1976, pg 112)(L39, 112)

Any minimum attempt of reconstruction of Wittgenstein proposals for various types of mathematical statements and their connection to Category Theory's notion of *morphism* would certainly grow to up into an entire volume.[36] This is clearly beyond the scope of our discussion on rule-following, here. But, before we leave behind our discussion of Wittgenstein's new conception of *rule* we should warn our reader though about a common *misconstrual* regarding Wittgenstein's handling of *relational rules*. On a first, initial reading, this construal could appear to be absurdly restricted: we only discussed very simple *singular statements*, identities such as $13 \times 13 = 169$ and $25^2 = 625$. What about general statements, such as multiplication's commutative law $a \times b = b \times a$? Once again, in a strategy reminiscent of category theory, the whole *foundational picture* of a tight structure organized in fixed levels commencing from a basic objetual ground floor is rejected. Instead, we have only different, quite independent, *Satzsysteme* (Shanker 1987, pg 6-7).

Just as in the case of *singular statements*, *general statements* such as $a \times b = b \times a$ should be understood as constraining directly *empirical allegations*, i.e., *empirical implementations* of o multiplications such as "4 × 5= ?" and "5 × 4= ?". Wittgenstein writes:

> The mere picture



36 (PORTO 2012) We also discuss more complex examples there, involving infinite expansions. Cf. also (PORTO 2009a, PORTO 2009b, RODYCH 1999, MARION and OKADA 2012)

> *regarded now as four rows of five dots, now as five columns of four dots, might convince someone of the commutative law. And he might thereupon carry out multiplications, now in the one direction, now in the other. One look at the pattern and pieces convinces him that he will be able to make them into that shape, i.e. he thereupon* undertakes *to do so.* (Wittgenstein 1998, pg 233)

Wittgenstein and the Problem of Pi's Uncalculated Decimals

We are at the very end of our investigations on Wittgenstein's *rule-following considerations*. But, before we finish, we should like to go back once more to our crucial challenge above: *Pi's uncalculated decimals* and the notion of *determination*. What about Pi's 5,000,000,000,001*st* decimal place and the notion of *determination*? What is exactly Wittgenstein's strategy for dealing with that challenge? We have seen above that he criticized both the classicist and the intuitionist for postulating some sort of abstract determination, either in the form of an *infinite abstract object*, the *set of ordered pairs* proposed by the classicist, or in the form of an *abstract potency*, the intuitionist's *method of obtainment*. But then what does he say about that digit?

Does he accept that it is *completely and utterly undetermined*? If this were so, how could we ever calculate it, say, in the future? After all, we do have a method for doing that. The "Japanese and US computer whizzes" above could have persevered just a tiny bit more. And if they had done that, would they be then *inventing some quite undetermined number*, or would they be *discovering which digit* "*the method assigns*" to that decimal place? What could the idea of a method be, if not precisely a way to fix ahead of us (albeit potentially) what would count as the correct application of that process to future arguments?

We are back to Dummett's idea of "*interposing an intermediary picture*" *between invention and discovery*. The idea may seem promising at first, but many authors have realized that, despite the initial plausibility of that suggestion, the task of offering a precise delineation of that intermediary position proves to be extremely elusive:

> … *if what Wittgenstein says at this stage is aimed at debunking the idea that how one means a rule is going to give you some notion of unconditioned determination of the correct continuation, then his remarks can make the carrying out of the steps look* groundless — *there being no grounds for*

> *going in one way rather than in another. (…) But I think it is better to try to read Wittgenstein as doing something else, which is admittedly hard to make out* … (Goldfarb 2012, pg 6)

So, in quite clear and direct terms: what does Wittgenstein have to say about our challenge: is that decimal *determined*, or not, after all? Let us go back to what we said at the end of our section on Wittgenstein's rejection of the intensional picture. We have emphasized there that he didn't have any problems with the idea of *potential behavior*, when understood as applying to a *particular empirical agents*, within specific *implementations*. So, according to Wittgenstein, we could very well wonder, say, what those computer whizzes would have calculated as being the digit of the Pi's 5,000,000,000,001st decimal place (if they had worked only a bit more). But, as we emphasized before, such dispositions for behavior *include dispositions for mistakes*.

Of course, this is not what our interlocutor wants. He is not interested at all in talking about *specific agents* (those guys) and *particular implementations* (the one in 2009). He wants to talk about "the correct way of implementing that method", a potentiality, not regarding the actual behavior of any agent, but of *correct behavior*. In other words, he wants to talk about an "*operation operated by no one in no particular occasion*". And this is precisely what Wittgenstein wishes to *reject*. According to his proposal, one can very well have *singular rules*, such as:

$$[\lambda n.\ \textit{Pi's } n^{th} \textit{ decimal}]\ (3) = 4$$

or even general ones such as:

$$[\lambda n.\ \textit{Leibniz's Pi expansion}(n)] = [\lambda n.\ \textit{Archimedes' Pi expansion}(n)]$$

(in the sense that we can use one expansion as furnishing *correcting criteria* for implementations of the other).[37] But, as we've seen above, to have a rule, one needs *both sides*, *the corrector side and the correcting side*. In Leibniz's and Archimedes' general rule above, this roles can be reversed, of course, but both are still there.

Now we can understand better what our opponent is suggesting. He is proposing the idea of a correct *potentiality,* not conceived in relation to an

37 Cf. (PORTO 2012) for a more detailed discussion of these rules.

independent method of obtainment, but in some sort of *absolute isolation*: a method abstracted from *all possible mistakes* (introduced by calculating agents in empirical situations) and *all reference to other methods*. Something like a "*pure abstract potentiality*", we could say. Wittgenstein complains:

> *A picture held us captive. And we could not get outside it, for it lay in our language and language seemed to repeat it to us inexorably.* (Wittgenstein 2001, § 115)

If we consider any algorithm in *isolation from other methods* and *all partial results*, the only way to answer the question "what it is to implement that formula?" is the *strictly communitary answer*: to imitate the agents that are supposed to implement it correctly:

> *We use the expression: "The steps are determined by the formula.....".* How *is it used? – We may perhaps refer to the fact that people are brought by their education (training) so to use the formula $y = x^2$, that they all work out the same value for y when they substitute the same number for x. Or we may say: "These people are so trained that they all take the same step at the same point when they receive the order 'add 3'"*. (Wittgenstein 2001, § 189)

That much was quite correct about the communitarist's proposal. But we still have the rules conceived in quite atemporal and *impersonal* terms:

> *"The rule, applied to these numbers, yields those" might mean: the expression of the rule, applied to a human being, makes him produce those numbers form these. One feels, quite rightly, that that would not be a mathematical proposition.*
> *"The justification of the proposition 25 x 25 = 625 is, naturally, that if anyone has been trained in such-and-such a way, then under normal circumstances he gets 625 as the result of multiplying 25 by 25. But the arithmetical proposition does not assert* that. … *It stipulates that the rule has been followed only when that is the result of the multiplication.* (Wittgenstein 1998, pgs 228, 325, Davidson 1991)
> *Certainly, the propositions "Human beings believe that 2×2 = 4" and "2×2 = 4" do not mean the same. The latter is a mathematical proposition; the other, if it makes sense at all, may perhaps mean: human beings have* arrived

> *at the mathematical proposition. The two propositions have entirely different uses.* (Wittgenstein 2001, pg 192-3)

In complete agreement with his general proposal, Wittgenstein writes about the very problem of Pi:

> *….when I calculate the expansion further, I am deriving new rules which the series obeys.*
> *However queer it sounds, the further expansion of an irrational number is a further expansion of mathematics.* (Wittgenstein 1998, pgs 269, 267)

It doesn't make sense to talk about a digit "waiting to be discovered" by some "abstract algorithmic method", as in Dummett's proposal, because it doesn't make sense to talk about potential behavior of a *method*, but only about potential behavior of an *agent*. Still, any further actual expansion of the sequence of digits is (or can be used as) a new rule "to judge proceedings".

Final Comment

It may appear strange that, all along our presentation on rule-following, we have failed to discuss that famous passage of §185 in which pupil is finally rehearsed in continuations of the series 0, 2, 4, 6,… beyond the limit 1000 and he goes: "1000, 1004, 1008, 1012". To most writers, *this* is the very kernel of the so-called *rule-following considerations*. Kripke even coined a term to be able to refer to these strange apprehensions of ordinary mathematical functions (including his own "*quaddition*"), he called them "*non-standard interpretations*". (KRIPKE 1982, 16) And, as we all know, these non-standard interpretations were the crucial weapons in his *skeptic assault* on the notions of "meaning and *intending* one *function rather than another*". (KRIPKE 1982, 13)

Let us try to explain our strange exclusion. In our construal, the very idea of *non-standard interpretations*, of strange *function-attributions* to calculating agents is engendered by the opponent's *insistence* of introducing an *abstract notion of operation*. As we have argued elsewhere (PORTO 2012), it is only

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38 Of course, many authors have made very similar remarks. Cf. for example (Diamond 1991, pg 13, Goldfarb 2012)

when we insist in postulating these *absolute standards* for fixing the *ultimate mathematical identities* of *abstractly conceived processes* that, all of a sudden, a very strange kind of skepticism, like Kripke's skepticism, appears reasonable.[38] This skepticism is very reminiscent of another famous kind of suspicion, this one about *private phenomena* such as the notion of *color-experience*.

> *The essential thing about private experience is really not that each person possesses his own exemplar, but that nobody knows whether other people also have this or something else. The assumption would thus be possible--though unverifiable – that one section of mankind had one sensation of red and another section another.* (Wittgenstein 2001, §272)

The key element in both cases is the idea that we have a definitive standard for the identification of sensations and mathematical processes, but these standards are *ineffable*: they can only take part "indirectly" in a *communicational exchange*. One cannot directly share one's red with one's interlocutor, just as one cannot offer him the "*infinite potentiality*" that supposedly was in one's mind. The point we want to stress is this: these are *counterattacks* Wittgenstein has in his disposal *against his opponent*, a kind of *reductio ad absurdum* of his proposals. But these problems only come up if we have priorly failed to reject, as Wittgenstein has recommend us, the *function-idea*. For us, this rejection is much more central than the reductio and is one of the two crucial components of the *rule-following considerations*. The other one is of course the *positive proposal* of a *new notion of a rule*.

References

American Physical Society. *Japanese and US whizzes claim news record for pi calculation*. Aug 2010. http://www.physorg.com/news200209829.html (accessed 2011).

Awodey, Steve. *Category Theory*. Oxford: Oxford Universtiy Press, 2010.

Baker, Gordon, and Peter Hacker. *Analytical Commentary on the Philosophical Investigations - Rules, Grammar and Necessity*. Oxford: Blacwell Publisher, 2000.

—. *Analytical Commentary on the Philosophical Investigations - Understanding and Meaning*. Vols. I,part I. Oxford: Blackwell Publishing, 2005.

Bangu, Sorin. "The Later Wittgenstein's Philosophy of Mathematics." *Internet Encyclopedia of Philosophy.* 2012. http://www.iep.utm.edu/wittmath/.

Bernays, Paul. "On Platonism in Mathematics." *In Philosophy of Mathematics: selected readings*, by Paul Benacerraf and Hilary Putnam. Cambridge: Cambridge University Press, 1983.

Boghossian, Paul. "The Rule-following considerations." In *Rule-following and Meaning,* by Alexander Miller and Crispin Wright, 141-187. Montreal: McGill-Queen's University Press, 2002.

Chomsky, Noam. *Syntatic Structures.* Berlin: Mouton de Gruyter, 2002.

Davidson, Donald. "Theories of Meaning and Learnable Languages." In *Truth and Interpretation*, by Donald Davidson. Oxford: Clarendon, 1991.

Diamond, Cora. The *Realistic Spirit*. Cambridge: The Mit Press, 1991.

Dummett, Michael. The *Logical Basis of Methaphysics*. Cambridge: Harvard University Press, 1991.

—. *Truth and Other Enigmas*. Cambridge: Harvard University Press, 1978.

Fodor, Jerry, and Ernest Lepore. *The Compositionality Papers*. Oxford: Clarendon Press, 2002.

Fodor, Jerry, and Jerrold Katz. "The structure of a Semantic Theory." In *Readings in the Philosophy of Language*, by Jay Rosenberg and Charles Travis. Englewood Cliffs: Prentice-Hall Inc, 1971.

Fogelin, Robert. *Wittgenstein*. London: Routledge & Kegan Paul, 1987.

Frascolla, Pasquale. *Wittgenstein's Philosophy of Mathematics*. London: Routledge, 1994.

Frege, Gottlob. *Logical Investigations*. Oxford: Basil Blackwell, 1977.

—. *The Foundations of Arithmetic*. Oxford: Basil Blackwell, 1978.

—. *Translations from the Philosophical Writings of Gottlob Frege*. Oxford: Basil Blackwell, 1977.

Goldfarb, Warren. "Rule-Following Revisited." *to appear*, 2012.

Hale, Bob. "Rule-following, objectiviyt and meaning." *In A Companion to the Philosophy of Language*, by Bob HALE and Crispin Wright, 369-96. Oxford: Blackwell Publishers, 1999.

Hodges, Wilfrid. "Necessity in Mathematics I." In *To appear as Proceedings of a conference on Necessity and Contingency*. 2007.

Kripke, Saul. *Wittgenstein on Rules and Private Language*. Oxford: Basil Blackwell, 1982.

Marion, Mathieu, and Mitsuhiro Okada. "Wittgenstein And Goodstein on The Equation Calculus and the Uniqueness Rule." to *appear*, 2012.

Martin-Löf, Per. "A Path from Logic to Methaphysics." In *Atti del Congresso Hovi problem dela Logica e della filosifa dela Scienza*. Bologna: Viareggio, 1990.

Martin-Löf, Per. *Intuitionistic Theory of Types*. Naples: Bibliopolis, 1984.

—. *Notes on Constructive Mathematics*. Uppsala: Alqvist & wiksells boktryckeri, 1970.

Mcdowell, John. "Wittgenstein on Following a Rule." In *Rule-following and Meaning*, by Alexander Miller and Crispin Wright. Montreal: McGill-Queen's University Press, 2002.

Mendelsohn, Richard. *The Philosophy of Gottlob Frege*. Cambridge: Cambridge University Press, 2005.

Miller, Alexander, and Crispin Wright. *Rule-Following and Meaning*. Montreal: McGill-Queen's University Press, 2002.

Parsons, Charles. *Mathematics in Philosophy*. Ithaca: Cornell Universtiy Press, 1983.

Porto, André. "Wittgenstein on Mathematical Identities." *Disputatio*: *International Journal of Philosophy*, November 2012.

—. "Wittgenstein sobre as provas indutivas." *Dois Pontos*, 2009b: 79-107.

—. "Composicionalismo Semântico, Predicação e o Automorfismo de Quine." *Philósophos*, 2005: 11-56.

—. "Wittgenstein e a Medida da Circunferência." *Philósophos*, 2009a: 55-85.

Prawitz, Dag. "Meaning and Proof: on the conflict between classical and intuitionistic logic." *Theoria*, 1977.

Quine, Willard. *From a Logical Point of View*. Cambridge: Harvard University Press, 1971.

Rodych, Victor. "Wittgenstein on Irrationals and Algorithmic Decidability." *Synthese*, 1999: 279-304.

Shanker, Stuart. *Wittgenstein and the Turning-point in the Philosophy of Mathematics*. London: Croom Helm, 1987.

Sommaruga, Giovanni. *History and Philosophy of Constructive Type Theory*. Dordrecht: Kluwer Academic Publishers, 2000.

Wittgenstein, Ludwig. *Big Typescript*. Oxford: Blacwell Publishing, 2005.

—. *Culture and Value*. Oxford: Basil Blackwell, 1977.

—. *Philosophical Grammar*. Oxford: Basil Blackwell, 1974.

—. *Philosophical Investigations*. Oxford: Blackwell Publishing, 2001.

—. *Philosophical Remarks*. Oxford: Basil Blackwell, 1975.

—. *Remarks on the Foundations of Mathematics*. London: The Athenaeum Press, 1998.

—. *The Big Typescript*. Oxford: Basil Blackwell, 2005.

—. *Tractatus Logico-Philosophicus*. London: Routeledge and Kegan Paul, 1971.

—. *Wittgenstein's Lectures: Cambridge 1932-1935*. Totowa: Rowman & Littlefield, 1979.

—. *Zettel*. Berkeley: University of California Press, 1970.

Wittgenstein, Ludwig, and Alice Ambrose (ed.). *Wittgenstein's Lectures: Cambridge*, 1932-1935. Totowa: Rowman and Littlefield, 1979.

Wittgenstein, Ludwig, and Cora Diamond (ed.). *Lectures on the Foundations of Mathematics*, *Cambridge*, *1939*. Chicago: The University of Chicago Press, 1976.

Wright, Crispin. "Rule-following without Reasons: Wittgenstein's Quietism and the Constitutive Question." Ratio, Dec 2007.