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ANGELA CAROLINA TUNUBALÁ SÁNCHEZ

Bifurcations in piecewise smooth dynamical systems and applications

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ATA DE DEFESA DE TESE

Ata nº 33 da sessão de Defesa de Tese de **Angela Carolina Tunubalá Sánchez** que confere o título de **Doutora** em Matemática, na área de concentração em **Sistemas Dinâmicos**.

Ao **segundo dia do mês de julho do ano de dois mil e vinte e seis**, a partir da 09h00, via **videoconferência**, realizou-se a sessão pública de Defesa de Tese intitulada “**Bifurcations on piecewise smooth dynamical systems and applications**”. Os trabalhos foram instalados pelo Orientador, Professor Doutor **Durval José Tonon - IME/UFG**, com a participação dos demais membros da Banca Examinadora: Professor Doutor **Rodrigo Donizete Euzébio - IME/UFG**, membro titular interno; Professor Doutor **Bruno Rodrigues de Freitas - IME/UFG**, membro titular interno; Professor Doutor **Tiago de Carvalho - IBILCE/UNESP** membro titular externo; Professor Doutor **Claudio Gomes Pessoa - IBILCE/UNESP**, membro titular externo. Durante a arguição os membros da banca **não fizeram** sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Tese tendo sido a candidata **aprovada** pelos seus membros. Proclamados os resultados pelo Professor Doutor **Durval José Tonon**, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos Membros da Banca Examinadora, ao **segundo dia do mês de julho do ano de dois mil e vinte e seis**.

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Bacharel em Matemática pela Universidad del Cauca, Colômbia. Mestrado pela Universidade Federal de Goiás–UFG, com bolsa CAPES, na área de Sistemas Dinâmicos. Doutorado também financiado pela CAPES, com período sanduíche no exterior pelo Programa de Doutorado-Sanduíche no Exterior (PDSE), sob o Código de Financiamento 001.

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*Por isso não temas, porque estou contigo; não te assustes,
porque sou o teu Deus; Eu te fortaleço, ajudo e sustento
com a mão direita da minha justiça.*

Isaías 41:10,
Bíblia Sagrada.

Resumo

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Este trabalho investiga o número máximo de ciclos limite em sistemas dinâmicos suaves por partes (PSDS) híbridos e não híbridos em $\mathbb{R}^2$, formados por subsistemas rígidos e separados por uma reta, bem como o efeito de uma perturbação sobre um sistema suave por partes linear em $\mathbb{R}^3$. Inicia-se com a análise de um sistema formado por um centro rígido linear e um centro polinomial homogêneo rígido de grau par, separados pela reta $x = 0$, após uma mudança afim de variáveis que preserva os centros rígidos. Esse resultado complementa o trabalho de [12], no qual se estuda o caso para n ímpar. O presente estudo avança na análise dessa família ao contemplar o caso de grau par, impondo condições específicas sobre a transformação afim.

A pesquisa estende-se a PSDS formados por dois sistemas rígidos de graus m e n , com $2 \leq m \leq n \leq 6$ e separados pela reta $y = 0$, estabelecendo que, enquanto sistemas de graus iguais possuem no máximo um ciclo limite, combinações de graus distintos podem apresentar cotas superiores específicas, chegando a até 8 ciclos limite no caso $(m, n) = (5, 6)$. Ao introduzir a dinâmica híbrida por meio de um mapa de reinicialização (*reset map*), observa-se que a inclusão da dinâmica discreta preserva as cotas superiores do sistema contínuo, com exceção dos casos $(m, n) = (3, 6)$ e $(m, n) = (4, 6)$, nos quais a cota aumenta de 5 e 6 para 6 e 10 ciclos, respectivamente. Finalmente, o estudo estende-se para o espaço tridimensional ao investigar o efeito de uma perturbação em sistemas com linhas de tangência paralelas, revelando que tal perturbação faz com que as linhas se interceptem em um ponto caracterizado como uma singularidade do tipo *two-fold* parabólica, o que resulta no surgimento de novos pontos de pseudo-equilíbrio sem alterar a estabilidade estrutural qualitativa do sistema original.

Palavras-chave

Sistemas Suaves por Partes, Ciclos Limite, Sistemas Rígidos, Teoría de Chebyshev, Systems Híbridos.

Abstract

Tunubalá Sánchez, Angela Carolina. **Bifurcation in piecewise smooth dynamical systems and applications**. Goiânia, 2026. 118p. PhD. Thesis. Programa de Pós-Graduação em Matemática, Instituto de Matemática e Estatística, Universidade Federal de Goiás.

This work investigates the maximum number of limit cycles in hybrid and non-hybrid piecewise smooth dynamical systems (PSDS) in $\mathbb{R}^2$, formed by rigid subsystems and separated by a straight line, as well as the effect of a perturbation on a piecewise linear smooth system in $\mathbb{R}^3$.

It begins with the analysis of a system formed by a linear rigid center and a rigid homogeneous polynomial center of even degree n , separated by the line $x = 0$, after an affine change of variables that preserves the rigid centers. This result complements the work of [12], in which the case for n odd is studied. The present study advances the analysis of this family by addressing the even-degree case under certain conditions on the affine transformation.

The research extends to PSDS formed by two rigid systems of degrees m and n , with $2 \leq m \leq n \leq 6$ and separated by the line $y = 0$, establishing that, while systems of equal degrees have at most one limit cycle, combinations of distinct degrees may exhibit specific upper bounds, reaching up to 8 limit cycles in the case $(m, n) = (5, 6)$. By introducing hybrid dynamics through a reset map, it is observed that the inclusion of discrete dynamics preserves the upper bounds of the continuous system, with the exception of the cases $(m, n) = (3, 6)$ and $(m, n) = (4, 6)$, in which the bound increases from 5 and 6 to 6 and 10 cycles, respectively.

Finally, the study extends to three-dimensional space by investigating the effect of a perturbation on systems with parallel tangency lines, revealing that such a perturbation causes the lines to intersect at a point characterized as a parabolic two-fold singularity, which results in the emergence of new pseudo-equilibrium points without altering the qualitative structural stability of the original system.

Keywords

Piecewise Smooth Dynamical Systems, Limit Cycles, Rigid Systems, Chebyshev Theory, Hybrid Systems.

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Introduction

The theory of dynamical systems is currently one of the fundamental pillars of mathematical modeling, with applications spanning a wide range of phenomena, both natural and abstract. In a rigorous sense, a dynamical system refers to systems whose intrinsic properties evolve over time, thus constituting a field with vast potential for application. Despite being a discipline that lies at the interface between applied phenomena and pure mathematics, establishing new results that ensure a strong connection between theory and practice remains a challenging and often non-trivial task. In this context, there is a clear and pressing need to develop new tools and achieve results that not only address but are fundamentally aligned with contemporary problems. These increasingly complex challenges continue to drive advances in mathematics and science as a whole, underscoring the central role of dynamical systems in tackling modern questions.

The study of limit cycles is central to dynamical system theory, see, for instance, [11, 12]. Its significance stems from Hilbert's 16th problem, which seeks to establish an upper bound on the number of limit cycles in planar polynomial vector fields of degree $n \geq 2$, a problem that remains unsolved (see [27]). A classical method for obtaining limit cycles involves suitably perturbing a differential system that has a center equilibrium point. According to [11] and [47], a center is a singularity $p \in \mathbb{R}^2$ of a planar analytic differential system for which there exists an open neighborhood U of p such that all solutions in $U \setminus \{p\}$ are periodic. Denote by T_q the period of the periodic orbit passing through the point $q \in U \setminus \{p\}$. The singularity p is called an *isochronous center* if T_q is constant for all $q \in U \setminus \{p\}$. An isochronous center is said to be *uniform* or *rigid* if the angular velocity of the vector field is the same for all periodic orbits in $U \setminus \{p\}$. In polar coordinates (r, θ) , given by $x = r \cos \theta$ and $y = r \sin \theta$, a rigid isochronous center can be written as $dr/dt = G(r, \theta)$ and $d\theta/dt = k$, where $k \neq 0$ is constant. Moreover, isochronicity has been widely studied due to its applications in physical phenomena, its relation to the stability of periodic solutions, and its role in the existence and uniqueness of solutions for various boundary-value, bifurcation, or perturbation problems; see, for instance, [1, 2, 21]. In the case of rigid centers, the isochronicity problem is closely related to the classical

center-focus problem, which remains open; see [14, 16, 23, 43, 54]. More recently, isochronicity has also been investigated for piecewise smooth dynamical systems (PSDS), particularly those involving the coupling of two or more rigid systems and the study of their dynamics; see [5, 13, 31, 39] and the references therein. PSDS has attracted significant attention from the mathematical community in recent decades, both for their theoretical richness and for their numerous applications in modeling phenomena with abrupt transitions, see [8, 26, 45, 50, 52]. They establish an interdisciplinary bridge connecting mathematics with fields such as biology, physics, and engineering. The mathematical study of such systems began with the pioneering works by Andronov [4], Filippov [20], and Utkin [51], motivated by applications. Numerous applications in relay systems, biological systems, and control theory can be found in [10, 18, 32, 46] and the references therein.

On the other hand, hybrid systems, where continuous dynamics interact with discrete events, serve as essential tools in fields such as computer science, control theory, and simulation languages (see [49] for more details). However, research on limit cycles in hybrid systems remains limited. As stated in [41], tools for analyzing their existence and the robustness of asymptotic stability are not yet well-developed. Notable progress includes extending Melnikov theory to discontinuous hybrid systems with periodic orbits divided by a straight line, as provided in [9], and the study of limit cycles in planar piecewise linear hybrid systems, addressing questions posed in [34] and [39]. These studies proved the existence of at most one limit cycle for systems separated by one straight line and formed by linear centers, and at most two for systems separated by two lines. Thus, the study of the existence of limit cycles remains an area of ongoing exploration. Such systems present substantial mathematical challenges, particularly concerning the development and application of tools to establish the existence and stability of limit cycles.

Furthermore, the study of limit cycle bifurcations in PSDS is of great importance, as it sheds light on how small variations in system parameters can lead to the creation, destruction, or modification of limit cycles, see for instance [17] and [48]. However, this area of research in three-dimensional systems is still developing, and general results remain scarce and often specific to particular cases. Progress has been limited, as the increase in dimensionality introduces additional challenges and complexities. Nevertheless, higher-dimensional systems enrich the dynamics, providing a broader range of phenomena and fostering greater scientific interest.

In Chapter 1, some basic theory about piecewise smooth dynamical systems, rigid systems, Chebyshev Theory and hybrid systems is presented. In Chapter 2, using the Chebyshev Theory, we study a piecewise smooth dynamical system composed

of a linear rigid center and a rigid homogeneous polynomial center of even degree n , separated by a straight line. The results obtained in this chapter complement the work of [12], which addressed the odd-degree case. Thus, this work advances the analysis of this family of systems by extending the even-degree case under certain affine transformations. This chapter gave rise to the paper [24].

Chapter 3 generated the paper [36], and the main purpose is to analyze the maximum number of limit cycles in a class of piecewise smooth dynamical systems formed by two rigid polynomial systems of degrees m and n separated by a straight line, where $m, n \in \{2, 3, 4, 5, 6\}$ and $m \leq n$. We study 14 representative cases (m, n) . Our main result establishes that when the degrees are equal ($m = n$), the system admits at most one limit cycle, and this bound is achieved. For $m = 2$ and $n = 3, 4, 5, 6$, the bounds are 2, 3, 4, and 5 respectively, with the bound for $(2, 3)$ being achieved; for $m = 3$ and $n = 4, 5, 6$ they are 4, 2 and 5 respectively, with the bound for $(3, 5)$ being achieved. For $m = 4$ and $n = 5, 6$ the bound is 6; and finally, for $m = 5$ and $n = 6$ it is 8. Thus, for these classes of systems we address the extended 16th Hilbert problem.

The main purpose of Chapter 4 is to introduce the study of limit cycles in a piecewise hybrid smooth dynamical system in $\mathbb{R}^2$. For this study, we consider systems similar to those defined in Chapter 3. Our main result states that for all cases (m, n) we obtain the same upper bounds for limit cycles as in the previous chapter, except for the cases $(m, n) = (3, 6)$ and $(m, n) = (4, 6)$, where the bounds are 6 and 10, respectively, which differ from the continuous case, where the bounds were 5 and 6, respectively. For these classes of piecewise systems, we address the extended 16th Hilbert problem. The results presented here gave rise to the paper [37].

In Chapter 5, we study the effect of a perturbation on a piecewise smooth dynamical system in $\mathbb{R}^3$. The original system has two parallel tangency lines, two cusp singularities, and a single pseudo-equilibrium point; the sliding region lies between these lines. Our main results are as follows. First, after the perturbation, the two tangency lines intersect at a point, and this intersection point is a parabolic two-fold. Second, three new pseudo-equilibrium points appear. One of them is the intersection point. Depending on the sign of a specific parameter, the other two pseudo-equilibrium points either move toward or away from the pseudo-equilibrium of the original system as the perturbation parameter tends to zero. Third, when a pseudo-equilibrium converges to the pseudo-equilibrium of the original system, its qualitative behavior remains the same as that of the original system. This analysis is based on the paper [17].

Finally, in Chapter [6](#), we summarize the main results of the thesis.

Preliminaries

The principal objective of this chapter is to present some definitions and fundamental results related to piecewise smooth dynamical systems, rigid systems, Chebyshev theory, and hybrid systems.

1.1 Piecewise smooth dynamical systems

Following [11, 12], let $U \subset \mathbb{R}^i$, $i = 2, 3$ be an open and connected subset containing the origin. Let $X^+, X^- \in \mathfrak{X}^r$, where $\mathfrak{X}^r$ denotes the space of vector field of class C^r , $r \geq 1$ defined on U . Consider a differential function $h : U \subset \mathbb{R}^i \rightarrow \mathbb{R}$ having 0 as a regular value. We define a switching boundary or discontinuity region Σ as being the set

$$\Sigma = h^{-1}(0) = \{\mathbf{x} \in U : h(\mathbf{x}) = 0\}.$$

Thus, Σ is a submanifold of codimension one in U that divides the open U into two open sets:

$$\Sigma^+ = \{\mathbf{x} \in U : h(\mathbf{x}) > 0\} \quad \text{and} \quad \Sigma^- = \{\mathbf{x} \in U : h(\mathbf{x}) < 0\}.$$

In this way, we define a piecewise smooth dynamical system or PSDS as

$$X(\mathbf{x}) = \begin{cases} X^+(\mathbf{x}), & \text{if } h(\mathbf{x}) \geq 0, \\ X^-(\mathbf{x}), & \text{if } h(\mathbf{x}) \leq 0, \end{cases} \quad (1-1)$$

which is denoted by $X = (X^+, X^-, \Sigma)$ or simply $X = (X^+, X^-)$ if the Σ is well understood. To establish the dynamics given by X , we introduce the Lie derivative as $X^\pm h(\mathbf{x}) = \langle \nabla h(\mathbf{x}), X^\pm(\mathbf{x}) \rangle$, where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^i$. The higher order Lie derivatives are given by $(X^\pm)^m h(\mathbf{x}) = \langle \nabla (X^\pm)^{m-1} h(\mathbf{x}), X^\pm(\mathbf{x}) \rangle$. This derivative defines the type of contact between the vector X^+ (or X^-) and the

switching manifold Σ . Thus, the following sets can be distinguished on Σ :

$$\begin{aligned}\Sigma^s &= \{\mathbf{x} \in \Sigma : X^+h(\mathbf{x}) < 0 < X^-h(\mathbf{x})\}, \\ \Sigma^e &= \{\mathbf{x} \in \Sigma : X^-h(\mathbf{x}) < 0 < X^+h(\mathbf{x})\}, \\ \Sigma^c &= \{\mathbf{x} \in \Sigma : X^+h(\mathbf{x})X^-h(\mathbf{x}) > 0\}.\end{aligned}$$

Now, we define the local trajectory passing through a point $p \in U$. If $p \in \Sigma^\pm$ the local trajectory through p is given by the vector field X^+ and X^- in the usual way. If $p \in \Sigma^c$, then the vectors fields point in the same direction; therefore, it is enough to concatenate the trajectories of X^+ and X^- through that point. In the sliding region $\Sigma^s \cup \Sigma^e$, the dynamics is described by the Filippov sliding vector field X^s , defined as

$$X^s(p) = (1 - \lambda)X^-(p) + \lambda X^+(p)$$

where X^s is a convex linear combination of $X^+(p)$ and $X^-(p)$ and for each $p \in \Sigma^s \cup \Sigma^e$, the value of λ is chosen such that $X^s h(p) = 0$. Consequently, the local trajectory through $p \in \Sigma^s \cup \Sigma^e$ is given by X^s . Thus, the trajectories of X are considered as a concatenation of trajectories of X^+ , X^- and X^s ; see [20]. When the trajectory of X^+ through $p \in \Sigma^c$ returns to Σ (for the first time) after a positive time $t_+(p)$, we define the half return map of X^+ by

$$\pi_+(p) = \phi_{X^+}(t_+(p), p) = p_1 \in \Sigma^c.$$

Similary, when the trajectory of X^- through $p_1 \in \Sigma$ returns to Σ^c (by the first time) after a positive time $t_-(p_1)$, we define the half return map of X^- by

$$\pi_-(p_1) = \phi_{X^-}(t_-(p_1), p_1) = p_2 \in \Sigma.$$

Thus, the first return map, or Poincaré returns map of $X = (X^+, X^-)$ is defined by the composition of these two half return maps, this is,

$$\pi(p) = \pi_- \circ \pi_+(p) = \phi_{X^-}(t_-(p_1), \phi_{X^+}(t_+(p), p))$$

or the reverse, applying first the flow of X^- .

Definition 1.1. *If F and G are first integrals of X^+ , X^- , respectively, we define the first integral of $X = (X^+, X^-)$ as the pair (F, G) .*

When the vector fields X^+ and X^- associated to X admits a first integral, the solution curves of the respective differential equations are contained in the level sets

of the first integral. In this way, the first return map can be computed by seeking points in Σ that are on the same level curves of these first integral functions [5, 11].

1.1.1 Two-fold singularities

According to [15], in their pioneering work on piecewise smooth dynamical systems, Teixeira and Filippov have proposed that, if the boundaries of the sliding region intersect transversely at a point called a two-fold, this can have a dramatic effect on the system dynamics. For particular configurations of the vector field, dynamics in a neighbourhood of such points can be structurally unstable, and the dynamics around these points can be rather intricate. Consider a three-dimensional piecewise smooth dynamical system of the form (1-1). A point $\mathbf{p} \in \mathbb{R}^3$ is said to be a two-fold singularity of system (1-1) if it is a point where Σ is smooth and the flows of the two vector fields X^+ and X^- are tangent to the switching manifold at $\mathbf{x} = \mathbf{p}$. These tangency points belong to the sets

$$T^+ = \{\mathbf{x} \in \Sigma : X^+h(\mathbf{x}) = 0\} \quad \text{and} \quad T^- = \{\mathbf{x} \in \Sigma : X^-h(\mathbf{x}) = 0\}$$

that delimit the sliding regions. Specifically we have the following definition, see [15].

Definition 1.2. *A point $\mathbf{p}$ is a two-fold singularity if*

$$X^+h(\mathbf{p}) = X^-h(\mathbf{p}) = 0, \quad \text{and} \quad (X^\pm)^2h(\mathbf{p}) \neq 0 \quad (1-2)$$

Moreover, the following nondegeneracy condition, is satisfied: the curves T^+ and T^- intersect transversely at $\mathbf{p}$. Mathematically, $\det(\mathbf{N}) \neq 0$, where $\mathbf{N}$ given by

$$\mathbf{N} = \begin{pmatrix} \nabla X^+h(\mathbf{p}) \\ \nabla X^-h(\mathbf{p}) \\ \nabla h(\mathbf{p}) \end{pmatrix} \quad (1-3)$$

The condition $(X^\pm)^2h(\mathbf{p}) \neq 0$ distinguishes between three qualitatively different singularities, see Figure 1.2:

- (a) *The Teixeira singularity or elliptic two fold, where $X^{+2}h(\mathbf{p}) < 0 < X^{-2}h(\mathbf{p})$.*
- (b) *The hiperbolic two-fold, where $X^{-2}h(\mathbf{p}) < 0 < X^{+2}h(\mathbf{p})$.*
- (c) *The parabolic two-fold, where $X^{-2}h(\mathbf{p})X^{+2}h(\mathbf{p}) > 0$.*

For simplicity, it is assumed in Figure 1.1 that $h(\mathbf{x})$ and $X^\pm h(\mathbf{x})$ are affine functions of $\mathbf{x}$. The left figure shows the case where, if $\mathbf{N}$ is singular, then either the nullclines $X^\pm h(\mathbf{x}) = 0$ are coplanar or their intersection with Σ is not a unique point. The

right figure shows the case where, if $\mathbf{N}$ is nonsingular, then the switching plane $h(\mathbf{x}) = 0$ and the nullcline planes $X^+h(\mathbf{x}) = 0$ and $X^-h(\mathbf{x}) = 0$ are in a generic position relative to each other, meaning that none are coplanar and that their three pairwise lines of intersection cross at a unique point: the two-fold singularity.

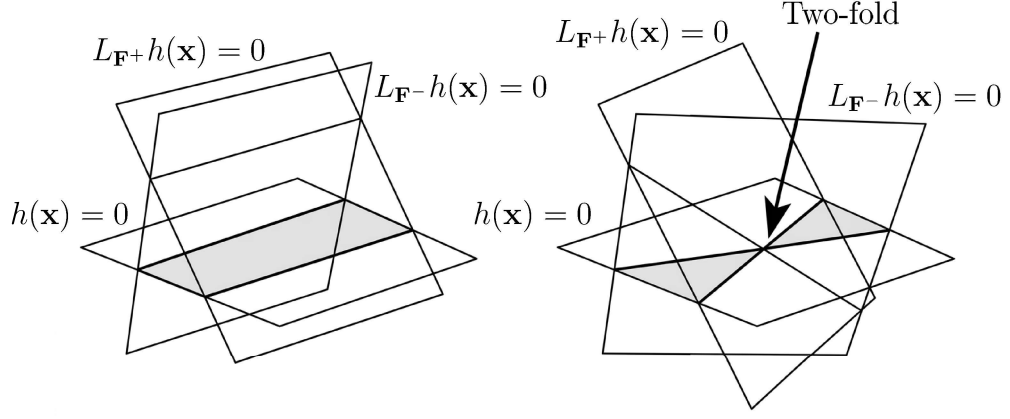


Figure 1.1: Critical surfaces and the two-fold singularity. Adapted from [15]

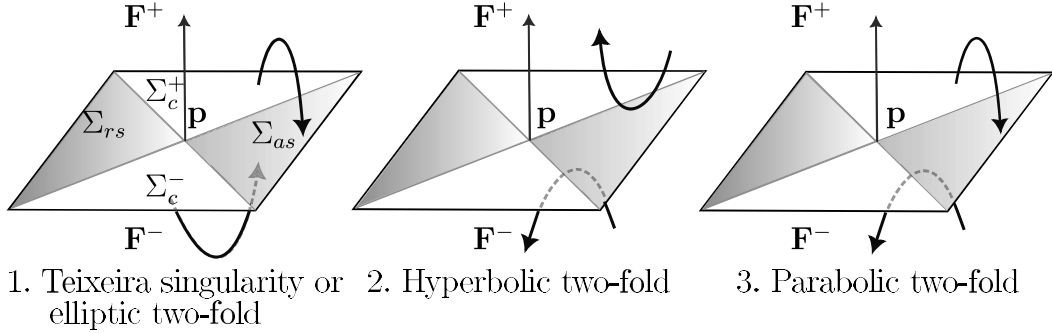


Figure 1.2: Two-folds singularities. Adapted from [15].

Definition 1.3. Consider a point $\mathbf{x} \in \mathbb{R}^3$.

- (a) Then, $\mathbf{x} \in \Sigma$ is said to be a fold point of X^+ (resp. X^-) if $X^+h(\mathbf{x}) = 0$ and $X^{+2}h(\mathbf{x}) \neq 0$ (resp. $X^-h(\mathbf{x}) = 0$ and $X^{-2}h(\mathbf{x}) \neq 0$). We say that $\mathbf{x}$ is a visible fold if $X^{+2}h(\mathbf{x}) > 0$, or invisible if $X^{+2}h(\mathbf{x}) < 0$, (resp. visible if $X^{-2}h(\mathbf{x}) < 0$ or invisible if $X^{-2}h(\mathbf{x}) > 0$).
- (b) $\mathbf{x} \in \Sigma$ is a cusp point of $X^\pm$ if $X^\pm h(\mathbf{x}) = X^{\pm 2}h(\mathbf{x}) = X^{\pm 3}h(\mathbf{x}) \neq 0$ and $\{\nabla h(\mathbf{x}), \nabla X^\pm h(\mathbf{x}), \nabla X^{\pm 2}h(\mathbf{x})\}$ is linearly independent.
- (c) $\mathbf{x} \in \Sigma^+ \cup \Sigma^-$ is a regular equilibrium of (1-1) if $X^\pm(\mathbf{x}) = \mathbf{0}$. Moreover, we say that $\mathbf{x}$ is real or admissible if $\mathbf{x} \in \Sigma^\pm$ and virtual (non-admissible) if $\mathbf{x} \in \Sigma^\mp$.
- (d) $\mathbf{x} \in \Sigma$ is a boundary equilibrium point of (1-1) if $X^\pm(\mathbf{x}) = \mathbf{0}$.
- (e) $\mathbf{x} \in \Sigma$ is a pseudo-equilibrium point if there exists $\alpha \in (0, 1)$ such that $(1 - \alpha)X^-(\mathbf{x}) + \alpha X^+(\mathbf{x}) = \mathbf{0}$, i.e., $X^\pm$ are anti-collinear at $\mathbf{x}$. Obviously,

$X^s(\mathbf{x}) = 0$. If $\mathbf{x}$ behaves as a node, a focus, or a saddle for the dynamics of X^s , we say that $\mathbf{x}$ is a pseudo-node, pseudo-focus, or pseudo-saddle, respectively.

1.2 Rigid centers

Consider a polynomial differential system in $\mathbb{R}^2$ of degree n

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y). \end{cases} \quad (1-4)$$

An equilibrium point p of the system (1-4) is a center if there exists an open neighborhood U of p where all the orbits contained in $U \setminus \{p\}$ are periodic. Without loss of generality we may assume that the equilibrium point is at the origin. According to [3, 29], this center is isochronous if all the periodic orbits contained in $U \setminus \{p\}$ have the same period. More specifically, let T_q be the period of the orbit through the point $q \in U \setminus \{p\}$. In this context, p is an isochronous center if T_q is constant for all $q \in U \setminus \{p\}$. However, this does not imply that the angular velocity of the vector $\vec{pq}$ is the same for all periodic orbits in $U \setminus \{p\}$. When this is the case, p is referred to as a *uniform isochronous center* or a *rigid center*. In other words, a center p is a rigid center if, in polar coordinates (r, θ) defined by $x = r \cos \theta$, $y = r \sin \theta$, it can be written as $\frac{dr}{dt} = G(r, \theta)$ and $\frac{d\theta}{dt} = k$, where $k \neq 0$ is constant.

Remark 1.4. Note that there are isochronous centers that are not uniform. In fact, consider the following polynomial system

$$\begin{cases} \dot{x} = \frac{1}{2} (x^7 - 5x^5y^2 - 13x^3y^4 - 7xy^6 - 2y), \\ \dot{y} = \frac{1}{2} (7x^6y + 13x^4y^3 + 5x^2y^5 + 2x - y^7). \end{cases} \quad (1-5)$$

Observe that system (1-5) satisfies condition (b.2) of Theorem 1 from [40], which implies that it possesses an isochronous center at the origin. However, in polar coordinates, system (1-5) takes the form

$$\begin{cases} \dot{r} = \frac{1}{2} r^7 \cos(2\theta), \\ \dot{\theta} = 1 + \frac{3}{2} r^6 \sin(2\theta). \end{cases}$$

Note that $\dot{\theta}$ is not a constant different from zero. Therefore, the origin is an isochronous center, but it is not a rigid center.

The following result determine conditions for a center in a planar polynomial system to be a rigid center. See [16, 30].

Proposition A. *Assume that a planar polynomial system given by (1-4) has a center at the origin of coordinates. Then, this center is uniform isochronous if and only if through a linear change of variables and a rescaling of time, it can be written in the form*

$$\dot{x} = -y + xf(x, y), \quad \dot{y} = x + yf(x, y), \quad (1-6)$$

where $f(x, y)$ is a polynomial of degree $n - 1$ in the variables x and y , with $f(0, 0) = 0$.

The following result about rigid centers was stated in [16].

Proposition B. *Let $f(x, y) = \sum_{i=0}^{n-1} a_i x^{n-i-1} y^i$ be a homogeneous polynomial of degree $n - 1$. Then the origin is a uniform isochronous center of (1-6) if n is even, or n is odd and*

$$\sum_{i=0}^{n-1} a_i \int_0^{2\pi} (\cos \theta)^{n-i-1} (\sin \theta)^i d\theta = \sum_{i=0}^{n-1} a_i I_{n-i-1, i} = 0.$$

Remark 1.5. *When n is even, the equality $\sum_{i=0}^{n-1} a_i I_{n-i-1, i} = 0$ always holds, since*

$$f(\cos(\theta + \pi), \sin(\theta + \pi)) = -f(\cos \theta, \sin \theta),$$

where

$$f(\cos \theta, \sin \theta) = \sum_{i=0}^{n-1} a_i (\cos \theta)^{n-i-1} (\sin \theta)^i \text{ and } I_{n-i-1, i} = \int_0^{2\pi} (\cos \theta)^{n-i-1} (\sin \theta)^i d\theta.$$

1.3 Chebyshev theory

Consider the set $\mathcal{F} = \{f_0(x), f_1(x), \dots, f_m(x)\}$ composed of $m + 1$ real-analytic and linearly independent functions defined on an open interval I and let $v \in \text{Span}(\mathcal{F})$ such that

$$v(x) = \sum_{j=0}^m a_j f_j(x), \quad a_j \in \mathbb{R},$$

where $\text{Span}(\mathcal{F})$ is the set (or space) of all linear combinations of the functions in $\mathcal{F}$. We denote by $\mathcal{Z}(v)$ the number of zeros in I , counted with their multiplicities, of a

function v and by

$$\mathcal{Z}(\mathcal{F}) = \max_{v \in \text{Span}(\mathcal{F}) \setminus \{0\}} \mathcal{Z}(v)$$

whenever they exist. A set of functions $\mathcal{F}$ for which $\mathcal{Z}(\mathcal{F}) = m$ is usually called an *Extended Chebyshev system* on I , or an ET-system, see [22]. The set $\mathcal{F}$ is an Extended Complete Chebyshev system or an ECT-system on I if and only if for any k with $0 \leq k \leq m$, $\{f_0(x), f_1(x), \dots, f_k(x)\}$ is an ET-system, see [44].

In what follows, we present a result from the literature that we apply in our context. See, for instance, [28, 33, 35].

Theorem A *Let $f_0, \dots, f_m$ be analytic functions defined on an open interval $I \subset \mathbb{R}$. Then, the set of functions $\{f_0, \dots, f_m\}$ is an ECT-system on I if and only if for each $0 \leq k \leq m$ and all $y \in I$, the Wronskian*

$$W(f_0, \dots, f_k)(y) = \begin{vmatrix} f_0(y) & f_1(y) & \dots & f_k(y) \\ f'_0(y) & f'_1(y) & \dots & f'_k(y) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(y) & f_1^{(k)}(y) & \dots & f_k^{(k)}(y) \end{vmatrix}$$

is different from zero.

1.4 Hybrid systems in $\mathbb{R}^2$

In this section, we give a brief introduction to hybrid systems (see, for instance, [38, 42]). According to [42], dynamical systems can be classified as follows:

- **Continuous:** if the state takes values in Euclidean space $\mathbb{R}^n$ for some $n \geq 1$. Thus, $x \in \mathbb{R}^n$ denotes the state of a continuous dynamical system.
- **Discrete:** if the state takes values in a countable or finite set $\{m_1, m_2, \dots\}$, where m denotes the state of a discrete system.
- **Hybrid:** if part of the state takes values in $\mathbb{R}^n$ while another part takes values in a finite set.

More specifically, hybrid systems involve interactions between variables in continuous and finite or countable sets; see [7, 34]. Continuous-time systems are modeled by differential equations or inclusions, parameterized by continuous time $t \in \mathbb{R}$ with $t \geq 0$, while discrete-time systems use difference equations or inclusions, parameterized by integers $n \in \mathbb{N}$; see [53].

The following example illustrates how continuous and discrete dynamics interact in a hybrid system.

Example 1.6 (Thermostat). *Consider a room being heated by a radiator controlled by a thermostat. Assume that when the radiator is off, the temperature, $x \in \mathbb{R}$, of the room decreases exponentially towards 0 degrees according to the differential equation*

$$\dot{x} = -ax \quad (1-7)$$

with $a > 0$. When the thermostat turns the heater on, the temperature increases exponentially, approaching 30 degrees, according to the differential equation

$$\dot{x} = -a(x - 30). \quad (1-8)$$

Thus, we have:

System states:

- *Continuous state (x): Represents the room temperature, which is a real variable, $x \in \mathbb{R}$.*
- *Discrete state (m): Indicates the radiator's operating mode, which can be either on (ON) or off (OFF).*

Continuous evolution (Flows): *The temperature evolves according to differential equations that depend on the radiator's state:*

- *Radiator off (OFF): The temperature decreases exponentially towards 0 degrees, following the equation $\dot{x} = -ax$, ($a > 0$).*
- *Radiator on (ON): The temperature increases exponentially towards 30 degrees, following the equation $\dot{x} = -a(x - 30)$.*

Control logic and transitions (Jumps): *The goal is to keep the temperature at approximately 20 degrees. To avoid the "chattering" phenomenon (turning the device on and off repeatedly in minimal intervals), the system uses safety thresholds:*

- *To turn on: The heater does not attempt to turn on until the temperature drops below 19 degrees, and it may reach 18 degrees before starting to heat.*
- *To turn off: The heater does not attempt to turn off until the temperature rises above 21 degrees, and it may reach 22 degrees before starting to cool down.*

The system's evolution is coupled since the discrete state determines which differential equation governs the temperature, while the temperature value (continuous state) enables or forces the change in the radiator's state. For more details, see [42].

Without loss of generality, consider $h(x) = x$. Following [38], let $\Sigma = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ be the switching manifold and $\mathcal{R} : \Sigma \rightarrow \Sigma$ the reset map, which is the

rule that defines the new initial state of the system at the exact instant of a jump, immediately after it reaches the switching manifold. Consider

$$\Sigma^+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}, \quad \Sigma^- = \{(x, y) \in \mathbb{R}^2 : x < 0\},$$

and let $\overline{\Sigma}^\pm$ denote the topological closure of $\Sigma^\pm$. Given $r \geq 1$, let $\overline{\mathfrak{X}}^r$ be the class of the hybrid dynamical systems $\mathcal{X} = (X^+, X^-, \Sigma, \mathcal{R})$, such that $X^\pm$ are nonlinear vector fields with a center singularity and $X^\pm$ is defined over $\overline{\Sigma}^\pm$.

1.4.1 Global orbit dynamics of $\mathcal{X}$

We define a global orbit $\Gamma(q, k, t)$ of $\mathcal{X}$ through an initial point $q \in \mathbb{R}^2$, with $k \in \mathbb{N}$, $t \geq 0$ and $\Gamma(q, 1, 0) = q$.

1. *The initial point does not lie on Σ ($q \notin \Sigma$):* Suppose $q \in \mathbb{R}^2 \setminus \Sigma$. In this case $\Gamma(q, k, t)$ is unique and follows these steps.

- *Continuous flow:* Assuming the system starts at Σ^- , that is, $q \in \Sigma^-$, let $\gamma_1(t)$ be the orbit of X^- with initial condition q . If γ_1 never intersects Σ , then $\Gamma(q, k, t) \equiv \Gamma(q, 1, t)$ and $\Gamma(q, 1, t) = \gamma_1(t)$ for all $t \geq 0$.
- *Discrete jump:* Suppose that there is a $t_1 \geq 0$ such that $q_1 = \gamma_1(t_1) \in \Sigma$ and $\gamma_1(t) \notin \Sigma$ for all $t \in [0, t_1)$. After reaching q_1 an instantaneous jump occurs to a new point q_2 , defined by the reset map $q_2 = \mathcal{R}(q_1)$. Let $\gamma_2(t)$ be the orbit of X^+ with initial condition q_2 .

After the jump to q_2 , the system evaluates whether it can enter the opposite region Σ^+ following γ_2 . Thus, exactly one of the following statements holds:

- (i) $\gamma_2(\varepsilon) \in \Sigma^+$ for sufficiently small $\varepsilon > 0$: then we can enter in Σ^+ by following $\gamma_2(t)$ and thus the process repeats. More precisely, we follow $\gamma_2(t)$ until we reach Σ again at a point $\bar{q}_2 = \gamma_2(t_2)$, $t_2 > 0$, and then we jump to $q_3 = \mathcal{R}(\bar{q}_2)$, try to enter in Σ^- , and so on. See Figure 1.3.
- (ii) $\gamma_2(\varepsilon) \in \overline{\Sigma}^-$ for sufficiently small $\varepsilon > 0$: then we cannot enter in Σ^+ by following $\gamma_2(t)$ and thus we jump to $q_3 = \mathcal{R}(q_2)$. Let $\gamma_3(t)$ be the orbit of X^- with initial condition q_3 . Similarly to the previous case, we now try to enter in Σ^- by following γ_3 . If it is possible, then we are done. If not, then we jump again and try to enter in Σ^+ , and so on, see Figure 1.4. In particular, observe that there is the possibility of never leaving Σ .

Therefore, $\Gamma(q, k, t)$ is defined as the ordered concatenation of these pieces of orbits of X^+ and X^- and the jumps between them. More precisely, let $\Gamma(q, 1, t) = \gamma_1(t)$ for $0 \leq t \leq t_1$, $\Gamma(q, 2, t) = \gamma_2(t)$ for $0 \leq t \leq t_2$, and so on (note

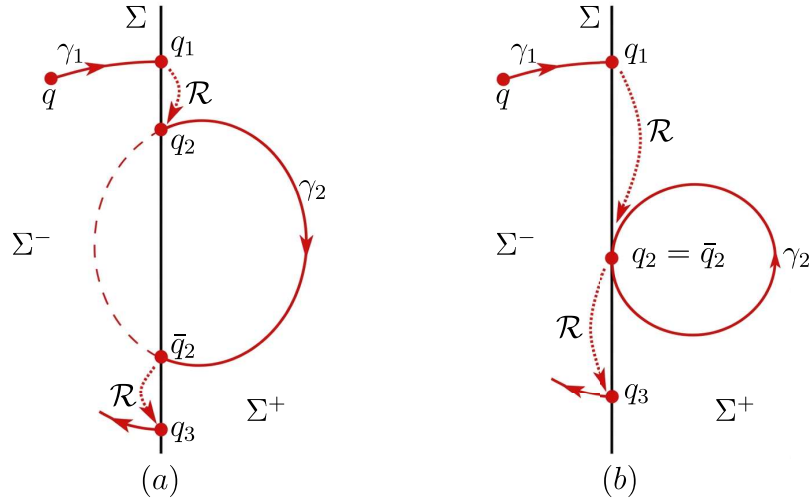


Figure 1.3: Orbits for the hybrid system $\mathcal{X}$, case (i).
Adapted from [38].

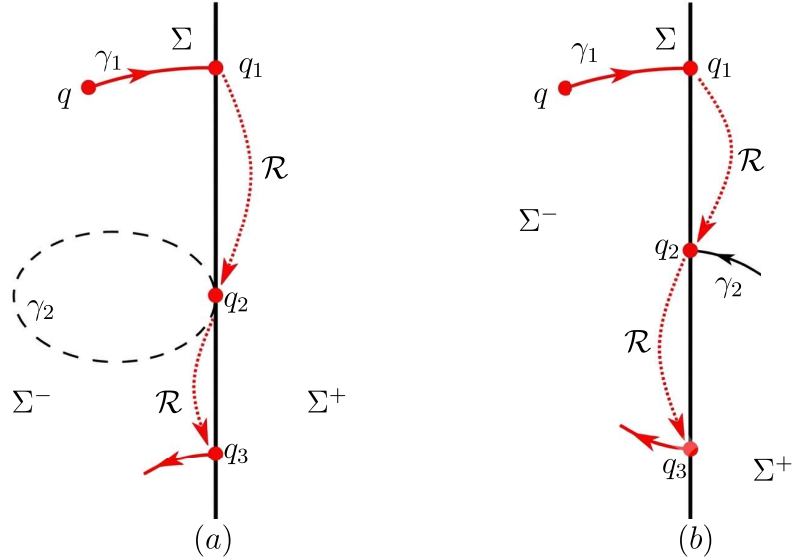


Figure 1.4: Orbits for the hybrid system $\mathcal{X}$, case (ii).
Adapted from [38].

that we may have $t_i = 0$), and define $\Gamma(q, k, t)$ as the ordered concatenation

$$\Gamma(q, k, t) = \Gamma(q, 1, t) \cup \Gamma(q, 2, t) \cup \cdots \cup \Gamma(q, j, t) \cup \cdots .$$

2. *The initial point lies on Σ ($q \in \Sigma$):* In this case $\mathcal{X}$ has two global orbits $\Gamma_1(q, k, t)$, $\Gamma_2(q, k, t)$ with initial condition q . The orbit Γ_1 is defined as follows. Let $\gamma_1(t)$ be the orbit of X^- with initial condition q . Suppose first that there is $t_1 < 0$ such that $\gamma_1(t) \in \Sigma^-$ for all $t \in [t_1, 0)$ and let $q_1 = \gamma_1(t_1)$. Then q belongs to $\Gamma(q_1, 1, t)$ and thus we just follow it. That is, we let $\Gamma_1(q, 1, t) = \{q_1\}$ and $\Gamma_1(q, j, t) = \Gamma(q_1, j, t)$ for $j \geq 2$.

Similarly, suppose there is $t_1 > 0$ such that $\gamma_1(t) \in \Sigma^-$ for all $t \in (0, t_1]$,

and let $q_1 = \gamma_1(t_1)$. In this case we concatenate the piece of γ_1 from q to q_1 with the orbit $\Gamma(q_1, k, t)$. More precisely, we let $\Gamma_1(q, 1, t) = \gamma_1(t)$ for $t \in [0, t_1]$, $\Gamma_1(q, 1, t) = \Gamma(q_1, 1, t - t_1)$ for $t \geq t_1$, and $\Gamma_1(q, j, t) = \Gamma(q_1, j, t)$ for $j \geq 2$. Similarly, one can define $\Gamma_2(q, k, t)$. Given $q \in \mathbb{R}^2$ we may refer to the orbits $\Gamma_1(q, k, t)$ and $\Gamma_2(q, k, t)$ of the hybrid system $\mathcal{X}$ with initial condition q . However, if $q \notin \Sigma$, then the orbit is unique, and consequently $\Gamma_1 = \Gamma_2$.

1.4.2 Periodic orbits, limit cycles and the first return map for the hybrid system $\mathcal{X}$

To study periodic orbits and limit cycles, we define a first return map $\bar{\pi} : \Sigma \rightarrow \Sigma$. Given $y \equiv (0, y) \in \Sigma$, its first return $\bar{\pi}(y) \in \Sigma$ can be defined in four different ways, depending on the orientation of the pieces of orbits γ_1 and γ_2 of $X^\pm$; (see Fig 1.5).

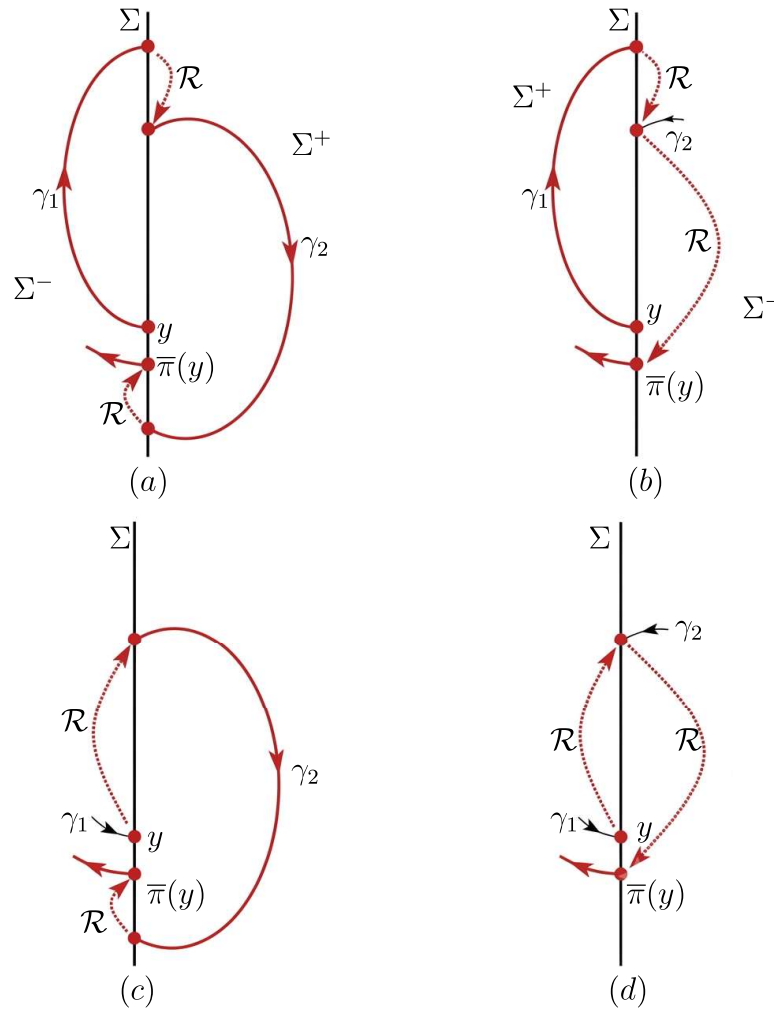


Figure 1.5: The first return map for the hybrid system $\mathcal{X}$.
Adapted from [38].

Definition 1.7. Consider the global orbit $\Gamma_1(y, k, t)$ with $y \in \Sigma$. Γ_1 is periodic if $\bar{\pi}^n(y) = y$ for some $n \geq 1$. In particular, if the orbit through y is an isolated periodic point, then we say that it is a limit cycle. Therefore, the limit cycles of $\mathcal{X}$ are in bijection with the isolated periodic points of $\bar{\pi}$. Furthermore, if the orientation of the pieces of orbits of X^+ and X^- agree, then we say that the limit cycle is regular, (See Figure 1.5.(a)); and if $n \in \mathbb{N}$ is the minimal n such that $\bar{\pi}^n(y) = y$, then we say that n is the period of the limit cycle, see Figure 1.6.

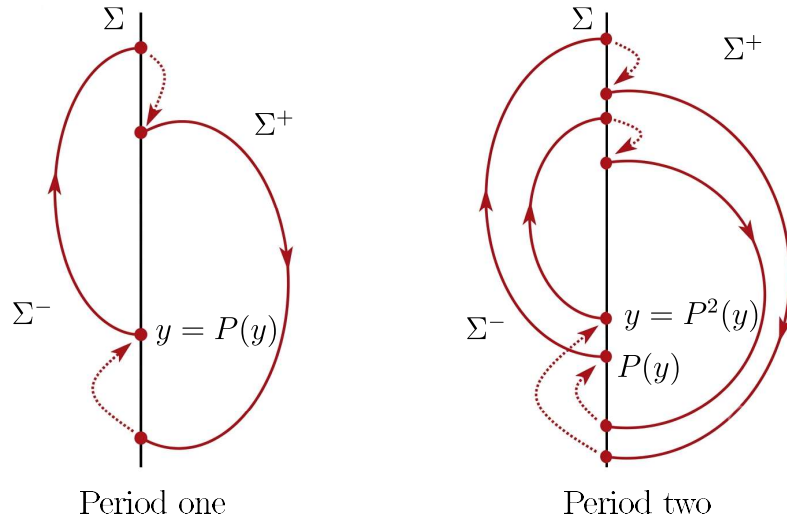


Figure 1.6: Regular limit cycles of the hybrid system $\mathcal{X}$.
Adapted from [38].

In the next chapters we present the results obtained.

Limit cycles in rigid piecewise smooth dynamical systems of even degree

This chapter, which is based on [24], establishes an upper bound for the number of crossing limit cycles in a class of piecewise smooth dynamical systems. The system is formed by a linear rigid center and a rigid center composed of a linear part plus a homogeneous polynomial of even degree n , separated by the straight line $x = 0$. Our results complement the work of [12], which addressed the odd-degree case.

We consider a PSDS whose switching boundary is given by a straight line. Let us denote the sets $\Sigma^+ = \{(x, y) \in \mathbb{R}^2 : x > 0\}$, $\Sigma^- = \{(x, y) \in \mathbb{R}^2 : x < 0\}$ and consider the PSDS given by

$$X_{1,n}(x, y) = \begin{cases} X_1^-(x, y), & \text{for } (x, y) \in \Sigma^-, \\ X_n^+(x, y), & \text{for } (x, y) \in \Sigma^+, \end{cases} \quad (2-1)$$

with $X_1^-(x, y) = (-y, x)$ and

$$X_n^+(x, y) = \left(-y + x \sum_{i=0}^{n-1} a_i x^{n-i-1} y^i, \quad x + y \sum_{i=0}^{n-1} a_i x^{n-i-1} y^i \right).$$

In [11], the authors proved that the PSDS (2-1) has no limit cycles. Furthermore, by applying an affine change of variables to (2-1) such that the centers remain rigid, they showed that the resulting PSDS has at most one limit cycle for $n = 2$ and $n = 3$, and this upper bound is reached. Consider the most general PSDS defined by

$$X_{m,n}(x, y) = \begin{cases} X_m^-(x, y), & \text{for } (x, y) \in \Sigma^-, \\ X_n^+(x, y), & \text{for } (x, y) \in \Sigma^+, \end{cases} \quad (2-2)$$

where

$$X_m^-(x, y) = \left(-y + x \sum_{i=0}^{m-1} s_i x^{m-i-1} y^i, \quad x + y \sum_{i=0}^{m-1} s_i x^{m-i-1} y^i \right).$$

Still in the same paper, the authors consider (2-2) and by applying an affine change of variables in Σ^+ , they proved that the resulting PSDS has at most one limit cycle for $m, n \in \{2, 3\}$, and that this upper bound of limit cycles is achieved. To establish a general result for arbitrary degrees, the authors of [12] analyzed (2-1) by applying an affine change of variables in $\Sigma^\pm$. Specifically, they used the transformation $(u, v) = (b_1x + b_2y + d_1, b_3x + b_4y + d_2)$, where $b_i, d_j \in \mathbb{R}$, in Σ^- , and $(u, v) = (c_1x + c_2y + M_1, c_3x + c_4y + M_2)$, where $c_i, M_j \in \mathbb{R}$, in Σ^+ , with $i = 1, 2, 3, 4$ and $j = 1, 2$. By establishing conditions on the parameters of the transformation so that the centers remain rigid, they proved that the resulting PSDS has at most $n - 2$ limit cycles for odd n .

The even-degree case of system (2-1) has proved to be particularly challenging. In this case, the first integral of system X_n in the new coordinates can be written as $F_2(u, v) = \frac{f_1(u, v)}{g_1(u, v)}$, where $f_1(u, v)$ is a polynomial function of degree $n - 1$ in u and v , and $g_1(u, v) = [(u - M_1)^2 + (v - M_2)^2]^{\frac{n-1}{2}}$. Unlike the odd-degree case, where g_1 is a polynomial whose zeros, corresponding to limit cycles, can be found directly, in the even-degree case g_1 is no longer a polynomial. Consequently, the corresponding equation is no longer a polynomial, and more refined techniques must be applied.

The main purpose of this chapter is to consider (2-1), apply the same affine change of variables in $\Sigma^\pm$ as in [12], and provide an upper bound for the maximum number of limit cycles when n is even. In this way, the present result represents an advancement in the study of this class of PSDS for even degrees, addressing a case that remained open in [12]. More precisely, the main result of this chapter is the following.

Theorem 2.1. *Consider the PSDS (2-1) with the affine change of variables $(u, v) = (b_1x + b_2y + d_1, b_3x + b_4y + d_2)$, $b_i \in \mathbb{R}, d_j \in \mathbb{R}$ in Σ^- and $(u, v) = (c_1x + c_2y + M_1, c_3x + c_4y + M_2)$, $c_i \in \mathbb{R}, M_j \in \mathbb{R}$ in Σ^+ , with $i = 1, 2, 3, 4$ and $j = 1, 2$. Assume that $M_1 \neq 0, c_1^2 + c_2^2 \neq 0$ and n is even.*

- (a) *If $d_2 \neq M_2, d_2 = 0$ and $n = 4$, then the resulting PSDS has at most 1 limit cycle and this upper bound is reached.*
- (b) *If $d_2 = M_2$, then the resulting PSDS has at most $\frac{n-2}{2}$ limit cycles.*

This chapter is organized as follows. Section 2.1 provides the expression for the closing function of the PSDS $X_{1,n}$. The proof of our main result is developed in Section 2.2. Section 2.3 presents numerical examples for $n = 4$ demonstrating that the upper bounds are attained, as well as an example of the case $d_2 \neq 0$ exhibiting three limit cycles. Finally, Section 2.4 contains some brief remarks on this chapter.

2.1 The closing function for the rigid center (2-1)

Consider the PSDS given in (2-1) whose switching manifold is the straight line $\Sigma = h^{-1}(0)$ with $h(x, y) = x$. Given that the vector fields X_1^- and X_n^+ associated with $X_{1,n} = (X_1^-, X_n^+)$ are integrable, the solution curves of the respective differential equation are contained in the level sets of their respective first integrals. In this way, the first return map associated with the PSDS $X_{1,n}$ can be computed by seeking points on the switching manifold Σ that lie on the same level curves of these first integrals. Consider (F_1, H_n) , the first integral of (2-1). Therefore, for a limit cycle of (2-1) that has two intersection points, $(0, y_0)$ and $(0, y_1)$, on the line of discontinuity $\Sigma = \{(x, y) \in U : x = 0\}$, the coordinates of these points must satisfy the following system of equations $F_1(0, y_0) = F_1(0, y_1)$, $H_n(0, y_0) = H_n(0, y_1)$, or, equivalently, $F_1(r_0, \frac{\pi}{2}) = F_1(R_0, \frac{3\pi}{2})$, $H_n(r_0, \frac{\pi}{2}) = H_n(R_0, \frac{3\pi}{2})$, where $(r_0, \frac{\pi}{2})$ and $(R_0, \frac{3\pi}{2})$ are the respective intersection points $(0, y_0)$ and $(0, y_1)$ in polar coordinates. More specifically, to analyze the existence of limit cycles of (2-1), we define the closing function $M(y_0)$ as the difference between the values of the first integral H_n evaluated at these intersection points: $M(y_0) = H_n(0, y_0) - H_n(0, y_1)$, where the return point y_1 is uniquely determined by the condition $F_1(0, y_0) = F_1(0, y_1)$. Consequently, the non-trivial zeros of $M(y_0)$ correspond precisely to the crossing limit cycles of the system. In what follows, we will obtain the explicit expression for the closing function.

By [12], we have that $F(x, y) = \sqrt{x^2 + y^2}$ and

$$H_n(x, y) = \frac{1}{(1-n)(x^2 + y^2)^{\frac{n-1}{2}}} - \sum_{i=0}^{n-1} a_i \int \frac{x^{n-i} y^i}{(x^2 + y^2)^{\frac{n+1}{2}}} dy.$$

In polar coordinates, we have

$$H_n(r, \theta) = \frac{1}{(1-n)r^{n-1}} - \sum_{i=0}^{n-1} a_i \int (\cos \theta)^{n-i-1} (\sin \theta)^i d\theta. \quad (2-3)$$

In order to obtain an explicit expression of H_n , we have to calculate the integral of expression (2-3). For $n \in \mathbb{N}$ even, we get

$$\begin{aligned} \sum_{i=0}^{n-1} a_i \int (\cos \theta)^{n-i-1} (\sin \theta)^i d\theta &= a_0 \int (\cos \theta)^{n-1} d\theta + \sum_{j=1}^{\frac{n-2}{2}} a_{2j-1} \int (\cos \theta)^{n-2j} (\sin \theta)^{2j-1} \\ &\quad d\theta + \sum_{j=1}^{\frac{n-2}{2}} a_{2j} \int (\cos \theta)^{n-1-2j} (\sin \theta)^{2j} d\theta + a_{n-1} \int (\sin \theta)^{n-1} d\theta. \end{aligned}$$

Using formulas 2.512(3), 2.512(4), 2.511(4), and 2.511(3) from [25], which are

respectively given by

$$\begin{aligned}
\int (\cos x)^{2l+1} dx &= \frac{\sin x}{2l+1} \left[\sum_{k=0}^{l-1} \frac{2^{k+1} (\cos x)^{2l-2k-2} \prod_{i=0}^k (l-i)}{\prod_{i=0}^k (2l-2i-1)} + (\cos x)^{2l} \right], \\
\int (\cos x)^p (\sin x)^{2l+1} dx &= -\frac{(\cos x)^{p+1}}{2l+p+1} \left[\sum_{k=1}^n \frac{2^k (\sin x)^{2l-2k} \prod_{i=1}^k (l-i+1)}{\prod_{i=1}^k (2l+p-2i+1)} \right. \\
&\quad \left. + (\sin x)^{2l} \right], \\
\int (\sin x)^p (\cos x)^{2l+1} dx &= \frac{(\sin x)^{p+1}}{2l+p+1} \left[\sum_{k=1}^l \frac{2^k (\cos x)^{2l-2k} \prod_{i=1}^k (l-i+1)}{\prod_{i=1}^k (2l+p-2i+1)} \right. \\
&\quad \left. + (\cos x)^{2l} \right], \\
\int (\sin x)^{2l+1} dx &= -\frac{\cos x}{2l+1} \left[\sum_{k=0}^{l-1} \frac{2^{k+1} (\sin x)^{2l-2k-2} \prod_{i=0}^k (l-i)}{\left(\prod_{i=1}^k (2l-2i+1) \right)} + (\sin x)^{2l} \right],
\end{aligned}$$

we have

$$\begin{aligned}
\bar{\Omega}_0 &= \int (\cos \theta)^{n-1} d\theta = \frac{\sin \theta}{n-1} \left[\sum_{k=0}^{\frac{n-4}{2}} \frac{2^{k+1} (\cos \theta)^{n-2k-4} \prod_{i=0}^k \left(\frac{n-2}{2} - i \right)}{\prod_{i=0}^k (n-2i-3)} \right. \\
&\quad \left. + (\cos \theta)^{n-2} \right], \\
\bar{\Omega}_{2j-1} &= \int (\cos \theta)^{n-2j} (\sin \theta)^{2j-1} d\theta = - \left[\sum_{k=1}^{j-1} \frac{2^k (\sin \theta)^{2j-2k-2} \prod_{i=1}^k (j-i)}{\prod_{i=1}^k (n-2i-1)} \right. \\
&\quad \left. + (\sin \theta)^{2j-2} \right] \frac{(\cos \theta)^{n-2j+1}}{n-1}, \\
\bar{\Omega}_{2j} &= \int (\cos \theta)^{n-1-2j} (\sin \theta)^{2j} d\theta = \left[\sum_{k=1}^{\frac{n-2j-2}{2}} \frac{2^k (\cos \theta)^{n-2j-2k-2} \prod_{i=0}^{k-1} \left(\frac{n-2j-2}{2} - i \right)}{\prod_{i=1}^k (n-2i-1)} \right. \\
&\quad \left. + (\cos \theta)^{n-2j-2} \right] \frac{(\sin \theta)^{2j+1}}{n-1}, \\
\bar{\Omega}_{n-1} &= \int (\sin \theta)^{n-1} d\theta = - \left[\sum_{k=0}^{\frac{n-4}{2}} \frac{2^{k+1} (\sin \theta)^{n-2k-4} \prod_{i=0}^k \left(\frac{n-2}{2} - i \right)}{\prod_{i=0}^k (n-2i-3)} \right. \\
&\quad \left. + (\sin \theta)^{n-2} \right] \frac{\cos \theta}{n-1}.
\end{aligned}$$

Therefore, the first integral $H_n(r, \theta)$ given by (2-3), for even n , can be written as

$$H_n(r, \theta) = \frac{1}{(1-n)r^{n-1}} - a_0 \bar{\Omega}_0 - \sum_{j=1}^{\frac{n-2}{2}} a_{2j-1} \bar{\Omega}_{2j-1} - \sum_{j=1}^{\frac{n-2}{2}} a_{2j} \bar{\Omega}_{2j} - a_{n-1} \bar{\Omega}_{n-1}.$$

In Cartesian coordinates, the expression of H_n becomes

$$\begin{aligned} H_n(x, y) = & \frac{1}{(x^2 + y^2)^{\frac{n-1}{2}}} \left[-\frac{a_0}{(n-1)} \sum_{k=0}^{\frac{n-4}{2}} \frac{2^{k+1} y x^{n-2k-4} (x^2 + y^2)^{k+1} \prod_{i=0}^k \left(\frac{n-2}{2} - i\right)}{\prod_{i=0}^k (n-2i-3)} \right. \\ & + \sum_{j=1}^{\frac{n-2}{2}} \frac{a_{2j-1}}{(n-1)} \sum_{k=1}^{j-1} \frac{2^k y^{2j-2k-2} x^{n-2j+1} (x^2 + y^2)^k \prod_{i=1}^k (j-i)}{\prod_{i=1}^k (n-2i-1)} \\ & + \sum_{j=1}^{\frac{n-2}{2}} a_{2j-1} \frac{y^{2j-2} x^{n-2j+1}}{(n-1)} - \sum_{j=1}^{\frac{n-2}{2}} a_{2j} \frac{y^{2j+1} x^{n-2j-2}}{(n-1)} \\ & - \sum_{j=1}^{\frac{n-2}{2}} \frac{a_{2j}}{(n-1)} \sum_{k=1}^{\frac{n-2j-2}{2}} \frac{2^k y^{2j+1} x^{n-2j-2k-2} (x^2 + y^2)^k \prod_{i=0}^{k-1} \left(\frac{n-2j-2}{2} - i\right)}{\prod_{i=1}^k (n-2i-1)} \\ & + a_{n-1} \frac{x y^{n-2}}{(n-1)} + \frac{a_{n-1}}{(n-1)} \sum_{k=0}^{\frac{n-4}{2}} \frac{2^{k+1} y^{n-2k-4} x (x^2 + y^2)^{k+1} \prod_{i=0}^k \left(\frac{n-2}{2} - i\right)}{\prod_{i=0}^k (n-2i-3)} \\ & \left. + \frac{-1}{(n-1)} - a_0 \frac{y x^{n-2}}{(n-1)} \right]. \end{aligned} \tag{2-4}$$

The following lemma provides information about how an affine change of coordinates modifies the expression of a smooth vector field presenting a rigid center at the singularity.

Lemma 2.2. *Consider the PSDS (2-1). The following statements hold.*

- (a) *Performing the affine change of coordinates $(u, v) = (b_1x + b_2y + d_1, b_3x + b_4y + d_2)$, with $b_i \in \mathbb{R}$, $d_j \in \mathbb{R}$, for $i = 1, 2, 3, 4$; $j = 1, 2$ to the smooth vector fields $X_1^-(x, y)$, the center located at the point (d_1, d_2) remains rigid if and only if $b_1 = b_4$ and $b_2 = -b_3$.*
- (b) *Performing the affine change of coordinates $(u, v) = (c_1x + c_2y + M_1, c_3x + c_4y + M_2)$, with $c_i \in \mathbb{R}$, $M_j \in \mathbb{R}$, for $i = 1, 2, 3, 4$; $j = 1, 2$, to the smooth vector fields $X_n^+(x, y)$, the center located at the point (M_1, M_2) remains rigid if and only if $c_4 = c_1$ and $c_3 = -c_2$.*

By [12] and Lemma 2.2, after the affine change of variables $(u, v) = (b_1x + b_2y + d_1, b_3x + b_4y + d_2)$, with $b_i \in \mathbb{R}$, $d_j \in \mathbb{R}$, for $i = 1, 2, 3, 4$; $j = 1, 2$,

and $(u, v) = (c_1x + c_2y + M_1, c_3x + c_4y + M_2)$, with $c_i \in \mathbb{R}$, $M_j \in \mathbb{R}$, for $i = 1, 2, 3, 4$; $j = 1, 2$, the expression of the PSDS (2-1) takes the form

$$X_1^-(u, v) = (-v + d_2, u - d_1), \quad (2-5)$$

and

$$X_n^+(u, v) = (X_n^{+,1}(u, v), X_n^{+,2}(u, v)), \quad (2-6)$$

with $X_n^{+,1}(u, v) = M_2 - v + (c_1^2 + c_2^2)^{1-n}(u - M_1) \cdot S$ and $X_n^{+,2}(u, v) = u - M_1 + (c_1^2 + c_2^2)^{1-n}(v - M_2) \cdot S$, where $S = \sum_{i=0}^{n-1} a_i(c_2(u - M_1) + c_1(v - M_2))^i(c_1(u - M_1) - c_2(v - M_2))^{n-i-1}$. The first integral of system (2-5) is given by $F_1(u, v) = \sqrt{(u - d_1)^2 + (v - d_2)^2}$, or, in polar coordinates, $F_1(r, \theta) = \sqrt{(r \cos \theta - d_1)^2 + (r \sin \theta - d_2)^2}$. Similarly, the first integral of system (2-6) is given by equation (2-4) evaluated at the change of variables, that is,

$$H_n \left(\frac{c_1(u - M_1) + c_2(M_2 - v)}{c_1^2 + c_2^2}, \frac{c_1(v - M_2) - c_2(M_1 - u)}{c_1^2 + c_2^2} \right).$$

Note that H_n can be written as $F_2(u, v) = \frac{f_1(u, v)}{g_1(u, v)}$, where $g_1(u, v) = [(u - M_1)^2 + (v - M_2)^2]^{\frac{n-1}{2}}$ is a function involving a root, and $f_1(u, v) = F_2(u, v)g_1(u, v)$ is a polynomial of degree $n - 1$ given by $f_1(u, v) = \beta_{00} + \beta_{10}u + \beta_{01}v + \beta_{20}u^2 + \beta_{11}uv + \beta_{02}v^2 + \dots + \beta_{(n-2)0}u^{n-2} + \beta_{(n-3)1}u^{n-3}v + \dots + \beta_{1(n-3)}uv^{n-3} + \beta_{0(n-2)}v^{n-2} + \beta_{(n-1)0}u^{n-1} + \beta_{(n-2)1}u^{n-2}v + \beta_{(n-3)2}u^{n-3}v^2 + \dots + \beta_{1(n-2)}uv^{n-2} + \beta_{0(n-1)}v^{n-1}$, with the real coefficients $\beta_{00}, \beta_{10}, \dots, \beta_{1(n-2)}, \beta_{0(n-1)}$ depending on the parameters c_1, c_2, M_1, M_2 and a_{ij} , with $0 \leq i, j \leq n - 1$. Thus, the limit cycles of the PSDS (2-1) are determined by the solutions of the following closing equations

$$\begin{cases} F_1(0, y_0) = F_1(0, y_1), \\ F_2(0, y_0) = F_2(0, y_1). \end{cases} \quad (2-7)$$

From the first equation of (2-7), we have $\sqrt{(-d_1)^2 + (y_0 - d_2)^2} = \sqrt{(-d_1)^2 + (y_1 - d_2)^2}$, which implies $y_1 = -y_0 + 2d_2$ for every $y_0, y_1 \in \mathbb{R}$ with $y_0 \neq y_1$. Now, from the second equation of (2-7), we obtain $0 = F_2(0, y_0) - F_2(0, y_1) = f_1(0, y_0)g_1(0, y_1) - g_1(0, y_0)f_1(0, y_1)$. Substituting the expression of y_1 in the equation above, we obtain the closing function $M(y_0)$ given by

$$\begin{aligned} M(y_0) &= f_1(0, y_0)g_1(0, -y_0 + 2d_2) - g_1(0, y_0)f_1(0, -y_0 + 2d_2) \\ &= [\alpha_0 + \alpha_1y_0 + \alpha_2y_0^2 + \dots + \alpha_{n-2}y_0^{n-2} + \alpha_{n-1}y_0^{n-1}] \gamma_2(y_0) \\ &\quad + [\alpha_0 + \alpha_1(-y_0 + 2d_2) + \alpha_2(-y_0 + 2d_2)^2 + \dots + \end{aligned} \quad (2-8)$$

$$\begin{aligned}
& +\alpha_{n-2}(-y_0 + 2d_2)^{n-2} + \alpha_{n-1}(-y_0 + 2d_2)^{n-1}] \gamma_1(y_0) \\
& = [\alpha_0 + \alpha_1 y_0 + \alpha_2 y_0^2 + \cdots + \alpha_{n-2} y_0^{n-2} + \alpha_{n-1} y_0^{n-1}] \gamma_2(y_0) \\
& + [\bar{\alpha}_0 + \bar{\alpha}_1 y_0 + \bar{\alpha}_2 y_0^2 + \cdots + \bar{\alpha}_{n-2} y_0^{n-2} + \bar{\alpha}_{n-1} y_0^{n-1}] \gamma_1(y_0) = 0,
\end{aligned}$$

where $\gamma_1(y_0) = g_1(0, y_0) = (M_1^2 + (y_0 - M_2)^2)^{\frac{n-1}{2}}$, $\gamma_2(y_0) = g_1(0, -y_0 + 2d_2) = (M_1^2 + (-y_0 + 2d_2 - M_2)^2)^{\frac{n-1}{2}}$, $\alpha_i = \beta_{0i}$ and $\bar{\alpha}_j = \sum_{i=j}^{n-1} \alpha_i \binom{i}{j} (2d_2)^{i-j} (-1)^j$; for $j = 0, 1, \dots, n-1$. If g_1 is not a polynomial function, then equation (2-8) is not polynomial. This fact differs from the odd-degree case studied in [12], where g_1 was a polynomial function and consequently equation (2-8) was also polynomial, so that its zeros could be determined directly. In contrast, in the even-degree case a different analysis is required. In particular, from equation (2-8), we have that the function $f_1(0, y_0) = \alpha_0 + \alpha_1 y_0 + \alpha_2 y_0^2 + \cdots + \alpha_{n-2} y_0^{n-2} + \alpha_{n-1} y_0^{n-1}$ is a polynomial of degree $n-1$ in the variable y_0 , which contains all monomials of degree q with $0 \leq q \leq n-1$. The same holds for $f_1(0, -y_0 + 2d_2) = \bar{\alpha}_0 + \bar{\alpha}_1 y_0 + \bar{\alpha}_2 y_0^2 + \cdots + \bar{\alpha}_{n-2} y_0^{n-2} + \bar{\alpha}_{n-1} y_0^{n-1}$. Therefore, we get that:

- (a) The product $f_1(0, y_0)g_1(0, -y_0 + 2d_2)$ is composed of n monomials times the rational power $\gamma_2(y_0)$,
- (b) The product $g_1(0, y_0)f_1(0, -y_0 + 2d_2)$ is composed of n monomials times the rational power $\gamma_1(y_0)$.

In this way, the function $M(y_0)$ given in (2-8) is generated by the set of functions

$$\mathcal{B}_n = \{\gamma_2(y_0), \gamma_2(y_0)y_0, \dots, \gamma_2(y_0)y_0^{n-1}, \gamma_1(y_0), \gamma_1(y_0)y_0, \dots, \gamma_1(y_0)y_0^{n-1}\}.$$

Therefore the problem of solving the system (2-8) is reduced to the number of zeros that the elements of the space generated by the linear combination of functions in $\mathcal{B}_n$ can have in a given interval. In order to estimate the number of zeros of any linear combination of elements of $\mathcal{B}_n$ we employ the theory of Chebyshev systems.

2.2 Proof of Theorem 2.1

2.2.1 Proof of Item (a) of Theorem 2.1

Assume $d_2 \neq M_2$ and $d_2 = 0$. Therefore, $M_2 \neq 0$. Thus, the functions $M(y_0)$ and $\gamma_2(y_0)$ can be rewritten

$$M(y_0) = \sum_{j=0}^{n-1} (\gamma_2(y_0) + \gamma_1(y_0)(-1)^{j+1}) \alpha_j y_0^j$$

and $\gamma_2(y_0) = (M_1^2 + (y_0 + M_2)^2)^{\frac{n-1}{2}}$, while $\gamma_1(y_0)$ remains unchanged. Since $M_1 \neq 0$, we have $\gamma_1(y_0) > 0$ and $\gamma_2(y_0) > 0$ for all $y_0 \in \mathbb{R}$. Consequently, the set $\mathcal{B}_n$ becomes $\mathcal{B}_n = \{\gamma_2(y_0) - \gamma_1(y_0), (\gamma_2(y_0) + \gamma_1(y_0))y_0, (\gamma_2(y_0) - \gamma_1(y_0))y_0^2, \dots, (\gamma_2(y_0) + \gamma_1(y_0))y_0^{n-1}\}$. In the following we prove that $\mathcal{B}_n$ is an ECT-system for $n = 4$. In fact, we have that $\mathcal{B}_4$ is given by $\mathcal{B}_4 = \{\gamma_2(y_0) - \gamma_1(y_0), (\gamma_2(y_0) + \gamma_1(y_0))y_0, (\gamma_2(y_0) - \gamma_1(y_0))y_0^2, (\gamma_2(y_0) + \gamma_1(y_0))y_0^3\}$, where $\gamma_1(y_0) = (M_1^2 + (y_0 - M_2)^2)^{\frac{3}{2}}$ and $\gamma_2(y_0) = (M_1^2 + (y_0 + M_2)^2)^{\frac{3}{2}}$. We denote by $f_0(y_0) = \gamma_2(y_0) - \gamma_1(y_0)$, $f_1(y_0) = (\gamma_2(y_0) + \gamma_1(y_0))y_0$, $f_2(y_0) = (\gamma_2(y_0) - \gamma_1(y_0))y_0^2$ and $f_3(y_0) = (\gamma_2(y_0) + \gamma_1(y_0))y_0^3$. The Wronskians are computed as follows

$$\begin{aligned} W(f_0)(y_0) &= \gamma_2(y_0) - \gamma_1(y_0) = 0 \Leftrightarrow \\ &(M_1^2 + (y_0 + M_2)^2)^{\frac{3}{2}} = (M_1^2 + (y_0 - M_2)^2)^{\frac{3}{2}} \Leftrightarrow \\ &M_1^2 + (y_0 + M_2)^2 = M_1^2 + (y_0 - M_2)^2 \Leftrightarrow \\ &(y_0 + M_2)^2 = (y_0 - M_2)^2 \Leftrightarrow \\ &y_0^2 + 2y_0M_2 + M_2^2 = y_0^2 - 2y_0M_2 + M_2^2 \Leftrightarrow \\ &4y_0M_2 = 0. \end{aligned}$$

Since by hypothesis $M_2 \neq 0$, $4y_0M_2 = 0$ if and only if $y_0 = 0$. Hence, to obtain $W(f_0)(y_0) \neq 0$, it follows that $y_0 \neq 0$. The second Wronskian is given by $W(f_0, f_1)(y_0) = 4y_0M_2W_2(y_0)$, where $W_2(y_0) = 3y_0^4 + 3(M_1^2 + M_2^2)^2 + 2y_0^2(3M_1^2 + 5M_2^2) + 3(\gamma_1(y_0))^{1/3}(\gamma_2(y_0))^{1/3}(y_0^2 - M_1^2 - M_2^2)$, and

$$(\gamma_1(y_0))^{1/3} = (M_1^2 + (y_0 - M_2)^2)^{1/2}, \quad (\gamma_2(y_0))^{1/3} = (M_1^2 + (y_0 + M_2)^2)^{1/2}. \quad (2-9)$$

Since $M_2 \neq 0$, $W(f_0, f_1)(y_0) = 0$ if and only if $y_0 = 0$ or $W_2(y_0) = 0$. Note that

$$\begin{aligned} W_2(y_0) &= 0 \Leftrightarrow 3y_0^4 + 3(M_1^2 + M_2^2)^2 + 2y_0^2(3M_1^2 + 5M_2^2) \\ &= -3(\gamma_1(y_0))^{1/3}(\gamma_2(y_0))^{1/3}(y_0^2 - M_1^2 - M_2^2). \end{aligned} \quad (2-10)$$

Squaring both sides of (2-10) and considering (2-9), we obtain:

$$\begin{aligned} (3y_0^4 + 3(M_1^2 + M_2^2)^2 + 2y_0^2(3M_1^2 + 5M_2^2))^2 &= 9(M_1^2 + (y_0 - M_2)^2)((y_0 + M_2)^2 \\ &+ M_1^2)(y_0^2 - M_1^2 - M_2^2)^2 \Leftrightarrow W_{21}(y_0) = W_{22}(y_0) \end{aligned} \quad (2-11)$$

where $W_{21}(y_0) = 9y_0^8 + 9(M_1^2 + M_2^2)^4 + 12y_0^2(M_1^2 + M_2^2)^2(3M_1^2 + 5M_2^2) + y_0^6(36M_1^2 + 60M_2^2) + 2y_0^4(27M_1^4 + 78M_1^2M_2^2 + 59M_2^4)$, and $W_{22}(y_0) = 9y_0^8 - 36y_0^6M_2^2 - 36y_0^2M_2^2(M_1^2 + M_2^2)^2 + 9(M_1^2 + M_2^2)^4 - 18y_0^4(M_1^4 - 2M_1^2M_2^2 - 3M_2^4)$. After subtracting $W_{22}(y_0)$ from $W_{21}(y_0)$ and simplifying, we find that the degree-8 terms cancel and the constant

term is zero. The result is a degree-6 polynomial of the form:

$$P_{26}(y_0) = C_{22}y_0^2 + C_{24}y_0^4 + C_{26}y_0^6,$$

where $C_{22} = 12(M_1^2 + M_2^2)^2(3M_1^2 + 8M_2^2)$, $C_{24} = 8(9M_1^4 + 15M_1^2M_2^2 + 8M_2^4)$, and $C_{26} = 36M_1^2 + 96M_2^2$. Provided that $M_1 \neq 0$, $M_2 \neq 0$, all coefficients C_{22}, C_{24}, C_{26} are strictly positive. Consequently, $P_{26}(y_0) > 0$ for all $y_0 \neq 0$, and $P_{26}(0) = 0$. Substituting $y_0 = 0$ into the original equation (2-10) yields $W_2(0) = 0$. Moreover, for $y_0 \neq 0$, $P_{26}(y_0) > 0$. Therefore, equation (2-11) has no nontrivial solution. Since any solution of (2-10) must satisfy (2-11), it follows that (2-10) has no solutions for $y_0 \neq 0$. Hence, $W_2(y_0) \neq 0$ for $y_0 \neq 0$, $M_1 \neq 0$ and $M_2 \neq 0$. Consequently, $W(f_0, f_1)(y_0) \neq 0$ under these conditions.

The Wronskian $W(f_0, f_1, f_2)(y_0)$ satisfies

$$W(f_0, f_1, f_2)(y_0) = \frac{y_0^3 M_2 W_3(y_0)}{(\gamma_1(y_0))^{1/3} (\gamma_2(y_0))^{1/3}},$$

where $W_3(y_0) = (\gamma_2(y_0))^{1/3}(y_0^4(-3M_1^2 - 8M_2^2) + y_0^3(3M_1^2M_2 + 8M_2^3) + y_0^2(-6M_1^4 - 4M_1^2M_2^2) + y_0M_2(M_1^2 + M_2^2)(3M_1^2 + 8M_2^2) - (M_1^2 + M_2^2)^2(3M_1^2 + 8M_2^2)) + (\gamma_1(y_0))^{1/3}(y_0^4(3M_1^2 + 8M_2^2) + y_0^3(3M_1^2M_2 + 8M_2^3) + y_0^2(6M_1^4 + 4M_1^2M_2^2) + y_0M_2(M_1^2 + M_2^2)(3M_1^2 + 8M_2^2) + (M_1^2 + M_2^2)^2(3M_1^2 + 8M_2^2))$. Note that $W(f_0, f_1, f_2)(y_0) = 0$ if and only if $y_0 = 0$ or $W_3(y_0) = 0$. We denote by $W_{31}(y_0) = (\gamma_2(y_0))^{1/3}(y_0^4(-3M_1^2 - 8M_2^2) + y_0^3(3M_1^2M_2 + 8M_2^3) + y_0^2(-6M_1^4 - 4M_1^2M_2^2) + y_0M_2(M_1^2 + M_2^2)(3M_1^2 + 8M_2^2) - (M_1^2 + M_2^2)^2(3M_1^2 + 8M_2^2))$, $W_{32}(y_0) = (\gamma_1(y_0))^{1/3}(y_0^4(3M_1^2 + 8M_2^2) + y_0^3(3M_1^2M_2 + 8M_2^3) + y_0^2(6M_1^4 + 4M_1^2M_2^2) + y_0M_2(M_1^2 + M_2^2)(3M_1^2 + 8M_2^2) + (M_1^2 + M_2^2)^2(3M_1^2 + 8M_2^2))$. Then, we have that

$$\begin{aligned} W_3(y_0) = 0 &\Leftrightarrow W_{31}(y_0) = W_{32}(y_0) \Leftrightarrow \\ &(W_{31}(y_0))^2 = (W_{32}(y_0))^2 \Leftrightarrow \\ &(W_{31}(y_0))^2 - (W_{32}(y_0))^2 = 0. \end{aligned} \quad (2-12)$$

From the last expression in (2-12), we obtain a degree-10 polynomial in y_0 , and, after simplification, it reduces to $-4y_0^3M_2^3P_{34}(y_0) = 0$, where $P_{34}(y_0) = C_{34}y_0^4 + C_{32}y_0^2 + C_{30}$, with $C_{34} = (-45M_1^4 - 144M_1^2M_2^2 - 64M_2^4)$, $C_{32} = -90M_1^6 - 54M_1^4M_2^2 + 160M_1^2M_2^4 + 128M_2^6$, and $C_{30} = -(M_1^2 + M_2^2)^2(45M_1^4 + 144M_1^2M_2^2 + 64M_2^4)$. Next, consider the change of variables $s = y_0^2$, so that $P_{34}(y_0)$ becomes $P_{34}(s) = C_{34}s^2 + C_{32}s + C_{30} = 0$ and the solutions of this equation are given by

$$s_{1,2} = \frac{-C_{32} \pm \sqrt{(C_{32})^2 - 4C_{34}C_{30}}}{2C_{34}}. \quad (2-13)$$

The discriminant of (2-13) is given by: $\Delta_1 = (C_{32})^2 - 4C_{34}C_{30} - M_1^2 M_2^2 (3M_1^2 + 4M_2^2)(9M_1^2 + 8M_2^2)^2 (15M_1^2 + 16M_2^2)$, and note that $\Delta_1 < 0$ for any nonzero values of M_1 and M_2 . Thus, $s_{1,2}$ are not real solutions of $P_{34}(s) = 0$, which implies that $P_{34}(y_0) = 0$ has no real roots. Therefore, the condition $-4y_0^3 M_2^3 P_{34}(y_0) = 0$ is satisfied if and only if $y_0 = 0$. Consequently, we conclude that $W(f_0, f_1, f_2)(y_0) \neq 0$ for $y_0 \neq 0$. Finally, consider $W(\mathcal{B}_4)(y_0)$. We have

$$W(\mathcal{B}_4)(y_0) = \frac{6144 y_0^6 M_2^4}{\gamma_1(y_0)\gamma_2(y_0)} W_4(y_0),$$

where $W_4(y_0) = y_0^6(15M_1^2 + 8M_2^2) + 3y_0^4(5M_1^4 - 7M_1^2 M_2^2 - 8M_2^4) - 3y_0^2(M_1^2 + M_2^2)(5M_1^4 - 7M_1^2 M_2^2 - 8M_2^4) - (M_1^2 + M_2^2)^3(15M_1^2 + 8M_2^2) + (\gamma_1(y_0))^{1/3}(\gamma_2(y_0))^{1/3}(y_0^4(15M_1^2 + 8M_2^2) - 2y_0^2(7M_1^2 M_2^2 + 8M_2^4) + (M_1^2 + M_2^2)^2(15M_1^2 + 8M_2^2))$ and $(\gamma_1(y_0))^{1/3}$ and $(\gamma_2(y_0))^{1/3}$ defined as in (2-9). Note that $W(\mathcal{B}_4)(y_0) = 0$ if and only if $y_0 = 0$ or $W_4(y_0) = 0$. Working in the same way as we have done in the previous case, we have

$$\begin{aligned} W_4(y_0) = 0 &\Leftrightarrow W_{41}(y_0) = W_{42}(y_0) \Leftrightarrow \\ (W_{41}(y_0))^2 &= (W_{42}(y_0))^2 \\ (W_{41}(y_0))^2 - (W_{42}(y_0))^2 &= 0, \end{aligned} \quad (2-14)$$

where, $W_{41}(y_0) = y_0^6(15M_1^2 + 8M_2^2) + 3y_0^4(5M_1^4 - 7M_1^2 M_2^2 - 8M_2^4) - 3y_0^2(M_1^2 + M_2^2)(5M_1^4 - 7M_1^2 M_2^2 - 8M_2^4) - (M_1^2 + M_2^2)^3(15M_1^2 + 8M_2^2)$, $W_{42}(y_0) = (\gamma_1(y_0))^{1/3}(\gamma_2(y_0))^{1/3}(y_0^4(15M_1^2 + 8M_2^2) - 2y_0^2(7M_1^2 M_2^2 + 8M_2^4) + (M_1^2 + M_2^2)^2(15M_1^2 + 8M_2^2))$. Solving (2-14) and simplifying, we obtain the expression $-4M_1^4 y_0^4 P_{44}(y_0) = 0$, where $P_{44}(y_0) = C_{44}y_0^4 + C_{42}y_0^2 + C_{40}$ with $C_{44} = 3(75M_1^4 + 160M_1^2 M_2^2 + 64M_2^4)$, $C_{42} = (450M_1^6 + 510M_1^4 M_2^2 - 320M_1^2 M_2^4 - 384M_2^6)$, and $C_{40} = 3(M_1^2 + M_2^2)^2(75M_1^4 + 160M_1^2 M_2^2 + 64M_2^4)$. Thus, consider the change of variable $s = y_0^2$, then $P_{44}(y_0)$ becomes $P_{44}(s) = C_{44}s^2 + C_{42}s + C_{40} = 0$. In this way, the solutions of $P_{44}(s) = 0$ are given by

$$s_{1,2} = \frac{-C_{42} \pm \sqrt{(C_{42})^2 - 4C_{44}C_{40}}}{2C_{44}},$$

whose discriminant is given by $\Delta_2 = -(810000M_1^{10}M_2^2 + 3225600M_1^8M_2^4 + 4807680M_1^6M_2^6 + 3178496M_1^4M_2^8 + 786432M_1^2M_2^{10})$. Note that $\Delta_2 < 0$ for all nonzero values of M_1 and M_2 ; therefore, $s_{1,2}$ are not real solutions of $P_{44}(s)$, which implies that $P_{44}(y_0) = 0$ has no real roots for $M_1 \neq 0$ and $M_2 \neq 0$. That is, $-4M_2^4 y_0^4 P_{44}(y_0) = 0$ if and only if $y_0 = 0$, concluding that for nonzero M_1 and M_2

we get $W(\mathcal{B}_4)(y_0) \neq 0$ since $y_0 \neq 0$. In summary, for $M_1 \neq 0$ and $M_2 \neq 0$, we have

$$W(f_0)(y_0) \neq 0, \quad W(f_0, f_1)(y_0) \neq 0, \quad W(f_0, f_1, f_2)(y_0) \neq 0, \quad \text{and} \quad W(\mathcal{B}_4)(y_0) \neq 0,$$

provided $y_0 \neq 0$. Therefore, by Theorem A, we conclude that the set $\mathcal{B}_4$ is an ECT-system on $\mathbb{R} \setminus \{0\}$. Additionally, the function M is given by

$$\begin{aligned} M(y_0) = & (\gamma_2(y_0) - \gamma_1(y_0))\alpha_0 + (\gamma_1(y_0) + \gamma_2(y_0))y_0\alpha_1 + (\gamma_2(y_0) - \gamma_1(y_0))y_0^2\alpha_2 \\ & - (\gamma_1(y_0) + \gamma_2(y_0))y_0^3\alpha_3, \end{aligned}$$

$$\begin{aligned} \text{where } \alpha_0 = & \frac{1}{3(c_1^2 + c_2^2)^{9/2}} \left[- (c_1^2 + c_2^2)^3 + a_2(c_2M_1 + c_1M_2)^3 - a_1(c_1M_1 - c_2M_2)^3 + \right. \\ & a_3(-3c_1c_2^2M_1^3 + 3c_1^2c_2M_2^3 + c_2^3M_2(3M_1^2 + 2M_2^2) - c_1^3(2M_1^3 + 3M_1M_2^2)) + a_0(3c_1^2c_2M_1^3 + \\ & 3c_1c_2^2M_2^3 + c_2^3M_1(2M_1^2 + 3M_2^2) + c_1^3(3M_1^2M_2 + 2M_2^3)) \left. \right], \alpha_1 = \frac{1}{(c_1^2 + c_2^2)^{9/2}} \left[a_0(-2c_2^3M_1M_2 - \right. \\ & 3c_1c_2^2M_2^2 - c_2^3(M_1^2 + 2M_2^2)) - a_1c_2(c_1M_1 - c_2M_2)^2 - a_2c_1(c_2M_1 + c_1M_2)^2 + a_3(2c_1^3M_1M_2 - \\ & 3c_1^2c_2M_2^2 - c_2^3(M_1^2 + 2M_2^2)) \left. \right], \alpha_2 = \frac{1}{(c_1^2 + c_2^2)^{9/2}} \left[a_0(c_2^3M_1 + 2c_1^3M_2 + 3c_1c_2^2M_2) + \right. \\ & a_1c_2^2(-c_1M_1 + c_2M_2) + a_2c_1^2(c_2M_1 + c_1M_2) + a_3(-c_1^3M_1 + 3c_1^2c_2M_2 + 2c_2^3M_2) \left. \right], \alpha_3 = \\ & \frac{a_0(-2c_1^3 - 3c_1c_2^2 - a_1c_2^3 - a_2c_1^3) + a_3(-3c_1^2c_2 - 2c_2^3)}{3(c_1^2 + c_2^2)^{9/2}}. \end{aligned}$$

Now consider the Jacobian matrix J of the coefficients α_j of $M(y_0)$ with respect to a_j , $j = 0, 1, 2, 3$, given by

$$J = \frac{1}{3(c_1^2 + c_2^2)^{9/2}} \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix},$$

where $e_{11} = 3c_1^2c_2M_1^3 + 3c_1c_2^2M_2^3 + c_2^3(2M_1^3 + 3M_1M_2^2) + c_1^3(3M_1^2M_2 + 2M_2^3)$, $e_{12} = -c_1^3M_1^3 + 3c_1^2c_2M_1^2M_2 - 3c_1c_2^2M_1M_2^2 + c_2^3M_2^3$, $e_{13} = c_2^3M_1^3 + 3c_1c_2^2M_1^2M_2 + 3c_1^2c_2M_1M_2^2 + c_1^3M_2^3$, $e_{14} = -3c_1c_2^2M_1^3 + 3c_1^2c_2M_2^3 - c_1^3(2M_1^3 + 3M_1M_2^2) + c_2^3(3M_1^2M_2 + 2M_2^3)$, $e_{21} = 3(-2c_2^3M_1M_2 - 3c_1c_2^2M_2^2 - c_1^3(M_1^2 + 2M_2^2))$, $e_{22} = -3c_2(c_1M_1 - c_2M_2)^2$, $e_{23} = 3(-c_1c_2^2M_1^2 - 2c_1^2c_2M_1M_2 - c_1^3M_2^2)$, $e_{24} = 3(-c_2^3M_1^2 + 2c_1^3M_1M_2 - 3c_1^2c_2M_2^2 - 2c_2^3M_2^2)$, $e_{31} = 3(c_2^3M_1 + 2c_1^3M_2 + 3c_1c_2^2M_2)$, $e_{32} = 3(-c_1c_2^2M_1 + c_2^3M_2)$, $e_{33} = 3(c_1^2(c_2M_1 + c_1M_2))$, $e_{34} = 3(-c_1^3M_1 + 3c_1^2c_2M_2 + 2c_2^3M_2)$, $e_{41} = -2c_1^3 - 3c_1c_2^2$, $e_{42} = -c_2^3$, $e_{43} = -c_1^3$, $e_{44} = -c_2(3c_1^2 + 2c_2^2)$. The determinant of the matrix J is given by $\det(J) = \frac{M_1^6}{9(c_1^2 + c_2^2)^{12}}$. Since by hypothesis $c_1^2 + c_2^2 \neq 0$ and $M_1 \neq 0$, it follows that $\det(J) \neq 0$. Moreover, this matrix is a 4×4 matrix of full rank. This demonstrates

that the coefficients α_j are linearly independent with respect to the parameters a_j for $j = 0, 1, 2, 3$.

In conclusion, since the set $\mathcal{B}_4$ is an ECT-system on $\mathbb{R} \setminus \{0\}$ and $M(y_0)$ is expressed as a linear combination of the elements of $\mathcal{B}_4$ with coefficients that are linearly independent with respect to the parameters a_j , we can conclude that $M(y_0)$ has at most 3 zeros. Moreover, there exist suitable choices of the parameters such that the upper bound is achieved.

However, $y_0 = 0$ is always a solution of $M(y_0)$ and this trivial solution does not correspond to a limit cycle. Therefore, $M(y_0)$ has at most 2 zeros. Note that if y_0 is a solution of M , then $y_1 = -y_0$ is also a solution of M . Moreover, each pair $(y_0, y_1 = -y_0)$ determines a limit cycle. Consequently, the PSDS $X_{1,4}$ can have at most 1 limit cycle and this upper bound is reached; see Example 2.4.

2.2.2 Proof of Item (b) of Theorem 2.1

Assume $d_2 = M_2$ with $M_1 \neq 0$. In this case $\gamma_1(y_0) = \gamma_2(y_0) \neq 0$, and consequently the function M is given by

$$M(y_0) = \gamma_1(y_0) \sum_{j=0}^{n-1} \omega_j y_0^j, \quad (2-15)$$

where $\omega_j = \left(\alpha_j + \sum_{i=j}^{n-1} \alpha_i \binom{i}{j} (2M_2)^{i-j} (-1)^{j+1} \right)$. Moreover, the set $\mathcal{B}_n$ becomes $\mathcal{B}_n = \gamma_1(y_0) \{1, y_0, y_0^2, \dots, y_0^{n-1}\}$, with $\gamma_1(y_0) = (M_1^2 + (y_0 - M_2)^2)^{\frac{n-1}{2}} > 0$. We claim that $\mathcal{B}_n$ is an ECT-system. To prove this, we apply Theorem A, which requires verifying that: under the above conditions, $\gamma_1(y_0)$ is smooth and non-vanishing on $I = \mathbb{R}$ and requires showing that the Wronskian $W(\mathcal{B}_k)(y_0) \neq 0$ for each $0 \leq k \leq n-1$ and all $y_0 \in I$. We proceed by induction on k . For $k = 2$:

$$W(1, y_0) = \begin{vmatrix} 1 & y_0 \\ 0 & 1 \end{vmatrix} \neq 0.$$

Assume that $W(1, y_0, \dots, y_0^{n-2}) \neq 0$ for $k = n-2$. This is always possible because this Wronskian equals the determinant of a triangular matrix, which is the product

of its diagonal entries, all nonzero. Then, for $k = n - 1$, we have

$$\begin{aligned}
 W(1, y_0, \dots, y_0^{n-1}) &= \begin{vmatrix} 1 & y_0 & \dots & y_0^{n-2} & y_0^{n-1} \\ 0 & 1 & \dots & (n-2)y_0^{n-3} & (n-1)y_0^{n-2} \\ 0 & 0 & \dots & (n-3)(n-2)y_0^{n-4} & (n-2)(n-1)y_0^{n-3} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & (n-2)! & (n-1)!y_0 \\ 0 & 0 & \dots & 0 & (n-1)! \end{vmatrix} \\
 &= W(1, y_0, \dots, y_0^{n-2})(n-1)! \neq 0,
 \end{aligned}$$

by the induction hypothesis. Therefore, by Theorem A, the ordered set of functions $\mathcal{B}_n$ is an ECT-system on I . Since M can be expressed as a linear combination of the elements of $\mathcal{B}_n$, we conclude that M has at most $n - 1$ zeros. However, similarly to the previous case, $y_0 = d_2$ is always a zero of $M(y_0)$, which does not correspond to a limit cycle. Hence, M has at most $n - 2$ zeros. We conclude that $X_{1,n}$ can have at most $\frac{n-2}{2}$ limit cycles for even n .

In the following we analyze the linear independence of the coefficients ω_j in terms of the original coefficients $a_j, j = 0, \dots, n - 1$. In fact, this is an important aspect for the rigorous application of Chebyshev theory to estimate the number of zeros of the closing function $M(y_0)$ is to establish that the perturbation parameters a_j , with $j = 0, \dots, n - 1$, produce linearly independent effects on the closing function coefficients. In what follows, we analyze this property. Note that the coefficients of the closing function given by (2-15) ω_j depend linearly on the perturbation parameters $a = (a_0, \dots, a_{n-1})$ and non-linearly on the geometric parameters $p = (c_1, c_2, d_2, M_1, M_2)$. Let $\text{Coef}_l(a, p)$, for $l = 1, \dots, n$, denotes the l -th coefficient of $M(y_0)$ as a linear combination of the n functions in $\mathcal{B}_n^* = \bigcup_{j=0}^{n-1} y_0^j$. Note that these coefficients are $\{\omega_0, \dots, \omega_{n-1}\}$. We define the $n \times n$ Jacobian matrix $A(p)$ given by

$$A(p) = (A_{lj}(p)) = \left(\frac{\partial \text{Coef}_l}{\partial a_j} \right)$$

that associates the parameter space $\mathbb{R}^n$ to the coefficient space $\mathbb{R}^n$. The effects of the parameters a_j on the closing function are linearly independent if and only if the rank of this Jacobian matrix is maximal, i.e., $\text{Rank}(A(p)) = n$, for geometric parameters p belonging to an open subset of the parameter space.

However, note that for each pair of zeros of the closing function $M(y_0)$ we have only one zero of the displacement map. Therefore, our purpose now is to prove that the $\text{Rank}(A(p)) = \frac{n}{2}$.

To justify this claim, we select the geometric parameter p_0 defined by $c_1 = 1$, $c_2 = 0$, $M_1 = 1$, $M_2 = 0$, and $d_2 = 0$. Evaluating the polynomial $f_1(0, y)$ under this choice of parameters yields the following expression:

$$\begin{aligned}
f_1(0, y) &= -\frac{1 + (-1)^n y a_0}{n-1} - \frac{a_0}{n-1} \sum_{k=0}^{\frac{n-4}{2}} \frac{2^{k+1} \left(\prod_{i=0}^k \left(\frac{n-2}{2} - i \right) \right) (-1)^{n-2k} y (1+y^2)^{1+k}}{\prod_{i=0}^k (n-2i-3)} \\
&+ \sum_{j=1}^{\frac{n-2}{2}} \frac{a_{2j-1}}{n-1} \left(\sum_{k=1}^{j-1} \frac{2^k \left(\prod_{i=1}^k (j-i) \right) (-1)^{n+1-2j} y^{2j-2k-2} (1+y^2)^k}{\prod_{i=1}^k (n-1-2i)} \right) \\
&+ \sum_{j=1}^{\frac{n-2}{2}} \frac{a_{2j-1} (-1)^{n+1-2j} y^{2j-2}}{n-1} + \frac{a_{n-1}}{n-1} (-y^{n-2}) \\
&- \sum_{j=1}^{\frac{n-2}{2}} \frac{a_{2j}}{n-1} \left(\sum_{k=1}^{\frac{n-2j-2}{2}} \frac{2^k \left(\prod_{i=0}^{k-1} \left(\frac{n-2j-2}{2} - i \right) \right) (-1)^{n-2(j+k)} y^{1+2j} (1+y^2)^k}{\prod_{i=1}^k (n-2i-1)} \right) \\
&+ \frac{a_{n-1}}{n-1} \sum_{k=0}^{\frac{n-4}{2}} \frac{2^{k+1} \left(\prod_{i=0}^k \left(\frac{n-2}{2} - i \right) \right) (-y^{n-4-2k}) (1+y^2)^{1+k}}{\prod_{i=0}^k (n-2i-3)} \\
&- \sum_{j=1}^{\frac{n-2}{2}} \frac{a_{2j} (-1)^{n-2j} y^{1+2j}}{n-1}.
\end{aligned} \tag{2-16}$$

Thus, from (2-16) we obtain the expression for $M(y_0) = F_2(0, y_0) - F_2(0, y_1) = [f_1(0, y_0) - f_1(0, -y_0 + 2d_2)]\gamma_1(y_0)$ in terms of the original parameters a_j ($j = 0, 1, \dots, n-1$):

$$\begin{aligned}
M(y_0) &= y_0 \left[\frac{2a_0}{n-1} + \frac{a_0}{n-1} \sum_{k=0}^{\frac{n-4}{2}} \frac{2^{k+2} \prod_{i=0}^k \left(\frac{n-2}{2} - i \right) \sum_{r=0}^{1+k} \binom{1+k}{r} y_0^{2r}}{\prod_{i=0}^k (n-2i-3)} + \sum_{j=1}^{\frac{n-2}{2}} \frac{2y_0^{2j} a_{2j}}{n-1} \right. \\
&+ \left. \sum_{j=1}^{\frac{n-2}{2}} \frac{a_{2j}}{n-1} \sum_{k=1}^{\frac{n-2j-2}{2}} \frac{2^{k+1} \prod_{i=0}^{k-1} \left(\frac{n-2j-2}{2} - i \right) (-1)^{n-2j-2k-2} \sum_{l=0}^k \binom{k}{l} y_0^{2(l+j)}}{\prod_{i=1}^k (n-2i-1)} \right] \\
&\times \gamma_1(y_0) = 0,
\end{aligned} \tag{2-17}$$

where, for this choice of parameters, $\gamma_1(y_0) = (1 + y_0^2)^{\frac{n-1}{2}}$. Excluding the trivial

solution $y_0 = 0$, a detailed structural analysis of (2-17) reveals which powers of y_0 are affected by each a_j . Consequently, the dependence of the coefficients ω_k on the parameters a_j leads to the conclusions summarized in Table 2.1. The analysis

Parameter a_k	Type of y_0 Power Affected	Range of y_0 Powers Affected
a_0	odd	$y^0, y^2, \dots, y^{n-2}$
a_1	none	- - -
a_2	odd	$y^2, y^4, \dots, y^{n-2}$
a_3	none	- - -
a_4	odd	$y^4, y^6, \dots, y^{n-2}$
$\vdots$	$\vdots$	$\vdots$
a_{n-2} (last even)	even	only y^{n-2}
a_{n-1} (last odd)	none	- - -

Table 2.1: Influence of perturbation parameters a_k on the powers of y_0 in $M(y_0)$ for even n .

reveals that the parameters a_j with even j only affect coefficients ω_k with k even, while parameters a_j with odd j do not affect any power, since they do not appear in the expression for $M(y_0)$, that is, the odd powers of y_0 vanish for this choice of parameters, possessing a triangular structure given by the minimum power affected by each a_j . Note that after eliminating the trivial solution in the equation for $M(y_0)$, the dimension of the matrix A reduces to $(n-1) \times n$. Furthermore, the matrix A can be rearranged, through permutations of rows and columns, into a block diagonal form,

$$A = \begin{pmatrix} A_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix},$$

where A_1 is square block of order $n/2 \times n/2$. Furthermore, the structure detailed in Table 2.1 regarding the minimum power of y_0 affected by each a_j ensures that the matrix A_1 is a triangular matrix with non-zero diagonal entries. Since the maximal rank $n/2$ is achieved for a specific non-degenerate parameter choice in $A = A(p_0)$, it holds for $A(p)$, on an open set of the parameter space.

We conclude that the parameters $a_0, \dots, a_{n-1}$ exert linearly independent effects on the $n/2$ coefficients of the displacement function for any even n . This ensures that the system possesses $n/2$ degrees of freedom given by the perturbation parameters. For the representative case $n = 4$, we can explicitly compute the Jacobian matrix $A(p_0)$ of order 3×4 given by

$$A(p_0) = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{pmatrix}.$$

It is straightforward to verify directly that the rank of $A(p_0)$ is two. However, following the general argument, we examine the 2×2 submatrix A_1 . Rearranging its rows and columns, we obtain the block triangular matrix

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This confirms the block structure given by the previous analysis where diagonal block is triangular with non-zero determinants ($\det(A_1) = 3$). Therefore $\text{Rank}(A_1) = 2$, and consequently, $\text{Rank}(A(p_0)) = 2$.

Remark 2.3. *Although the preceding analysis was carried out for a specific choice of geometric parameters, the result holds for the general $n \times n$ case (or equivalently, for the $(n-1) \times n$ matrix A), as illustrated by the following example with $n = 4$. The general $n-1 \times n$ matrix $A(p)$ for this case is explicitly given by:*

$$A(p) = \begin{pmatrix} c_1(3c_1^2M_1^2 + 2c_1^2M_2^2 + 3c_2^2M_2^2) & c_2(3c_1^2M_1^2 + c_2^2M_2^2) & c_1(3c_2^2M_1^2 + c_1^2M_2^2) & c_2(3c_2^2M_1^2 + 3c_1^2M_2^2 + 2c_2^2M_2^2) \\ -2c_1(2c_1^2 + 3c_2^2)M_2 & -2c_2^3M_2 & -2c_1^3M_2 & -2c_2(3c_1^2 + 2c_2^2)M_2 \\ c_1(2c_1^2 + 3c_2^2) & c_2^3 & c_1^3 & c_2(3c_1^2 + 2c_2^2) \end{pmatrix}.$$

A direct computation yields that the rank of the matrix $A(p)$ is 2. In fact, note that the second line is equal the product of the third line by $-2M_2$. In this way, we conclude that two of the coefficients ω_j ($j = 0, 1, 2$) are linearly independent in relation to the parameters a_0, a_1, a_2, a_3 for $n = 4$.

2.3 Some examples

We present two numerical examples for each case of Theorem 2.1 for $n = 4$.

Example 2.4. *Consider the PSDS*

$$X_{1,4}(x, y) = \begin{cases} X_1^-(x, y) = (-y, x + 2), & x \leq 0, \\ X_4^+(x, y) = (X_4^{+,1}(x, y), X_4^{+,2}(x, y)), & x \geq 0, \end{cases} \quad (2-18)$$

where

$$X_4^{+,1}(x, y) = -y + 5 - \frac{(x+5)(3x+2y+5)(19x^2 - 5x(y-43) + y(7y-95) + 775)}{17576},$$

and

$$X_4^{+,2}(x, y) = x + 5 - \frac{(y-5)(3x+2y+5)(19x^2 - 5x(y-43) + y(7y-95) + 775)}{17576}.$$

System (2-18) corresponds to system (2-6) with the parameter choices $d_1 = -2$, $d_2 = 0$, $c_1 = 2$, $c_2 = -10$, $a_0 = -1$, $a_1 = 1$, $a_2 = -3$, $a_3 = 3$, $M_1 = -5$, and $M_2 = 5$. By solving the first equation of the respective system (2-7), we obtain $y_1 = -y_0$. Furthermore, the function

$$F_2(x, y) = \frac{1}{10967424\sqrt{26}(x(x+10) + (y-10)y + 50)^{3/2}} \left(-3(14x + 695)y^2 + 3(x(57x + 710) + 5250)y - 3(x(x(17x + 540) + 4475) + 22067) + 125y^3 \right)$$

is the first integral of X_4^+ . Thus, the solutions of equation $F_2(0, y_0) = F_2(0, -y_0)$, that is, the zeros of the aforementioned function $M(y_0)$, are given by $y_0^1 = 0$, $y_0^2 = -6.00236$, and $y_0^3 = 6.00236$. Note that y_0^1 is the tangency point of X_1^- with the separation line. See Figure 2.1. The other two solutions correspond to the crossing limit cycle intersecting the separation line at these points.

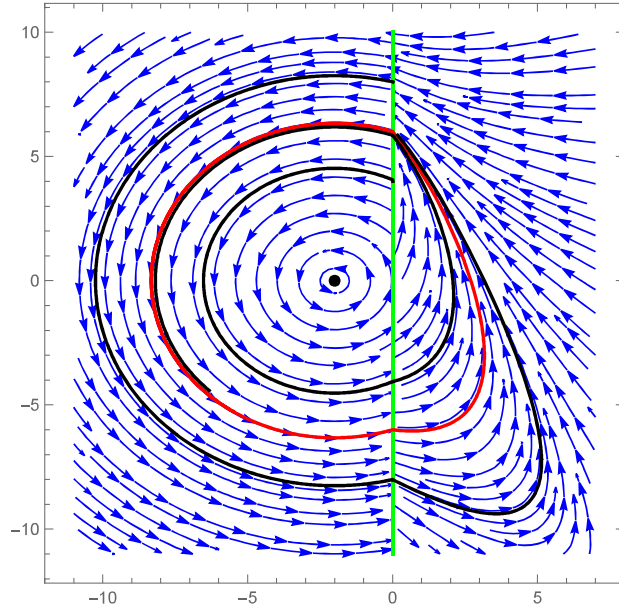


Figure 2.1: The limit cycle of the PSDS (2-18), where $d_2 = 0$.

Unlike Item (a) of Theorem 2.1, if $d_2 \neq 0$ there exists PSDS with more than one limit cycle. In the following example we present another choice of parameters a and p such that the PSDS presents three limit cycles.

Example 2.5. Consider the PSDS

$$X_{1,4}(x, y) = \begin{cases} X_1^-(x, y) = (-y + 2, x + 2), & x \leq 0, \\ X_4^+(x, y) = (X_4^{+,1}(x, y), X_4^{+,2}(x, y)), & x \geq 0, \end{cases}$$

where

$$X_4^{+,1}(x, y) = -y + \frac{1}{121945} \left(2(-4+x)(8629(-4+x)^3 + 1976(-4+x)^2y - 13359(-4+x)y^2 + 37254y^3) \right),$$

and

$$X_4^{+,2}(x, y) = \frac{1}{121945} \left(121945(-4+x) + 17258(-4+x)^3y + 3952(-4+x)^2y^2 - 26718(-4+x)y^3 + 74508y^4 \right).$$

Observe that the above system corresponds to system (2-6) with $d_1 = -2$, $d_2 = 2$, $c_1 = -\frac{7}{2}$, $c_2 = \frac{3}{2}$, $a_0 = -7$, $a_1 = -\frac{1}{5}$, $a_2 = -30$, $a_3 = -30$, $M_1 = 4$, and $M_2 = 0$. By solving the first equation of the respective system (2-7), we get $y_1 = 4 - y_0$. Furthermore, the function

$$F_2(x, y) = \frac{2\sqrt{2}}{10609215\sqrt{29}((x-4)^2 + y^2)^{3/2}} \left(223524(x-4)y^2 - 51774(x-8)xy \right. \\ \left. + 152968x((x-12)x + 48) - 7798y^3 - 828384y - 9911897 \right)$$

is the first integral of X_4^+ . Thus, the solutions of equation $F_2(0, y_0) = F_2(0, 4 - y_0)$, that is, the solutions of $M(y_0)$, are given by $y_0^1 = 11.327008825003787$, $y_0^2 = -7.327008825067321$, $y_0^3 = 5.1418638050573655$, $y_0^4 = 4.027216667769918$, $y_0^5 = -1.1418638050577559$, $y_0^6 = -0.027216667769995536$, and $y_0^7 = 2$. Note that $F_2(0, y_0^1) = F_2(0, y_0^2) = -0.0041509418886195625$, $F_2(0, y_0^3) = F_2(0, y_0^4) = -0.006960423929763933$, and $F_2(0, y_0^5) = F_2(0, y_0^6) = -0.007649789524607148$. Therefore, there are three limit cycles passing through the pairs of points $(0, y_0^1)$ and $(0, y_0^2)$, $(0, y_0^3)$ and $(0, y_0^4)$, and $(0, y_0^5)$ and $(0, y_0^6)$. See Figure 2.2.

The next example illustrates the upper bound for the number of limit cycles for item (b) of Theorem 2.1 for the case $n = 4$, presenting a PSDS with exactly one limit cycle.

Example 2.6. Consider the PSDS

$$\overline{X}_{1,4}(x, y) = \begin{cases} X_1^-(x, y) = (-y + 2, x + 2), & x \leq 0, \\ \overline{X}_4^+(x, y) = \left(\overline{X}_4^{+,1}(x, y), \overline{X}_4^{+,2}(x, y) \right), & x \geq 0, \end{cases}$$

where

$$\overline{X}_4^{+,1}(x, y) = 2 - y + \frac{1}{175}(x-1) \left(1175 + 23x^3 - 1245y + 427y^2 - 52y^3 - 23x^2(7y-11) - 23x(45-34y+4y^2) \right),$$

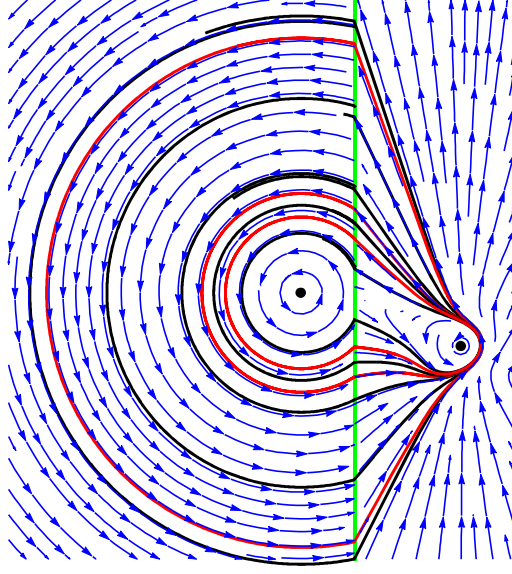


Figure 2.2: The limit cycles of the PSDS $X_{1,4}$ for $d_2 \neq M_2$.
Note that both the outer and inner limit cycles are repelling, while the middle one is attracting.

and

$$\bar{X}_4^{+,2}(x, y) = -1 + x + \frac{1}{175}(y-2) \left(-1175 - 23x^3 + 1245y - 427y^2 + 52y^3 + 23x^2(-11 + 7y) + 23x(45 - 34y + 5y^2) \right).$$

Observe that the above system corresponds to system (2-6) with $d_1 = -2$, $d_2 = 2$, $c_1 = -3$, $c_2 = 1$, $a_0 = -7$, $a_1 = -\frac{257}{7}$, $a_2 = -5$, $a_3 = -5$, $M_1 = 1$ and $M_2 = 2$. By solving the first equation of the respective system (2-7) we get $y_1 = 4 - y_0$. Furthermore, we have that

$$\bar{F}_2(x, y) = \frac{1}{5250\sqrt{10}(5 - 2x + x^2 - 4y + y^2)^{3/2}} \left(-650 + 256x^3 - 135y + 258y^2 - 69y^3 + x^2(69y - 933) + 3x(565 - 254y + 52y^2) \right)$$

is the first integral of $\bar{X}_4^+$. Then, the solutions of the equation $\bar{F}_2(0, y_0) = \bar{F}_2(0, 4 - y_0)$ are given by $y_0^1 = 1$, $y_0^2 = 2$ and $y_0^3 = 3$. Note that $\bar{F}_2(0, 1) = \bar{F}_2(0, 3) = -\frac{149}{5250\sqrt{5}}$. Therefore, there exists a limit cycle through the points $(0, 1)$ and $(0, 3)$, as shown in Figure 2.3.

2.4 Brief conclusion

In this chapter, we investigated the maximum number of crossing limit cycles for a PSDS composed of two rigid centers separated by the line $x = 0$.

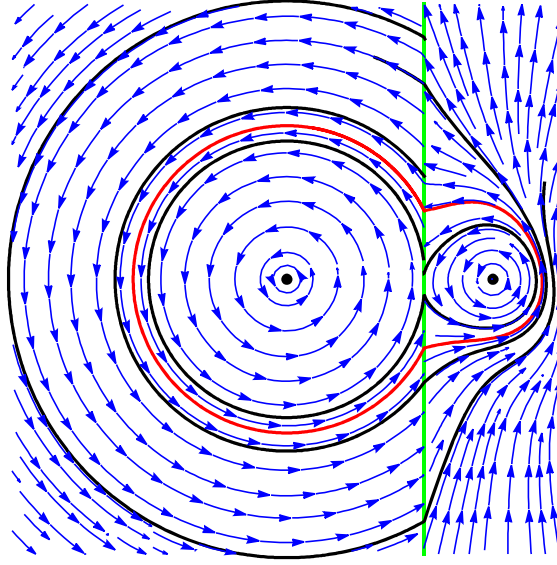


Figure 2.3: *The unstable limit cycle in red of the PSDS $X_{1,4}$ for $d_2 = M_2$.*

For an even degree n , we proved that the system admits at most $(n - 2)/2$ limit cycles provided that $d_2 = M_2$. Furthermore, specifically for $n = 4$ with $d_2 = 0$ and $d_2 \neq M_2$, we established that the sharp upper bound is 1. These results extend the study developed in [12]. Finally, Example 2.5 demonstrates that the upper bound obtained in Item (a) is sensitive to the condition $d_2 = 0$. By taking $d_2 \neq 0$, a system with $n = 4$ can exhibit three crossing limit cycles, thereby surpassing the bound established for the restricted case.

In the next chapter, we will continue studying the maximum number of limit cycles in rigid piecewise smooth dynamical systems. We will consider a system with a discontinuity at $y = 0$, composed of two rigid differential systems of degrees m and n , respectively, where $m \leq n$. In this system, P and Q are homogeneous polynomials of degrees $m - 1$ and $n - 1$, respectively.

Limit cycles of piecewise smooth dynamical systems formed by two rigid systems separated by a straight line

Based on [36], in this chapter we study the maximum number of limit cycles in some classes of discontinuous piecewise differential systems formed by two rigid systems separated by a straight line. These rigid systems are formed by a linear center plus a homogeneous polynomial of degree 2, 3, 4, 5, 6. Therefore, for these classes of systems we address the extended 16th Hilbert problem.

Without loss of generality, we set the straight line as $y = 0$ and we consider the piecewise smooth dynamical system formed by rigid systems (3-1) and (3-2):

$$\begin{aligned}\dot{x} &= -y + x(\lambda + P(x, y)) = X_m^1(x, y), \\ \dot{y} &= x + y(\lambda + P(x, y)) = X_m^2(x, y),\end{aligned}\tag{3-1}$$

where $P(x, y)$ is a homogeneous polynomial of degree $m - 1$, and

$$\begin{aligned}\dot{x} &= -y + x(\lambda + Q(x, y)) = Y_n^1(x, y), \\ \dot{y} &= x + y(\lambda + Q(x, y)) = Y_n^2(x, y),\end{aligned}\tag{3-2}$$

where $Q(x, y)$ is a homogeneous polynomial of degree $n - 1$. As usual, the dot denotes derivative with respect to the time t .

Thus, our main result is the following one.

Theorem 3.1. *Consider the PSDS separated by the straight line $y = 0$ and formed by the two rigid polynomial differential systems (3-1) and (3-2) of degree m and n respectively, with $m \leq n$, where P and Q are homogeneous polynomials of degree $m - 1$ and $n - 1$, respectively. Then the PSDS has*

- (a) *at most 1 limit cycle if $m = n = 2$ and the bound is reached;*

- (b) at most 2 limit cycles if $m = 2$ and $n = 3$ and the bound is reached;
- (c) at most 3 limit cycles if $m = 2$ and $n = 4$;
- (d) at most 4 limit cycles if $m = 2$ and $n = 5$;
- (e) at most 5 limit cycles if $m = 2$, and $n = 6$;
- (f) at most 1 limit cycle if $m = n = 3$ and the bound is reached;
- (g) at most 4 limit cycles if $m = 3$ and $n = 4$;
- (h) at most 2 limit cycles if $m = 3$ and $n = 5$ and the bound is reached;
- (i) at most 5 limit cycles if $m = 3$ and $n = 6$;
- (j) at most 1 limit cycle if $m = 4$ and $n = 4$ and the bound is reached;
- (k) at most 6 limit cycles if $m = 4$ and $n = 5$;
- (l) at most 6 limit cycles if $m = 4$ and $n = 6$;
- (m) at most 1 limit cycle if $m = n = 5$ and the bound is reached;
- (n) at most 8 limit cycles if $m = 5$ and $n = 6$.

The rest of the chapter is dedicated to prove Theorem 3.1.

3.1 Proof of Theorem 3.1

If the degrees of the polynomials $P(x, y)$ and $Q(x, y)$ are ≥ 2 , then the eigenvalues of the linear part of the differential systems (3-1) and (3-2) at the origin of coordinates are $\lambda \pm i$. So the origin is a hyperbolic focus if $\lambda \neq 0$, and it is a weak focus or a center if $\lambda = 0$.

3.1.1 Case $(m, n) = (2, 2)$

When $(m, n) = (2, 2)$, the homogeneous polynomials $P(x, y)$ and $Q(x, y)$ have degree one, and we have the PSDS separated by $y = 0$ and formed by the two rigid systems defined by the vector fields

$$X_2(x, y) = \begin{cases} X_2^1(x, y) = -y + x(\lambda + a_1x + a_2y), \\ X_2^2(x, y) = x + y(\lambda + a_1x + a_2y), \end{cases} \quad (3-3)$$

and

$$Y_2(x, y) = \begin{cases} Y_2^1(x, y) = -y + x(\lambda + b_1x + b_2y), \\ Y_2^2(x, y) = x + y(\lambda + b_1x + b_2y). \end{cases} \quad (3-4)$$

We write these two differential systems in polar coordinates (r, θ) , where $x = r \cos \theta$ and $y = r \sin \theta$. Then system (3-3) becomes

$$\dot{r} = r\lambda + r^2(a_1 \cos \theta + a_2 \sin \theta), \quad \dot{\theta} = 1.$$

or equivalently

$$r' = \frac{dr}{d\theta} = r\lambda + r^2(a_1 \cos \theta + a_2 \sin \theta).$$

Thus, the solution $r_1(\theta)$ of the differential equation $r'(\theta)$ such $r_1(0) = r_0$ is

$$r_1(\theta) = \frac{e^{\theta\lambda} r_0 (1 + \lambda^2)}{1 - a_2 r_0 + a_1 r_0 \lambda + \lambda^2 + e^{(\theta\lambda)} r_0 (a_2 - a_1 \lambda) \cos \theta - e^{(\theta\lambda)} r_0 (a_1 + a_2 \lambda) \sin \theta}. \quad (3-5)$$

Similarly system (3-4) in polar coordinates taken θ as the new independent variable writes

$$r' = \frac{dr}{d\theta} = r\lambda + r^2(b_1 \cos \theta + b_2 \sin \theta).$$

and its solution $r_2(\theta)$ such $r_2(\pi) = r_1$ is

$$r_2(\theta) = \frac{e^{\theta\lambda} r_1 (1 + \lambda^2)}{e^{\pi\lambda} (1 + b_2 r_1 - b_1 r_1 \lambda + \lambda^2) - e^{(\theta\lambda)} r_1 ((-b_2 + b_1 \lambda) \cos \theta + (b_1 + b_2 \lambda) \sin \theta)}. \quad (3-6)$$

The trajectory $r_1(\theta)$ of the vector field X_2 such that $r_1(0) = r_0$ intersects the straight line $y = 0$ at the point $(r_1(\pi), 0)$. Then since we want to concatenate the solution $r_1(\theta)$ with a trajectory of vector field Y_2 , we consider the solution $r_2(\theta)$ such that $r_2(\pi) = r_1(\pi)$. Now we want that after a time π the solution $r_2(\theta)$ intersects the straight line $y = 0$ at r_0 , in this way the PSDS will have a periodic orbit. If this periodic orbit is isolated in the set of all periodic orbits of the PSDS it will be a limit cycle.

In summary, for studying the limit cycles of the PSDS formed by the vector fields X_2 and Y_2 we must solve the equation $r_2(2\pi) = r_0$ for r_0 , i.e. the equation

$$\frac{e^{2\pi\lambda} (\lambda^2 + 1) r_0}{1 + \lambda^2 + r_0(1 + e^{\pi\lambda})(a_1 \lambda - a_2 + e^{\pi\lambda}(b_2 - b_1 \lambda))} = r_0. \quad (3-7)$$

The solutions of this equation are

$$r_{0,1} = 0 \quad \text{and} \quad r_{0,2} = \frac{(1 - e^{\pi\lambda}) (\lambda^2 + 1)}{a_2 - a_1 \lambda + e^{\pi\lambda} (b_1 \lambda - b_2)}. \quad (3-8)$$

Since the origin is always an equilibrium point of the system, we only have the solution $r_{0,2}$ providing a periodic orbit. More precisely, if $\lambda = 0$, then $r_{0,2} = 0$ and if $(a_2 - a_1 \lambda + e^{\pi\lambda} (b_1 \lambda - b_2)) = 0$, then $r_{0,2}$ does not exist. So, in order that $r_{0,2}$ be positive and provide a periodic orbit we must have $(1 - e^{\pi\lambda}) (a_2 - a_1 \lambda + e^{\pi\lambda} (b_1 \lambda - b_2)) > 0$, and then since the PSDS has only one periodic orbit, this is a limit cycle. Statement (a) of Theorem 3.1 is proved.

3.1.2 Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{2}, \mathbf{3})$

Now the PSDS is formed by the vector field X_2 defined in the previous case and the vector field (3-2) with $n = 3$, i.e.

$$Y_3(x, y) = \begin{cases} Y_3^1(x, y) = -y + x(\lambda + b_1x^2 + b_2xy + b_3y^2), \\ Y_3^2(x, y) = x + y(\lambda + b_1x^2 + b_2xy + b_3y^2). \end{cases} \quad (3-9)$$

This vector field in polar coordinates (3-9) provides the differential system

$$\dot{r} = r\lambda + \frac{1}{2}r^3(b_1 + b_3 + (b_1 - b_3)\cos(2\theta) + b_2\sin(2\theta)), \quad \dot{\theta} = 1, \quad (3-10)$$

or the equivalent differential equation

$$r' = \frac{dr}{d\theta} = r\lambda + \frac{1}{2}r^3(b_1 + b_3 + (b_1 - b_3)\cos(2\theta) + b_2\sin(2\theta)).$$

The solution $r_2(\theta)$ of this differential equation such that $r_2(\pi) = r_1$ is

$$r_2(\theta) = \frac{e^{\theta\lambda}\sqrt{2\lambda(1+\lambda^2)}}{\sqrt{E_{11}/r_1^2 + E_{12}}} \quad (3-11)$$

where

$$\begin{aligned} E_{11} &= e^{2\pi\lambda}((b_1 + b_3)r_1^2 + (2 - b_2r_1^2)\lambda + 2b_1r_1^2\lambda^2 + 2\lambda^3), \\ E_{12} &= -e^{2\theta\lambda}((b_1 + b_3)(1 + \lambda^2) + \lambda(-b_2 + (b_1 - b_3)\lambda)\cos(2\theta) + \lambda(b_1 - b_3 \\ &\quad + b_2\lambda)\sin(2\theta)). \end{aligned}$$

In this way, the limit cycles satisfy the equation $(r_2(2\pi))^2 - r_0^2 = 0$, then we consider the numerator of the equation $(r_2(2\pi))^2 - r_0^2 = 0$, and we obtain $(1 + e^{\pi\lambda})r_0^2p_1(r_0)$, where $p_1(r_0)$ is the following polynomial of degree 2 in r_0

$$p_1(r_0) = C_{0,1} + C_{1,1}r_0 + C_{2,1}r_0^2, \quad (3-12)$$

where

$$\begin{aligned} C_{0,1} &= 2(e^{\pi\lambda} - e^{2\pi\lambda} + e^{3\pi\lambda} - 1)\lambda(\lambda^2 + 1)^2, \\ C_{1,1} &= 4\lambda(\lambda^2 + 1)(a_2 - a_1\lambda), \\ C_{2,1} &= -\lambda(2a_1^2(e^{\pi\lambda} + 1)\lambda^2 - 4a_1a_2(e^{\pi\lambda} + 1)\lambda + 2a_2^2(e^{\pi\lambda} + 1) + b_2e^{2\pi\lambda}(e^{\pi\lambda} - 1) \\ &\quad (\lambda^2 + 1)) + b_1e^{2\pi\lambda}(e^{\pi\lambda} - 1)(2\lambda^4 + 3\lambda^2 + 1) + b_3e^{2\pi\lambda}(e^{\pi\lambda} - 1)(\lambda^2 + 1). \end{aligned}$$

The polynomial $p_1(r_0)$ can have two real positive roots. For instance, for the values $\lambda = -1$, $a_1 = 10$, $a_2 = 15$, $b_1 = 14$, $b_2 = 12$ and $b_3 = -2$ it has two real positive roots given by $r_0^1 = 0.0757703$ and $r_0^2 = 0.0776237$, see Figure 3.1. For these values of the variables and of the roots the denominator of the equation $(r_2(2\pi))^2 - r_0^2$ is different from zero. So the PSDS separated by the straight line $y = 0$ with the vector fields X_2 and Y_3 can have two limit cycles. Statement (b) of Theorem 3.1 is proved.

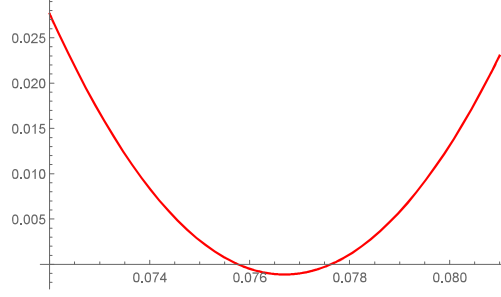


Figure 3.1: Real positive roots for case $(\mathbf{m}, \mathbf{n}) = (2, 3)$.

3.1.3 Case $(\mathbf{m}, \mathbf{n}) = (2, 4)$

Consider the PSDS separated by the straight line $y = 0$ and formed by vector field X_2 defined in (3-3) and the vector field (3-2) with $n = 4$, i.e.

$$Y_4(x, y) = \begin{cases} Y_4^1(x, y) = -y + x(\lambda + b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3), \\ Y_4^2(x, y) = x + y(\lambda + b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3). \end{cases} \quad (3-13)$$

The vector field Y_4 provides in polar coordinates the differential system

$$\dot{r}_2 = r\lambda + r^4(b_1 \cos^3 \theta + b_2 \cos^2 \theta \sin \theta + b_3 \cos \theta \sin^2 \theta + b_4 \sin^3 \theta), \quad \dot{\theta} = 1,$$

or equivalently the differential equation

$$r' = \frac{dr}{d\theta} = r\lambda + r^4(b_1 \cos^3 \theta + b_2 \cos^2 \theta \sin \theta + b_3 \cos \theta \sin^2 \theta + b_4 \sin^3 \theta).$$

Then the solution $r_2(\theta)$ of this differential equation such that $r_2(\pi) = r_1$ is

$$r_2(\theta) = \frac{2^{\frac{2}{3}} e^{\theta\lambda} r_1 ((1 + \lambda^2)(1 + 9\lambda^2))^{\frac{1}{3}}}{E_2^{\frac{1}{3}}}, \quad (3-14)$$

with

$$\begin{aligned} E_2 = & 4e^{3\pi\lambda}(1 + \lambda^2)(1 + 9\lambda^2) + 4e^{3\pi\lambda}r_1^3(b_2 + 2b_4 + 3b_2\lambda^2 - \lambda(7b_1 + 2b_3 + 9b_1\lambda^2)) \\ & - e^{3\theta\lambda}r_1^3(-3(b_2 + 3b_4 - 3(3b_1 + b_3)\lambda)(1 + \lambda^2)\cos\theta + (-b_2 + b_4 + b_1\lambda - b_3\lambda)) \end{aligned}$$

$$(1 + 9\lambda^2) \cos 3\theta + 3(3b_1 + b_3 + 3b_2\lambda + 9b_4\lambda)(1 + \lambda^2) \sin \theta + (1 + 9\lambda^2) \sin 3\theta \\ (b_1 - b_3 + b_2\lambda - b_4\lambda).$$

Thus, the limit cycles satisfy the equation $(r_2(2\pi))^3 - r_0^3 = 0$. Then we consider the numerator of equation $(r_2(2\pi))^3 - r_0^3 = 0$ and we obtain the following polynomial of degree 3 in r_0

$$p_2(r_0) = r_0^3(C_{0,2} + C_{1,2}r_0 + C_{2,2}r_0^2 + C_{3,2}r_0^3), \quad (3-15)$$

where

$$\begin{aligned} C_{0,2} &= -(e^{6\pi\lambda} - 1)(1 + \lambda^2)^3(1 + 9\lambda^2), \\ C_{1,2} &= 3(1 + e^{\pi\lambda})(-a_2 + a_1\lambda)(1 + \lambda^2)^2(1 + 9\lambda^2), \\ C_{2,2} &= 3(1 + e^{\pi\lambda})^2(a_2 - a_1\lambda)^2(1 + \lambda^2)^2(1 + 9\lambda^2), \\ C_{3,2} &= e^{3\pi\lambda} \left(\lambda \left(\lambda^3 (9a_1^3\lambda + b_2 (3\lambda^2 + 7) + 2b_4) + \lambda (a_1^3\lambda + 5b_2 + 4b_4) - b_1 (\lambda^2 + 1)^2 \right. \right. \\ &\quad \left. \left. (9\lambda^2 + 7) - 2b_3 (\lambda^2 + 1)^2 \right) - 3a_1^2a_2\lambda^2 (9\lambda^2 + 1) + 3a_1a_2^2\lambda (9\lambda^2 + 1) \right. \\ &\quad \left. - (a_2^3 (9\lambda^2 + 1)) + b_2 + 2b_4 \right) + 3e^{\pi\lambda} (9\lambda^2 + 1) (a_1\lambda - a_2)^3 + 3e^{2\pi\lambda} (9\lambda^2 + 1) \\ &\quad (a_1\lambda - a_2)^3 + (9\lambda^2 + 1) (a_1\lambda - a_2)^3 + e^{6\pi\lambda} (\lambda^2 + 1)^2 (-\lambda (9b_1\lambda^2 + 7b_1 + 2b_3) \\ &\quad + 3b_2\lambda^2 + b_2 + 2b_4). \end{aligned}$$

The polynomial $p_2(r_0)$ can have at most three real positive roots. That is, the equation $(r_2(2\pi))^3 - r_0^3 = 0$ has at most three positive real roots, or equivalently, $r_2(2\pi) - r_0 = 0$ has at most three positive real roots. So the PSDS separated by the straight line $y = 0$ with the vector fields X_2 and Y_4 can have at most three limit cycles. Statement (c) of Theorem 3.1 is proved.

3.1.4 Case $(m, n) = (2, 5)$

We consider the PSDS separated by straight line $y = 0$ and formed by two vector fields X_2 defined in (3-3) and Y_5

$$Y_5(x, y) = \begin{cases} Y_5^1(x, y) = -y + x(\lambda + b_1x^4 + b_2x^3y + b_3x^2y^2 + b_4xy^3 + b_5y^4), \\ Y_5^2(x, y) = x + y(\lambda + b_1x^4 + b_2x^3y + b_3x^2y^2 + b_4xy^3 + b_5y^4). \end{cases} \quad (3-16)$$

The vector field (3-16) in polar coordinates becomes

$$\begin{aligned} \dot{r}_2 &= r\lambda + r^5 (b_1 \cos^4 \theta + b_2 \cos^3 \theta \sin \theta + b_3 \cos^2 \theta \sin^2 \theta + b_4 \cos \theta \sin^3 \theta + b_5 \sin^4 \theta), \\ \dot{\theta} &= 1, \end{aligned}$$

or equivalently

$$r' = \frac{dr}{d\theta} = r\lambda + r^5 (b_1 \cos^4 \theta + b_2 \cos^3 \theta \sin \theta + b_3 \cos^2 \theta \sin^2 \theta + b_4 \cos \theta \sin^3 \theta + b_5 \sin^4 \theta).$$

Thus, the solution $r_2(\theta)$ of equation $r'(\theta)$ such that $r_2(\pi) = r_1$ is

$$r_2(\theta) = \frac{2^{3/4} e^{\theta\lambda} \lambda^{1/4} (1 + 5\lambda^2 + 4\lambda^4)^{1/4} r_1}{E_3^{1/4}},$$

where

$$\begin{aligned} E_3 = & 8e^{4\pi\lambda}(\lambda + 5\lambda^3 + 4\lambda^5) - r_1^4 (-e^{4\pi\lambda} (b_3 + 3b_5 - (5b_2 + 3b_4)\lambda + 4(b_3 - 2b_2\lambda)\lambda^2 \\ & + b_1(3 + 32\lambda^2 + 32\lambda^4)) + e^{4\theta\lambda} ((3b_1 + b_3 + 3b_5)(1 + 5\lambda^2 + 4\lambda^4) \\ & - 4\lambda(b_2 + b_4 + 4(-b_1 + b_5)\lambda)(1 + \lambda^2) \cos(2\theta) + \lambda(-b_2 + b_4 + (b_1 - b_3 + b_5)\lambda) \\ & (1 + 4\lambda^2) \cos(4\theta) + 8\lambda(b_1 - b_5 + (b_2 + b_4)\lambda)(1 + \lambda^2) \sin(2\theta) + \lambda(b_1 - b_3 \\ & + b_5 + b_2\lambda - b_4\lambda)(1 + 4\lambda^2) \sin(4\theta))) . \end{aligned}$$

Then assuming $r_1 = r_1(\pi) = r_2(\pi)$, where $r_1(\theta)$ is the solution of (3-3) defined in (3-5). Now consider the numerator of the equation $(r(2\pi))^4 - r_0^4 = 0$, and we obtain $(1 + e^{\pi\lambda})r_0^4 p_3(r_0)$, being $p_3(r_0)$ the following polynomial of degree 4 in r_0

$$p_3(r_0) = C_{0,3} + C_{1,3}r_0 + C_{2,3}r_0^2 + C_{3,3}r_0^3 + C_{4,3}r_0^4,$$

where

$$\begin{aligned} C_{0,3} = & 8(-1 + e^{\pi\lambda})(1 + e^{2\pi\lambda})(1 + e^{4\pi\lambda})\lambda(1 + \lambda^2)^4(1 + 4\lambda^2), \\ C_{1,3} = & -32\lambda(a_1\lambda - a_2)(1 + \lambda^2)^3(1 + 4\lambda^2), \\ C_{2,3} = & -48(1 + e^{\pi\lambda})\lambda(a_2 - a_1\lambda)^2(1 + \lambda^2)^2(1 + 4\lambda^2), \\ C_{3,3} = & 32(1 + e^{\pi\lambda})^2\lambda(a_2 - a_1\lambda)^3(1 + 5\lambda^2 + 4\lambda^4), \\ C_{4,3} = & -8(1 + e^{\pi\lambda})^3\lambda(a_2 - a_1\lambda)^4(1 + 4\lambda^2) + e^{4\pi\lambda}(-1 + e^{\pi\lambda})(1 + e^{2\pi\lambda})(1 + \lambda^2)^3 \\ & (3b_1 + b_3 + 3b_5 - (5b_2 + 3b_4)\lambda + (32b_1 + 4b_3)\lambda^2 - 8b_2\lambda^3 + 32b_1\lambda^4). \end{aligned}$$

Now we shall use the Descartes' Theorem (see [6]).

Theorem 3.2 (Descartes' Theorem). *Consider the real polynomial $p(x) = a_{i_1}x^{i_1} + a_{i_2}x^{i_2} + \dots + a_{i_r}x^{i_r}$ with $0 \leq i_1 < i_2 < \dots < i_r$ and $a_{i_j} \neq 0$ real constants for $j \in \{1, 2, \dots, r\}$. When $a_{i_j}a_{i_{j+1}} < 0$, we say that a_{i_j} and $a_{i_{j+1}}$ have a variation of sign. If the number of variations of signs is m , then $p(x)$ has at most m positive real roots. Moreover, if the coefficients of the polynomial $p(x)$ are independent, then it is always possible to choose the coefficients of $p(x)$ in such a way that $p(x)$ has*

exactly m positive real roots.

By Descartes' Theorem the polynomial $p_3(r_0)$ can have at most four real positive roots. For instance, with the conditions, $a_1 = -0.1$, $a_2 = 1$, $b_1 = -2$, $b_2 = 0$, $b_3 = 10$, $b_4 = -1$, $b_5 = -\frac{4}{3}$ and $\lambda = -1$ we have four variations of sign. Therefore the PSDS separated by the straight line $y = 0$ with the vector fields X_2 and Y_5 can have at most four limit cycles. Statement (d) of Theorem 3.1 is proved.

3.1.5 Case $(\mathbf{m}, \mathbf{n}) = (2, 6)$

We consider the PSDS separated by the straight line $y = 0$ and formed by the vector field X_2 defined in (3-3) and the vector field (3-2) for $n = 6$

$$Y_6(x, y) = \begin{cases} Y_6^1(x, y) = -y + x(\lambda + b_1x^5 + b_2x^4y + b_3x^3y^2 + b_4x^2y^3 + b_5xy^4 + b_6y^5), \\ Y_6^2(x, y) = x + y(\lambda + b_1x^5 + b_2x^4y + b_3x^3y^2 + b_4x^2y^3 + b_5xy^4 + b_6y^5). \end{cases} \quad (3-17)$$

The vector field (3-17) in polar coordinates becomes

$$\begin{aligned} \dot{r}_2 &= r\lambda + r^6 (b_1 \cos^5 \theta + b_2 \cos^4 \theta \sin \theta + b_3 \cos^3 \theta \sin^2 \theta + b_4 \cos^2 \theta \sin^3 \theta \\ &\quad + b_5 \cos \theta \sin^4 \theta + b_6 \sin^5 \theta), \\ \dot{\theta} &= 1, \end{aligned}$$

or equivalently

$$\begin{aligned} r' = \frac{dr}{d\theta} &= r\lambda + r^6 (b_1 \cos^5 \theta + b_2 \cos^4 \theta \sin \theta + b_3 \cos^3 \theta \sin^2 \theta + b_4 \cos^2 \theta \sin^3 \theta \\ &\quad + b_5 \cos \theta \sin^4 \theta + b_6 \sin^5 \theta). \end{aligned}$$

Thus, the solution $r_2(\theta)$ of equation $r'(\theta)$ such that $r_2(\pi) = r_1$ is

$$r_2(\theta) = \frac{2^{\frac{4}{5}} e^{\theta\lambda} ((1 + \lambda^2)(1 + 25\lambda^2)(9 + 25\lambda^2))^{\frac{1}{5}} r_1}{E_4^{\frac{1}{5}}}, \quad (3-18)$$

where

$$\begin{aligned} E_4 &= 16e^{5\pi\lambda} (9 + 3(3b_2 + 2b_4 + 8b_6)r_1^5 - (149b_1 + 26b_3 + 24b_5)r_1^5\lambda \\ &\quad + (259 + 10(11b_2 + 3b_4)r_1^5)\lambda^2 - 50(15b_1 + b_3)r_1^5\lambda^3 + 125(7 + b_2r_1^5)\lambda^4 \\ &\quad - 625b_1r_1^5\lambda^5 + 625\lambda^6) - e^{5\theta\lambda} r_1^5 (-10(b_2 + b_4 + 5b_6 - 5(5b_1 + b_3 + b_5)\lambda) \\ &\quad (1 + \lambda^2)(9 + 25\lambda^2) \cos \theta - 5(1 + \lambda^2)(1 + 25\lambda^2) (5((-5b_1 + b_3 + 3b_5)\lambda - 3b_6) \\ &\quad + 9b_2 + 3b_4) \cos 3\theta + (b_4 - b_2 - b_6 + (b_1 - b_3 + b_5)\lambda)(9 + 25\lambda^2 + 625\lambda^4) \cos 5\theta \\ &\quad + 10(5b_1 + b_3 + b_5 + 5(b_2 + b_4 + 5b_6)\lambda)(1 + \lambda^2)(9 + 25\lambda^2) \sin \theta + 5(1 + \lambda^2) \end{aligned}$$

$$(15b_1 - 3b_3 - 9b_5 + 5(3b_2 + b_4 - 5b_6)\lambda)(1 + 25\lambda^2) \sin 3\theta + (9 + 250\lambda^2 + 625\lambda^4) \\ (b_1 - b_3 + b_5 + (b_2 - b_4 + b_6)\lambda) \sin 5\theta).$$

Now consider $r_1 = r_1(\pi) = r_2(\pi)$ with $r_1(\theta)$ defined in (3-5). Then consider the numerator the equation $r_2(2\pi)^5 - r_0^5 = 0$ and we obtain $(1 + e^{\pi\lambda})r_0^5 p_4(r_0)$, being $p_4(r_0)$ the following polynomial of degree 5 in r_0

$$p_4(r_0) = C_{0,4} + C_{1,4}r_0 + C_{2,4}r_0^2 + C_{3,4}r_0^3 + C_{4,4}r_0^4 + C_{5,4}r_0^5,$$

where

$$\begin{aligned} C_{0,4} &= (1 - e^{\pi\lambda} + e^{2\pi\lambda} - e^{3\pi\lambda} + e^{4\pi\lambda})(1 - e^{5\pi\lambda})(1 + \lambda^2)^5(9 + 250\lambda^2 + 625\lambda^4), \\ C_{1,4} &= 5(a_1\lambda - a_2)(1 + \lambda^4)(9 + 250\lambda^2 + 625\lambda^4), \\ C_{2,4} &= 10(1 + e^{\pi\lambda})(a_2 - a_1\lambda)^2(1 + \lambda^2)^3(9 + 250\lambda^2 + 625\lambda^4), \\ C_{3,4} &= 10(1 + e^{\pi\lambda})^2(a_1\lambda - a_2)^3(1 + \lambda^2)^2(9 + 250\lambda^2 + 625\lambda^4), \\ C_{4,4} &= 5(1 + e^{\pi\lambda})^3(a_2 - a_1\lambda)^4(1 + \lambda^2)(9 + 250\lambda^2 + 625\lambda^4), \\ C_{5,4} &= (1 + e^{\pi\lambda})^4(a_1\lambda - a_2)^5(9 + 250\lambda^2 + 625\lambda^4) + e^{5\pi\lambda} (1 - e^{\pi\lambda} + e^{2\pi\lambda} \\ &\quad - e^{3\pi\lambda} + e^{4\pi\lambda}) (1 + \lambda^2)^4 (24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) \\ &\quad - \lambda (149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4)). \end{aligned}$$

The polynomial $p_4(r_0)$ can have at most 5 positive real roots. For example, consider $\lambda = -1$, $a_1 = 10$, $a_2 = 15$, $b_1 = b_2 = b_3 = b_4 = 0$, $b_5 = -14$ and $b_6 = -12$, with these values there are 5 variations of sign. So By Descartes' Theorem it can have at most 5 positive real roots, is that, the PSDS formed by the two vector fields X_2 and Y_6 and separated by the straight line $y = 0$ can have at most 5 limit cycles. Statement (e) of Theorem 3.1 is proved.

3.1.6 Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{3})$

Consider the PSDS separated by the straight line $y = 0$ and formed by the vector field Y_3 defined in (3-9) and the vector field

$$X_3(x, y) = \begin{cases} x_3^1(x, y) = -y + x(\lambda + a_1x^2 + a_2xy + a_3y^2), \\ x_3^2(x, y) = x + y(\lambda + a_1x^2 + a_2xy + a_3y^2). \end{cases} \quad (3-19)$$

Thus the solution $r_1(\theta)$ of the vector field X_3 in polar coordinates such that $r_1(0) = r_0$ is

$$r_1(\theta) = \frac{e^{\theta\lambda} \sqrt{2\lambda(1 + \lambda^2)}}{\sqrt{2(\lambda + \lambda^3)/r_0^2 + E_5}}, \quad (3-20)$$

where

$$E_5 = e^{2\theta\lambda} (-(a_1 + a_3)(1 + \lambda^2) + \lambda(a_2 + (-a_1 + a_3)\lambda) \cos(2\theta) - \lambda(a_1 - a_3 + a_2\lambda) \sin(2\theta)) + \lambda(-a_2 + 2a_1\lambda) + a_1 + a_3.$$

The solution $r_2(\theta)$ of the vector field Y_3 such that $r_2(\pi) = r_1$ is given in (3-11). Then solving the equation $r_2(2\pi) = r_0$ for r_0 we have the following solution

$$r_0 = \sqrt{\frac{-2\lambda(1 + e^{2\pi\lambda})(1 + \lambda^2)}{a_1 + a_3 - a_2\lambda + 2a_1\lambda^2 + e^{2\pi\lambda}(b_1 + b_3 - b_2\lambda + 2b_1\lambda^2)}}.$$

When $r_0 > 0$ the PSDS separated by the straight line $y = 0$ and formed by the vector fields X_3 and Y_3 can have one limit cycle. Statement (f) of Theorem 3.1 is proved.

3.1.7 Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{4})$

We consider the PSDS separated by the straight line $y = 0$ and formed by the two vector fields X_3 defined in (3-19) and Y_4 defined in (3-13). We know their solutions $r_1(\theta)$ such that $r_1(0) = r_0$ and $r_2(\theta)$ such that $r_2(\pi) = r_1(\pi)$. Then the numerator of the equation $r_2(2\pi)^3 - r_0^3 = 0$ is $A + B = 0$ where

$$\begin{aligned} A &= 2\sqrt{2}e^{3\pi\lambda}\lambda^{3/2}(\lambda^2 + 1)^{3/2}r_0^3(9b_1\lambda^3 + 7b_1\lambda - b_2(3\lambda^2 + 1) + 2b_3\lambda - 2b_4) \\ &\quad + 2\sqrt{2}e^{6\pi\lambda}\lambda^{3/2}(\lambda^2 + 1)^{3/2}(\lambda r_0^3(7b_1 + 2b_3) + 9b_1\lambda^3r_0^3 - r_0^3(b_2 + 2b_4) \\ &\quad + \lambda^2(10 - 3b_2r_0^3) + 9\lambda^4 + 1), \\ B &= (\lambda^2 + 1)(9\lambda^2 + 1)r_0((e^{2\pi\lambda} - 1)r_0^2(2a_1\lambda^2 + a_1 - a_2\lambda + a_3) - 2(\lambda^3 + \lambda)) \\ &\quad \sqrt{a_1 + a_3 + \lambda(-a_2 + 2a_1\lambda) - e^{2\pi\lambda}(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2) + 2(\lambda^3 + \lambda)/r_0^2}. \end{aligned}$$

Now we consider the equation $A^2 - B^2 = 0$ and we obtain the polynomial

$$p_5(r_0) = C_{0,5} + C_{2,5}r_0^2 + C_{3,5}r_0^3 + C_{4,5}r_0^4 + C_{6,5}r_0^6,$$

where

$$\begin{aligned} C_{0,5} &= 8(e^{12\pi\lambda} - 1)\lambda^3(\lambda^2 + 1)^5(9\lambda^2 + 1)^2, \\ C_{2,5} &= 12(e^{2\pi\lambda} - 1)(\lambda^2 + 1)^4(9\lambda^3 + \lambda)^2(2a_1\lambda^2 + a_1 - a_2\lambda + a_3), \\ C_{3,5} &= 16e^{9\pi\lambda}(e^{3\pi\lambda} + 1)\lambda^3(\lambda^2 + 1)^4(9\lambda^2 + 1)(9b_1\lambda^3 + 7b_1\lambda - b_2(3\lambda^2 + 1) \\ &\quad + 2b_3\lambda - 2b_4), \\ C_{4,5} &= 6(1 - e^{2\pi\lambda})^2\lambda(\lambda^2 + 1)^3(9\lambda^2 + 1)^2(2a_1\lambda^2 + a_1 - a_2\lambda + a_3)^2, \end{aligned}$$

$$\begin{aligned}
C_{6,5} = & a_1^3 (2\lambda^2 + 1)^3 (9\lambda^4 + 10\lambda^2 + 1)^2 (e^{2\pi\lambda} - 1)^3 + 3 (18\lambda^6 + 29\lambda^4 + 12\lambda^2 + 1)^2 \\
& (e^{2\pi\lambda} - 1)^3 (a_3 - a_2\lambda)a_1^2 + 3a_1 (2\lambda^2 + 1) (9\lambda^4 + 10\lambda^2 + 1)^2 (e^{2\pi\lambda} - 1)^3 \\
& (a_3 - a_2\lambda)^2 + (\lambda^2 + 1)^2 \left(\lambda^3 \left((-9b_1\lambda^3 - 7b_1\lambda + 3b_2\lambda^2 + b_2 - 2b_3\lambda + 2b_4)^2 \right. \right. \\
& \left. \left. 8e^{6\pi\lambda} (e^{3\pi\lambda} + 1)^2 (\lambda^2 + 1) - a_2^3 (e^{2\pi\lambda} - 1)^3 (9\lambda^2 + 1)^2 \right) + 3a_2^2 a_3 \lambda^2 (9\lambda^2 + 1)^2 \right. \\
& \left. (e^{2\pi\lambda} - 1)^3 - 3a_2\lambda (e^{2\pi\lambda} - 1)^3 (9a_3\lambda^2 + a_3)^2 + a_3^3 (9\lambda^2 + 1)^2 (e^{2\pi\lambda} - 1)^3 \right).
\end{aligned}$$

By Descartes' Theorem the polynomial $p_5(r_0)$ can have at most four positive real roots. So the PSDS separated by the straight line $y = 0$ and formed by the two vector fields X_3 and Y_4 can have at most four limit cycles. Statement (g) of Theorem 3.1 is proved.

3.1.8 Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{5})$

We consider the PSDS separated by the straight line $y = 0$ and formed by the two vector fields X_3 defined in (3-19) and Y_5 defined in (3-16). We know their solutions $r_1(\theta)$ such that $r_1(0) = r_0$ and $r_2(\theta)$ such that $r_2(\pi) = r_1(\pi)$. Then we consider the numerator of the equation $(r_2(2\pi))^4 - r_0^4 = 0$ and we get $(-1 + e^{2\pi\lambda})r_0^4 p_6(r_0)$, where $p_6(r_0)$ is the following polynomial of degree 4 in r_0

$$p_6(r_0) = C_{0,6} + C_{2,6}r_0^2 + C_{4,6}r_0^4,$$

with

$$\begin{aligned}
C_{0,6} = & 8(1 + e^{2\pi\lambda})(1 + e^{4\pi\lambda})\lambda^2(1 + \lambda^2)^2(1 + 4\lambda^2), \\
C_{2,6} = & 8\lambda(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)(1 + 5\lambda^2 + 4\lambda^4), \\
C_{4,6} = & 2(1 - e^{2\pi\lambda})(1 + 4\lambda^2)(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)^2 + e^{4\pi\lambda}(1 + e^{2\pi\lambda})(\lambda + \lambda^3) \\
& (b_3 + 3b_5 - 5b_2\lambda - 3b_4\lambda + 4\lambda^2(b_3 - 2b_2\lambda) + b_1(3 + 32(\lambda^2 + \lambda^4))).
\end{aligned}$$

By Descartes' Theorem the polynomial $p_6(r_0)$ can have at most two positive real roots. For example, for the values $\lambda = 0.01$, $a_1 = -4.95409$, $a_2 = 10$, $a_3 = 5$, $b_1 = b_2 = b_3 = 0$, $b_4 = 250000$ and $b_5 = 2500.02$ it has two real positive roots, $r_0^1 = 1$ and $r_0^2 = 1$. See Figure 3.2. For these values the denominator of the equation $(r_2(2\pi))^4 - r_0^4$ is different from zero. Moreover, with these values of the parameters, r_0^1 and $r_0^2 = 2$ are also roots of equation $r_2(2\pi) - r_0 = 0$. So the PSDS separated by the straight line $y = 0$ with the vector fields X_3 and Y_5 can have two limit cycles. Statement (h) of Theorem 3.1 is proved.

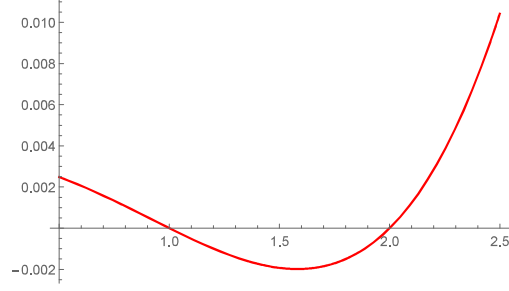


Figure 3.2: Real positive roots for case $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{5})$.

3.1.9 Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{6})$

We consider the PSDS separated by the straight line $y = 0$ and formed by the two vector fields X_3 defined in (3-19) and Y_6 defined in (3-17). We know their solutions $r_1(\theta)$ such that $r_1(0) = r_0$ and $r_2(\theta)$ such that $r_2(\pi) = r_1(\pi)$. Then the numerator the equation $r_2(2\pi)^5 - r_0^5 = 0$ is $A + B = 0$ where

$$\begin{aligned} A &= -r_0(1 + \lambda^2)(1 + 25\lambda^2)\sqrt{(1 - e^{2\pi\lambda})(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2) + 2(\lambda + \lambda^3)/r_0^2} \\ &\quad (9 + 25\lambda^2), \\ B &= 4\sqrt{2}e^{5\pi\lambda}\lambda^{\frac{5}{2}}(1 + \lambda^2)^{\frac{5}{2}}(e^{5\pi\lambda}(9 + 259\lambda^2 + 875\lambda^4 + 625\lambda^6) - (1 + e^{5\pi\lambda})r_0^5 \\ &\quad (24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda(149b_1 + 26b_3 + 24b_5 \\ &\quad + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4))) . \end{aligned}$$

Now consider the equation $A^2 - B^2 = 0$ and we obtain $(1 + e^{\pi\lambda})(1 + \lambda^2)p_7(r_0)$, where $p_7(r_0)$ is a polynomial of degree 10 in r_0

$$p_7(r_0) = C_{0,7} + C_{2,7}r_0^2 + C_{4,7}r_0^4 + C_{5,7}r_0^5 + C_{6,7}r_0^6 + C_{8,7}r_0^8 + C_{10,7}r_0^{10},$$

where

$$\begin{aligned} C_{0,7} &= 32(1 - e^{\pi\lambda})(1 + e^{2\pi\lambda})(1 + e^{4\pi\lambda} + e^{8\pi\lambda} + e^{12\pi\lambda} + e^{16\pi\lambda})\lambda^5(9 + 250\lambda^2 + 625\lambda^4)^2 \\ &\quad (1 + \lambda^2)^5, \\ C_{2,7} &= 80(1 - e^{\pi\lambda})(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)(\lambda + \lambda^3)^4(9 + 250\lambda^2 + 625\lambda^4)^2, \\ C_{4,7} &= 80(-1 + e^{\pi\lambda})^2(1 + e^{\pi\lambda})(9 + 250\lambda^2 + 625\lambda^4)^2(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)^2 \\ &\quad \lambda^3(1 + \lambda^2)^3, \\ C_{5,7} &= 64e^{15\pi\lambda}(1 - e^{\pi\lambda} + e^{2\pi\lambda} - e^{3\pi\lambda} + e^{4\pi\lambda})\lambda^5(1 + \lambda^2)^4(9 + 250\lambda^2 + 625\lambda^4) \\ &\quad (24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda(149b_1 + 26b_3 + 24b_5 \\ &\quad + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4)), \end{aligned}$$

$$\begin{aligned}
C_{6,7} &= 40(1 - e^{2\pi\lambda})^3(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)^3(1 + \lambda^2)^2(9 + 250\lambda^2 + 625\lambda^4)^2 \\
&\quad (1 + e^{\pi\lambda})^2\lambda^2, \\
C_{8,7} &= 10(-1 + e^{\pi\lambda})^4(1 + e^{\pi\lambda})^3(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)^4(9 + 250\lambda^2 + 625\lambda^4)^2 \\
&\quad \lambda(1 + \lambda^2), \\
C_{10,7} &= (-1 - e^{\pi\lambda})((-1 + e^{\pi\lambda})^5(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)^5(9 + 250\lambda^2 + 625\lambda^4)^2 \\
&\quad (1 + e^{\pi\lambda})^3 + 32e^{10\pi\lambda}(1 - e^{\pi\lambda} + e^{2\pi\lambda} - e^{3\pi\lambda} + e^{4\pi\lambda})^2\lambda^5(1 + \lambda^2)^3 \\
&\quad (24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda(14b_1 + 26b_3 + 24b_5 \\
&\quad + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4))^2).
\end{aligned}$$

Note that the signs of the coefficients C_0, C_4 and C_8 are determined only by the sign of λ . Assume $\lambda < 0$, then $C_0 < 0$, $C_4 < 0$ and $C_8 < 0$. In this way, between C_0, C_2 and C_4 there are at most two variations of sign, between C_4, C_5, C_6 and C_8 there are at most two variations of sign and between C_8 and C_{10} there is at most one variation of sign. Therefore, there are at most five variations of sign between the coefficient of $p_7(r_0)$. More specifically, the sign of the coefficient C_2 is determined by the sign of $A_1 = (a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)$. We assume $A_1 > 0$ in order to $C_2 > 0$. The sign of C_5 is given by

$$\begin{aligned}
A_2 &= (24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda(149b_1 + 26b_3 + 24b_5 \\
&\quad + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4)),
\end{aligned}$$

then assuming $A_2 < 0$ we have $C_5 > 0$. Now as $A_1 > 0$ then $C_6 > 0$. Finally as $A_1 > 0$ we have $C_{10} > 0$. Thus, we obtain five variations of sign, i.e $C_0 < 0$, $C_2 > 0$, $C_4 < 0$, $C_5 > 0$, $C_6 > 0$, $C_8 < 0$ and $C_{10} > 0$ for $\lambda < 0$. On the other hand, assuming $\lambda > 0$ we have $C_0 < 0$, $C_4 > 0$ and $C_8 > 0$. Thus between C_0, C_2 and C_4 there is at most one variation of sign and between C_4, C_5, C_6 and C_8 there are at most two variations of sign and between C_8 and C_{10} there is one variation. In this way, for $\lambda > 0$ there are at most four variations of sign. For instance, consider $\lambda < 0$, $a_1 > 0$, $a_2 \geq 0$, $a_3 \geq -a_1$, $b_1 = b_2 = 0$, $b_3 < 0$, $b_4 < 0$, $b_5 = -\frac{13b_3}{12}$ and $b_6 < 0$, with this conditions we have five variations of sign. In conclusion by Descartes' Theorem the polynomial $p_7(r_0)$ can have at most five positive real roots. In this way, it follows that the PSDS separated by the straight line $y = 0$ and formed by the two vector fields X_3 and Y_6 can have at most five limit cycles. Statement (i) of Theorem 3.1 is proved.

3.1.10 Case $(\mathbf{m}, \mathbf{n}) = (4, 4)$

Now consider the PSDS separated by the straight line $y = 0$ and formed by vector field Y_4 defined in (3-13) and the vector field (3-1) for $m = 4$, i.e

$$X_4(x, y) = \begin{cases} X_4^1(x, y) = -y + x(\lambda + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3), \\ X_4^2(x, y) = x + y(\lambda + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3). \end{cases} \quad (3-21)$$

The vector field (3-21) in polar coordinates provides the differential system

$$\dot{r}_1(\theta) = r\lambda + r^4(a_1 \cos^3 \theta + a_2 \cos^2 \theta \sin \theta + a_3 \cos \theta \sin^2 \theta + a_4 \sin^3 \theta), \quad \dot{\theta} = 1$$

or equivalently the differential equation

$$r' = \frac{dr}{d\theta} = r\lambda + r^4(a_1 \cos^3 \theta + a_2 \cos^2 \theta \sin \theta + a_3 \cos \theta \sin^2 \theta + a_4 \sin^3 \theta).$$

Then the solution $r_1(\theta)$ of the equation $r'(\theta)$ such that $r_1(0) = r_0$ is

$$r_1(\theta) = \frac{2^{\frac{2}{3}} e^{\theta\lambda} r_0 ((1 + \lambda^2)(1 + 9\lambda^2))^{\frac{1}{3}}}{E_6^{\frac{1}{3}}}, \quad (3-22)$$

where

$$\begin{aligned} E_6 = & -e^{3\theta\lambda} r_0^3 \left(-3(a_2 + 3a_4 - 3(3a_1 + a_3)\lambda)(1 + \lambda^2) \cos \theta + (-a_2 + a_4 + a_1\lambda - a_3\lambda) \right. \\ & (1 + 9\lambda^2) \cos 3\theta + 3(3a_1 + a_3 + 3a_2\lambda + 9a_4\lambda)(1 + \lambda^2) \sin \theta + (a_1 - a_3 + a_2\lambda \\ & - a_4\lambda)(1 + 9\lambda^2) \sin 3\theta) - 4r_0^3(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)) \\ & \left. + 4(1 + \lambda^2)(1 + 9\lambda^2) \right). \end{aligned}$$

Then assuming $r_1 = r_1(\pi) = r_2(\pi)$, where $r_2(\theta)$ is the solution of (3-13) defined in (3-14), we solve $r_2(2\pi) = r_0$ for r_0 . More specifically we consider the numerator of equation $(r_2(2\pi))^3 - r_0^3 = 0$ and we get $(1 + e^{3\pi\lambda})r_0^3 p_8(r_0)$, with $p_8(r_0)$ being the following polynomial of degree 3 in r_0

$$p_8(r_0) = C_{0,8} + C_{3,8}r_0^3,$$

where

$$\begin{aligned} C_{0,8} = & (1 - e^{3\pi\lambda})(1 + 10\lambda^2 + 9\lambda^4), \\ C_{3,8} = & 9a_1\lambda^3 - a_2(1 + 3\lambda^2) + e^{3\pi\lambda}(b_2 + 2b_4 - (7b_1 + 2b_3)\lambda + 3b_2\lambda^2 - 9b_1\lambda^3) - 2a_4 \\ & + (7a_1 + 2a_3)\lambda. \end{aligned}$$

The polynomial $p_8(r_0)$ can have at most one real positive root $r_0 = \left(\frac{-C_0}{C_3}\right)^{1/3}$. Therefore the PSDS, separated by the straight line $y = 0$, formed by the vector fields X_4 and Y_4 , can have one limit cycle. Statement (j) of Theorem 3.1 is proved.

3.1.11 Case $(\mathbf{m}, \mathbf{n}) = (4, 5)$

Now consider the PSDS separated by the straight line $y = 0$ and formed by the two vector fields X_4 defined in (3-21) and Y_5 defined in (3-16). We consider their solutions $r_1(\theta)$ such that $r_1(0) = r_0$ and $r_2(\theta)$ such that $r_2(\pi) = r_1(\pi)$. Then the numerator of the equation $(r_2(2\pi))^4 - r_0^4 = 0$ is $A - B = 0$ where

$$\begin{aligned} A &= -8\lambda(1 + 5\lambda^2 + 4\lambda^4) \left((1 + \lambda^2)(1 + 9\lambda^2) - (1 + e^{\pi\lambda})(1 - e^{\pi\lambda} + e^{2\pi\lambda})r_0^3 \right. \\ &\quad \left. (a_2 + 2a_4 - 7a_1\lambda - 2a_3\lambda + 3a_2\lambda^2 - 9a_1\lambda^3) \right) \left((-1 - \lambda^2)(1 + 9\lambda^2) + (e^{\pi\lambda} + 1) \right. \\ &\quad \left. (1 - e^{\pi\lambda} + e^{2\pi\lambda})r_0^3(a_2 + 2a_4 - 7a_1\lambda - 2a_3\lambda + 3a_2\lambda^2 - 9a_1\lambda^3) \right)^{1/3}, \\ B &= e^{4\pi\lambda}((1 + \lambda^2)(1 + 9\lambda^2))^{4/3} \left(8e^{4\pi\lambda}(\lambda + 5\lambda^3 + 4\lambda^5) + (-1 + e^{\pi\lambda})(1 + e^{\pi\lambda}) \right. \\ &\quad \left. (1 + e^{2\pi\lambda})r_0^4(b_3 + 3b_5 - (5b_2 + 3b_4)\lambda + 4b_3\lambda^2 - 8b_2\lambda^3 + b_1(3 + 32\lambda^2 + 32\lambda^4)) \right). \end{aligned}$$

Now we consider the equation $A^3 - B^3$ and we obtain $(1 + e^{\pi\lambda})(1 + \lambda^2)^3 p_9(r_0)$, being $p_9(r_0)$ the following polynomial of degree 12 in r_0

$$p_9(r_0) = C_{0,9} + C_{3,9}r_0^3 + C_{4,9}r_0^4 + C_{6,9}r_0^6 + C_{8,9}r_0^8 + C_{9,9}r_0^9 + C_{12,9}r_0^{12},$$

where

$$\begin{aligned} C_{0,9} &= -512(1 - e^{\pi\lambda} + e^{2\pi\lambda})(-1 + e^{3\pi\lambda})(1 + e^{6\pi\lambda})(1 + e^{12\pi\lambda})\lambda^3(1 + \lambda^2)^4(1 + 4\lambda^2)^3 \\ &\quad (1 + 9\lambda^2)^4, \\ C_{3,9} &= -2048(1 - e^{\pi\lambda} + e^{2\pi\lambda})(\lambda + 14\lambda^3 + 49\lambda^5 + 36\lambda^7)^3 (a_2 + 2a_4 + 3a_2\lambda^2 \\ &\quad - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)), \\ C_{4,9} &= -192e^{20\pi\lambda}(-1 + e^{\pi\lambda})(1 + e^{2\pi\lambda})\lambda^2(1 + \lambda^2)^3(1 + 9\lambda^2)^4 (b_3 + 3b_5 - (5b_2 + 3b_4)\lambda \\ &\quad + 4(b_3 - 2b_2\lambda)\lambda^2 + b_1(3 + 32\lambda^2 + 32\lambda^4))(1 + 4\lambda^2)^2, \\ C_{6,9} &= 3072(1 + e^{\pi\lambda})(1 - e^{\pi\lambda} + e^{2\pi\lambda})^2(\lambda + 4\lambda^3)^3(1 + 10\lambda^2 + 9\lambda^4)^2 (a_2 + 2a_4 + 3a_2\lambda^2 \\ &\quad - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^2, \\ C_{8,9} &= -24e^{16\pi\lambda}(1 + e^{\pi\lambda})(-1 + e^{\pi\lambda} - e^{2\pi\lambda} + e^{3\pi\lambda})^2\lambda(1 + \lambda^2)^2(1 + 4\lambda^2)(1 + 9\lambda^2)^4 \\ &\quad (b_3 + 3b_5 - (5b_2 + 3b_4)\lambda + 4(b_3 - 2b_2\lambda)\lambda^2 + b_1(3 + 32\lambda^2 + 32\lambda^4))^2, \\ C_{9,9} &= -2048(1 + e^{\pi\lambda})^2(1 - e^{\pi\lambda} + e^{2\pi\lambda})^3\lambda^3(1 + 4\lambda^2)^3(1 + 10\lambda^2 + 9\lambda^4)(a_2 + 2a_4 \\ &\quad + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^3, \end{aligned}$$

$$C_{12,9} = - (1 + e^{\pi\lambda})^2 (-512(1 + e^{\pi\lambda})(1 - e^{\pi\lambda} + e^{2\pi\lambda})^4(\lambda + 4\lambda^3)^3 (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^4 + e^{12\pi\lambda}(-1 + e^{\pi\lambda})^3(1 + e^{2\pi\lambda})^3(1 + \lambda^2)(1 + 9\lambda^2)^4 (b_3 + 3b_5 - (5b_2 + 3b_4)\lambda + 4(b_3 - 2b_2\lambda)\lambda^2 + b_1(3 + 32\lambda^2 + 32\lambda^4))^3).$$

By Descartes' Theorem the polynomial $p_9(r_0)$ can have at most six positive real roots. For example, for the values $\lambda = 0.1$, $a_1 = 1$, $a_2 = 0$, $a_3 = -2$, $a_4 = -1$, $b_1 = 1$, $b_2 = b_3 = 0$, $b_4 = -1$, $b_5 = 1$ it has six variations of sign, that is, $p_9(r_0)$ can have at most six positive real roots. So the PSDS separated by the straight line $y = 0$ with the vector fields X_4 and Y_5 can have at most six limit cycles. Statament (k) of Theorem (3.1) is proved.

3.1.12 Case $(\mathbf{m}, \mathbf{n}) = (4, 6)$

We consider the PSDS separated by the straight line $y = 0$ and formed by the two vector fields X_4 defined in (3-21) and Y_6 defined in (3-17). We know their solutions $r_1(\theta)$ such that $r_1(0) = r_0$ and $r_2(\theta)$ such that $r_2(\pi) = r_1(\pi)$. Then the numerator of the equation $r_2(2\pi)^5 - r_0^5 = 0$ is $r_0^5(A - B) = 0$ where

$$A = (1 + 10\lambda^2 + 9\lambda^4 - (1 + e^{3\pi\lambda})r_0^3(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))) (9 + 259\lambda^2 + 875\lambda^4 + 625\lambda^6) (-1 - 10\lambda^2 - 9\lambda^4 + (1 + e^{3\pi\lambda})r_0^3(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)))^{2/3},$$

$$B = e^{5\pi\lambda}(1 + \lambda^2)^{2/3}(1 + 9\lambda^2)^{2/3}(1 + 10\lambda^2 + 9\lambda^4) (e^{5\pi\lambda}(1 + \lambda^2)(1 + 25\lambda^2)(9 + 25\lambda^2) + (r_0^5 + e^{5\pi\lambda}r_0^5) (-6(b_4 + 4b_6) + (149b_1 + 26b_3 + 24b_5)\lambda - b_2(9 + 110\lambda^2 + 125\lambda^4) + 5\lambda^2(-6b_4 + 5\lambda(2b_3 + 5b_1(6 + 5\lambda^2))))).$$

Then we consider the equation $A^3 - B^3 = 0$ and we get the polynomial

$$p_{10}(r_0) = C_{0,10} + C_{3,10}r_0^3 + C_{5,10}r_0^5 + C_{6,10}r_0^6 + C_{9,10}r_0^9 + C_{10,10}r_0^{10} + C_{12,10}r_0^{12} + C_{15,10}r_0^{15},$$

where

$$C_{0,10} = (1 - e^{30\pi\lambda})(1 + \lambda^2)^8(1 + 9\lambda^2)^5(9 + 250\lambda^2 + 625\lambda^4)^3,$$

$$C_{3,10} = -5(1 + e^{3\pi\lambda})(1 + \lambda^2)^7(1 + 9\lambda^2)^4(9 + 250\lambda^2 + 625\lambda^4)^3 (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)),$$

$$C_{5,10} = 3e^{25\pi\lambda}(1 + e^{5\pi\lambda})(1 + \lambda^2)^7(1 + 9\lambda^2)^5(9 + 250\lambda^2 + 625\lambda^4)^2(24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda(149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4)),$$

$$\begin{aligned}
C_{6,10} &= 10(1 + e^{3\pi\lambda})^2(1 + \lambda^2)^6(1 + 9\lambda^2)^3(9 + 250\lambda^2 + 625\lambda^4)^3 (a_2 + 2a_4 + 3a_2\lambda^2 \\
&\quad - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^2, \\
C_{9,10} &= -10(1 + e^{3\pi\lambda})^3(1 + \lambda^2)^5(1 + 9\lambda^2)^2(9 + 250\lambda^2 + 625\lambda^4)^3 (a_2 + 2a_4 + 3a_2\lambda^2 \\
&\quad - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^3, \\
C_{10,10} &= -3e^{20\pi\lambda}(1 + e^{5\pi\lambda})^2(1 + \lambda^2)^6(1 + 9\lambda^2)^5(9 + 250\lambda^2 + 625\lambda^4) \\
&\quad (24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda(149b_1 + 26b_3 + 24b_5 \\
&\quad + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4))^2, \\
C_{12,10} &= 5(1 + e^{3\pi\lambda})^4(1 + \lambda^2)^4(1 + 9\lambda^2)(9 + 250\lambda^2 + 625\lambda^4)^3 (a_2 + 2a_4 + 3a_2\lambda^2 \\
&\quad - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^4, \\
C_{15,10} &= -(1 + e^{\pi\lambda})(1 + \lambda^2)^3 ((1 + e^{3\pi\lambda})^2(1 - e^{\pi\lambda} + e^{2\pi\lambda})^5(9 + 250\lambda^2 + 625\lambda^4)^3 \\
&\quad (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^5 + (1 - e^{\pi\lambda} + e^{2\pi\lambda} - e^{3\pi\lambda} \\
&\quad + e^{4\pi\lambda})^3 (- (24b_6 + 6b_4(1 + \lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda(149b_1 + 26b_3 \\
&\quad + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4)))^3 e^{15\pi\lambda}(1 + \lambda^2)^2(1 + 9\lambda^2)^5).
\end{aligned}$$

Since we are assuming that C_i is not null for all i , then we have that $C_6 > 0$, $C_{10} < 0$ and $C_{12} > 0$. In this way, note that between C_0 , C_3 , C_5 and C_6 there are at most three variations of sign. Now between C_6 , C_9 , C_{10} and C_{12} there are at most two variations of sign and between C_{12} and C_{15} there is at most one variation. Therefore there are at most six variations of sign. For example, consider $\lambda = 0.1$, $a_1 = 0$, $a_2 = -2$, $a_3 = 1$, $a_4 = 1$, $b_1 = b_2 = 0$, $b_3 = 1$, $b_4 = -0.5$, $b_5 = -13/12$ and $b_6 = -1$, with this conditions there are six variations of sign. So by the Descartes' Theorem the polynomial $p_{10}(r_0)$ can have at most six positive real roots. Thus, the PSDS separated by the straight line $y = 0$ and formed by the two vector fields X_4 and Y_6 can have at most 6 limit cycles. Statement (I) of Theorem 3.1 is proved.

3.1.13 Case $(\mathbf{m}, \mathbf{n}) = (5, 5)$

Now consider the PSDS separated by the straight line $y = 0$ and formed by the two vector fields Y_5 defined in (3-16) and

$$X_5(x, y) = \begin{cases} X_5^1(x, y) = -y + x(\lambda + a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4), \\ X_5^2(x, y) = x + y(\lambda + a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4). \end{cases} \quad (3-23)$$

Thus the solution $r_1(\theta)$ of the vector field X_5 in polar coordinates such that $r_1(0) = r_0$ is

$$r_1(\theta) = \frac{2^{3/4}e^{\theta\lambda}\lambda^{1/4}(1 + 5\lambda^2 + 4\lambda^4)^{1/4}r_0}{E_7^{1/4}}, \quad (3-24)$$

where

$$\begin{aligned} E_7 = & 8(\lambda + 5\lambda^3 + 4\lambda^5) - r_0^4 (e^{4\pi\lambda} (-3a_5 - a_3(1 + 4\lambda^2) - a_1(3 + 32\lambda^2 + 32\lambda^4) \\ & + \lambda(3a_4 + a_2(5 + 8\lambda^2))) + e^{4\theta\lambda} (3a_1 + a_3 + 3a_5 + (3a_1 + a_3 + 3a_5)\lambda^2(5 + 4\lambda^2) \\ & - 4\lambda(a_2 + a_4 + 4(-a_1 + a_5)\lambda)(1 + \lambda^2) \cos 2\theta + \lambda(-a_2 + a_4 + (a_1 - a_3 + a_5)\lambda) \\ & (1 + 4\lambda^2) \cos 4\theta + 8\lambda(a_1 - a_5 + (a_2 + a_4)\lambda)(1 + \lambda^2) \sin 2\theta + \lambda(a_1 - a_3 \\ & + a_5 + a_2\lambda - a_4\lambda)(1 + 4\lambda^2) \sin 4\theta)). \end{aligned}$$

We consider the solution $r_2(\theta)$ of Y_5 such that $r_2(\pi) = r_1$ and (3-24), then we assume that $r_1 = r_1(\pi) = r_2(\pi)$. Thus we consider the numerator of the equation $(r_2(2\pi))^4 - r_0^4 = 0$ and we obtain $(1 - e^{4\pi\lambda})r_0^4 p_{11}(r_0)$, being $p_{11}(r_0)$ the following polynomial of degree 4 in r_0

$$p_{11}(r_0) = C_{0,11} + C_{4,11}r_0^4,$$

where

$$\begin{aligned} C_{0,11} = & 8(1 + e^{4\pi\lambda})(\lambda + 5\lambda^3 + 4\lambda^5); \\ C_{4,11} = & 3a_1 + a_3 + 3a_5 - (5a_2 + 3a_4)\lambda + (32a_1 + 4a_3)\lambda^2 - 8a_2\lambda^3 + 32a_1\lambda^4 \\ & + e^{4\pi\lambda} (3b_1 + b_3 + 3b_5 - (5b_2 + 3b_4)\lambda + (32b_1 + 4b_3)\lambda^2 - 8b_2\lambda^3 + 32b_1\lambda^4). \end{aligned}$$

The polynomial $p_{11}(r_0)$ can have one real positive root. For instance, for the values $\lambda = -0.1$, $a_1 = 2.83501$, $a_2 = 9$, $a_3 = -9$, $a_4 = 1$, $a_5 = -1$, $b_1 = b_2 = b_3 = b_4 = 0$ and $b_5 = -1$ it has one real positive root $r_0 = 1$, see Figure 3.3. With theses values the denominator of the equation $(r_2(2\pi))^4 - r_0^4$ is different from zero. So the PSDS by the straight line $y = 0$ with the vector fields X_5 and Y_5 can have one limit cycle. Statement (m) of Theorem 3.1 is proved.

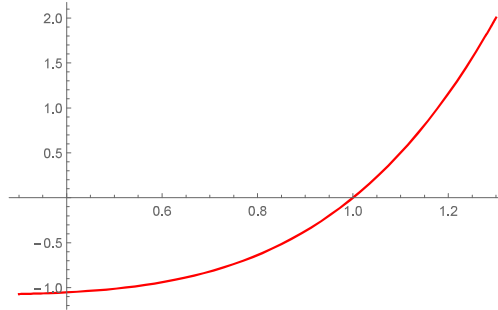


Figure 3.3: Real positive root for case $(\mathbf{m}, \mathbf{n}) = (\mathbf{5}, \mathbf{5})$.

3.1.14 Case $(\mathbf{m}, \mathbf{n}) = (5, 6)$

Now consider the PSDS separated by the straight line $y = 0$ and formed by the two vector fields X_5 defined in (3-23) and Y_6 defined in (3-17). We consider their solutions $r_1(\theta)$ such that $r_1(0) = r_0$ and $r_2(\theta)$ such that $r_2(\pi) = r_1$. Then the numerator of the equation $(r_2(2\pi))^5 - r_0^5 = 0$ is $r_0^5(A - B) = 0$ where

$$\begin{aligned} A &= 8 \times 2^{3/4} e^{10\pi\lambda} \lambda^{5/4} (1 + \lambda^2)^2 (1 + 5\lambda^2 + 4\lambda^4)^{5/4} (9 + 286\lambda^2 + 1625\lambda^4 + 2500\lambda^6) \\ &\quad + 8 \times 2^{3/4} e^{5\pi\lambda} (1 + e^{5\pi\lambda}) r_0^5 \lambda^{5/4} (1 + 5\lambda^2 + 4\lambda^4)^{5/4} ((149b_1 + 26b_3 + 24b_5)\lambda \\ &\quad - 6(b_4 + 4b_6) - b_2(9 + 110\lambda^2 + 125\lambda^4) + 5\lambda^2(-6b_4 + 5\lambda(2b_3 + 5b_1(6 + 5\lambda^2))))), \\ B &= (9 + 259\lambda^2 + 875\lambda^4 + 625\lambda^6) (8\lambda + 40\lambda^3 + 32\lambda^5 + (1 - e^{4\pi\lambda})r_0^4(a_3 + 3a_5 \\ &\quad - (5a_2 + 3a_4)\lambda + 4a_3\lambda^2 - 8a_2\lambda^3 + a_1(3 + 32\lambda^2 + 32\lambda^4)))^{5/4}. \end{aligned}$$

Now consider the equation $A^4 - B^4 = 0$ and we obtain the polynomial

$$\begin{aligned} p_{12}(r_0) &= C_{0,12} + C_{4,12}r_0^4 + C_{5,12}r_0^5 + C_{8,12}r_0^8 + C_{10,12}r_0^{10} + C_{12,12}r_0^{12} + C_{15,12}r_0^{15} \\ &\quad + C_{16,12}r_0^{16} + C_{20,12}r_0^{20}, \end{aligned}$$

where

$$\begin{aligned} C_{0,12} &= 32768(-1 + e^{40\pi\lambda})(1 + \lambda^2)^9(\lambda + 4\lambda^3)^5(9 + 250\lambda^2 + 625\lambda^4)^4, \\ C_{4,12} &= 20480(-1 + e^{4\pi\lambda})\lambda^4(1 + \lambda^2)^8(1 + 4\lambda^2)^4(9 + 250\lambda^2 + 625\lambda^4)^4(a_3 + 3a_5 - \\ &\quad \lambda(5a_2 + 3a_4) + 4a_3\lambda^2 - 8a_2\lambda^3 + a_1(3 + 32\lambda^2 + 32\lambda^4)), \\ C_{5,12} &= -131072e^{35\pi\lambda}(1 + e^{5\pi\lambda})(1 + \lambda^2)^8(9 + 250\lambda^2 + 625\lambda^4)^3(24b_6 + 6b_4(1 + 5\lambda^2) \\ &\quad + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda(149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 \\ &\quad + 625b_1\lambda^4))(\lambda + 4\lambda^3)^5, \\ C_{8,12} &= -5120(-1 + e^{4\pi\lambda})^2(1 + \lambda^2)^7(\lambda + 4\lambda^3)^3(9 + 250\lambda^2 + 625\lambda^4)^4(a_3 + 3a_5 - \lambda \\ &\quad (5a_2 + 3a_4) + 4a_3\lambda^2 - 8a_2\lambda^3 + a_1(3 + 32\lambda^2 + 32\lambda^4))^2, \\ C_{10,12} &= 196608e^{30\pi\lambda}(1 + e^{5\pi\lambda})^2(1 + \lambda^2)^7(24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 \\ &\quad + 125\lambda^4) - \lambda(149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4))^2 \\ &\quad (9 + 250\lambda^2 + 625\lambda^4)^2(\lambda + 4\lambda^3)^5, \\ C_{12,12} &= 640(-1 + e^{4\pi\lambda})^3(1 + \lambda^2)^6(9 + 250\lambda^2 + 625\lambda^4)^4(a_3 + 3a_5 - (5a_2 + 3a_4)\lambda \\ &\quad + 4a_3\lambda^2 - 8a_2\lambda^3 + a_1(3 + 32\lambda^2 + 32\lambda^4))^3(\lambda + 4\lambda^3)^2, \\ C_{15,12} &= -131072e^{25\pi\lambda}(24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) \\ &\quad - \lambda(149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4))^3(1 + e^{5\pi\lambda})^3 \\ &\quad (9 + 250\lambda^2 + 625\lambda^4)(1 + \lambda^2)^6(\lambda + 4\lambda^3)^5, \end{aligned}$$

$$\begin{aligned}
C_{16,12} &= -40(-1 + e^{4\pi\lambda})^4(1 + 4\lambda^2)(9 + 250\lambda^2 + 625\lambda^4)^4(a_3 + 3a_5 - (5a_2 + 3a_4)\lambda \\
&\quad + 4a_3\lambda^2 - 8a_2\lambda^3 + a_1(3 + 32\lambda^2 + 32\lambda^4))^4\lambda(1 + \lambda^2)^5, \\
C_{20,12} &= (-1 + e^{4\pi\lambda})^5(1 + \lambda^2)^4(9 + 250\lambda^2 + 625\lambda^4)^4(a_3 + 3a_5 - (5a_2 + 3a_4)\lambda + 4a_3\lambda^2 \\
&\quad - 8a_2\lambda^3 + a_1(3 + 32\lambda^2 + 32\lambda^4))^5 + 32768e^{20\pi\lambda}(1 + e^{5\pi\lambda})^4(1 + 5\lambda^2 + 4\lambda^4)^5 \\
&\quad \lambda^5(24b_6 + 6b_4(1 + 5\lambda^2) + b_2(9 + 110\lambda^2 + 125\lambda^4) - \lambda(149b_1 + 26b_3 + 24b_5 \\
&\quad + 50(15b_1 + b_3)\lambda^2 + 625b_1\lambda^4))^4.
\end{aligned}$$

By Descartes' Theorem the polynomial $p_{12}(r_0)$ can have at most eight positive real roots. For instance, for the values $\lambda = 1$, $a_1 = -1$, $a_2 = 0$, $a_3 = 0$, $a_4 = 1$, $a_5 = -1$, $b_1 = b_3 = b_4 = 0$, $b_2 = b_6 = -1$ and $b_5 = 1$ there are eight variations of sign between the coefficients, that is, $p_{12}(r_0)$ can have at most eight positive real roots. Therefore the PSDS separated by the straight line $y = 0$ with the vector fields X_5 and Y_6 can have at most eight limit cycles. Statement (n) of Theorem 3.1 is proved.

3.2 Brief conclusion

We study the maximum number of limit cycles that the PSDS formed by (3-1) and (3-2) of degree m and n , respectively, with $m \leq n$ can have. We specifically analyze the cases where $m, n \in \{2, 3, 4, 5, 6\}$. To this end, we study the number of zeros of the equation $r_1(\theta) - r_0 = 0$. Using Descartes' Theorem, we determine the maximum number of positive real roots that the polynomial corresponding to the equation $r_1(\theta) - r_0 = 0$ can have. These roots correspond to the limit cycles of the system.

In Chapter 4, considering the previous system, but now with PSDS being hybrid, we will study an upper bound for the maximum number of limit cycles that can appear. Additionally, a more general version of the previous system will be analyzed, where the parameters in the expressions of P and Q are different.

Limit cycles of discontinuous piecewise hybrid rigid systems separated by a straight line

This chapter is based on [37], and our objective is to provide an upper bound for the maximum number of limit cycles in piecewise hybrid planar smooth dynamical systems separated by $y = 0$; i.e., to address the extended 16th Hilbert problem for that classes of planar hybrid piecewise smooth dynamical systems.

Here, we consider a piecewise hybrid dynamical system $(X_m, Y_n, \Sigma, \mathcal{R})$ in $\mathbb{R}^2$, formed by two rigid systems, $X_m = (X_m^1(x, y), X_m^2(x, y))$ and $Y_n = (Y_n^1(x, y), Y_n^2(x, y))$ of degrees m and n respectively. Σ is given by $\Sigma = \{(x, y) \in \mathbb{R}^2 : y = 0\}$, and we consider the reset map given by: $\mathcal{R}(x) = sx$ for $x < 0$ and $\mathcal{R}(x) = ux$ for $x > 0$, with $s, u \in (0, 1)$, see [34]. For this hybrid system we define the set of admissible orbits as follows: starting at $(x_1, 0) \in \Sigma$, with $x_1 > 0$, an orbit follows X_m in the half-plane $y \geq 0$ until reaching $(x_2, 0)$ with $x_2 < 0$. The reset rule $\mathcal{R}$ moves the system to $(x_3, 0) = (sx_2, 0)$. The orbit then follows Y_n in $y \leq 0$, returning to $(x_4, 0)$ with $x_4 > 0$, followed by the reset rule $\mathcal{R}$ to $(x_5, 0) = (ux_4, 0)$. This process repeats indefinitely. If $x_5 = x_1$, then the admissible orbit is periodic, see Figure 4.1. If a periodic orbit is isolated in the set of all periodic orbits, it is a limit cycle.

We deal with the discontinuous piecewise hybrid systems formed by the following two rigid polynomial differential systems

$$\begin{aligned} \dot{x} &= -y + x(\lambda + P(x, y)) = X_m^1(x, y), \\ \dot{y} &= x + y(\lambda + P(x, y)) = X_m^2(x, y), \end{aligned} \tag{4-1}$$

in $y \geq 0$, where $P(x, y)$ is a homogeneous polynomial of degree $m - 1$, and:

$$\begin{aligned} \dot{x} &= -y + x(\mu + Q(x, y)) = Y_n^1(x, y), \\ \dot{y} &= x + y(\mu + Q(x, y)) = Y_n^2(x, y), \end{aligned} \tag{4-2}$$

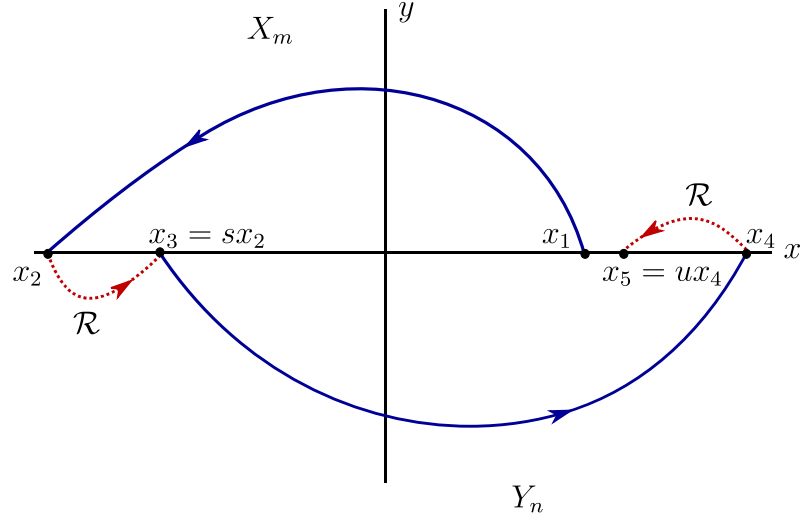


Figure 4.1: The admissible orbit of the hybrid system $\mathcal{X}$.

in $y \leq 0$, where $Q(x, y)$ is a homogeneous polynomial of degree $n - 1$. Our main result is the following one.

Theorem 4.1. *Consider the discontinuous piecewise hybrid systems separated by the straight line $y = 0$ and formed by the two rigid polynomial differential systems (4-1) and (4-2) of degrees m and n respectively, with $m \leq n$. Then the piecewise hybrid system has*

- (a) at most 1 limit cycle if $m = n = 2$;
- (b) at most 2 limit cycles if $m = 2$ and $n = 3$;
- (c) at most 3 limit cycles if $m = 2$ and $n = 4$;
- (d) at most 4 limit cycles if $m = 2$ and $n = 5$;
- (e) at most 5 limit cycles if $m = 2$, and $n = 6$;
- (f) at most 1 limit cycle if $m = n = 3$;
- (g) at most 4 limit cycles if $m = 3$ and $n = 4$;
- (h) at most 2 limit cycles if $m = 3$ and $n = 5$;
- (i) at most 6 limit cycles if $m = 3$ and $n = 6$;
- (j) at most 1 limit cycle if $m = 4$ and $n = 4$;
- (k) at most 6 limit cycles if $m = 4$ and $n = 5$;
- (l) at most 10 limit cycle if $m = 4$ and $n = 6$;
- (m) at most 1 limit cycles if $m = n = 5$;
- (n) at most 8 limit cycles if $m = 5$ and $n = 6$.

These results are summarized in Table 4.1.

$m \backslash n$	2	3	4	5	6
2	1	2	3	4	5
3	X	1	4	2	6
4	X	X	1	6	10
5	X	X	X	1	8

Table 4.1: Upper bounds for the maximum number of limit cycles of the piecewise hybrid differential systems formed by systems (4-1) and (4-2).

Remark 4.2. The symbol “X” denotes cases where $m \geq n$, which are omitted due to the symmetry of the systems. The study of the case (n, m) is equivalent to the case (m, n) by switching systems (4-1) and (4-2).

The rest of the chapter is dedicated to prove Theorem 4.1.

4.1 Proof of Theorem 4.1

If the degrees of the polynomials $P(x, y)$ and $Q(x, y)$ are ≥ 2 , then the eigenvalues of the linear part of the differential systems (4-1) and (4-2) at the origin of coordinates are $\lambda \pm i$ and $\mu \pm i$, respectively. Then the origin of these systems is a hyperbolic focus if $\lambda \neq 0$ and $\mu \neq 0$, and it is a weak focus or a center if $\lambda = 0$ and $\mu = 0$.

Let $r_1(\theta)$ (resp. $r_2(\theta)$) be the solution of system (4-1) (resp. (4-2)) in polar coordinates. Note that in polar coordinates $(x_1, 0) = (r_1(0) \cos 0, r_1(0) \sin 0)$ with $x_1 = r_1(0)$. The point $(x_2, 0) = (r_1(\pi) \cos \pi, r_1(\pi) \sin \pi)$ satisfies $x_2 = -r_1(\pi) < 0$. Then $x_3 = sx_2$ and $(x_3, 0) = (r_2(\pi) \cos \pi, r_2(\pi) \sin \pi)$, with $x_3 = -r_2(\pi) < 0$. Now the point $(x_4, 0) = (r_2(2\pi) \cos(2\pi), r_2(2\pi) \sin(2\pi))$ satisfies $x_4 = r_2(2\pi) > 0$, and $(x_5, 0) = (ux_4, 0)$.

4.1.1 Case $(m, n) = (2, 2)$

Consider the piecewise hybrid system separated by $y = 0$ and formed by the two rigid systems defined by the vector fields

$$X_2(x, y) = \begin{cases} X_2^1(x, y) = -y + x(\lambda + a_1x + a_2y), \\ X_2^2(x, y) = x + y(\lambda + a_1x + a_2y), \end{cases} \quad (4-3)$$

and

$$Y_2(x, y) = \begin{cases} Y_2^1(x, y) = -y + x(\mu + b_1x + b_2y), \\ Y_2^2(x, y) = x + y(\mu + b_1x + b_2y). \end{cases} \quad (4-4)$$

In polar coordinates (r, θ) , where $x = r \cos \theta$ and $y = r \sin \theta$, system (4-3) becomes

$$\dot{r} = r\lambda + r^2(a_1 \cos \theta + a_2 \sin \theta), \quad \dot{\theta} = 1,$$

or equivalently, the differential equation

$$r' = \frac{dr}{d\theta} = r\lambda + r^2(a_1 \cos \theta + a_2 \sin \theta).$$

Thus, the solution $r_1(\theta)$ of this last differential equation such that $r_1(0) = x_1$ is

$$r_1(\theta) = \frac{e^{\theta\lambda}x_1(1 + \lambda^2)}{1 - a_2x_1 + \lambda(a_1x_1 + \lambda) + e^{\theta\lambda}x_1((a_2 - a_1\lambda) \cos \theta - (a_1 + a_2\lambda) \sin \theta)}.$$

Similarly, system (4-4) in polar coordinates, taking θ as the new independent variable, can be written as

$$r' = \frac{dr}{d\theta} = r\mu + r^2(b_1 \cos \theta + b_2 \sin \theta),$$

and its solution $r_2(\theta)$ such $r_2(\pi) = x_3 = sx_2 = sr_1(\pi)$ is

$$r_2(\theta) = \frac{e^{\theta\mu}x_3(1 + \mu^2)}{e^{\pi\mu}(1 + b_2x_3 - b_1x_3\mu + \mu^2) + e^{\theta\mu}x_3((b_2 - b_1\mu) \cos \theta - (b_1 + b_2\mu) \sin \theta)}.$$

In order to study the limit cycles of the piecewise hybrid system formed by the vector fields X_2 and Y_2 we must solve the equation $x_5 - x_1 = ux_4 - x_1 = ur_2(2\pi) - x_1 = 0$, i.e.,

$$\frac{e^{\pi(\lambda+\mu)}sux_1(\lambda^2 + 1)(\mu^2 + 1)}{(1 + \lambda^2)(1 + \mu^2) + x_1(e^{\pi\lambda}(1 + e^{\pi\mu})s(1 + \lambda^2)(b_2 - b_1\mu) - (1 + e^{\pi\lambda})(a_2 - a_1\lambda)(1 + \mu^2))} - x_1 = 0,$$

in terms of x_1 . The solutions of this equation are

$$x_1^1 = 0 \quad \text{and} \quad x_1^2 = \frac{(1 - e^{\pi(\lambda+\mu)}su)(\lambda^2 + 1)(\mu^2 + 1)}{(1 + e^{\pi\lambda})(a_2 - a_1\lambda)(1 + \mu^2) + e^{\pi\lambda}(1 + e^{\pi\mu})s(b_1\mu - b_2)(1 + \lambda^2)}.$$

Since the origin is always an equilibrium point of the system, we only have the solution x_1^2 providing a periodic orbit when $x_1^2 > 0$. For instance, for values $\lambda = \mu = 0$, $u = s = 99/100$, $a_1 = 0$, $a_2 = 3$, $b_1 = b_2 = 1$, we have $x_5 - x_1 = 0$, with

$x_1^2 = 0.00495025 > 0$. Since, the piecewise hybrid differential system has only one periodic orbit, this is a limit cycle and statement (a) of Theorem 3.1 is proved.

4.1.2 Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{2}, \mathbf{3})$

Consider the piecewise hybrid system formed by the vector fields X_2 and

$$Y_3(x, y) = \begin{cases} Y_3^1(x, y) = -y + x(\mu + b_1x^2 + b_2xy + b_3y^2), \\ Y_3^2(x, y) = x + y(\mu + b_1x^2 + b_2xy + b_3y^2). \end{cases} \quad (4-5)$$

This vector field in polar coordinates provides the differential system

$$\dot{r} = r\mu + \frac{1}{2}r^3(b_1 + b_3 + (b_1 - b_3)\cos(2\theta) + b_2\sin(2\theta)), \quad \dot{\theta} = 1,$$

or, equivalently, the differential equation

$$r' = \frac{dr}{d\theta} = r\mu + \frac{1}{2}r^3(b_1 + b_3 + (b_1 - b_3)\cos(2\theta) + b_2\sin(2\theta)).$$

The solution $r_2(\theta)$ of this differential equation that such $r_2(\pi) = x_3 = sx_2 = sr_1(\pi)$ is

$$r_2(\theta) = e^{\theta\mu} \sqrt{\frac{2\mu(1 + \mu^2)}{H_{11}/x_3^2 + H_{12}}}, \quad (4-6)$$

with

$$\begin{aligned} H_{11} &= e^{2\pi\mu}((b_1 + b_3)x_3^2 + (2 - b_2x_3^2)\mu + 2b_1x_3^2\mu^2 + 2\mu^3), \\ H_{12} &= -e^{2\theta\mu}((b_1 + b_3)(1 + \mu^2) + \mu(-b_2 + (b_1 - b_3)\mu)\cos(2\theta) + \mu(b_1 - b_3 \\ &\quad + b_2\mu)\sin(2\theta)). \end{aligned}$$

To study the maximum number of limit cycles, we examine the zeros of the equation $x_5 - x_1 = 0$. In this case, we will equivalently study the numerator of the equation $x_5^2 - x_1^2 = (x_5 - x_1)(x_5 + x_1)$, since the upper bound for the number of solutions of the equation $x_5^2 - x_1^2 = 0$ also provides an upper bound for the number of solutions of the equation $x_5 - x_1 = 0$.

Thus, we have that the limit cycles satisfy the equation $x_5^2 - x_1^2 = 0$, where $x_5 = ux_4 = ur_2(2\pi)$. We consider its numerator, which is equal to $-x_1^2P_1(x_1)$, where $P_1(x_1) = C_{0,1} + C_{1,1}x_1 + C_{2,1}x_1^2$, with

$$\begin{aligned} C_{0,1} &= 2(e^{2\pi\mu} - e^{2\pi(\lambda+2\mu)}s^2u^2)\mu(\lambda^2 + 1)^2(\mu^2 + 1), \\ C_{1,1} &= 4e^{2\pi\mu}\mu(1 + e^{\pi\lambda})(a_1\lambda - a_2)(\lambda^2 + 1)(\mu^2 + 1), \end{aligned}$$

$$C_{2,1} = 2e^{2\pi\mu}\mu(1 + e^{\pi\lambda})^2(a_2 - a_1\lambda)^2(1 + \mu^2) - e^{2\pi(\lambda+\mu)}(-1 + e^{2\pi\mu})s^2(1 + \lambda^2)^2(b_1 + b_3 - b_2\mu + 2b_1\mu^2).$$

The polynomial $P_1(x_1)$ can have two real positive roots. So the piecewise hybrid system separated by the straight line $y = 0$ with the vector fields X_2 and Y_3 can have at most two limit cycles. Statement (b) in Theorem 4.1 was proved.

4.1.3 Case $(\mathbf{m}, \mathbf{n}) = (2, 4)$

Now consider the piecewise hybrid system separated by the straight line $y = 0$ and formed by the vector field X_2 and the vector field

$$Y_4(x, y) = \begin{cases} Y_4^1(x, y) = -y + x(\mu + b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3), \\ Y_4^2(x, y) = x + y(\mu + b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3). \end{cases} \quad (4-7)$$

The vector field Y_4 in polar coordinates is given by the differential system

$$\dot{r} = r\mu + r^4(b_1 \cos^3 \theta + b_2 \cos^2 \theta \sin \theta + b_3 \cos \theta \sin^2 \theta + b_4 \sin^3 \theta), \quad \dot{\theta} = 1,$$

or equivalently the differential equation

$$r' = \frac{dr}{d\theta} = r\mu + r^4(b_1 \cos^3 \theta + b_2 \cos^2 \theta \sin \theta + b_3 \cos \theta \sin^2 \theta + b_4 \sin^3 \theta).$$

Then the solution $r_2(\theta)$ of this differential equation such that $r_2(\pi) = x_3 = sx_2 = r_1(\pi)$ is

$$r_2(\theta) = \frac{2^{2/3}e^{\theta\mu}x_3((1 + \mu^2)(1 + 9\mu^2))^{1/3}}{H_2^{1/3}}, \quad (4-8)$$

with

$$\begin{aligned} H_2 = & 4e^{3\pi\mu}(1 + \mu^2)(1 + 9\mu^2) + 4e^{3\pi\mu}x_3^3(b_2 + 2b_4 + 3b_2\mu^2 - \mu(7b_1 + 2b_3 + 9b_1\mu^2)) \\ & - e^{3\theta\mu}x_3^3(-3(b_2 + 3b_4 - 3(3b_1 + b_3)\mu)(1 + \mu^2) \cos \theta + (-b_2 + b_4 + b_1\mu - b_3\mu) \\ & (1 + 9\mu^2) \cos 3\theta + 3(3b_1 + b_3 + 3b_2\mu + 9b_4\mu)(1 + \mu^2) \sin \theta \\ & + (b_1 - b_3 + b_2\mu - b_4\mu)(1 + 9\mu^2) \sin 3\theta). \end{aligned}$$

The limit cycles satisfy the equation $x_5^3 - x_1^3 = 0$, where $x_5 = ux_4 = ur_2(2\pi)$. Then we consider the numerator of this equation and we obtain the polynomial $x_1^3 P_2(x_1) = x_1^3(C_{0,2} + C_{1,2}x_1 + C_{2,2}x_1^2 + C_{3,2}x_1^3)$, where

$$\begin{aligned} C_{0,2} = & (e^{3\pi\mu} - e^{3\pi(\lambda+2\mu)}s^3u^3)(1 + \lambda^2)^3(1 + \mu^2)(1 + 9\mu^2), \\ C_{1,2} = & -3(e^{3\pi\mu} + e^{\pi(\lambda+3\mu)})(a_2 - a_1\lambda)(1 + \lambda^2)^2(1 + \mu^2)(1 + 9\mu^2), \end{aligned}$$

$$\begin{aligned}
C_{2,2} &= 3e^{3\pi\mu}(1 + e^{\pi\lambda})^2(a_2 - a_1\lambda)^2(1 + \lambda^2)(1 + \mu^2)(1 + 9\mu^2), \\
C_{3,2} &= -3e^{3\pi\mu}(1 + e^{\pi\lambda})^3(a_1\lambda - a_2)^3(1 + 10\mu^2 + 9\mu^2) + (e^{3\pi(\lambda+\mu)} + e^{3\pi(\lambda+2\mu)})s^3 \\
&\quad (1 + \lambda^2)^3(b_2 + 2b_4 + 3b_2\mu^2 - \mu(7b_1 + 2b_3 + 9b_1\mu^2)).
\end{aligned}$$

The polynomial $P_2(x_1)$ can have at most three real positive roots. Therefore the piecewise hybrid system separated by the straight line $y = 0$ and formed by the vector fields X_2 and Y_4 can have at most three limit cycles. Statement (c) of Theorem 4.1 is proved.

4.1.4 Case $(\mathbf{m}, \mathbf{n}) = (2, 5)$

We consider the piecewise hybrid system separated by straight line $y = 0$ and formed by the two vector fields X_2 and

$$Y_5(x, y) = \begin{cases} Y_5^1(x, y) = -y + x(\mu + b_1x^4 + b_2x^3y + b_3x^2y^2 + b_4xy^3 + b_5y^4), \\ Y_5^2(x, y) = x + y(\mu + b_1x^4 + b_2x^3y + b_3x^2y^2 + b_4xy^3 + b_5y^4). \end{cases} \quad (4-9)$$

The vector field (4-9) provides in polar coordinates the differential system

$$\begin{aligned}
\dot{r} &= r\mu + r^5(b_1 \cos^4 \theta + b_2 \cos^3 \theta \sin \theta + b_3 \cos^2 \theta \sin^2 \theta + b_4 \cos \theta \sin^3 \theta + b_5 \sin^4 \theta) \\
\dot{\theta} &= 1
\end{aligned}$$

or equivalently

$$r' = \frac{dr}{d\theta} = r\mu + r^5(b_1 \cos^4 \theta + b_2 \cos^3 \theta \sin \theta + b_3 \cos^2 \theta \sin^2 \theta + b_4 \cos \theta \sin^3 \theta + b_5 \sin^4 \theta).$$

The solution $r_2(\theta)$ of this differential equation such that $r_2(\pi) = x_3$ is

$$r_2(\theta) = \frac{2^{3/4}e^{\theta\mu}\mu^{1/4}(1 + 5\mu^2 + 4\mu^4)^{1/4}x_3}{H_3^{1/4}} \quad (4-10)$$

where

$$\begin{aligned}
H_3 &= 8e^{4\pi\mu}(\mu + 5\mu^3 + 4\mu^5) - x_3^4(-e^{4\pi\mu}(b_3 + 3b_5 - (5b_2 + 3b_4)\mu + 4(b_3 - 2b_2\mu)\mu^2 \\
&\quad + b_1(3 + 32\mu^2 + 32\mu^4)) + e^{40\mu}((3b_1 + b_3 + 3b_5)(1 + 5\mu^2 + 4\mu^4) \\
&\quad - 4\mu(b_2 + b_4 + 4(-b_1 + b_5)\mu)(1 + \mu^2) \cos(2\theta) + \mu(-b_2 + b_4 + (b_1 - b_3 + b_5)\mu) \\
&\quad (1 + 4\mu^2) \cos(4\theta) + 8\mu(b_1 - b_5 + (b_2 + b_4)\mu)(1 + \mu^2) \sin(2\theta) + \mu(b_1 - b_3 \\
&\quad + b_5 + b_2\mu - b_4\mu)(1 + 4\mu^2) \sin(4\theta))).
\end{aligned}$$

Recall that $x_2 = r_1(\pi)$, $x_3 = sx_2 = r_2(\pi)$. Considering the numerator of the equation $x_5^4 - x_1^4 = 0$, where $x_5 = ux_4 = ur_2(2\pi)$, we obtain $x_1^4 P_3(x_1)$, where

$P_3(x_1) = C_{0,3} + C_{1,3}x_1 + C_{2,3}x_1^2 + C_{3,3}x_1^3 + C_{4,3}x_1^4$ and

$$\begin{aligned} C_{0,3} &= 8(e^{4\pi(\lambda+2\mu)}s^4u^4 - e^{4\pi\mu})\mu(1+\lambda^2)^4(1+5\mu^2+4\mu^4), \\ C_{1,3} &= 32\mu e^{4\pi\mu}(1+e^{\pi\lambda})(a_2-a_1\lambda)(1+\lambda^2)^3(1+\mu^2)(1+4\mu^2), \\ C_{2,3} &= -48e^{4\pi\mu}\mu(1+e^{\pi\lambda})^2(a_2-a_1\lambda)^2(1+\lambda^2)^2(1+\mu^2)(1+4\mu^2), \\ C_{3,3} &= 32e^{4\pi\mu}\mu(1+e^{\pi\lambda})^3(a_2-a_1\lambda)^3(1+\lambda^2)(1+\mu^2)(1+4\mu^2), \\ C_{4,3} &= -8e^{4\pi\mu}\mu(1+e^{\pi\lambda})^4(a_2-a_1\lambda)^4(1+\mu^2)(1+4\mu^2) + e^{4\pi(\lambda+\mu)}(e^{4\pi\mu}-1)(1+\lambda^2)^4 \\ &\quad (b_3+3b_5-(5b_2+3b_4)\mu+(4b_3-8b_2\mu)\mu^2+b_1(3+32\mu^2+32\mu^4))s^4. \end{aligned}$$

The polynomial $P_3(x_1)$ can have at most four real positive roots. So, the piecewise hybrid system separated by the straight line $y = 0$ and formed by the vector fields X_2 and Y_5 can have at most four limit cycles. Statement (d) of Theorem 4.1 is proved.

4.1.5 Case $(\mathbf{m}, \mathbf{n}) = (2, 6)$

We consider the piecewise hybrid system separated by the straight line $y = 0$ and formed by the vector field X_2 and the vector field

$$Y_6(x, y) = \begin{cases} Y_6^1(x, y) = -y + x(\mu + b_1x^5 + b_2x^4y + b_3x^3y^2 + b_4x^2y^3 + b_5xy^4 + b_6y^5), \\ Y_6^2(x, y) = x + y(\mu + b_1x^5 + b_2x^4y + b_3x^3y^2 + b_4x^2y^3 + b_5xy^4 + b_6y^5). \end{cases} \quad (4-11)$$

The vector field (4-11) in polar coordinates becomes

$$\begin{aligned} \dot{r} &= r\mu + r^6(b_1\cos^5\theta + b_2\cos^4\theta\sin\theta + b_3\cos^3\theta\sin^2\theta + b_4\cos^2\theta\sin^3\theta \\ &\quad + b_5\cos\theta\sin^4\theta + b_6\sin^5\theta), \\ \dot{\theta} &= 1, \end{aligned}$$

or equivalently

$$\begin{aligned} r' = \frac{dr}{d\theta} &= r\mu + r^6(b_1\cos^5\theta + b_2\cos^4\theta\sin\theta + b_3\cos^3\theta\sin^2\theta + b_4\cos^2\theta\sin^3\theta \\ &\quad + b_5\cos\theta\sin^4\theta + b_6\sin^5\theta). \end{aligned}$$

The solution $r_2(\theta)$ of this differential equation such that $r_2(\pi) = x_3 = sx_2 = sr_1(\pi)$ is

$$r_2(\theta) = \frac{2^{4/5}e^{\theta\mu}((1+\mu^2)(1+25\mu^2)(9+25\mu^2))^{1/5}x_3}{H_4^{1/5}}, \quad (4-12)$$

where

$$\begin{aligned}
H_4 = & - \left(-16e^{5\pi\mu} (9 + 3(3b_2 + 2b_4 + 8b_6)x_3^5 - (149b_1 + 26b_3 + 24b_5)x_3^5\mu \right. \\
& + (259 + 10(11b_2 + 3b_4)x_3^5)\mu^2 - 50(15b_1 + b_3)x_3^5\mu^3 + 125(7 + b_2x_3^5)\mu^4 \\
& - 625b_1x_3^5\mu^5 + 625\mu^6) + e^{5\theta\mu}x_3^5(-10(b_2 + b_4 + 5b_6 - 5(5b_1 + b_3 + b_5)\mu) \\
& (1 + \mu^2)(9 + 25\mu^2)\cos\theta - 5(1 + \mu^2)(1 + 25\mu^2)(9b_2 + 3b_4 + 5((-5b_1 + b_3 \\
& + 3b_5)\mu - 3b_6))\cos(3\theta) + (b_4 - b_2 - b_6 + (b_1 - b_3 + b_5)\mu)(9 + 250\mu^2 \\
& + 625\mu^4)\cos(5\theta) + 10(5b_1 + b_3 + b_5 + 5(b_2 + b_4 + 5b_6)\mu)(1 + \mu^2) \\
& (9 + 25\mu^2)\sin\theta + 5(15b_1 - 3b_3 - 9b_5 + 5(3b_2 + b_4 - 5b_6)\mu)(1 + \mu^2)(1 + 25\mu^2) \\
& \left. \sin(3\theta) + (b_1 - b_3 + b_5 + (b_2 - b_4 + b_6)\mu)(9 + 250\mu^2 + 625\mu^4)\sin(5\theta) \right) .
\end{aligned}$$

We consider the numerator of the equation $x_5^5 - x_1^5 = 0$, where $x_5 = ux_4 = ur_2(2\pi)$, and we obtain $x_1^5 P_4(x_1)$, being $P_4(x_1) = C_{0,4} + C_{1,4}x_1 + C_{2,4}x_1^2 + C_{3,4}x_1^3 + C_{4,4}x_1^4 + C_{5,4}x_1^5$, where

$$\begin{aligned}
C_{0,4} &= (e^{5\pi(\lambda+\mu)}s^2u^2 - 1)(1 + \lambda^2)^5(1 + \mu^2)(1 + 25\mu^2)(9 + 25\mu^2), \\
C_{1,4} &= 5(1 + e^{\pi\lambda})(a_2 - a_1\lambda)(1 + \lambda^2)^4(1 + \mu^2)(1 + 25\mu^2)(9 + 25\mu^2), \\
C_{2,4} &= -10(1 + e^{\pi\lambda})^2(a_2 - a_1\lambda)^2(1 + \lambda^2)^3(1 + \mu^2)(1 + 25\mu^2)(9 + 25\mu^2), \\
C_{3,4} &= 10(1 + e^{\pi\lambda})^3(a_2 - a_1\lambda)^3(1 + \lambda^2)^2(1 + \mu^2)(1 + 25\mu^2)(9 + 25\mu^2), \\
C_{4,4} &= -5(1 + e^{\pi\lambda})^4(a_2 - a_1\lambda)^4(1 + \lambda^2)(1 + \mu^2)(1 + 25\mu^2)(9 + 25\mu^2), \\
C_{5,4} &= (1 + e^{\pi\lambda})^5(a_2 - a_1\lambda)^5(1 + \mu^2)(1 + 25\mu^2)(9 + 25\mu^2) - (e^{5\pi\lambda} + e^{5\pi(\lambda+\mu)})s^5 \\
& \quad (24b_6 + 6b_4(1 + 5\mu^2) + b_2(9 + 110\mu^2 + 125\mu^4) + \mu(24b_5 + b_3(26 + 50\mu^2) \\
& \quad + b_1(149 + 750\mu^2 + 625b_1\mu^4))) (1 + \lambda^2)^5.
\end{aligned}$$

The polynomial $P_4(x_1)$ can have at most 5 positive real roots. That is, the piecewise hybrid system formed by the two vector fields X_2 and Y_6 and separated by the straight line $y = 0$, can have at most 5 limit cycles. Statement (e) of Theorem 4.1 is proved.

4.1.6 Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{3})$

Now we consider the piecewise hybrid system separated by the straight line $y = 0$ and formed by the vector field Y_3 and the vector field

$$X_3(x, y) = \begin{cases} x_3^1(x, y) = -y + x(\lambda + a_1x^2 + a_2xy + a_3y^2), \\ x_3^2(x, y) = x + y(\lambda + a_1x^2 + a_2xy + a_3y^2). \end{cases} \quad (4-13)$$

The solution $r_1(\theta)$ of the vector field X_3 in polar coordinates such that $r_1(0) = x_1$ is

$$r_1(\theta) = e^{\theta\lambda} \sqrt{\frac{2\lambda(1+\lambda^2)}{2(\lambda+\lambda^3)/x_1^2 + H_5}}, \quad (4-14)$$

where $H_5 = e^{2\theta\lambda}(-(a_1 + a_3)(1 + \lambda^2) + \lambda(a_2 + (-a_1 + a_3)\lambda) \cos(2\theta) - \lambda(a_1 - a_3 + a_2\lambda) \sin(2\theta)) + \lambda(-a_2 + 2a_1\lambda) + a_1 + a_3$. The solution $r_2(\theta)$ of the vector field Y_3 such that $r_2(\pi) = x_3$ is given in (4-6). Then, solving the equation $x_5 - x_1 = 0$ in terms of the variable x_1 , where $x_5 = ux_4 = ur_2(2\pi)$ we obtain the following solution

$$x_1 = \sqrt{\frac{(-2 + 2e^{2\pi(\lambda+\mu)}s^2u^2)\lambda(1+\lambda^2)\mu(1+\mu^2)}{R_1}},$$

where $R_1 = (1 - e^{2\pi\lambda})(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)(\mu + \mu^3) + e^{2\pi\lambda}(1 - e^{2\pi\mu})(b_1 + b_3 - b_2\mu + 2b_1\mu^2)s^2(\lambda + \lambda^3)$. This solution yields a periodic orbit if $x_1 > 0$. Therefore, the piecewise hybrid system formed by the two hybrid vector fields X_3 and Y_3 and separated by the straight line $y = 0$ can have at most one limit cycle, and statement (f) of Theorem 4.1 is proved. Furthermore, for values $\lambda = \mu = 1$, $s = u = 9/100$, $a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = -10$ we have that $x_5 - x_1 = 0$ and $x_1 = 0.0288542 > 0$.

4.1.7 Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{4})$

Consider the piecewise hybrid systems formed by the two vectors fields X_3 and Y_4 . Their solutions $r_1(\theta)$ such that $r_1(0) = x_1$, and $r_2(\theta)$ such that $r_2(\pi) = x_3 = sx_2$ are defined in (4-14) and (4-8), respectively. Then the numerator of the equation $x_5^3 - x_1^3 = 0$, where $x_5 = ux_4 = r_2(2\pi)$, is $A_0 + B_0 = 0$ where

$$\begin{aligned} A_0 &= 2\sqrt{2}s^{1/3}\lambda^{3/2}(1+\lambda^2)^{3/2} \left(e^{3\pi(\lambda+2\mu)}u^3(1+\mu^2)(1+9\mu^2) - e^{3\pi(\lambda+\mu)}(1+e^{3\pi\mu})x_1^3 \right. \\ &\quad \left. (b_2 + 2b_4 + 3b_2\mu^2 - \mu(7b_1 + 2b_3 + 9b_1\mu^2)) \right), \\ B_0 &= e^{3\pi\mu}x_1 \left(-2\lambda(1+\lambda^2) + (e^{2\pi\lambda} - 1)x_1^2(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2) \right) (1+\mu^2)(1+9\mu^2) \\ &\quad \sqrt{(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)(1 - e^{2\pi\lambda}) + \frac{2(\lambda + \lambda^3)}{x_1^2}}. \end{aligned}$$

So we consider the equation $A_0^2 - B_0^2 = 0$ and we obtain the following polynomial $P_5(x_1) = C_{0,5} + C_{2,5}x_1^2 + C_{3,5}x_1^3 + C_{4,5}x_1^4 + C_{6,5}x_1^6$, with

$$\begin{aligned} C_{0,5} &= -8(e^{6\pi\mu} - e^{6\pi(\lambda+2\mu)}s^6u^6)(\lambda + \lambda^3)^3(1 + 10\mu^2 + 9\mu^4)^2, \\ C_{2,5} &= 12e^{6\pi\mu}(e^{2\pi\lambda} - 1)\lambda^2(1 + \lambda)^2(a_1 + a_3 - a_2\lambda + 2a_1\lambda^2)(1 + \lambda^2)^2(1 + 9\mu^2)^2, \end{aligned}$$

$$\begin{aligned}
C_{3,5} &= -16e^{3\pi(2\lambda+3\mu)}(1+e^{\pi\mu})(1-e^{\pi\mu}+e^{2\pi\mu})s^6u^3\lambda^3(1+\mu^2)(1+9\mu^2) \\
&\quad (b_2+2b_4+3b_2\mu^2-\mu(7b_1+2b_3+9b_1\mu^2))(1+\lambda^2)^3, \\
C_{4,5} &= -6e^{6\pi\mu}(e^{2\pi\lambda}-1)^2\lambda(1+\lambda^2)(a_1+a_3+a_2\lambda+2a_1\lambda^2)^2(1+\mu^2)^2(1+9\mu^2)^2, \\
C_{6,5} &= e^{6\pi\mu}(e^{2\pi\lambda}-1)^3(a_1+a_3+a_2\lambda+2a_1\lambda^2)^3(1+10\mu^2+9\mu^4)^2+8e^{6\pi(\lambda+\mu)} \\
&\quad (1+e^{3\pi\mu})^2s^6(\lambda+\lambda^3)^3(b_2+2b_4+3b_2\mu^2-\mu(7b_1+2b_3+9b_1\mu^2))^2.
\end{aligned}$$

By Descartes' Theorem the polynomial $P_5(x_1)$ can have at most 4 variations of sign. That is, the piecewise hybrid system formed by two vector fields X_3 and Y_4 and separated by the straight line $y = 0$ can have at most four limit cycles. This concludes the proof of statement (g) of Theorem 4.1.

4.1.8 Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{5})$

We consider now the piecewise hybrid systems formed by the vector fields X_3 and Y_5 defined in the previous cases. The solutions of this system satisfying $r_1(0) = x_1$, $r_1(\pi) = x_2$, $r_2(\pi) = x_3 = sx_2$, $r_2(2\pi) = x_4$ and $x_5 = ux_4$, with $s, u \in (0, 1)$, where r_1, r_2 are defined in (4-14) and (4-10), respectively. From the equation $x_5^4 - x_1^4 = 0$, whose numerator is defined by the expression $x_1^4 P_6(x_1)$, where $P_6(x_1) = C_{0,6} + C_{2,6}x_1^2 + C_{4,6}x_1^4$, with

$$\begin{aligned}
C_{0,6} &= -8(e^{4\pi\mu} - e^{4\pi(\lambda+2\mu)}s^4u^4)(\lambda+\lambda^3)^2\mu(1+5\mu^2+4\mu^4), \\
C_{2,6} &= 8e^{4\pi\mu}(e^{2\pi\lambda}-1)\lambda(1+\lambda^2)(a_1+a_3-a_2\lambda+2a_1\lambda^2)\mu(1+5\mu^2+4\mu^4), \\
C_{4,6} &= -2e^{4\pi\mu}(-1+e^{2\pi\lambda})^2(a_1+a_3-a_2\lambda+2a_1\lambda^2)^2\mu(1+5\mu^2+4\mu^2)+e^{4\pi(\lambda+\mu)}s^4 \\
&\quad (e^{4\pi\mu}-1)(b_3+3b_5-\mu(5b_2+3b_4)+4b_3\mu^2-8b_2\mu^3+b_1(3+32\mu^2+32\mu^4)) \\
&\quad (\lambda+\lambda^3)^2.
\end{aligned}$$

By Descartes' Theorem, this polynomial can have at most two positive real roots. Therefore the piecewise hybrid system formed by the two vector fields X_3 and Y_5 separated by the straight line $y = 0$ can have at most 2 limit cycles. Hence statement (h) of Theorem 4.1 is proved.

4.1.9 Case $(\mathbf{m}, \mathbf{n}) = (\mathbf{3}, \mathbf{6})$

Here we consider the piecewise hybrid systems formed by the vector fields X_3 and Y_6 . We know their solutions, given in (4-14) and (4-12), respectively. The limit cycles of these piecewise hybrid systems satisfy the equation $0 = x_5^2 - x_1^2 = A_1 + B_1$,

where

$$\begin{aligned}
A_1 &= -x_1(2\lambda(1+\lambda^2) + (1-e^{2\pi\lambda})x_1^2(a_1+a_3-a_2\lambda+2a_1\lambda^2))^2(1+\mu^2)(1+25\mu^2) \\
&\quad (9+25\mu^2)\sqrt{(1-e^{2\pi\lambda})(a_1+a_3-a_2\lambda+2a_1\lambda^2) + 2(\lambda+\lambda^2)/x_1^2}, \\
B_1 &= 4\sqrt{2}e^{5\pi(\lambda+\mu)}s^5u^5\lambda^{5/2}(1+\lambda^2)^{5/2}(1+\mu^2)(1+25\mu^2)(9+25\mu^2) - 4\sqrt{2}e^{5\pi\lambda}s^5x_1^5 \\
&\quad (1+e^{5\pi\mu})\lambda^{5/2}(1+\lambda^2)^{5/2}(24b_6+6b_4(1+5\mu^2)+b_2(9+110\mu^2+125\mu^4) \\
&\quad -\mu(149b_1+26b_3+24b_5+50(15b_1+b_3)\mu^2+625b_1\mu^4)).
\end{aligned}$$

Since $A_1^2 - B_1^2 = (A_1 + B_1)(A_1 - B_1)$, then the upper bound for the number of zeros of $A_1^2 - B_1^2$ also provides an upper bound for the number of zeros of $A_1 + B_1$. The expression $A_1^2 - B_1^2 = P_7(x_1)$, where $P_7(x_1) = C_{0,7} + C_{2,7}x_1^2 + C_{4,7}x_1^4 + C_{5,7}x_1^5 + C_{6,7}x_1^6 + C_{8,7}x_1^8 + C_{10,7}x_1^{10}$, with

$$\begin{aligned}
C_{0,7} &= 32(1-e^{10\pi(\lambda+\mu)}s^{10}u^{10})(\lambda+\lambda^3)^5(9+259\mu^2+875\mu^4+625\mu^6)^2, \\
C_{2,7} &= 80(1-e^{\pi\lambda}(1+e^{\pi\lambda}))\lambda^4(1+\lambda^2)^4\beta_1(1+\mu^2)^2(1+25\mu^2)^2(9+25\mu^2)^2, \\
C_{4,7} &= 80(-1+e^{2\pi\lambda})^2\beta_1^2(\lambda+\mu^3)^3(1+\mu^2)^2(1+25\mu^2)^2(9+25\mu^2)^2, \\
C_{5,7} &= 64e^{5\pi(2\pi+\mu)}(1+e^{\pi\mu})(1-e^{\pi\mu}+e^{2\pi\mu}-e^{3\pi\mu}+e^{4\pi\mu})s^{10}u^5\lambda^5(1+\lambda^2)^5 \\
&\quad (1+\mu^2)(1+25\mu^2)(9+25\mu^2)\delta_1, \\
C_{6,7} &= -40(-1+e^{\pi\lambda})^3(1+e^{\pi\mu})^3\lambda^2(1+\lambda^2)^2\beta_1^3(1+\mu^2)^2(1+25\mu^2)^2(9+25\mu^2)^2, \\
C_{8,7} &= 10(e^{\pi\lambda}-1)^4(1+e^{\pi\lambda})^4\lambda(1+\lambda^2)\beta_1^4(1+\mu^2)^2(1+25\mu^2)^2(9+25\mu^2)^2, \\
C_{10,10} &= -(-1+e^{2\pi\lambda})^5\beta_1^5(9+259\mu^2+875\mu^4+625\mu^6)^2 - 32e^{10\pi\lambda}(1+e^{\pi\mu})^2 \\
&\quad s^{10}(\lambda+\lambda^3)^5\delta_1^2,
\end{aligned}$$

where $\beta_1 = a_1 + a_3 - a_2\lambda + 2a_1\lambda^2$ and $\delta_1 = 24b_6 + 6b_4(1+\mu^2) + b_2(9+110\mu^2+125\mu^4) - \mu(149b_1+26b_3+24b_5+50(15b_1+b_3)\mu^2+625b_1\mu^2)$.

By Descartes' Theorem the polynomial $P_7(x_1)$ can have at most six positive real roots. So the piecewise hybrid system formed by the two vector fields X_3 and Y_6 separated by the straight line $y = 0$ can have at most 6 limit cycles. Hence statement (i) of Theorem 4.1 is proved.

4.1.10 Case $(\mathbf{m}, \mathbf{n}) = (4, 4)$

We Now we consider the discontinuous piecewise hybrid system formed by the rigid vector fields Y_4 and X_4

$$X_4(x, y) = \begin{cases} X_4^1(x, y) = -y + x(\lambda + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3), \\ X_4^2(x, y) = x + y(\lambda + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3). \end{cases} \quad (4-15)$$

We can rewritten this system in polar coordinates

$$\dot{r} = r\lambda + r^4(a_1 \cos^3 \theta + a_2 \cos^2 \theta \sin \theta + a_3 \cos \theta \sin^2 \theta + a_4 \sin^3 \theta), \quad \dot{\theta} = 1.$$

Similarly to the previous cases

$$r' = \frac{dr}{d\theta} = r\lambda + r^4(a_1 \cos^3 \theta + a_2 \cos^2 \theta \sin \theta + a_3 \cos \theta \sin^2 \theta + a_4 \sin^3 \theta).$$

Then the solution of this differential equation $r_1(0) = x_1$ is

$$r_1(\theta) = \frac{2^{2/3}x_1 e^{\theta\lambda}((1+\lambda^2)(1+9\lambda^2))^{1/3}}{H_6^{1/3}},$$

where

$$\begin{aligned} H_6 = & -(-4(1+\lambda^2)(1+9\lambda^2) - 4x_1^3(-2a_4 + 2a_3\lambda - a_2(1+3\lambda^2) + a_1\lambda(7+9\lambda^2)) \\ & + e^{3\theta\lambda}x_1^3(-3(a_2+3a_4-3(3a_1+a_3)\lambda)(1+\lambda^2)\cos\theta + (-a_2+a_4+a_1\lambda-a_3\lambda) \\ & (1+9\lambda^2)\cos(3\theta) + 3(3a_1+a_3+3a_2\lambda+9a_4\lambda)(1+\lambda^2)\sin\theta + (a_1-a_3+a_2\lambda \\ & -a_4\lambda)(1+9\lambda^2)\sin(3\theta))). \end{aligned}$$

As in the previous cases, we assume that $r_1(\pi) = x_2$, $r_2(\pi) = x_3 = sx_2$, $r_2(2\pi) = x_4$ and $x_5 = ux_4$, where $r_2(\theta)$ is the solution of Y_4 defined in (4-8). Then, the limit cycles satisfy the equation $x_5 - x_1 = 0$. We consider the denominator of equation $x_5^3 - x_1^3 = x_1^3 P_8(x_1) = 0$, where $P_8(x_1) = C_{0,8} + C_{3,8}x_1^3$, with

$$\begin{aligned} C_{0,8} = & (e^{3\pi(\lambda+\mu)}s^3u^3 - 1)(1+10\lambda^2+9\lambda^4)(1+10\mu^2+9\mu^4) \\ C_{3,8} = & (1+e^{3\pi\lambda})(a_2+2a_4+3a_2\lambda^2-\lambda(7a_1+2a_3+9a_1\lambda^2))(1+10\mu^2+9\mu^4) \\ & - e^{3\pi\lambda}(1+e^{3\pi\mu})s^3(1+10\lambda^2+9\lambda^4)(b_2+2b_4+3b_2\mu^2-\mu(7b_1+2b_3+9b_1\mu^2)). \end{aligned}$$

The polynomial $P_8(x_1)$ can have at most one positive real root. This concludes the proof of statement (j) of Theorem 4.1.

4.1.11 Case $(\mathbf{m}, \mathbf{n}) = (4, 5)$

Here we study the discontinuous piecewise hybrid systems formed by the two rigid vector fields X_4 and Y_5 . We know their solutions $r_1(\theta)$ and $r_2(\theta)$. Then we consider the numerator of the equation $x_5^4 - x_1^4 = x_1^4(A_2 + B_2) = 0$, where

$$\begin{aligned} A_2 = & -8e^{4\pi\mu}(1+10\lambda^2+9\lambda^2 - (1+e^{3\pi\lambda})x_1^3(a_2+2a_4+3a_2\lambda^2-\lambda(7a_1+2a_3 \\ & +9a_1\lambda^2)))^{4/3}(\mu+5\mu^3+4\mu^5), \end{aligned}$$

$$B_2 = 8e^{8\pi\mu}e^{4\pi\lambda}s^4u^4(1+10\lambda^2+9\lambda^2)^{4/3}(\mu+5\mu^3+4\mu^5)+(1+10\lambda^2+9\lambda^2)^{4/3}e^{4\pi\lambda}s^4(b_3+3b_5-(5b_2+3b_4)\mu+4\mu^2(b_3-2b_2\mu)+b_1(3+32(\mu^2+\mu^4)))e^{4\pi\mu}x_1^4(e^{4\pi\mu}-1).$$

As $A_2^3 + B_2^3 = (A_2 + B_2)(A_2^2 + B_2^2 - A_2B_2)$ then the upper bound for the number of solutions of equation $A_2^3 + B_2^3 = 0$ provides also an upper bound for the number of solution of the equation $A_2 + B_2 = 0$. Note that,

$$A_2^3 + B_2^3 = P_9(x_1) = C_{0,8} + C_{3,8}x_1^3 + C_{4,8}x_1^4 + C_{6,8}x_1^6 + C_{8,8}x_1^8 + C_{9,8}x_1^9 + C_{12,9}x_1^{12},$$

with

$$\begin{aligned} C_{0,8} &= 512e^{12\pi\mu}(e^{12\pi(\lambda+\mu)}s^{12}u^{12}-1)(1+10\lambda^2+9\lambda^4)^4(\mu+5\mu^3+4\mu^5)^3, \\ C_{3,8} &= 2048e^{12\pi\mu}(1+e^{\pi\lambda})(1-e^{\pi\lambda}+e^{2\pi\lambda})(1+\lambda^2)^3(1+9\lambda^2)^3\mu^3(1+\mu^2)^3 \\ &\quad (a_2+2a_4+3a_2\lambda^2-\lambda(7a_1+2a_3+9a_1\lambda^2))(1+4\mu^2)^3, \\ C_{4,8} &= 192e^{4\pi(3\lambda+5\mu)}(e^{4\pi\mu}-1)s^{12}u^8(1+\lambda^2)^4(1+9\lambda^2)^4(\mu+5\mu^3+4\mu^5)^2 \\ &\quad (b_3+3b_5-(5b_2+3b_4)\mu+4b_3\mu^2(b_3-2b_2\mu)+b_1(3+32\mu^2+32\mu^4)), \\ C_{6,8} &= -3072e^{12\pi\mu}(1+e^{3\pi\lambda})^2(1+\lambda^2)^2(1+9\lambda^2)^2(a_2+2a_4+3a_2\lambda^2-\lambda(7a_1+2a_3+9a_1\lambda^2))^2(\mu+5\mu^3+4\mu^5)^3, \\ C_{8,8} &= 24e^{4\pi(3\lambda+4\mu)}(e^{4\pi\mu}-1)^2s^{12}u^4(1+10\lambda^2+9\lambda^4)^4(\mu+5\mu^3+4\mu^5) \\ &\quad (b_3+3b_5-(5b_2+3b_4)\mu+4b_3\mu^2(b_3-2b_2\mu)+b_1(3+32\mu^2+32\mu^4))^2, \\ C_{9,8} &= 2048e^{12\pi\mu}(1+e^{3\pi\lambda})^3(1+10\lambda^2+9\lambda^4)(a_2+2a_4+3a_2\lambda^2-\lambda(7a_1+2a_3+9a_1\lambda^2))^3(\mu+5\mu^3+4\mu^5)^3, \\ C_{12,8} &= (-512(1+e^{3\pi\lambda})^4(a_2+2a_4+3a_2\lambda^2-\lambda(7a_1+2a_3+9a_1\lambda^2))^4(\mu+5\mu^3+4\mu^5)^3 \\ &\quad +e^{12\pi\lambda}(b_3+3b_5-(5b_2+3b_4)\mu+4b_3\mu^2(b_3-2b_2\mu)+b_1(3+32\mu^2+32\mu^4))^3(e^{4\pi\mu}-1)^3s^{12}(1+10\lambda^2+9\lambda^4)^4)e^{12\pi\mu}. \end{aligned}$$

By Descartes's Theorem, the polynomial $P_9(x_1)$ can have at most six variations of sign, i.e., $P_9(x_1)$ can have at most six positive real roots. Therefore, the discontinuous piecewise hybrid system formed by the two rigid vector fields X_4 and Y_5 can have at most six limit cycles, and statement (k) of Theorem 4.1 was proved.

4.1.12 Case $(\mathbf{m}, \mathbf{n}) = (4, 6)$

The hybrid piecewise smooth dynamical system The discontinuous piecewise hybrid system formed by the two rigid vector fields X_4 and Y_6 has solutions $r_1(\theta)$ and $r_2(\theta)$, respectively. Assuming that $r_1(0) = x_1$, $r_1(\pi) = x_2$, $r_2(\pi) = x_3 = sx_2$, $r_2(2\pi) = x_4$, and $x_5 = ux_4$, we have that the possible limit cycles satisfy the following equation $x_5 - x_1 = 0$. Working in a similar way as in the previous case, since $x_5^5 - x_1^5 = (x_5 - x_1)(x_5^4 + x_5^3x_1 + x_5^2x_1^2 + x_5x_1^3 + x_1^4)$, the upper bound for the

number of solutions of equation $x_5^5 - x_1^5 = 0$ provides also an upper bound for the number of solution of the equation $x_5 - x_1 = 0$. We consider the numerator of $x_5^5 - x_1^5 = x_1^5(A_3 + B_3)$, where

$$\begin{aligned} A_3 &= - (1 + 10\lambda^2 + 9\lambda^4) + (1 + e^{3\pi\lambda}) (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^2 \\ &\quad x_1^3(1 + \mu^2)(1 + 25\mu^2)(9 + 25\mu^2), \\ B_3 &= e^{5\pi\lambda}s^5 (1 + 10\lambda^2 + 9\lambda^4 - (1 + e^{3\pi\lambda})x_1^3(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 \\ &\quad + 9a_1\lambda^2)))^{1/3} (1 + 10\lambda^2 + 9\lambda^4)^{5/3} (e^{5\pi\mu}u^5(1 + \mu^2)(1 + 25\mu^2)(9 + 25\mu^2) \\ &\quad + (1 + e^{5\pi\mu})x_1^5((149b_1 + 26b_3 + 24b_5)\mu + 5\mu^2(5\mu(2b_3 + 5b_1(6 + 5\mu^2)) - 6b_4) \\ &\quad - b_2(9 + 110\mu^2 + 125\mu^4) - 6(b_4 + 4b_6))) . \end{aligned}$$

Then we consider $A_3^3 + B_3^3 = (A_3 + B_3)(A_3^2 + B_3^2 - A_3B_3)$ and obtain the following polynomial of degree 18: $P_{10}(x_1) = C_{0,10} + C_{3,10}x_1^3 + C_{5,10}x_1^5 + C_{6,10}x_1^6 + C_{8,10}x_1^8 + C_{9,10}x_1^9 + C_{10,10}x_1^{10} + C_{12,10}x_1^{12} + C_{13,10}x_1^{13} + C_{15,10}x_1^{15} + C_{18,10}x_1^{18}$, where,

$$\begin{aligned} C_{0,10} &= (e^{15\pi(\lambda+\mu)}s^{15}u^{15} - 1)(1 + 10\lambda^2 + 9\lambda^4)^6(9 + 259\mu^2 + 875\mu^4 + 625\mu^6)^3, \\ C_{3,10} &= (e^{15\pi(\lambda+\mu)}s^{15}u^{15} - 6)(1 + 10\lambda^2 + 9\lambda^4)^5(9 + 259\mu^2 + 875\mu^4 + 625\mu^6)^3 \\ &\quad (- (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)))(1 + e^{3\pi\lambda}), \\ C_{5,10} &= 3e^{5\pi(3\lambda+2\mu)}(1 + e^{5\pi\mu})s^{15}u^{10}(1 + 10\lambda^2 + 9\lambda^4)^6(9 + 259\mu^2 + 875\mu^4 \\ &\quad + 625\mu^6)^2((149b_1 + 26b_3 + 24b_5)\mu - b_2(9 + 110\mu^2 + 125\mu^4) \\ &\quad + 5\mu^2(5\mu(2b_3 + 5b_1(6 + 5\mu^2)) - 6b_4) - 6(b_4 + 4b_6)), \\ C_{6,10} &= -15(1 + 10\lambda^2 + 9\lambda^4)^4(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^2 \\ &\quad (9 + 259\mu^2 + 875\mu^4 + 625\mu^6)^3(1 + e^{3\pi\lambda})^2, \\ C_{8,10} &= 3e^{5\pi(3\lambda+2\mu)}(1 + e^{3\pi\mu})(1 + 10\lambda^2 + 9\lambda^4)^5(9 + 259\mu^2 + 875\mu^4 + 625\mu^6)^2 \\ &\quad (a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))((-149b_1 - 26b_3 - 24b_5)\mu \\ &\quad + b_2(9 + 110\mu^2 + 125\mu^4) + 5\mu^2(6b_4 - 5\mu(2b_3 + 5b_1(6 + 5\mu^2))) \\ &\quad + 6(b_4 + 4b_6))(1 + e^{5\pi\mu})s^{15}u^{10}, \\ C_{9,10} &= 20(1 + 10\lambda^2 + 9\lambda^4)^3(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^3 \\ &\quad (9 + 259\mu^2 + 875\mu^4 + 625\mu^6)^3(1 + e^{3\pi\lambda})^3, \\ C_{10,10} &= 3e^{5\pi(3\lambda+2\mu)}(1 + e^{5\pi\mu})^2s^{15}u^5(1 + 10\lambda^2 + 9\lambda^4)^6(9 + 259\mu^2 + 875\mu^4 + 625\mu^6) \\ &\quad ((149b_1 + 26b_3 + 24b_5)\mu + 5\mu^2(5\mu(2b_3 + 5b_1(6 + 5\mu^2)) - 6b_4) \\ &\quad - 6(b_4 + 4b_6) - b_2(9 + 110\mu^2 + 125\mu^4))^2, \\ C_{12,10} &= -15(1 + 10\lambda^2 + 9\lambda^4)^2(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^4 \\ &\quad (9 + 259\mu^2 + 875\mu^4 + 625\mu^6)^3(1 + e^{3\pi\lambda})^4, \\ C_{13,10} &= -3e^{5\pi(3\lambda+2\mu)}(1 + e^{3\pi\mu})(1 + e^{5\pi\mu})^2s^{15}u^5(1 + 10\lambda^2 + 9\lambda^4)^5(9 + 259\mu^2 \\ &\quad + 875\mu^4 + 625\mu^6)(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2)) \\ &\quad ((149b_1 + 26b_3 + 24b_5)\mu - b_2(9 + 110\mu^2 + 125\mu^4) + 5\mu^2(5\mu(2b_3 \\ &\quad + 5b_1(6 + 5\mu^2)) - 6b_4) - 6(b_4 + 4b_6))^2, \end{aligned}$$

$$\begin{aligned}
C_{15,10} = & 6(1 + 10\lambda^2 + 9\lambda^4)(1 + e^{3\pi\lambda})^5(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^5 \\
& (9 + 259\mu^2 + 875\mu^4 + 625\mu^6)^3 - e^{15\pi\lambda}(1 + e^{5\pi\mu})^3 s^{15}(1 + 10\lambda^2 + 9\lambda^4)^6 \\
& (-(149b_1 + 26b_3 + 24b_5)\mu + b_2(9 + 110\mu^2 + 125\mu^4) + 5\mu^2(6b_4 - 5\mu(2b_3 \\
& + 5b_1(6 + 5\mu^2))) + 6(b_4 + 4b_6))^3, \\
C_{18,10} = & -(1 + e^{3\pi\lambda})(a_2 + 2a_4 + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))((a_2 + 2a_4 \\
& + 3a_2\lambda^2 - \lambda(7a_1 + 2a_3 + 9a_1\lambda^2))^5(1 + e^{3\pi\lambda})^5(9 + 259\mu^2 + 875\mu^4 + 625\mu^6)^3 \\
& - e^{15\pi\lambda}(1 + e^{5\pi\mu})^3 s^{15}(1 + 10\lambda^2 + 9\lambda^4)^5(24b_6 + 6b_4(1 + 5\mu^2) + b_2(9 + 110\mu^2 \\
& + 125\mu^4) - \mu(149b_1 + 26b_3 + 24b_5 + 50(15b_1 + b_3)\mu^2 + 625b_1\mu^4))^3).
\end{aligned}$$

Therefore, by Descartes' Theorem, the polynomial $P_{10}(x_1)$ can have at most ten variations of sign, or equivalently, ten positive real roots. Consequently, the equation $x_5^5 - x_1^5 = 0$ or the original equation $x_5 - x_1 = 0$ can have at most ten positive real roots, meaning that piecewise hybrid system formed by the two rigid vector fields X_4 and Y_6 can have at most ten limit cycles, and statement (l) of Theorem 4.1 was proved.

4.1.13 Case $(\mathbf{m}, \mathbf{n}) = (5, 5)$

Consider the discontinuous piecewise hybrid system formed by the two rigid vector fields X_5 and Y_5 where

$$X_5(x, y) = \begin{cases} X_5^1(x, y) = -y + x(\lambda + a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4), \\ X_5^2(x, y) = x + y(\lambda + a_1x^4 + a_2x^3y + a_3x^2y^2 + a_4xy^3 + a_5y^4). \end{cases} \quad (4-16)$$

In polar coordinates system (4-16) becomes

$$\begin{aligned}
\dot{r} = & r\lambda + r^5 (a_1 \cos^4 \theta + a_2 \cos^3 \theta \sin \theta + a_3 \cos^2 \theta \sin^2 \theta + a_4 \cos \theta \sin^3 \theta + a_5 \sin^4 \theta), \\
\dot{\theta} = & 1
\end{aligned}$$

or, equivalently

$$r' = \frac{dr}{d\theta} = r\lambda + r^5 (a_1 \cos^4 \theta + a_2 \cos^3 \theta \sin \theta + a_3 \cos^2 \theta \sin^2 \theta + a_4 \cos \theta \sin^3 \theta + a_5 \sin^4 \theta).$$

Its solution $r_1(\theta)$ such that $r_1(0) = x_1$ is

$$r_1(\theta) = \frac{2^{3/4} e^{\theta\lambda} \lambda^{1/4} (1 + 5\lambda^2 + 4\lambda^4)^{1/4} x_1}{H_7^{1/4}}$$

where $H_7 = 8(\lambda + 5\lambda^3 + 4\lambda^5) - x_1^4(-3a_5 - a_3(1 + 4\lambda^2) - a_1(3 + 32\lambda^2 + 32\lambda^4) + \lambda(3a_4 + a_2(5 + 8\lambda^2))) + e^{4\theta\lambda}((3a_1 + a_3 + 3a_5)(1 + 5\lambda^2 + 4\lambda^4) - 4\lambda(a_2 + a_4 + 4(-a_1 + a_5)\lambda)(1 +$

$$\lambda^2) \cos(2\theta) + \lambda(-a_2 + a_4 + (a_1 - a_3 + a_5)\lambda)(1 + 4\lambda^2) \cos(4\theta) + 8\lambda(a_1 - a_5 + (a_2 + a_4)\lambda)(1 + \lambda^2) \sin(2\theta) + \lambda(a_1 - a_3 + a_5 + a_2\lambda - a_4\lambda)(1 + 4\lambda^2) \sin(4\theta))).$$

We assume $r_1(0) = x_1$, $r_1(\pi) = x_2$, $r_2(\pi) = x_3 = sx_2$, $r_2(2\pi) = x_4$, and $x_5 = ux_4$, where $s, u \in (0, 1)$, and $r_1(\theta)$ and $r_2(\theta)$ are the solutions of X_5 and Y_5 (see eq. (4-9)), resp.. Thus, the possible limit cycles satisfy the equation $x_5 - x_1 = 0$. Therefore, consider the numerator of the expression $x_5^4 - x_1^4 = x_1^4 P_{11}(x_1) = 0$, being $P_{11}(x_1) = C_{0,11} + C_{4,11}x_1^4$, with

$$\begin{aligned} C_{0,11} &= 8(-1 + e^{4\pi(\lambda+\mu)} s^4 u^4) \lambda(1 + 5\lambda^2 + 4\lambda^4)(\mu + 5\mu^3 + 4\mu^5), \\ C_{4,11} &= \mu(e^{4\pi\lambda} - 1)(a_3 + 3(a_1 + a_5) + (-5a_2 - 3a_4)\lambda + 4(8a_1 + a_3)\lambda^2 - 8a_2\lambda^3 \\ &\quad + 32a_1\lambda^4)(1 + 5\mu^2 + 4\mu^4) - \mu e^{4\pi\lambda}(-1 + e^{4\pi\mu}) s^4 \lambda(1 + 5\lambda^2 + 4\lambda^4) \\ &\quad (3b_4 + b_2(5 + 8\mu^2)) + e^{4\pi\lambda}(b_3 + 3b_5 + 4b_3\mu^2 + b_1(3 + 32(\mu^2 + \mu^4))) \\ &\quad (e^{4\pi\mu} - 1) s^4 \lambda(1 + 5\lambda^2 + 4\lambda^4). \end{aligned}$$

By Descartes' Theorem, the polynomial $P_{11}(x_1)$ can have at most one variation of sign, meaning that it can have at most one positive real root. Therefore, the discontinuous piecewise hybrid system formed by X_5 and Y_5 can have at most one limit cycle, proving the statement (m) of Theorem 4.1.

4.1.14 Case $(\mathbf{m}, \mathbf{n}) = (5, 6)$

Finally, we consider the discontinuous piecewise hybrid system given by the two vector fields X_5 and Y_6 defined in (4-16) and (4-11), respectively. We know their solutions $r_1(\theta)$ and $r_2(\theta)$ such that, $r_1(0) = x_1$, $r_1(\pi) = x_2$, $r_2(\pi) = x_3 = sx_2$, $r_2(2\pi) = x_4$, and $x_5 = ux_4$ with $s, u \in (0, 1)$. For this system, the possible limit cycles satisfy the equation $x_5 - x_1 = 0$ or, equivalently, the numerator the equation $x_5^5 - x_1^5 = x_1^5(A_4 + B_4)$, where

$$\begin{aligned} A_4 &= (8\lambda + 40\lambda^3 + 32\lambda^5 + (1 - e^{4\pi\lambda})x_1^4(a_3 + 3a_5 - (5a_2 + 3a_4)\lambda + 4a_3\lambda^2 - 8a_2\lambda^3 \\ &\quad + a_1(3 + 32\lambda^2 + 32\lambda^4)))^{1/4} (-8(\lambda + 5\lambda^3 + 4\lambda^5) + (e^{4\pi\lambda} - 1)x_1^4(a_3 + 3a_5 \\ &\quad - (5a_2 + 3a_4)\lambda + 4a_3\lambda^2 - 8a_2\lambda^3 + a_1(3 + 32\lambda^2 + 32\lambda^4))) (9 + 259\mu^2 + 875\mu^4 \\ &\quad + 625\mu^6), \\ B_4 &= 8 \times 2^{3/4} s^5 \lambda^{5/4} (1 + 5\lambda^2 + 4\lambda^4)^{5/4} (e^{5\pi(\lambda+\mu)} u^5 (9 + 259\mu^2 + 875\mu^4 + 625\mu^6) - x_1^5 \\ &\quad (1 + e^{5\pi\mu}) e^{5\pi\lambda} (24b_6 + 6b_4(1 + 5\mu^2) + b_2(9 + 110\mu^2 + 125\mu^4) - \mu(149b_1 + 26b_3 \\ &\quad + 24b_5 + 50(15b_1 + b_3)\mu^2 + 625b_1\mu^4))) , \end{aligned}$$

We consider equation $A_4^4 - B_4^4 = 0$. In this way, we obtain a polynomial of degree 20 given by $P_{10}(x_1) = C_{0,12} + C_{4,10}x_1^4 + C_{5,10}x_1^5 + C_{8,12}x_1^8 + C_{10,12}x_1^{10} + C_{12,12}x_1^{12} +$

$C_{15,12}x_1^{15} + C_{16,12}x_1^{16} + C_{20,12}x_1^{20}$, where

$$\begin{aligned}
C_{0,12} &= -32768(-1 + e^{20\pi(\lambda+\mu)}s^{20}u^{20})(\lambda + 5\lambda^3 + 4\lambda^5)^5(9 + 259\mu^2 + 875\mu^4 \\
&\quad + 625\mu^6)^4, \\
C_{4,12} &= -20480(-1 + e^{4\pi\lambda})\lambda^4(1 + 5\lambda^2 + 4\lambda^4)^4(9 + 259\mu^2 + 875\mu^4 + 625\mu^6)^4 \\
&\quad (a_3 + 3a_5 - 5a_2\lambda - 3a_4\lambda + 4a_3\lambda^2 - 8a_2\lambda^3 + a_1(3 + 32\lambda^2 + 32\lambda^4)), \\
C_{5,12} &= 131072e^{5\pi(4\lambda+3\mu)}(1 + e^{5\pi\mu})(\lambda + 5\lambda^3 + 4\lambda^5)^5(9 + 259\mu^2 + 875\mu^4 + 625\mu^6)^3 \\
&\quad (24b_6 + 6b_4(1 + 5\mu^2) + b_2(9 + 110\mu^2 + 125\mu^4) - \mu(149b_1 + 26b_3 + 24b_5 \\
&\quad + 625b_1\mu^4 + 50(15b_1 + b_3)\mu^2))s^{20}u^{15}, \\
C_{8,12} &= 5120(a_3 + 3a_5 - 5a_2\lambda - 3a_4\lambda + 4a_3\lambda^2 - 8a_2\lambda^3 + a_1(3 + 32\lambda^2 + 32\lambda^4))^2 \\
&\quad (\lambda + 5\lambda^3 + 4\lambda^5)^3(9 + 259\mu^2 + 875\mu^4 + 625\mu^6)^4(e^{4\pi\lambda} - 1)^2, \\
C_{10,12} &= -196608e^{10\pi(2\lambda+\mu)}(1 + e^{5\pi\mu})^2(\lambda + 5\lambda^3 + 4\lambda^5)^5(9 + 259\mu^2 + 875\mu^4 \\
&\quad + 625\mu^6)^2(24b_6 + 6b_4(1 + 5\mu^2) + b_2(9 + 110\mu^2 + 125\mu^4) - \mu(149b_1 + 26b_3 \\
&\quad + 24b_5 + 625b_1\mu^4 + 50(15b_1 + b_3)\mu^2))^2s^{20}u^{10}, \\
C_{12,12} &= -640(a_3 + 3a_5 - 5a_2\lambda - 3a_4\lambda + 4a_3\lambda^2 - 8a_2\lambda^3 + a_1(3 + 32\lambda^2 + 32\lambda^4))^3 \\
&\quad (\lambda + 5\lambda^3 + 4\lambda^5)^2(9 + 259\mu^2 + 875\mu^4 + 625\mu^6)^4(e^{4\pi\lambda} - 1)^3, \\
C_{15,12} &= 131072e^{5\pi(4\lambda+\mu)}(1 + e^{5\pi\mu})^3(\lambda + 5\lambda^3 + 4\lambda^5)^5(9 + 259\mu^2 + 875\mu^4 + 625\mu^6) \\
&\quad (24b_6 + 6b_4(1 + 5\mu^2) + b_2(9 + 110\mu^2 + 125\mu^4) - \mu(149b_1 + 26b_3 + 24b_5 \\
&\quad + 625b_1\mu^4 + 50(15b_1 + b_3)\mu^2))^3s^{20}u^5, \\
C_{16,12} &= 40(a_3 + 3a_5 - (5a_2 + 3a_4)\lambda + 4a_3\lambda^2 - 8a_2\lambda^3 + a_1(3 + 32\lambda^2 + 32\lambda^4))^4 \\
&\quad (\lambda + 5\lambda^3 + 4\lambda^5)(9 + 259\mu^2 + 875\mu^4 + 625\mu^6)^4(e^{4\pi\lambda} - 1)^4.
\end{aligned}$$

We do not provide the explicit expression of $C_{20,12}$ because it is very long. By Descartes's Theorem, the polynomial $P_{12}(x_1)$ can have at most eight positive real roots. So the discontinuous hybrid differential system formed by two rigid vector fields X_5 and Y_6 can have at most eight limit cycles, and the statement (n) of Theorem 4.1 was proved.

4.2 Brief conclusion

Here we investigate the maximum number of limit cycles for a discontinuous piecewise hybrid system separated by the straight line $y = 0$. We compare our results with those of the continuous systems studied previous chapter (see [36]), where we study systems (3-1) and (3-2) with $\lambda = \mu$, no jumps, and the same separation line $y = 0$. We consider the same cases as in Chapter 3 and obtain the same maximum

number of limit cycles in all of them, except for $(m = 3, n = 6)$ and $(m = 4, n = 6)$. For these two cases, the maximum number is 5 and 6 in the continuous system, whereas in the hybrid system the upper bounds increase to 6 and 10, respectively. Furthermore, in Chapter 3 we were able to exhibit parameter values for which this bound is actually attained in the cases where $m = n$ and $m = 3, n = 5$. In the hybrid case, however, achieving the upper bound is generally a non-trivial problem; taking $\lambda \neq \mu$ makes the expressions for each pair (m, n) considerably more complex and difficult to manipulate. Nevertheless, we were able to show that the bound is attained when $m = n$, except for the cases $m = n = 4$ and $m = n = 5$.

This suggests that for low values of m and n specifically, $m, n \in \{2, 3, 4, 5, 6\}$ there is not a substantial difference between the maximum number of limit cycles in the hybrid and continuous cases. We believe that for higher values of m and n , more significant variations might appear, which could help determine whether the addition of discrete dynamics increases the number of limit cycles. However, since the main objective of this chapter is to illustrate a method for calculating limit cycles in such systems, we have limited our study to fourteen representative cases, as calculations become excessively complex for more cases.

In the next chapter, unlike the previous ones, we study a piecewise smooth dynamical system defined in $\mathbb{R}^3$ that is neither rigid nor hybrid. This study is based on the paper [17], where the authors classify the dynamics associated with the Hopf-Like bifurcations $HLB - 7$ and $HLB - 10$ applied to a special class of piecewise smooth systems in $\mathbb{R}^3$.

Bifurcations in piecewise smooth vector fields

A fundamental class within nonsmooth dynamical systems is constituted by three-dimensional piecewise linear vector fields whose tangency sets consist of two parallel straight lines. This class is particularly significant due to its frequent appearance in switching systems, hysteresis loops, relay control, dry friction oscillators, and power electronics. Such systems with parallel tangency lines appear naturally in engineering, control systems, and biological phenomena. Moreover, such systems of the form $\dot{x} = A^\pm x + b^\pm$ have received significant attention due to their rich dynamical behavior, including sliding motions, invariant crossing surfaces, and bifurcation phenomena, see [48] and the references therein.

In [17], the authors consider a class of piecewise linear systems in $\mathbb{R}^3$ given by (5-1), having two parallel tangency lines on the switching boundary, two cusp singularities, and a pseudo-equilibrium point. For this class of systems, they show the occurrence of two distinct types of Hopf-like bifurcations, each giving rise to a crossing limit cycle. For this system, we analyze how a perturbation affects its dynamics. Our main results are as follows. First, after the perturbation, the two tangency lines intersect at a point, which is a parabolic two-fold. Second, three new pseudo-equilibrium points appear, one of which is this intersection point. Depending on the sign of a specific parameter, the other two points either move toward or away from the pseudo-equilibrium of the original system as the perturbation parameter tends to zero. Third, when one of these two new points converges to the original pseudo-equilibrium, its qualitative behavior remains the same as that of the original system.

This chapter is organized as follows. Section 5.1 introduces the original system, its canonical form, and the Proposition 1 (see [17], page 8). The perturbed system is obtained in Section 5.2. Section 5.3 presents its characterization. In section 5.4, we study the dynamics of this system when it exhibits a parabolic two-fold singularity,

two cusp singularity points, and three pseudo-equilibrium points. Finally, Section 5.5 contains the conclusions of this chapter.

5.1 The piecewise linear system with two parallel tangency lines

In this section, we present the model (5-1) introduced in [17] and some results related to it. This is the model that will be perturbed in the next section, and its perturbed version will be studied throughout the remainder of this chapter. Thus, consider a piecewise linear system of the form

$$\dot{\mathbf{x}} = \begin{cases} P\mathbf{x} + \mathbf{n}^-, & \text{if } h(\mathbf{x}) < 0 \\ P\mathbf{x} + \mathbf{n}^+, & \text{if } h(\mathbf{x}) > 0, \end{cases} \quad (5-1)$$

where the dot denotes derivative respect to the time t , $\mathbf{x} = (x, y, z) \in \mathbb{R}^3$ is the state vector, $P = (p_{ij})_{3 \times 3}$ for $i, j \in \{1, 2, 3\}$ and $\mathbf{n}^\pm = (n_1^\pm, n_2^\pm, n_3^\pm)$ are matrix and vector parameters respectively, and h is a switching scalar function. The switching boundary Σ of (5-1) is defined as being the third coordinate plane:

$$\Sigma = \{(x, y, z) \in \mathbb{R}^3 : h(\mathbf{x}) = z = 0\}, \quad (5-2)$$

where $\mathbb{R}^3$ -space is formed by the union of all the set Σ , $\Sigma^- = \{(x, y, z) \in \mathbb{R}^3 : z < 0\}$ and $\Sigma^+ = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$. The linear vector field is defined in the zones Σ^- and Σ^+ by

$$\mathcal{F}^-(\mathbf{x}) = P\mathbf{x} + \mathbf{n}^- \quad \text{and} \quad \mathcal{F}^+(\mathbf{x}) = P\mathbf{x} + \mathbf{n}^+, \quad (5-3)$$

respectively, with both vector fields appropriately extend to Σ . The system (5-1) satisfies the following hypotheses:

(H1) There are two tangency lines on Σ , namely

$$T^\pm = \{(x, y, z) \in \Sigma : p_{31}x + p_{32}y + n_3^\pm = 0\} \quad (5-4)$$

is that, $p_{31}^2 + p_{32}^2 \neq 0$, one for each vector field $\mathcal{F}^\pm(\mathbf{x})$. These tangency lines are parallel if $n_3^- \neq n_3^+$ or coincident if $n_3^- = n_3^+$.

(H2) There are two points $\mathbf{q}^+ \in T^+$, $\mathbf{q}^- \in T^-$ such that

$$h(\mathbf{q}^\pm) = L_{\mathcal{F}^\pm} h(\mathbf{q}^\pm) = L_{\mathcal{F}^\pm}^2 h(\mathbf{q}^\pm) = 0,$$

where $= L_{\mathcal{F}^\pm} h$ is the Lie derivative of the vector fields $\mathcal{F}^\pm$, $\det(\mathbf{M}) \neq 0$, and

$$\mathbf{M} = \begin{pmatrix} 0 & 0 & 1 \\ p_{31} & p_{32} & p_{33} \\ p_{21}p_{32} + p_{31}(p_{11} + p_{33}) & p_{12}p_{31} + p_{32}(p_{22} + p_{33}) & p_{13}p_{31} + p_{23}p_{32} + p_{33}^2 \end{pmatrix}.$$

Equivalently, these points $\mathbf{q}^\pm$ correspond to cusp singularities in $\mathcal{F}^\pm$.

(H3) Assume que $\det(R) \neq 0$, where

$$R = \begin{pmatrix} p_{11} & p_{12} & n_1^+ - n_1^- \\ p_{21} & p_{22} & n_2^+ - n_2^- \\ p_{31} & p_{32} & n_3^+ - n_3^- \end{pmatrix}.$$

This hypothesis implies that there is only one point $\mathbf{x}_p \in \Sigma$ where the vector fields $\mathcal{F}^+$ and $\mathcal{F}^-$ are anti-collinear to Σ , i.e., the linear equations system

$$\begin{aligned} (1 - \alpha)\mathcal{F}^-(x, y, z) + \alpha\mathcal{F}^+(x, y, z) &= P\mathbf{x} + \alpha\mathbf{n}^+ + (1 - \alpha)\mathbf{n}^- = 0, \\ h(x, y, z) &= z = 0, \end{aligned}$$

has a single solution for (x, y, z, α) . That is, the system (5-1) has a single pseudo-equilibrium, denoted by $\mathbf{x}_p$.

5.1.1 Canonical form of the system (5-1)

Under the hypotheses **(H1)**-**(H3)** the system (5-1) can be rewritten in a canonical form by

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{F}^-(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \alpha\mathbf{b}, & \text{if } z < 0, \\ \mathbf{F}^+(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + (\alpha - 1)\mathbf{b}, & \text{if } z > 0 \end{cases} \quad (5-5)$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} \nu \\ \beta \\ \mu \end{pmatrix}$$

where $a_{11}, a_{12}, a_{13}, \alpha, \nu, \beta$, and μ are real coefficients. Moreover, the switching boundary Σ is the same as defined in (5-2). These results are summarized in the following proposition.

Proposition 5.1. *Consider the hypotheses **(H1)**-**(H3)** with respect to system (5-1). The following statements hold.*

- (a) *There is an affine transformation of coordinates such that the phase portrait of (5-1) is mapped onto the phase portrait of (5-5), preserving the orientation of all orbits including sliding orbits.*
- (b) *The coefficients of the canonical form (5-5) are related to the coefficients of the general form (5-1) as described below:*

$$\begin{aligned}
 a_{11} &= \text{tr}(P), \quad a_{12} = -\text{tr}(\text{adj}(P)), \quad a_{13} = \det(P), \\
 \alpha &= \frac{L_{\mathcal{F}^-}h(\mathbf{x}_p) - L_{\mathcal{F}^+}h(\mathbf{x}_p)}{L_{\mathcal{F}^-}h(\mathbf{x}_p) - L_{\mathcal{F}^+}h(\mathbf{x}_p)} = -\frac{\det(Q)}{\det(R)}, \\
 \nu &= L_{\mathcal{F}^-}^3h(\mathbf{x}) - L_{\mathcal{F}^+}^3h(\mathbf{x}) = \begin{pmatrix} p_{31} & p_{32} & p_{33} \end{pmatrix} P \begin{pmatrix} n_1^- & -n_1^+ \\ n_2^- & -n_2^+ \\ n_3^- & -n_3^+ \end{pmatrix} \\
 \beta &= L_{\mathcal{F}^-}^2h(\mathbf{x}) - L_{\mathcal{F}^+}^2h(\mathbf{x}) = p_{31}(n_1^- - n_1^+) + p_{32}(n_2^- - n_2^+) + p_{33}(n_3^- - n_3^+), \\
 \mu &= L_{\mathcal{F}^-}h(\mathbf{x}_p) - L_{\mathcal{F}^+}h(\mathbf{x}_p) = n_3^- - n_3^+,
 \end{aligned}$$

where $\mathbf{x}_p$ is the pseudo-equilibrium point of (5-1), adj stands for the adjugate matrix and

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}, \quad Q = \begin{pmatrix} p_{11} & p_{12} & n_1^- \\ p_{21} & p_{22} & n_2^- \\ p_{31} & p_{32} & n_3^- \end{pmatrix}, \quad R = \begin{pmatrix} p_{11} & p_{12} & n_1^+ & -n_1^- \\ p_{21} & p_{22} & n_2^+ & -n_2^- \\ p_{31} & p_{32} & n_3^+ & -n_3^- \end{pmatrix}$$

According to [17], the parameter α controls whether the pseudo-equilibrium lies in the sliding or crossing region. The parameter μ controls whether there is an attractive or repulsive sliding region, as well as the distance between the tangency lines in Σ . The parameter β controls whether there is an invisible or visible two-fold segment when $\mu = 0$, as well as the length of that two-fold segment. The parameter ν controls the position of the regular equilibria with respect to Σ (whether it lies above or below Σ) when $\mu = \beta = 0$.

5.2 The perturbed system

We now seek to introduce a perturbation into system (5-1) such that the resulting perturbed system satisfies the following conditions: its tangency lines intersect transversally at a point; its sliding vector field is well-defined; and its canonical form is as simple as possible. Moreover, as the perturbation parameter tends to zero, this perturbed system recovers the original system, and its canonical form recovers that of the original system. In what follows, we explicitly construct this perturbation and present the corresponding calculations.

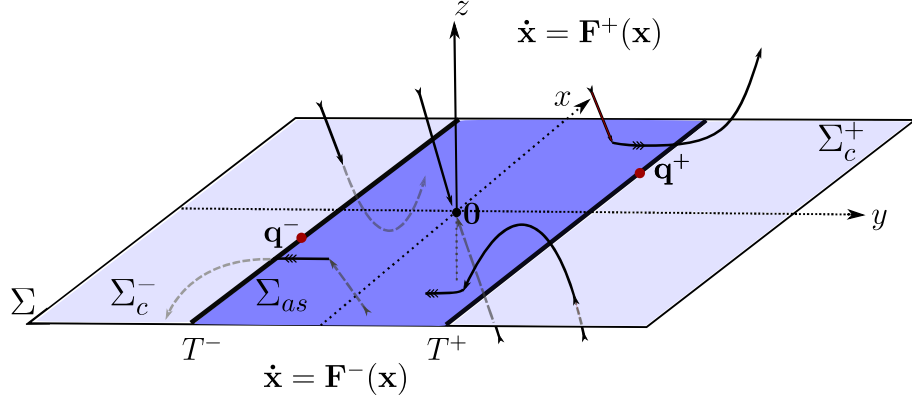


Figure 5.1: *Switching boundary of system (5-1) for $\mu > 0$, $0 < \alpha < 1$ and $\beta > 0$. The points $\mathbf{q}^+$ and $\mathbf{q}^-$ indicate the cusp points of $\mathbf{F}^+$ and $\mathbf{F}^-$, respectively. Adapted from [17].*

We consider $n_3^+ \neq n_3^-$. Then the tangency lines T^+ and T^- are parallel (see hypothesis **(H1)** and Figure 5.1). Thus, it is sufficient to rotate one of them so that they intersect transversally at a point. We will therefore keep the line T^+ fixed, meaning the field $\mathcal{F}^+(\mathbf{x})$ remains unchanged, and rotate the line T^- , that is, we will perturb the system $\mathcal{F}^-(\mathbf{x})$.

Thus, consider $p_{22} = (-p_{12}p_{31} - p_{32}p_{33})/p_{32}$. This choice simplifies our calculations and does not alter the original conditions of the system (5-1). Furthermore, because the original hypotheses guarantee $p_{32} \neq 0$, the expression is well-defined. Then, consider the matrix P and its perturbation P_ε

$$P = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}, \quad P_\varepsilon = \begin{pmatrix} p_{11\varepsilon} & p_{12\varepsilon} & p_{13\varepsilon} \\ p_{21\varepsilon} & p_{22\varepsilon} & p_{23\varepsilon} \\ p_{31\varepsilon} & p_{32\varepsilon} & p_{33\varepsilon} \end{pmatrix},$$

the vectors $n^+ = \{n_1^+, n_2^+, n_3^+\}$, $n^- = \{n_1^-, n_2^-, n_3^-\}$, the perturbed vector of n^- , denoted by $n_\varepsilon^- = \{n_{1\varepsilon}^-, n_{2\varepsilon}^-, n_{3\varepsilon}^-\}$. Furthermore, consider the vector fields given in (5-3) and perturbed vector field of $\mathcal{F}^-$, denoted by $\mathcal{F}_\varepsilon^-(\mathbf{x}) = P_\varepsilon \mathbf{x} + n_\varepsilon^-$, together with the tangency line T^- defined in (5-4). The perturbation of T^- is denoted by $T_\varepsilon^- = \{(x, y, z) \in \Sigma : p_{31\varepsilon}x + p_{32\varepsilon}y + p_{33\varepsilon}z + n_{3\varepsilon}^- = 0\}$. As mentioned before, we now rotate the tangency line T^- , obtaining the new line T_ε^- . Thus, setting $z = 0$ in the equation corresponding to T^- , we obtain,

$$y = -\frac{p_{31}x - n_3^-}{p_{32}}.$$

This expression can be written in the form $y = mx + b$, or equivalently as $y = \tan(\theta)x + b$, where θ denotes the angle between the line T^- and the x -axis.

Hence,

$$m = \tan(\theta) = -\frac{p_{31}}{p_{32}}, \quad b = -\frac{n_3^-}{p_{32}} \Rightarrow y = -\frac{p_{31}}{p_{32}}x - \frac{n_3^-}{p_{32}}.$$

We now rotate the line so that the new angle is ε -close to the original one, with $\varepsilon > 0$ sufficiently small. To this end, we use the expansion

$$\tan(\theta + \varepsilon) = \tan(\theta) + (1 + \tan^2(\theta))\varepsilon + \mathcal{O}(\varepsilon^2).$$

Consequently, the expression of the perturbed tangency curve is

$$y_{\theta\varepsilon} = \tan(\theta + \varepsilon)x + \beta_\varepsilon = \tan \theta x + (1 + \tan^2(\theta))x\varepsilon + \beta_\varepsilon. \quad (5-6)$$

Next, we calculate

$$y - y_{\theta\varepsilon} = 0 \Rightarrow x = -\frac{p_{32}(n_3^- + p_{32}\beta_\varepsilon)}{(p_{31}^2 + p_{32}^2)\varepsilon}.$$

Furthermore, we impose that the intersection point of the tangency curves occurs at $y = 0$. Substituting the value of x into equation (5-6) and calculating $y_{\theta\varepsilon} = 0$, we have

$$\beta_\varepsilon = n_3^- \left(\frac{\varepsilon}{p_{31}} + \frac{-p_{32} + p_{31}\varepsilon}{p_{32}^2} \right),$$

with $p_{31} \neq 0$. Thus, the tangential curve of the perturbed field is

$$y_{\theta\varepsilon} = \left(-\frac{p_{31}}{p_{32}} + \left(\frac{p_{31}^2 + p_{32}^2}{p_{32}^2} \right) \varepsilon \right) x + \left(\frac{1}{p_{31}} + \frac{p_{31}}{p_{32}^2} \right) \varepsilon - \frac{n_3^-}{p_{32}}. \quad (5-7)$$

Now we determine the parameter values of the perturbed vector field that yield this perturbed tangency curve. Thus, consider the equation that defines the line T_ε^-

$$y_\varepsilon = -\frac{p_{31\varepsilon}}{p_{32\varepsilon}}x - \frac{n_{3\varepsilon}^-}{p_{32\varepsilon}}, \quad (5-8)$$

where $p_{32\varepsilon} \neq 0$, and the equation (5-7). Then from

$$y_\varepsilon - y_{\theta\varepsilon} = \left(\frac{p_{31}}{p_{32}} - \frac{p_{31\varepsilon}}{p_{32\varepsilon}} + \left(-1 - \frac{p_{31}^2}{p_{32}^2} \right) \varepsilon \right) x - n_3^- \left(\frac{1}{p_{31}} + \frac{p_{31}}{p_{32}^2} \right) \varepsilon + \frac{n_3^-}{p_{32}} - \frac{n_{3\varepsilon}^-}{p_{32\varepsilon}} = 0 \quad (5-9)$$

we obtain

$$p_{31\varepsilon} = -\frac{p_{32\varepsilon}(-p_{31}p_{32} + (p_{31}^2 + p_{32}^2)\varepsilon)}{p_{32}^2}.$$

Substituting this expression in (5-9) we get

$$n_3^- \varepsilon \left(-\frac{p_{31}}{p_{32}^2} - \frac{1}{p_{31}} \right) + \frac{n_3^-}{p_{32}} - \frac{n_{3\varepsilon}^-}{p_{32\varepsilon}} = 0,$$

from which it follows that

$$n_{3\varepsilon}^- = n_3^- p_{32\varepsilon} \left(\frac{p_{32} - p_{31}\varepsilon}{p_{32}^2} - \frac{\varepsilon}{p_{31}} \right)$$

Therefore the equations (5-8) and (5-7) are equal and they are given by

$$y_{\theta\varepsilon} = y_\varepsilon = \frac{1}{p_{31}p_{32}^2} (p_{31}x + n_3^-) (-p_{31}p_{32} + p_{31}^2\varepsilon + p_{32}^2\varepsilon). \quad (5-10)$$

Thus, the tangency line T_ε^- , which is transversal to T^+ , is defined by expression (5-10). In this way, with $p_{31\varepsilon}$ and $n_{3\varepsilon}^-$ defined as above, the vector field $\mathcal{F}_\varepsilon^-(\mathbf{x})$ is given by

$$\begin{aligned} \mathcal{F}_\varepsilon^-(\mathbf{x}) = & (p_{11\varepsilon}x + p_{12\varepsilon}y + p_{13\varepsilon}z + n_{1\varepsilon}^-, p_{21\varepsilon}x + p_{22\varepsilon}y + p_{23\varepsilon}z + n_{2\varepsilon}^-, \\ & \frac{1}{p_{31}p_{32}^2} (p_{31}p_{32}(p_{32\varepsilon}(n_3^- + p_{31}x + p_{32}y) + p_{32}p_{33\varepsilon}z) \\ & - p_{32\varepsilon}(p_{31}^2 + p_{32}^2)(n_3^- + p_{31}x)\varepsilon) \Big). \end{aligned}$$

Next, we consider the sliding vector fields given by

$$F^s(\mathbf{x}) = \frac{L_{\mathcal{F}^-}h(\mathbf{x})\mathcal{F}^+(\mathbf{x}) - L_{\mathcal{F}^+}h(\mathbf{x})\mathcal{F}^-(\mathbf{x})}{L_{\mathcal{F}^-}(\mathbf{x}) - L_{\mathcal{F}^+}(\mathbf{x})} = \{f^1(\mathbf{x}), f^2(\mathbf{x}), 0\},$$

where

$$f^1(\mathbf{x}) = \frac{((p_{11}x + p_{12}y + n_1^+)(p_{31}x + p_{32}y + n_3^-) - (p_{11}x + p_{12}y + n_1^-)(p_{31}x + p_{32}y + n_3^+))}{n_3^- - n_3^+},$$

$$f^2(\mathbf{x}) = \frac{((p_{21}x + p_{22}y + n_2^+)(p_{31}x + p_{32}y + n_3^-) - (p_{21}x + p_{22}y + n_2^-)(p_{31}x + p_{32}y + n_3^+))}{n_3^- - n_3^+},$$

and

$$F_\varepsilon^s(\mathbf{x}) = \frac{L_{\mathcal{F}_\varepsilon^-}h(\mathbf{x})\mathcal{F}_\varepsilon^+(\mathbf{x}) - L_{\mathcal{F}_\varepsilon^+}h(\mathbf{x})\mathcal{F}_\varepsilon^-(\mathbf{x})}{L_{\mathcal{F}_\varepsilon^-}(\mathbf{x}) - L_{\mathcal{F}_\varepsilon^+}(\mathbf{x})} = \{f_\varepsilon^1(\mathbf{x}), f_\varepsilon^2(\mathbf{x}), 0\},$$

where $f_\varepsilon^1 = f_\varepsilon^{11}/f_\varepsilon^{12}$, $f_\varepsilon^2 = f_\varepsilon^{21}/f_\varepsilon^{22}$ and

$$\begin{aligned}
f_\varepsilon^{11}(\mathbf{x}) &= p_{31}p_{32}^2(p_{11\varepsilon}x + p_{12\varepsilon}y + n_{1\varepsilon}^-)(p_{31}x + p_{32}y + n_3^+) + p_{32\varepsilon}(p_{11}x + p_{12}y + n_1^+) \\
&\quad ((p_{31}^2 + p_{32}^2)(p_{31}x + n_3^-)\varepsilon - p_{31}p_{32}(p_{31}x + p_{32}y + n_3^-)), \\
f_\varepsilon^{12}(\mathbf{x}) &= p_{31}p_{32}(n_3^+p_{32} - n_3^-p_{32\varepsilon} + (p_{32} - p_{32\varepsilon})(p_{31}x + p_{32}x_2)) + (p_{31}^2 + p_{32}^2) \\
&\quad p_{32\varepsilon}(n_3^- + p_{31}x_1)\varepsilon, \\
f_\varepsilon^{21}(\mathbf{x}) &= p_{31}p_{32}^3(p_{21\varepsilon}x + p_{22\varepsilon}y + n_{2\varepsilon}^-)(p_{31}x + p_{32}y + n_3^+) - (n_2^+p_{32} + p_{21}p_{32}x - (p_{12}p_{31} \\
&\quad + p_{32}p_{33})y)(p_{31}p_{32}p_{32\varepsilon}(p_{31}x + p_{32}y + n_3^-) - p_{32\varepsilon}(p_{31}^2 + p_{32}^2)(p_{31}x + n_3^-)\varepsilon), \\
f_\varepsilon^{22}(\mathbf{x}) &= -p_{31}p_{32}^2p_{32\varepsilon}(p_{31}x + p_{32}y + n_3^-) + p_{31}p_{32}^3(p_{31}x + p_{32}y + n_3^+) + (p_{31}^2 + p_{32}^2) \\
&\quad p_{32\varepsilon}p_{32}(p_{31}x + n_3^-)\varepsilon.
\end{aligned}$$

In order to obtain the dynamics of the perturbed sliding vector field on Σ , we consider the normalized sliding vector fields $\hat{F}^s(\mathbf{x})$ and $\hat{F}_\varepsilon^s(\mathbf{x})$, and the expression $\hat{F}^s(\mathbf{x}) - \hat{F}_\varepsilon^s(\mathbf{x}) = 0$, where $\hat{F}^s(\mathbf{x}) = (L_{\mathcal{F}^-}h(\mathbf{x}) - L_{\mathcal{F}^+}h(\mathbf{x}))F^s(\mathbf{x})$ and $\hat{F}_\varepsilon^s(\mathbf{x}) = (L_{\mathcal{F}_\varepsilon^-}h(\mathbf{x}) - L_{\mathcal{F}_\varepsilon^+}h(\mathbf{x}))F_\varepsilon^s$. We then determine the values of parameters $p_{ij\varepsilon}$ and $n_{\varepsilon j}^-$ for which the pseudo-equilibria and the sliding vector field are well defined. We obtain

$$\begin{aligned}
p_{11\varepsilon} &= p_{11} \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right), \quad p_{21\varepsilon} = p_{21} \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right), \quad p_{22\varepsilon} = p_{22} \\
p_{31\varepsilon} &= p_{31} \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right), \quad n_{3\varepsilon}^- = n_3^- \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right), \quad p_{12\varepsilon} = p_{12}, \\
p_{32\varepsilon} &= p_{32}, \quad n_{1\varepsilon}^- = n_1^-, \quad n_{2\varepsilon}^- = n_2^-.
\end{aligned}$$

Note that with these value of parameters we have $\lim_{\varepsilon \rightarrow 0} F_\varepsilon^s = F^s$ and two pseudo-equilibriums given by $\mathbf{x}_\varepsilon^0 = \{x_\varepsilon^0, y_\varepsilon^0, 0\}$ and $\mathbf{x}_\varepsilon^1 = \{x_\varepsilon^1, y_\varepsilon^1, 0\}$, with $x_\varepsilon^1 = \frac{x_\varepsilon^{11}}{x_\varepsilon^{12}}$ and $y_\varepsilon^1 = \frac{y_\varepsilon^{11}}{y_\varepsilon^{12}}$, where

$$\begin{aligned}
x_\varepsilon^0 &= \frac{(n_3^- - n_3^+)p_{32}}{(p_{31}^2 + p_{32}^2)\varepsilon} - \frac{n_3^-}{p_{31}} \quad \text{and} \quad y_\varepsilon^0 = (n_3^- - n_3^+) \left(\frac{1}{p_{32}} - \frac{p_{31}}{(p_{31}^2 + p_{32}^2)\varepsilon} \right), \\
x_\varepsilon^{11} &= p_{31}p_{32}(n_2^-p_{32}(n_1^+p_{32} - n_3^+p_{12}) + n_3^- (n_1^+p_{12}p_{31} + n_2^+p_{12}p_{32} + n_1^+p_{32}p_{33}) \\
&\quad - n_1^-(n_3^+p_{12}p_{31} + n_2^+p_{32}^2 + n_3^+p_{32}p_{33})) - n_3^-(p_{31}^2 + p_{32}^2)(n_2^+p_{12}p_{32} + n_1^+(p_{12}p_{31} \\
&\quad + p_{32}p_{33})\varepsilon), \\
x_\varepsilon^{12} &= ((n_1^+ - n_1^-)(p_{12}p_{31}^2 + p_{32}(p_{21}p_{32} + p_{31}p_{33})) + p_{32}(n_2^- - n_2^+)(p_{12}p_{31} - p_{11}p_{32}) \\
&\quad + (n_3^+ - n_3^-)(p_{11}p_{12}p_{31} + p_{12}p_{21}p_{32} + p_{11}p_{32}p_{33}))p_{31}p_{32} + (p_{31}^2 + p_{32}^2)(n_2^+p_{32} \\
&\quad (p_{12}p_{31} - p_{11}p_{32}) + (n_3^+ - n_3^-)(p_{11}p_{12}p_{31} + p_{12}p_{21}p_{32} + p_{11}p_{32}p_{33}))
\end{aligned}$$

$$\begin{aligned}
& +n_1^+(p_{12}p_{31}^2 + p_{32}(p_{21}p_{32} + p_{31}p_{33}))\varepsilon, \\
y_\varepsilon^{11} &= (p_{31}n_3^-(n_1^+p_{21} - n_2^+p_{11}) + p_{31}n_2^-(n_3^+p_{11} - n_1^+p_{31}) + p_{31}n_1^-(n_2^+p_{31} - n_3^+p_{21}))p_{32}^2 \\
& + n_3^-(n_2^+p_{11} - n_1^+p_{21})p_{32}(p_{31}^2 + p_{32}^2)\varepsilon, \\
y_\varepsilon^{12} &= p_{31}p_{32}(n_1^- - n_1^+)(p_{12}p_{31}^2 + p_{32}(p_{21}p_{32} + p_{31}p_{33})) + (n_2^- - n_2^+)(p_{12}p_{31}^2 - p_{11}p_{32}) \\
& p_{31}p_{32}^2 - (n_3^- - n_3^+)(p_{11}p_{12}p_{31} + p_{12}p_{21}p_{32} + p_{11}p_{32}p_{33}) + n_3^-(p_{31}^2 + p_{32}^2) \\
& (p_{11}p_{12}p_{31} + p_{12}p_{21}p_{32} + p_{11}p_{32}p_{33})\varepsilon.
\end{aligned}$$

Moreover, $\lim_{\varepsilon \rightarrow 0} \mathbf{x}_\varepsilon^1 = \mathbf{x}_\mathbf{p}$, where $\mathbf{x}_\mathbf{p}$ is the pseudo-equilibrium of the original system and, $\mathbf{x}_\varepsilon^0$ is the intersection point of the tangency lines. Now, it remains to determine the values for the following parameters: $p_{13\varepsilon}$, $p_{23\varepsilon}$ and $p_{13\varepsilon}$. Thus, the matrix P_ε and the vector n_ε^- are given by

$$P_\varepsilon = \begin{pmatrix} p_{11} \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right) & p_{12} & p_{13\varepsilon} \\ p_{21} \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right) & p_{22} & p_{23\varepsilon} \\ p_{31} \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right) & p_{32} & p_{33\varepsilon} \end{pmatrix}, \quad n_\varepsilon^- = \left\{ n_1^-, n_2^-, n_3^- \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right) \right\},$$

where $p_{22} = \frac{-p_{12}p_{32} - p_{32}p_{33}}{p_{32}}$. In what follows, we explicitly construct the matrix A_ε ; that is, we obtain the canonical form of matrix P_ε . For this purpose, we consider the matrix T given in [17] and compute $TP_\varepsilon T^{-1}$. Thus

$$A_\varepsilon = \begin{pmatrix} A_{11\varepsilon} & A_{12\varepsilon} & A_{13\varepsilon} \\ A_{21} & 0 & 0 \\ A_{31\varepsilon} & 1 & 0 \end{pmatrix}, \quad (5-11)$$

where, $A_{11\varepsilon} = a_{11} + \varepsilon A_{11}$, $A_{12\varepsilon} = a_{12}$, $A_{13\varepsilon} = a_{13} + \varepsilon A_{13}$, $A_{21\varepsilon} = 1 + \varepsilon A_{21}$, $A_{22\varepsilon} = 0$, $A_{23\varepsilon} = 0$, $A_{31\varepsilon} = \varepsilon A_{31}$, $A_{32\varepsilon} = 1$, $A_{33\varepsilon} = 0$, with

$$\begin{aligned}
A_{11} &= \frac{(p_{31}^2 + p_{32}^2)}{p_{31} \det M} (p_{11}^2 p_{31} + p_{11}(p_{21}p_{32} + p_{31}p_{33} + p_{31}(p_{13}p_{31} + p_{23}p_{32} + p_{33}^2))), \\
A_{12} &= 0, \quad A_{22} = 0, \quad A_{23} = 0, \quad A_{32} = 0, \quad A_{33} = 0, \\
A_{13} &= -\frac{(p_{31}^2 + p_{32}^2)}{p_{31} \det M} (p_{11}^2 p_{31} + p_{11}(p_{21}p_{32} + p_{31}p_{33} + p_{31}(p_{13}p_{31} + p_{23}p_{32} + p_{33}^2))) \\
& (p_{13}p_{31} + p_{23}p_{32} + p_{33}^2), \\
A_{21} &= -\frac{p_{31}^2 + p_{32}^2}{p_{31}p_{32}}, \quad A_{31} = \frac{(p_{31}^2 + p_{32}^2)}{\det M}.
\end{aligned}$$

The coefficients a_{ij} denote the entries of the original matrix A from [17]. Further-

more, note that $A_{21} \neq 0$ and $A_{31} \neq 0$, and the entries of A_ε are obtained by setting

$$\begin{aligned} p_{13\varepsilon} &= p_{13} + \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}(\det M)^2} p_{32}(p_{13}p_{31} + p_{23}p_{32} + p_{33}^2)^2\varepsilon, \\ p_{23\varepsilon} &= p_{23} - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}(\det M)^2} (p_{13}p_{31} + p_{23}p_{32} + p_{33}^2) (p_{11}^2p_{31}^2 + p_{13}p_{31}^3 + p_{23}p_{31}^2p_{32} + p_{21}^2p_{32}^2 \\ &\quad + p_{21}p_{31}p_{32}p_{33} + p_{31}^2p_{33}^2 + p_{11}p_{31}(2p_{21}p_{32} + p_{31}p_{33})) , \\ p_{33\varepsilon} &= p_{33} - \frac{(p_{31}^2 + p_{32}^2)(p_{13}p_{31} + p_{23}p_{32} + p_{33}^2)\varepsilon}{\det M}, \end{aligned}$$

with $\det M = -p_{32}(p_{21}p_{32} + p_{31}(p_{11} + p_{33})) \neq 0$.

Now, we compute $T(P_\varepsilon X_\varepsilon^1 + n_\varepsilon^-) = \alpha_\varepsilon \mathbf{b}_\varepsilon$, where $\alpha_\varepsilon = \frac{\alpha_\varepsilon^1}{\alpha_\varepsilon^2}$ with

$$\begin{aligned} \alpha_\varepsilon^1 &= p_{31}p_{32} \left(-n_3^-(p_{12}p_{21} - p_{11}p_{22}) + n_2^-(p_{12}p_{31} - p_{11}p_{32}) - n_1^-(p_{22}p_{31} - p_{21}p_{32}) \right) \\ &\quad + \varepsilon n_3^-(p_{12}p_{21} - p_{11}p_{22}) (p_{31}^2 + p_{32}^2), \\ \alpha_\varepsilon^2 &= p_{31}p_{32} \left((n_3^+ - n_3^-)(p_{12}p_{21} - p_{11}p_{22}) + (n_2^- - n_2^+)(p_{12}p_{31} - p_{11}p_{32}) \right. \\ &\quad \left. - (n_1^- - n_1^+)(p_{22}p_{31} - p_{21}p_{32}) \right) + \varepsilon n_3^-(p_{12}p_{21} - p_{11}p_{22}) (p_{31}^2 + p_{32}^2) \end{aligned}$$

and, expanding the expression above in the Taylor series to first order in ε , gives $T(P_\varepsilon X_\varepsilon^1 + n_\varepsilon^-) = \alpha \mathbf{b} + \varepsilon \mathbf{b}_1$, where $\mathbf{b}_1 = \{\nu_1, \beta_1, \mu_1\}$ and ν_1, β_1, μ_1 depend on the originals parameters of system (5-1). Therefore, the system to be study is given by

$$\begin{aligned} \mathbf{F}^+ &= TPT^{-1}\mathbf{y} + T(PX_p + \mathbf{n}^+) = \mathbf{A}\mathbf{y} + (\alpha - 1)\mathbf{b}, \\ \mathbf{F}_\varepsilon^- &= TP_\varepsilon T^{-1}\mathbf{y} + T(P_\varepsilon X_\varepsilon^1 + \mathbf{n}_\varepsilon^-) = \mathbf{A}_\varepsilon \mathbf{y} + \alpha \mathbf{b} + \varepsilon \mathbf{b}_1. \end{aligned} \tag{5-12}$$

In the next section, we summarize the results obtained.

5.3 Characterization of the perturbed system

Consider the piecewise system of the form

$$\dot{\mathbf{x}}_\varepsilon = \begin{cases} \mathcal{F}^+(\mathbf{x}) = P\mathbf{x} + \mathbf{n}^+, & h(\mathbf{x}) > 0 \\ \mathcal{F}_\varepsilon^-(\mathbf{x}) = P_\varepsilon \mathbf{x} + \mathbf{n}_\varepsilon^-, & h(\mathbf{x}) < 0. \end{cases} \tag{5-13}$$

where the matrix P_ε and the vector $\mathbf{n}_\varepsilon^-$ are given by

$$P_\varepsilon = \begin{pmatrix} p_{11\varepsilon} & p_{12} & p_{13\varepsilon} \\ p_{21\varepsilon} & p_{22} & p_{23\varepsilon} \\ p_{31\varepsilon} & p_{32} & p_{33\varepsilon} \end{pmatrix}, \quad \mathbf{n}_\varepsilon^- = \{n_1^-, n_2^-, n_{3\varepsilon}^-\},$$

and

$$\begin{aligned}
p_{11\varepsilon} &= p_{11} \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right), & p_{21\varepsilon} &= p_{21} \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right), \\
p_{31\varepsilon} &= p_{31} \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right), & n_{3\varepsilon}^- &= n_3^- \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right), \\
p_{13\varepsilon} &= p_{13} + \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}(\det M)^2} p_{32} (p_{13}p_{31} + p_{23}p_{32} + p_{33}^2)^2 \varepsilon, \\
p_{23\varepsilon} &= p_{23} - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}(\det M)^2} (p_{13}p_{31} + p_{23}p_{32} + p_{33}^2) (p_{11}^2 p_{31}^2 + p_{13}p_{31}^3 + p_{23}p_{31}^2 p_{32} \\
&\quad + p_{21}^2 p_{32}^2 + p_{21}p_{31}p_{32}p_{33} + p_{31}^2 p_{33}^2 + p_{11}p_{31}(2p_{21}p_{32} + p_{31}p_{33})), \\
p_{33\varepsilon} &= p_{33} - \frac{(p_{31}^2 + p_{32}^2)(p_{13}p_{31} + p_{23}p_{32} + p_{33}^2)\varepsilon}{\det M}, & p_{22} &= \frac{-p_{12}p_{31} - p_{32}p_{33}}{p_{32}}.
\end{aligned}$$

We assume the following hypotheses described below with respect to system (5-13):

($\overline{\mathbf{H1}}$) Assume $p_{31}p_{32} \neq 0$ and $\varepsilon > 0$ sufficiently small. This hypothesis implies the existence of two tangency lines on Σ

$$\begin{aligned}
T^+ &= \{(x, y, z) \in \Sigma : p_{31}x + p_{32}y + n_3^+\}, \\
T_\varepsilon^- &= \left\{ (x, y, z) \in \Sigma : p_{31} \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right) x + p_{32}y + n_3^- \left(1 - \frac{(p_{31}^2 + p_{32}^2)\varepsilon}{p_{31}p_{32}} \right) \right\},
\end{aligned}$$

one for each vector field $\mathcal{F}^+(\mathbf{x})$ and $\mathcal{F}_\varepsilon^-(\mathbf{x})$, respectively. These tangency lines intersect at a point $\mathbf{x}_\varepsilon^0$.

($\overline{\mathbf{H2}}$) Assume $\det M \neq 0$ and $\det M_\varepsilon \neq 0$, where

$$\begin{aligned}
\mathbf{M} &= \begin{pmatrix} 0 & 0 & 1 \\ p_{31} & p_{32} & p_{33} \\ p_{21}p_{32} + p_{31}(p_{11} + p_{33}) & p_{12}p_{31} + p_{32}(p_{22} + p_{33}) & p_{13}p_{31} + p_{23}p_{32} + p_{33}^2 \end{pmatrix}, \\
\mathbf{M}_\varepsilon &= \begin{pmatrix} 0 & 0 & 1 \\ p_{31\varepsilon} & p_{32} & p_{33\varepsilon} \\ p_{21\varepsilon}p_{32} + p_{31\varepsilon}(p_{11\varepsilon} + p_{33\varepsilon}) & p_{12}p_{31\varepsilon} + p_{32}(p_{22} + p_{33\varepsilon}) & p_{13\varepsilon}p_{31\varepsilon} + p_{23\varepsilon}p_{32} + p_{33\varepsilon}^2 \end{pmatrix}
\end{aligned}$$

and the “det” indicates the determinant. This hypothesis implies the existence of two points: $\mathbf{q}^+ \in T_+$, $\mathbf{q}_\varepsilon^- \in T_\varepsilon^-$ such that

$$h(\mathbf{q}^+) = L_{\mathcal{F}^+} h(\mathbf{q}^+) = L_{\mathcal{F}^+}^2 h(\mathbf{q}^+) = 0, \quad h(\mathbf{q}_\varepsilon^-) = L_{\mathcal{F}_\varepsilon^-} h(\mathbf{q}_\varepsilon^-) = L_{\mathcal{F}_\varepsilon^-}^2 h(\mathbf{q}_\varepsilon^-) = 0.$$

These points $\mathbf{q}^+$, $\mathbf{q}_\varepsilon^-$ are cusp singularities of $\mathcal{F}^+$ and $\mathcal{F}_\varepsilon^-$.

($\overline{\mathbf{H3}}$) The system (5-13) have two points of pseudo-equilibriums denote by $\mathbf{x}_\varepsilon^0$ and $\mathbf{x}_\varepsilon^1$.

5.3.1 Canonical form of the perturbed system

Under hypotheses $(\overline{\mathbf{H1}})$ - $(\overline{\mathbf{H3}})$, system $\mathcal{F}_\varepsilon^-(\mathbf{x})$ in (5-13) can be rewritten in a canonical form described by

$$\mathbf{F}_\varepsilon^-(\mathbf{x}) = \mathbf{A}_\varepsilon(\mathbf{x}) + \alpha \mathbf{b} + \varepsilon \mathbf{b}_1 \quad (5-14)$$

where

$$A_\varepsilon = \begin{pmatrix} A_{11\varepsilon} & A_{12\varepsilon} & A_{13\varepsilon} \\ A_{21\varepsilon} & 0 & 0 \\ A_{31\varepsilon} & 1 & 0 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \nu \\ \beta \\ \mu \end{pmatrix}, \quad \text{and} \quad \mathbf{b}_1 = \begin{pmatrix} \nu_1 \\ \beta_1 \\ \mu_1 \end{pmatrix},$$

where $A_{ij\varepsilon}, \alpha_\varepsilon, \nu, \beta, \mu, \nu_1, \beta_1 \in \mu_1$ are reals coefficients. In addition, the switching boundary Σ , occurring at $z = 0$, is the same as the one defined in (5-2). These statements are summarized in the following proposition, which is analogous to Proposition 5.1.

Proposition 5.2. *Consider hypotheses $(\overline{\mathbf{H1}})$ - $(\overline{\mathbf{H3}})$ for system $\mathcal{F}_\varepsilon^-(\mathbf{x})$ in (5-13). The following statements hold.*

- (a) *There is an affine coordinate transformation that maps the phase portrait of $\mathcal{F}_\varepsilon^-(\mathbf{x})$ onto the phase portrait of (5-14), preserving the orientation of all orbits, including the sliding ones.*
- (b) *The coefficients of the canonical form (5-14) are given in terms of the coefficients of the original system as described below.*

$$\begin{aligned} A_{11\varepsilon} &= a_{11} + \varepsilon A_{11}, & A_{12\varepsilon} &= a_{12}, & A_{13\varepsilon} &= a_{13} + \varepsilon A_{13}, \\ A_{21\varepsilon} &= 1 + \varepsilon A_{21}, & A_{22\varepsilon} &= 0, & A_{23\varepsilon} &= 0, & A_{31\varepsilon} &= \varepsilon A_{31}, \\ A_{32\varepsilon} &= 1, & A_{33\varepsilon} &= 0, \end{aligned}$$

where $a_{11}, a_{12}, a_{13}, \nu, \beta, \mu$ are defined in the Proposition 5.1, and the parameters ν_1, β_1 and μ_1 are given in the original terms of system (5-1). Besides,

$$\lim_{\varepsilon \rightarrow 0} A_\varepsilon = A, \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} \alpha \mathbf{b} + \varepsilon \mathbf{b}_1 = \alpha \mathbf{b},$$

that is, $\lim_{\varepsilon \rightarrow 0} \mathbf{F}_\varepsilon^-(\mathbf{x}) = \mathbf{F}^-(\mathbf{x})$.

Therefore, from the second system in (5-5) and (5-14), it follows that the canonical form of perturbed system (5-13) is given by

$$\mathbf{X} = \begin{cases} \mathbf{F}^+(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + (\alpha - 1)\mathbf{b}, & \text{if } z > 0, \\ \mathbf{F}_\varepsilon^-(\mathbf{x}) = \mathbf{A}_\varepsilon(\mathbf{x}) + \alpha \mathbf{b} + \varepsilon \mathbf{b}_1, & \text{if } z < 0. \end{cases} \quad (5-15)$$

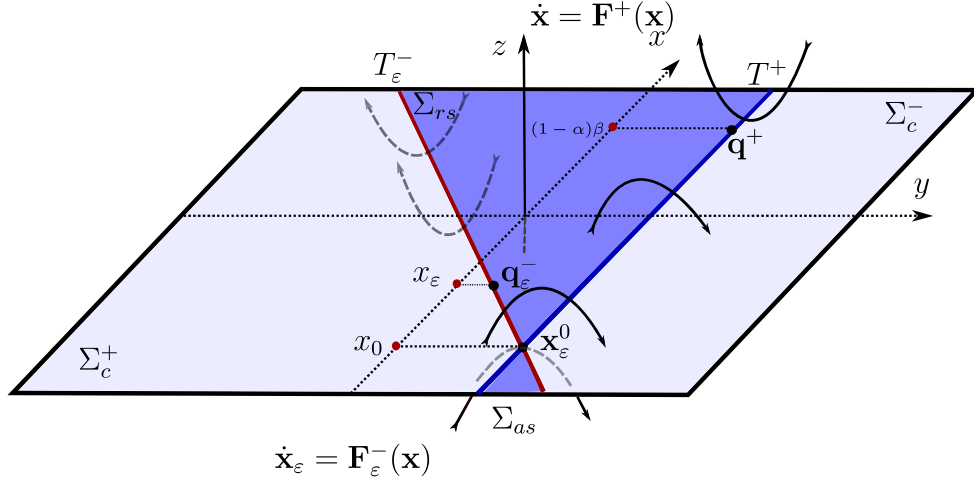


Figure 5.2: *Switching boundary of perturbed system (5-15)*
 $\mu > 0$, $0 < \alpha < 1$, $\beta > 0$, $\varepsilon > 0$ and $A_{31} > 0$.

5.4 Dynamics on the switching boundary Σ of the perturbed system

To study the dynamics of system (5-15) on Σ , we consider the Lie derivatives. Since for $z > 0$ the system F^+ in (5-15) is the same as that in (5-5), the Lie derivatives for this system are given as in [17]. Moreover, we adopt the following hypotheses from the same work for the system (5-15): $0 < \alpha < 1$, $\beta > 0$, $a_{11} \neq 0$, $\mu \neq 0$ and $\gamma \neq 0$. Thus, the Lie derivatives of $F^+(x)$ are given by

$$L_{F^+}h(\Sigma) = y + (\alpha - 1)\mu, \quad L_{F^+}^2h(T_+) = x + (\alpha - 1)\beta, \quad \text{and} \quad L_{F^+}^3h(q^+) = (\alpha - 1)\gamma$$

with $\gamma = \nu - a_{11}\beta - a_{12}\mu \neq 0$.

On the other hand, the Lie derivatives of $F_\varepsilon^-(x)$ expanded in a first-order Taylor series in ε are given by

$$\begin{aligned} L_{F_\varepsilon^-}h(\Sigma) &= y + \alpha\mu + \varepsilon(A_{31}x + \mu_1), \\ L_{F_\varepsilon^-}^2h(T_\varepsilon^-) &= x + \alpha\beta + \varepsilon(\beta_1 - \alpha A_{31}(a_{12}\mu - \nu) + x(A_{21} + a_{11}A_{31})), \\ L_{F_\varepsilon^-}^3h(q_\varepsilon^-) &= \alpha\gamma + \varepsilon(\nu_1 - a_{11}b_{11} - a_{12}\mu_1 - \alpha(A_{11}\beta - a_{12}A_{31}\beta + a_{12}A_{21}\mu - A_{21}\nu)). \end{aligned}$$

The change from an invisible to a visible fold on T^+ corresponds to a cusp singularity of F^+ whenever $\alpha \neq 1$, see [17]. Now, for T_ε^- it corresponds to a cusp singularity of F_ε^- provided that $\alpha \neq 0$ and $\varepsilon \neq 0$. The coordinates of these points are given by

$$q^+ = (1 - \alpha)(\beta, \mu, 0) \quad \text{and} \quad q_\varepsilon^- = (q_\varepsilon^1, q_\varepsilon^2, 0),$$

where

$$\begin{aligned} q_\varepsilon^1 &= -\alpha\beta + \varepsilon[\alpha(A_{21}\beta + A_{31}(a_{11}\beta + a_{12}\mu + \nu)) - \beta_1], \\ q_\varepsilon^2 &= -\alpha\mu + \varepsilon(\alpha A_{31}\beta - u_1). \end{aligned}$$

In addition, $\lim_{\varepsilon \rightarrow 0} L_{\mathbf{F}_\varepsilon^-} h(\Sigma) = L_{\mathbf{F}^-} h(\Sigma)$, $\lim_{\varepsilon \rightarrow 0} L_{\mathbf{F}_\varepsilon^-}^2 h(T_\varepsilon^-) = L_{\mathbf{F}^-}^2 h(T^-)$, $\lim_{\varepsilon \rightarrow 0} L_{\mathbf{F}_\varepsilon^-}^3 h(\mathbf{q}_\varepsilon^-) = L_{\mathbf{F}^-}^3 h(\mathbf{q}^-)$, and $\lim_{\varepsilon \rightarrow 0} \mathbf{q}_\varepsilon^- = \mathbf{q}^-$, with $\mathbf{q}^- = -\alpha\{\beta, \mu, 0\}$. Furthermore, the intersection point between the tangency lines is $\mathbf{q}_0 = (x_0, y_0, 0) = \left(-\frac{\mu + \varepsilon\mu_1}{\varepsilon A_{31}}, (1 - \alpha)\mu, 0\right)$. From the zeros of the first-order Lie derivatives, we explicitly obtain the tangency lines: $T^+ = \{(x, y, z) \in \Sigma : y = (1 - \alpha)\mu\}$ and $T_\varepsilon^- = \{(x, y, z) \in \Sigma : y = -\alpha\mu - \varepsilon(A_{31}x + \mu_1)\}$, where the vector fields $\mathbf{F}^+$ and $\mathbf{F}_\varepsilon^-$ are tangent to Σ . Using item (a) of Definition 1.3, we introduce the following propositions.

Proposition 5.3. *For all $x < (1 - \alpha)\beta$ the points in T^+ are invisible folds and for all $x > (1 - \alpha)\beta$ are visible folds, see [17].*

The following proposition provides a characterization of the points on the tangency line T_ε^- .

Proposition 5.4. *Assume that $0 < \alpha < 1$, $\beta > 0$, $A_{31} \neq 0, \gamma \neq 0$, and $\varepsilon > 0$ is a sufficiently small parameter. If $x > -\alpha\beta + \varepsilon(\alpha(A_{21}\beta - A_{31}\gamma) - \beta_1)$, then the points on T_ε^- are invisible folds; if $x < -\alpha\beta + \varepsilon(\alpha(A_{21}\beta - A_{31}\gamma) - \beta_1)$, then the points on T_ε^- are visible folds.*

Remark 5.5. *By item (a) of definition 1.3 and the fact that the inequality $A_{21} > \frac{-1 - \varepsilon a_{11} A_{31}}{\varepsilon}$ always holds, the proof of the previous proposition follows.*

Proposition 5.6. *For sufficiently small $\varepsilon > 0$, the point $\mathbf{q}_0$, which is the intersection of the tangency lines, corresponds to a parabolic two-fold.*

Proof. In fact, consider the point $\mathbf{q}_0 = (x_0, y_0, 0) = \left(-\frac{\mu + \varepsilon\mu_1}{A_{31}\varepsilon}, (1 - \alpha)\mu, 0\right)$. We must prove that one of two inequalities holds

$$-\alpha\beta + \varepsilon(\alpha(A_{21}\beta - A_{31}\gamma) - \beta_1) < x_0 < (1 - \alpha)\beta,$$

or

$$-\alpha\beta + \varepsilon(\alpha(A_{21}\beta - A_{31}\gamma) - \beta_1) > x_0 > (1 - \alpha)\beta.$$

From the expression of $x_0 = -\frac{\mu + \varepsilon\mu_1}{A_{31}\varepsilon} = -\frac{\mu}{\varepsilon A_{31}} - \frac{\mu_1}{A_{31}}$, note that

$$\lim_{\varepsilon \rightarrow 0^+} x_0 = \begin{cases} -\infty & \text{if } \frac{\mu}{A_{31}} > 0, \\ +\infty & \text{if } \frac{\mu}{A_{31}} < 0. \end{cases}$$

- (1) If $\mu A_{31} > 0$, then $x_0 < 0$ (it is a large negative number); furthermore, as $0 < \alpha < 1$ and $\beta > 0$, we have $(1 - \alpha)\beta > 0$. In this way, $x_0 < (1 - \alpha)\beta$ holds for a sufficiently small $\varepsilon > 0$.
- (2) If $\mu A_{31} < 0$, then $x_0 > 0$ (x_0 is a large positive number). Therefore, for a sufficiently small $\varepsilon > 0$, $x_0 > (1 - \alpha)\beta > 0$.

- (3) If $\mu A_{31} > 0$, we have $x_0 < 0$ (it is a large negative number); then for a sufficiently small $\varepsilon > 0$ we have $x_0 < x_\varepsilon$.
- (4) If $\mu A_{31} < 0$, we have that $x_0 > 0$. So, for a sufficiently small $\varepsilon > 0$, $x_0 > x_\varepsilon$.

Additionally, the sliding attractive region is given by $\Sigma_{as} = \{(x, y, 0) \in \Sigma : L_{\mathbf{F}+h}(\Sigma) < 0 \text{ and } L_{\mathbf{F}_\varepsilon}h(\Sigma) > 0\}$. By definition of the first Lie derivative, we

have $y < (1 - \alpha)\mu$ and $y > -\alpha\mu - \varepsilon(A_{31}x + \mu_1)$, if and only if,

$$\begin{aligned} -\alpha\mu - \varepsilon(A_{31}x + \mu_1) < y < (1 - \alpha)\mu &\Leftrightarrow -\alpha\mu - \varepsilon A_{31}x - \epsilon\mu_1 < \mu - \alpha\mu \\ &\Leftrightarrow -\varepsilon A_{31}x - \epsilon\mu_1 < \mu \\ &\Leftrightarrow -\varepsilon A_{31}x < \mu + \epsilon\mu_1 \end{aligned}$$

If $A_{31} < 0$, then $x < -\frac{\mu + \varepsilon\mu_1}{\varepsilon A_{31}} = x_0$. Furthermore, from the proof of Proposition 5.6, for sufficiently small $\varepsilon > 0$ and $\mu > 0$, we obtain $x_0 > 0$. Now, if $A_{31} > 0$, then $x > -\frac{\mu + \varepsilon\mu_1}{\varepsilon A_{31}} = x_0$, and for $\mu > 0$ it follows that $x_0 < 0$. Therefore, assume that $\mu > 0$ and

- if $A_{31} < 0$, $\Sigma_{as}(\mu, A_{31}) = \{(x, y, 0) \in \Sigma : x < x_0 > 0, y < (1 - \alpha)\mu, y > -\alpha\mu - \varepsilon(A_{31}x + \mu_1)\}$,
- if $A_{31} > 0$, then $\Sigma_{as}(\mu, A_{31}) = \{(x, y, 0) \in \Sigma : x > x_0 < 0, y < (1 - \alpha)\mu, y > -\alpha\mu - \varepsilon(A_{31}x + \mu_1)\}$.

On the other hand, the repulsive sliding region is defined by $\Sigma_{ar} = \{(x, y, 0) \in \Sigma : L_{\mathcal{F}^+}h(\Sigma) > 0, L_{\mathcal{F}^-}h(\Sigma) < 0\}$. Thus, $y > (1 - \alpha)\mu$ and $y < -\alpha\mu - \varepsilon(A_{31}x + \mu_1)$ if and only if

$$\begin{aligned} (1 - \alpha)\mu < y < \alpha\mu - \varepsilon(A_{31}x + \mu_1) &\Leftrightarrow (1 - \alpha)\mu < -\alpha\mu - \varepsilon(A_{31}x + \mu_1) \\ &\Leftrightarrow \mu < -\varepsilon A_{31}x - \varepsilon\mu_1 \\ &\Leftrightarrow \mu + \varepsilon\mu_1 < -\varepsilon A_{31}x. \end{aligned}$$

If $A_{31} < 0$, then $x > -\frac{\mu + \varepsilon\mu_1}{\varepsilon A_{31}} = x_0$. Moreover, from the proof of Proposition 5.6, if $\mu < 0$ and $\varepsilon > 0$ sufficiently small, then $x_0 < 0$. If $A_{31} > 0$, then $x < -\frac{\mu + \varepsilon\mu_1}{\varepsilon A_{31}} = x_0$ and, for $\mu < 0$ we have $x_0 > 0$. Assume that $\mu < 0$ and

- if $A_{31} < 0$, $\Sigma_{ar}(\mu, A_{31}) = \{(x, y, 0) \in \Sigma : x > x_0 < 0, y > (1 - \alpha)\mu, y < -\alpha\mu - \varepsilon(A_{31}x + \mu_1)\}$,
- if $A_{31} > 0$, then $\Sigma_{ar}(\mu, A_{31}) = \{(x, y, 0) \in \Sigma : x < x_0 > 0, y > (1 - \alpha)\mu, y < -\alpha\mu - \varepsilon(A_{31}x + \mu_1)\}$.

Finally, the crossing regions are defined by $\Sigma_c^-(\mu) = \{(x, y, 0) \in \Sigma : y < (1 - \alpha)\mu, \text{ and } y < -\alpha\mu - \varepsilon(A_{31}x + \mu_1)\}$, $\Sigma_c^+(\mu) = \{(x, y, 0) \in \Sigma : y > (1 - \alpha)\mu \text{ and } y > -\alpha\mu - \varepsilon(A_{31}x + \mu_1)\}$ for all μ .

5.4.1 Equilibrium points of the vector fields $\mathbf{F}^+$ and $\mathbf{F}_\varepsilon^-$

Here we study the equilibrium points of the systems $\mathbf{F}^+$ and $\mathbf{F}_\varepsilon^-$. From [17], we assume that $a_{13} \neq 0$ and $\gamma \neq 0$. Thus the vector field $\mathbf{F}^+$ have an equilibrium

point given by

$$E^+ = (1 - \alpha) \left(\beta, \mu, \frac{\gamma}{a_{13}} \right),$$

and the equilibrium point of the perturbed vector field $\mathbf{F}_\varepsilon^-$ is $E_\varepsilon^- = (e_\varepsilon^1, e_\varepsilon^2, e_\varepsilon^3)$, where

$$\begin{aligned} e_\varepsilon^1 &= -\alpha\beta + (\alpha A_{21}\beta - \beta_1)\varepsilon, & e_\varepsilon^2 &= -\alpha\mu + (\alpha A_{31}\beta - \mu_1)\varepsilon, \\ e_\varepsilon^3 &= -\frac{\alpha\gamma}{a_{13}} + \frac{(a_{13}\alpha\beta(A_{11} - a_{11}A_{21} - a_{12}A_{31}) + a_{13}(+a_{11}\beta_1 + a_{12}\mu_1 - \nu_1) + \alpha A_{13}\gamma)\varepsilon}{a_{13}^2}. \end{aligned}$$

and $\lim_{\varepsilon \rightarrow 0} E_\varepsilon^- = e^-$, being $e^- = -\alpha(\beta, \mu, \frac{\gamma}{a_{13}})$ the equilibrium point of the unperturbed system $\mathbf{F}^-$. Similarly to Proposition 2. in [17] we have

Proposition 5.7. *Consider the equilibria E^+ and E_ε^- of the systems (5-5), and (5-15) respectively.*

1. *The equilibrium point E^+ is said to be admissible (resp. non-admissible) if $\frac{(1-\alpha)}{a_{13}} > 0$ (resp. < 0). If $\alpha = 1$, then E^+ becomes a boundary equilibrium.*
2. *The equilibrium E_ε^- is said to be admissible (resp. non-admissible) when $-\frac{\alpha\gamma}{a_{13}} < 0$ (resp. > 0) for sufficiently small $\varepsilon > 0$. If $\alpha = \nu_1 = \beta_1 = \mu_1 = 0$, then E_ε^- becomes a boundary equilibrium.*
3. *For a sufficient small $\varepsilon > 0$, the equilibria E^+ and E_ε^- have the same stability, are asymptotically stable iff*

$$a_{11} < 0, \quad a_{13} < 0, \quad \text{and} \quad a_{11}a_{12} + a_{13}.$$

Proof. The proof of item (1) is given in [17]. For item (2), note that for a sufficiently small $\varepsilon > 0$, $\lim_{\varepsilon \rightarrow 0^+} e_\varepsilon^3 = -\alpha\gamma/a_{13}$; that is, the sign of e_ε^3 is determined by $-\alpha\gamma/a_{13}$, and, according to Proposition 2 in [17], we have $-\alpha\gamma/a_{13} < 0$. Therefore, by Definition 1.3, E_ε^- is admissible. Furthermore, by Proposition 2 in [17] we assume $\alpha = 0$; then $e_\varepsilon^- = 0$ if $b_1 = u_1 = \nu_1 = 0$. Thus $E_\varepsilon^- \in \Sigma$, that is, E_ε^- is a boundary equilibrium for sufficiently small $\varepsilon > 0$. For item (3), consider characteristic polynomial of $\mathbf{A}$ given by $-\lambda^3 + a_{11}\lambda^2 + a_{12}\lambda + a_{13}$, then by Routh-Hurwitz stability criterion we have E^+ is asymptotically stable if and only if $a_{11} < 0$, $a_{13} < 0$ and $a_{11}a_{12} + a_{13} > 0$ (see [17]). Similarly, consider the characteristic polynomial of matrix $\mathbf{A}_\varepsilon$ given by $c_3\lambda^3 + c_1\lambda^2 + \lambda + c_0$, where

$$\begin{aligned} c_0 &= a_{13} + (A_{13} + a_{13}A_{21})\varepsilon, & c_1 &= a_{12} + (a_{12}A_{21} + a_{13}A_{31})\varepsilon, & c_2 &= a_{11} + A_{21}\varepsilon, \\ \text{and} \quad c_3 &= -1. \end{aligned}$$

Since $c_3 = -1$, if $c_0 < 0$, $c_2 < 0$ and $c_1c_2 - c_0c_3 > 0$, then E_ε^- is asymptotically

stable by the Routh-Hurwitz criterion. In fact, for $\varepsilon > 0$ sufficiently small we have

$$\begin{aligned} c_0 &= a_{13} + (A_{13} + a_{13}A_{21})\varepsilon \Rightarrow a_{13} + (A_{13} + a_{13}A_{21})\varepsilon < 0 \Leftrightarrow a_{13} < 0, \\ c_2 &= a_{11} + A_{21}\varepsilon \Rightarrow a_{11} + A_{21}\varepsilon < 0 \Leftrightarrow a_{11} < 0, \\ c_1c_2 - c_0c_{3>0} &= a_{11}a_{12} + a_{13} + (a_{12}(A_{11} + a_{11}A_{21}) + a_{13}(A_{21} + a_{11}A_{31}) + A_{13})\varepsilon < 0 \\ &\Leftrightarrow a_{11}a_{12} + a_{13} > 0. \end{aligned}$$

Thus, the signs of c_0 , c_2 , and $c_1c_2 - c_0c_3$ are determined by a_{13} , a_{11} , and $a_{11}a_{12} + a_{13}$, respectively; and since $a_{11} < 0$, $a_{13} < 0$ and $a_{11}a_{12} + a_{13} > 0$, then $c_0 < 0$, $c_2 < 0$, and $c_1c_2 - c_0c_3 > 0$. $\square$

5.4.2 Pseudo-equilibria of the sliding vector field

This section introduces the pseudo-equilibria of the sliding vector field of the perturbed system (5-15). Unlike that of the original system (5-1), which is linear, the perturbed sliding vector field is nonlinear and possesses three pseudo-equilibrium points.

Consider the sliding vector field of the nonperturbed system, given by the equation (6) in [17]

$$\mathbf{F}^s(x, y, 0) = \begin{pmatrix} a_{11}x + \left(a_{12} - \frac{\nu}{\mu}\right)y \\ x - \frac{\beta}{\mu}y \\ 0 \end{pmatrix}. \quad (5-16)$$

Thus, the sliding vector field of perturbed system (5-15), is defined by

$$\mathbf{F}_\varepsilon^s(x, y, 0) = \mathbf{F}^s(x, y, 0) + \varepsilon(f_\varepsilon^1(x, y, 0), f_\varepsilon^2(x, y, 0), 0) \quad (5-17)$$

where

$$\begin{aligned} f_\varepsilon^1(x, y, 0) &= \frac{1}{\mu^2} [(1 - \alpha)\mu(A_{11}\mu - A_{31}\nu)x + (\mu_1\nu - \mu\nu_1)y + (A_{31}\nu - A_{11}\mu)xy \\ &\quad + (\alpha - 1)\mu(\mu_1\nu - \mu\nu_1)], \\ f_\varepsilon^2(x, y, 0) &= \frac{1}{\mu^2} [(\alpha - 1)\mu(A_{31}\beta - A_{21}\mu)x + (\beta\mu_1 - \beta_1\mu)y + (A_{31}\beta - A_{21}\mu)xy \\ &\quad + (\alpha - 1)\mu(\beta\mu_1 - \beta_1\mu)]. \end{aligned}$$

This field is nonlinear and satisfies $\lim_{\varepsilon \rightarrow 0} \mathbf{F}_\varepsilon^s = \mathbf{F}^s$. Next, to find the pseudo-equilibria, we solve $\mathbf{F}_\varepsilon^s(x, y, 0) = 0$. Doing this, we obtain:

$$\mathbf{X}_0 = \left(-\frac{\mu + \varepsilon\mu_1}{\varepsilon A_{31}}, (1 - \alpha)\mu, 0 \right), \quad \mathbf{X}_\varepsilon^1 = (x_\varepsilon^1, y_\varepsilon^1, 0), \quad \text{and} \quad \mathbf{X}_\varepsilon^2 = (x_\varepsilon^2, y_\varepsilon^2, 0)$$

where

$$x_\varepsilon^1 = \frac{1}{2\varepsilon(A_{11} - a_{11}A_{21} - a_{12}A_{31})} \left[(\alpha_{11}\beta_1 + \alpha_{12}\mu_1 - (\alpha - 1)((A_{11} - \alpha_{12}A_{31})\beta + A_{21}(\alpha_{12}u - \nu)) - \nu_1)\varepsilon + \alpha_{11}\beta + \alpha_{12}u - \nu - (4(1 - \alpha)(A_{11} - a_{11}A_{21} - a_{12}A_{31})(a_{12}\beta_1u - a_{12}\beta\mu_1 - \beta_1\nu + \beta\nu_1)\varepsilon^2 + (-a_{11}\beta - a_{12}u + \nu + (\nu_1 - a_{11}\beta_1 - a_{12}\mu_1 + (\alpha - 1)((A_{11} - a_{12}A_{31})\beta + A_{21}(a_{12}u - \nu)))\varepsilon)^2)^{1/2} \right],$$

$$x_\varepsilon^2 = \frac{1}{2\varepsilon(A_{11} - a_{11}A_{21} - a_{12}A_{31})} \left[(\alpha_{11}\beta_1 + \alpha_{12}\mu_1 - (\alpha - 1)((A_{11} - \alpha_{12}A_{31})\beta + A_{21}(\alpha_{12}u - \nu)) - \nu_1)\varepsilon + \alpha_{11}\beta + \alpha_{12}u - \nu + (4(1 - \alpha)(A_{11} - a_{11}A_{21} - a_{12}A_{31})(a_{12}\beta_1u - a_{12}\beta\mu_1 - \beta_1\nu + \beta\nu_1)\varepsilon^2 + (-a_{11}\beta - a_{12}u + \nu + (\nu_1 - a_{11}\beta_1 - a_{12}\mu_1 + (\alpha - 1)((A_{11} - a_{12}A_{31})\beta + A_{21}(a_{12}u - \nu)))\varepsilon)^2)^{1/2} \right].$$

Note that the point $\mathbf{X}_0$ coincides with $\mathbf{q}_0$, i.e., it is the intersection point of the tangency lines. Besides, the expressions for y_ε^1 and y_ε^2 are not written out explicitly here because they are very large. Now, to simplify the expression of the pseudo-equilibria, we expand in a first-order Taylor series in ε . Consider $\gamma = \nu - a_{11}\beta - a_{12}\mu \neq 0$. Thus, we have

$$x_\varepsilon^1 = \frac{1}{2(A_{11} - a_{11}A_{21} - a_{12}A_{31})\gamma} \left((-\gamma - \sqrt{\gamma^2})(a_{12}\mu_1 + a_{11}\beta_1 - (\alpha - 1)(A_{11}\beta + a_{12}(A_{21}u - A_{31}\beta) - A_{21}\nu) - \nu_1) \right) + \frac{-\gamma - \sqrt{\gamma^2}}{2\varepsilon(A_{11} - a_{11}A_{21} - a_{12}A_{31})} + \frac{(\alpha - 1)((a_{12}\mu - \nu)\beta_1 - (a_{12}\mu_1 - \nu_1)\beta)\varepsilon}{\sqrt{\gamma^2}},$$

$$x_\varepsilon^2 = \frac{1}{2(A_{11} - a_{11}A_{21} - a_{12}A_{31})\gamma} \left((-\gamma + \sqrt{\gamma^2})(a_{12}\mu_1 + a_{11}\beta_1 - (\alpha - 1)(A_{11}\beta + a_{12}(A_{21}u - A_{31}\beta) - A_{21}\nu) - \nu_1) \right) + \frac{-\gamma + \sqrt{\gamma^2}}{2\varepsilon(A_{11} - a_{11}A_{21} - a_{12}A_{31})} - \frac{(\alpha - 1)((a_{12}\mu - \nu)\beta_1 - (a_{12}\mu_1 - \nu_1)\beta)\varepsilon}{\sqrt{\gamma^2}}.$$

Furthermore, to simplify these expressions, we analyzed the sign of γ . From this analysis, we have

1. If $\gamma < 0$, then $\mathbf{X}_\varepsilon^1 = (x_\varepsilon^1, y_\varepsilon^1, 0) = (x_\varepsilon, y_\varepsilon, 0) = X_\varepsilon$, where

$$x_\varepsilon = \frac{(1 - \alpha)(a_{12}\beta_1\mu - a_{12}\beta\mu_1 - \beta_1\nu + \beta\nu_1)\varepsilon}{\gamma} = \frac{(1 - \alpha)(\beta\gamma_1 - \beta_1\gamma)\varepsilon}{\gamma},$$

$$y_\varepsilon^1 = \frac{(1 - \alpha)(-a_{11}\beta_1\mu + a_{11}\beta\mu_1 - \mu_1\nu + \mu\nu_1)\varepsilon}{\gamma} = \frac{(1 - \alpha)(\mu\gamma_1 - \mu_1\gamma)\varepsilon}{\gamma},$$

$\gamma = \nu - a_{11}\beta - a_{12}\mu$, and $\gamma_1 = \nu_1 - a_{11}\beta_1 - a_{12}\mu_1$. So that the pseudo-equilibrium $\mathbf{X}_\varepsilon^1 = (x_\varepsilon^1, y_\varepsilon^1, 0)$ satisfies $\lim_{\varepsilon \rightarrow 0^+} \mathbf{X}_\varepsilon^1 = (0, 0, 0)$, where $\mathbf{x}_p = (0, 0, 0)$ is the pseudo-equilibrium of the unperturbed system. Moreover, $\mathbf{X}_\varepsilon^2 = \bar{\mathbf{X}}_\varepsilon = (\bar{x}_\varepsilon, \bar{y}_\varepsilon, 0)$, where

$$\begin{aligned} \bar{x}_\varepsilon &= \frac{1}{-A_{11} + a_{11}A_{21} + a_{12}A_{31}} \left((\alpha - 1)(\beta(A_{11} - a_{12}A_{31}) + A_{21}(a_{12}\mu - \nu)) \right. \\ &\quad \left. - a_{11}\beta_1 - a_{12}\mu_1 + \nu_1 \right) + \frac{(1 - \alpha)(-a_{12}\beta_1\mu + a_{12}\beta\mu_1 + \beta_1\nu - \beta\nu_1)\varepsilon}{\gamma} \\ &\quad - \frac{\gamma}{(A_{11} - a_{11}A_{21} - a_{12}A_{31})\varepsilon}, \\ \bar{y}_\varepsilon &= \frac{1}{Q_{14}} \left(Q_{11} + \frac{Q_{12}}{\varepsilon} + Q_{13}\varepsilon \right), \end{aligned}$$

where Q_{1i} ($i = 1, 2, 3, 4$) depend on the system parameters, and as $\varepsilon \rightarrow 0^+$, $\mathbf{X}_\varepsilon^2$ diverges to either $+\infty$ or $-\infty$.

2. If $\gamma > 0$, then

$$\begin{aligned} x_\varepsilon^1 &= \frac{1}{-A_{11} + a_{11}A_{21} + a_{12}A_{31}} \left((\alpha - 1)(\beta(A_{11} - a_{12}A_{31}) + A_{21}(a_{12}\mu - \nu)) \right. \\ &\quad \left. - a_{11}\beta_1 - a_{12}\mu_1 + \nu_1 \right) + \frac{(1 - \alpha)(-a_{12}\beta_1\mu + a_{12}\beta\mu_1 + \beta_1\nu - \beta\nu_1)\varepsilon}{\gamma} \\ &\quad - \frac{\gamma}{(A_{11} - a_{11}A_{21} - a_{12}A_{31})\varepsilon}, \\ y_\varepsilon^1 &= \frac{1}{Q_{14}} \left(Q_{11} + \frac{Q_{12}}{\varepsilon} + Q_{13}\varepsilon \right), \end{aligned}$$

that is, $\mathbf{X}_\varepsilon^1 = \bar{\mathbf{X}}_\varepsilon$. Hence, as $\varepsilon \rightarrow 0^+$, $\mathbf{X}_\varepsilon^1$ diverges to either $+\infty$ or $-\infty$, while

$$\begin{aligned} x_\varepsilon^2 &= \frac{(1 - \alpha)(a_{12}\beta_1\mu - a_{12}\beta\mu_1 - \beta_1\nu + \beta\nu_1)\varepsilon}{\gamma}, \\ y_\varepsilon^2 &= \frac{(1 - \alpha)(-a_{11}\beta_1\mu + a_{11}\beta\mu_1 - \mu_1\nu + \mu\nu_1)\varepsilon}{\gamma}, \end{aligned}$$

or equivalently, $\mathbf{X}_\varepsilon^2 = \mathbf{X}_\varepsilon$. Thus, for the pseudo-equilibrium $\mathbf{X}_\varepsilon^2 = (x_\varepsilon^2, y_\varepsilon^2, 0)$ we have $\lim_{\varepsilon \rightarrow 0^+} \mathbf{X}_\varepsilon^2 = (0, 0, 0) = \mathbf{x}_p$.

Therefore, γ determines the behavior of the pseudo-equilibria as $\varepsilon \rightarrow 0^+$: one of them converges to $\mathbf{x}_p = (0, 0, 0)$, while the other diverges to either $+\infty$ or $-\infty$. When $\gamma = 0$, however, no pseudo-equilibria exist, since they would not be well-defined.

5.4.3 Stability of the pseudo-equilibrium

We consider the perturbed sliding vector field (5-15) to study the sliding dynamics for $\mu \neq 0$ and $\varepsilon > 0$. This system has three pseudo-equilibrium points: $\mathbf{X}_0$,

$\mathbf{X}_\varepsilon^1$, and $\mathbf{X}_\varepsilon^2$, where $\mathbf{X}_0$ corresponds to the intersection point of the tangency lines. If $\gamma < 0$, then as $\varepsilon \rightarrow 0^+$, $\mathbf{X}_\varepsilon^1$ converges to $\mathbf{x}_p$ while $\mathbf{X}_\varepsilon^2$ diverges to either $+\infty$ or $-\infty$; conversely, if $\gamma > 0$, the opposite behavior occurs. When the pseudo-equilibrium diverges to either $+\infty$ or $-\infty$, it does not belong to the sliding region and therefore is not considered in the present study. Therefore, we present the following Theorem, analogous to Proposition 3 in [17].

Theorem 5.8. *Consider $\mu \neq 0$, $0 < \alpha < 1$, $A_{31} \neq 0$, $\gamma \neq 0$, and $\varepsilon > 0$. The following statements hold.*

1. *If $\mu < 0$ and $\mu A_{31} \neq 0$, then $X_\varepsilon \in \Sigma_{rs}$ and it is unstable.*
2. *If $\mu > 0$ and $\mu A_{31} \neq 0$, then $X_\varepsilon \in \Sigma_{as}$ and it is stable. Moreover, assume $\gamma > 0$, then*
 - (a) *$X_\varepsilon = \mathbf{X}_\varepsilon^2$ is a stable pseudo-node if $a_{11} \leq \frac{\beta}{\mu} - \sqrt{\frac{4\gamma}{\mu} - \varepsilon\Delta_1}$ or an unstable pseudo-node if $a_{11} \geq \frac{\beta}{\mu} + \sqrt{\frac{4\gamma}{\mu} - \varepsilon\Delta_1}$.*
 - (b) *$X_\varepsilon = \mathbf{X}_\varepsilon^2$ is a stable pseudo-focus if $\frac{\beta}{\mu} - \sqrt{\frac{4\gamma}{\mu} - \varepsilon\Delta_1} < a_{11} < \frac{\beta}{\mu}$ or an unstable pseudo-focus if $\frac{\beta}{\mu} < a_{11} < \frac{\beta}{\mu} + \sqrt{\frac{4\gamma}{\mu} - \varepsilon\Delta_1}$.*
3. *If $\mu\gamma < 0$, then X_ε is a pseudo-saddle.*

Proof. Consider the pseudo-equilibrium point $X_\varepsilon = (x_\varepsilon, y_\varepsilon, 0)$, where $X_\varepsilon = \mathbf{X}_\varepsilon^1$ if $\gamma < 0$ and $X_\varepsilon = \mathbf{X}_\varepsilon^2$ if $\gamma > 0$. Moreover,

$$x_\varepsilon = \frac{(1-\alpha)(\beta\gamma_1 - \beta_1\gamma)\varepsilon}{\gamma}, \quad y_\varepsilon = \frac{(1-\alpha)(\mu\gamma_1 - \mu_1\gamma)\varepsilon}{\gamma},$$

with $\gamma = \nu - a_{12}\mu - a_{11}\beta$ and $\gamma_1 = \nu_1 - a_{12}\mu_1 - a_{11}\beta_1$.

1. Assume that $\mu < 0$ and $\mu A_{31} \neq 0$. Then we must prove that $x_\varepsilon > x_0$ and $(1-\alpha)\mu < y_\varepsilon < -\alpha\mu - \varepsilon(A_{31}x + \mu_1)$.

Indeed, suppose $A_{31} < 0$. Then $x_0 = -\frac{\mu + \mu_1\varepsilon}{A_{31}\varepsilon} \rightarrow -\infty$ and $x_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0^+$. Therefore, for sufficiently small ε , we have $x_\varepsilon > x_0$ for any values of γ , γ_1 , β , and β_1 .

On the other hand, note that as $\varepsilon \rightarrow 0^+$, we have $y_\varepsilon \rightarrow 0$, while $(1-\alpha)\mu < 0$ remains constant. Moreover, $-\alpha\mu - \varepsilon(A_{31}x + \mu_1) \rightarrow -\alpha\mu > 0$ since $\mu < 0$. Thus, the inequality $(1-\alpha)\mu < y_\varepsilon < -\alpha\mu - \varepsilon(A_{31}x + \mu_1)$ holds for any values of γ , γ_1 , β , and β_1 .

For $A_{31} > 0$, we have $x_0 \rightarrow +\infty$ and $x_\varepsilon \rightarrow 0$. Hence, for sufficiently small $\varepsilon > 0$, we obtain $x_\varepsilon < x_0$. Consequently, $X_\varepsilon \in \Sigma_{rs}$.

2. By an analysis similar to that in item (1), for $\mu > 0$ and $\mu A_{31} \neq 0$, we have $X_\varepsilon \in \Sigma_{as}$ for any values of γ , γ_1 , β , and β_1 .

Now, consider the sliding vector field (5-17) and its Jacobian matrix

$$J_\varepsilon(x, y) = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}$$

where

$$\begin{aligned} J_{11} &= a_{11} - \frac{(A_{11}\mu - A_{31}\nu)(y - (1 - \alpha)\mu)\varepsilon}{\mu^2}, \\ J_{12} &= a_{12} + \frac{-\mu\nu + (\mu_1\nu - \mu\nu_1 + (A_{31}\nu - A_{11}\mu)x)\varepsilon}{\mu^2}, \\ J_{21} &= 1 + \frac{1}{\mu^2} \left(\mu(-1 + \alpha)(A_{31}\beta - A_{21}\mu) + (A_{31}\beta - A_{21}\mu)y \right) \varepsilon, \\ J_{22} &= \frac{-\beta\mu + (\beta\mu_1 - \beta_1\mu + (A_{31}\beta - A_{21}\mu)x)\varepsilon}{\mu^2}, \end{aligned}$$

and $\lim_{\varepsilon \rightarrow 0^+} J_\varepsilon = J$, with

$$J(x, y) = \begin{pmatrix} a_{11} & a_{12} - \frac{\nu}{\mu} \\ 1 & -\frac{\beta}{\mu} \end{pmatrix}$$

being the Jacobian matrix of the original sliding vector field (5-16). Evaluating the matrix J_ε at X_ε and taking the Taylor series up to first order in ε , we obtain

$$\begin{aligned} \det(J_\varepsilon) &= \frac{\gamma}{\mu} + \frac{\varepsilon}{\mu^2} \left((\nu_1 - a_{11}\beta_1)\mu + (a_{11}\beta - \nu)\mu_1 - (1 - \alpha)\mu((A_{11} - a_{12}A_{31})\beta \right. \\ &\quad \left. + (a_{12}\mu - \nu)A_{21}) \right), \\ &= \det(J) + \frac{\varepsilon}{\mu^2} \left((\nu_1 - a_{11}\beta_1)\mu + (a_{11}\beta - \nu)\mu_1 - (1 - \alpha)\mu((A_{11} - a_{12}A_{31})\beta \right. \\ &\quad \left. + (a_{12}\mu - \nu)A_{21}) \right), \\ \operatorname{tr}(J_\varepsilon) &= a_{11} - \frac{\beta}{\mu} + \varepsilon \left(\frac{-\beta_1\mu + \beta\mu_1}{\mu^2} - \frac{(\alpha - 1)(A_{11}\mu - A_{31}\nu)}{\mu} \right), \\ &= \operatorname{tr}(J) + \varepsilon \left(\frac{-\beta_1\mu + \beta\mu_1}{\mu^2} - \frac{(\alpha - 1)(A_{11}\mu - A_{31}\nu)}{\mu} \right), \\ \Delta(J_\varepsilon) &= \operatorname{tr}(J_\varepsilon)^2 - 4\det(J_\varepsilon) = \Delta(J) + \varepsilon\Delta_1, \end{aligned}$$

where $\Delta_1 = \left(a_{11} - \frac{\beta}{\mu} \right) \frac{2}{\mu^2} (-\beta_1\mu + \beta\mu_1 + (1 - \alpha)\mu(A_{11}\mu - A_{31}\nu)) - \frac{4}{\mu^2} (\mu_1(a_{11}\beta - \nu) + \mu(\nu_1 - a_{11}\beta_1)(1 - \alpha)\mu((a_{12}A_{31} - A_{11})\beta - A_{21}(a_{12}\mu - \nu)))$. Thus, the

eigenvalues are given by

$$\lambda_{1,2} = \frac{tr(J_\varepsilon) \pm \sqrt{tr(J_\varepsilon)^2 - 4det(J_\varepsilon)}}{2}.$$

- (a) Assume $\gamma > 0$ and $\mu > 0$, so that $X_\varepsilon = X_\varepsilon^2$. Note that, for sufficiently small $\varepsilon > 0$, the sign of $det(J_\varepsilon)$ is the same as that of $det(J) = \gamma/\mu > 0$. Hence, $det(J_\varepsilon) > 0$ for small ε . Now, consider

$$a_{11} \leq \frac{\beta}{\mu} - \sqrt{\frac{4\gamma}{\mu} - \varepsilon\Delta_1} \Leftrightarrow a_{11} - \frac{\beta}{\mu} \leq -\sqrt{\frac{4\gamma}{\mu} - \varepsilon\Delta_1} < 0.$$

For sufficiently small $\varepsilon > 0$, we have $\sqrt{\frac{4\gamma}{\mu} - \varepsilon\Delta_1} > 0$, then $tr(J) = a_{11} - \frac{\beta}{\mu} < 0$, and since the sign of $tr(J_\varepsilon)$ is the same as that of $tr(J)$ for small ε , it follows that $tr(J_\varepsilon) < 0$. Consequently

$$\begin{aligned} a_{11} - \frac{\beta}{\mu} \leq -\sqrt{\frac{4\gamma}{\mu} - \varepsilon\Delta_1} &\Rightarrow \left| a_{11} - \frac{\beta}{\mu} \right| \geq \sqrt{\frac{4\gamma}{\mu} - \varepsilon\Delta_1} \Leftrightarrow \left(a_{11} - \frac{\beta}{\mu} \right)^2 \geq \frac{4\gamma}{\mu} - \varepsilon\Delta_1 \\ &\Leftrightarrow \left(a_{11} - \frac{\beta}{\mu} \right)^2 - \frac{4\gamma}{\mu} + \varepsilon\Delta_1 \geq 0, \end{aligned}$$

That is, $\Delta(J_\varepsilon) \geq 0$. Therefore, for sufficiently small $\varepsilon > 0$ we have $det(J_\varepsilon) > 0$, $\Delta(J_\varepsilon) \geq 0$, and $tr(J_\varepsilon) < 0$; hence X_ε is a stable pseudo-node. Analogously, we shows that X_ε is an unstable pseudo-node.

- (b) Assume $\frac{\beta}{\mu} - \sqrt{\frac{4\gamma}{\mu} - \varepsilon\Delta_1} < a_{11} < \frac{\beta}{\mu}$. As $a_{11} < \frac{\beta}{\mu}$, then $tr(J) = a_{11} - \frac{\beta}{\mu} < 0$, consequently $tr(J_\varepsilon) < 0$ for small $\varepsilon > 0$. Moreover, as

$$\begin{aligned} \frac{\beta}{\mu} - \sqrt{\frac{4\gamma}{\mu} - \varepsilon\Delta_1} < a_{11} &\Leftrightarrow a_{11} - \frac{\beta}{\mu} > -\sqrt{\frac{4\gamma}{\mu} - \varepsilon\Delta_1} \Leftrightarrow \left(a_{11} - \frac{\beta}{\mu} \right)^2 < \frac{4\gamma}{\mu} - \varepsilon\Delta_1 \\ &\Leftrightarrow \left(a_{11} - \frac{\beta}{\mu} \right)^2 - \frac{4\gamma}{\mu} + \varepsilon\Delta_1 < 0, \end{aligned}$$

then $\Delta(J_\varepsilon) < 0$. Therefore, for small ε we have $det(J_\varepsilon) > 0$, $\Delta(J_\varepsilon) < 0$, and $tr(J_\varepsilon) < 0$, that is, X_ε is a stable pseudo-focus. Similarly, we shows that X_ε is an unstable pseudo-focus.

3. Assume $\mu\gamma < 0$. Then, for sufficiently small $\varepsilon > 0$, we have $det(J_\varepsilon) < 0$. Hence, X_ε is a pseudo-saddle.

□

5.4.4 Previous results of unperturbed systems

As shown in Section 5.1, the authors in [17] study a piecewise linear system in $\mathbb{R}^3$ given by (5-1). Through an affine change of coordinates, the system is reduced to the canonical form (5-5). For this system, the local asymptotic stability analysis is first carried out (see [17], p.10). Here, Proposition 2 characterizes the regular equilibria $\mathbf{e}^\pm$ of the vector fields $\mathbf{F}^\pm$. Admissibility conditions are established to determine whether each equilibrium is real or virtual, and it is proved that under certain parameter conditions, both equilibria are asymptotically stable.

Next, the sliding vector field given by (5-16) is considered. In Proposition 3, (see [17], p.11), is shown that the pseudo-equilibrium at the origin belongs to the attractive sliding region if $\mu > 0$, and to the repulsive sliding region if $\mu < 0$. Moreover, under the assumptions $\mu > 0$ and $\gamma > 0$, this pseudo-equilibrium can be either a pseudo-node or a pseudo-focus. In the case $\mu\gamma < 0$, the origin is a pseudo-saddle.

Furthermore, the authors study two distinct types of bifurcations. The Type I bifurcation. This bifurcation is associated with a change in stability of the sliding region, from attractive to repulsive or vice versa. It occurs due to the variation of the parameter μ , with $\mu = 0$ being the bifurcation point. As shown in Theorem 1 (see [17], p.11), this bifurcation gives rise to a crossing limit cycle around the sliding region. Such a limit cycle can be stable or unstable (saddle or node), depending on the parameter conditions of the system. There are four possible unfoldings for this bifurcation.

The Type II bifurcation occurs when the tangency lines of the system coincide, i.e., $\mu = 0$, and the parameter ν is varied, with $\nu = 0$ being the bifurcation point. Theorem 2 in [17] shows that this bifurcation gives rise to a crossing limit cycle that emerges directly from the origin. This limit cycle can be stable or unstable (saddle or node), depending on the parameter conditions of the system. As with the previous bifurcation, there are four possible unfoldings.

Finally, by analyzing the parameters μ and ν simultaneously, the authors establish a bifurcation diagram in the (μ, ν) -plane. Under certain conditions, the system can exhibit two crossing limit cycles coexisting simultaneously. Moreover, the existence of a saddle-node bifurcation curve emanating from the origin is also proved. Along this curve, the two limit cycles collide and disappear.

Note that after perturbing the system (5-1), our results, such as Proposition 5.7 and Theorem 5.8 are analogous to those of Propositions 2 and 3 in [17] for the pseudo-equilibrium of the perturbed field. Moreover, since the crossing limit cycles arising from the Hopf-like bifurcations mentioned above are hyperbolic (saddle or node),

the ε -perturbation performed in this chapter will not affect those results of [17], as they are structurally stable.

5.5 Brief conclusion

Here, we studied an ε -perturbation of the piecewise smooth dynamical system in $\mathbb{R}^3$ given by (5-1). In the resulting perturbed system (5-13), the two tangency lines intersect at a point, and this intersection point is a parabolic two-fold. This system is reduced to the canonical form (5-15), and its sliding vector field (5-17) possesses three new pseudo-equilibrium points. One of them is the intersection point. Depending on the sign of the parameter γ , the other two points either move toward or away from the pseudo-equilibrium of the original system as the perturbation parameter $\varepsilon > 0$ tends to zero. Moreover, when the pseudo-equilibrium converges to that of the original system, its qualitative behavior remains the same; that is, it can be a pseudo-node, a pseudo-focus, or a pseudo-saddle. This occurs because the original pseudo-equilibrium at the origin is hyperbolic, and therefore it persists under small perturbations in other words, it is structurally stable. Thus, from the previous remark, we conclude that the ε -perturbation performed in this chapter will not affect the results of [17], since they are structurally stable.

Conclusion

In this thesis, we study the maximum number of limit cycles in a class of piecewise smooth dynamical systems in $\mathbb{R}^2$ primarily using Chebyshev Theory and Descartes' Theorem. In $\mathbb{R}^3$, we describe System (5-1) after an appropriate perturbation and we analyze its dynamics.

In Chapter 2, we investigate the maximum number of limit cycles that a piecewise smooth dynamical system composed by two rigid centers can exhibit. These centers are governed by even-degree polynomials given by a specific equation. Since the system is integrable, we consider the first integrals of each system, denoted by H_n and F . From these integrals, we construct the closing function $M(y_0) = 0$ for the system (2-1). This function, is no longer a polynomial function and it is generated by a set of functions $\mathcal{B}_n$. Thus, since $M(y_0)$ is no longer a polynomial, the problem of solving system (2-8) reduces to the number of zeros that the elements of the space generated by the linear combination of functions in $\mathcal{B}_n$ can have in a given interval. To estimate the number of zeros of any linear combination of elements of $\mathcal{B}_n$, we use the theory of Chebyshev systems. In this way, for an even degree n , we proved that the system admits at most $(n-2)/2$ limit cycles provided that $d_2 = M_2$. Furthermore, specifically for $n = 4$ with $d_2 = 0$ and $d_2 \neq M_2$, we established that the sharp upper bound is 1. These results extend the study developed in [12]. Finally, Example 2.5 demonstrates that the upper bound obtained in Item (a) is sensitive to the condition $d_2 = 0$. By taking $d_2 \neq 0$, a system with $n = 4$ can exhibit three crossing limit cycles, thereby surpassing the bound established for the restricted case.

In Chapter 3, similarly to the previous case, we study the maximum number of limit cycles in a piecewise smooth dynamical system composed by two rigid systems separated by the straight line $y = 0$. We analyze 14 cases (m, n) , examining the positive roots of the equation $r_1(\theta) - r_0 = 0$ using Descartes' Theorem. These roots correspond to the limit cycles of the system. Thus, we proved that when the degrees are equal $m = n$, the system admits at most one limit cycle, and this bound is

achieved. For $m = 2$ and $n = 3, 4, 5, 6$, the bounds are 2, 3, 4, and 5 respectively, with the bound for (2, 3) being achieved; for $m = 3$ and $n = 4, 5, 6$ they are 4, 2 and 5 respectively, with the bound for (3, 5) being achieved. For $m = 4$ and $n = 5, 6$ the bound is 6; and finally, for $m = 5$ and $n = 6$ it is 8. Therefore, for these classes of systems we addressed the extended 16th Hilbert problem.

Following a similar approach to Chapter 4, we consider a generalization of the system formed by (3-1) and (3-2) but in this case, the system is a hybrid system. We analyze the cases when $m, n \in \{2, 3, 4, 5, 6\}$ with $m \leq n$, studying the positive real roots of the equation $x_5 - x_1 = 0$, which determine the limit cycles of the system. To obtain the desired results, we once again apply Descartes' Theorem. Here, we consider the same cases as in Chapter 3 and obtain the same maximum number of limit cycles in all of them, except for $(m = 3, n = 6)$ and $(m = 4, n = 6)$. For these two cases, the maximum number is 5 and 6 in the continuous case, whereas in the hybrid case the upper bounds increase to 6 and 10, respectively. In Chapter 3 we exhibit parameter values for which this bound is actually attained in the cases where $m = n$ and $m = 3, n = 5$. In the hybrid case, however, achieving the upper bound is generally a non-trivial problem; taking $\lambda \neq \mu$ makes the expressions considerably more complex. Nevertheless, we were able to show that the bound is attained when $m = n$, except for $m = n = 4$ and $m = n = 5$. This suggests that for low values of m and n specifically, $m, n \in \{2, 3, 4, 5, 6\}$ there is not a substantial difference between the maximum number of limit cycles in the hybrid and continuous cases. We believe that for higher values of m and n , more significant variations might appear, which could help determine whether the addition of discrete dynamics increases the number of limit cycles. Therefore, for these classes of systems we addressed the extended 16th Hilbert problem

In Chapter 5 we study a piecewise linear system of the form (5-1) that has two parallel tangency lines on the switching boundary. Each line has a cusp singularity, and the sliding vector field has a single pseudo-equilibrium point.

After a perturbation given by a sufficiently small parameter $\varepsilon > 0$, the parallel tangency lines intersect at a point, which corresponds to a parabolic two-fold singularity. This perturbed system is transformed, via an affine change of coordinates, into a canonical form (5-15). For this system we analyze the dynamics on Σ after the perturbation.

Unlike the original sliding vector field, which is linear and possesses a single pseudo-equilibrium, the sliding vector field of the perturbed system is nonlinear and has three pseudo-equilibria. One of them corresponds to the intersection point. For the other two, if $\gamma < 0$, the pseudo-equilibrium $\mathbf{X}_\varepsilon^1 = X_\varepsilon$ converges to that of the original

system (5-5) as $\varepsilon \rightarrow 0^+$, while $\mathbf{X}_\varepsilon^2$ diverges to $+\infty$ or $-\infty$; for $\gamma > 0$, the opposite occurs.

For sufficiently small $\varepsilon > 0$, when the pseudo-equilibrium converges to the original pseudo-equilibrium point, it belongs to the sliding region. Specifically, if $\mu < 0$ and $\mu A_{31} \neq 0$, then $X_\varepsilon \in \Sigma_{rs}$ (repulsive sliding); if $\mu > 0$ and $\mu A_{31} \neq 0$, then $X_\varepsilon \in \Sigma_{as}$ (attractive sliding). Conversely, when the pseudo-equilibrium diverges to $+\infty$ or $-\infty$, it does not belong to the sliding region and therefore is not considered in the present study.

Moreover, according to the conditions established in Theorem 5.8, this equilibrium can be a pseudo-node, a pseudo-focus, or a pseudo-saddle. This result shows that the qualitative behavior of the converging pseudo-equilibrium is the same as that of the original system. This occurs because the original pseudo-equilibrium at the origin is hyperbolic; hence it persists under small perturbations, i.e., it is structurally stable. Consequently, the perturbation studied here preserves the results obtained in [17], as they are structurally stable.

In the previous paragraphs we have summarized the results obtained in this tesis, where Chapters 2, 3 and 4 resulted in the papers [24], [36] and [37], respectively. In this way, we believe that they contribute to a better understanding of the study of limit cycles in piecewise smooth dynamical systems. In particular, as mentioned in [37], there are still only a few works on hybrid piecewise smooth dynamical systems. With the study carried out here, we are contributing to the understanding of this type of system, which has various applications in different sciences. Therefore, in order to further contribute to this topic, a future study would focus on heteroclinic connections in hybrid piecewise smooth dynamical systems and in perturbed hybrid piecewise smooth dynamical systems. In the final chapter, to complement the study of the sliding dynamics of the perturbed system, we propose as future work the analysis of parameter conditions leading to saddle-node, Hopf, pitchfork, and transcritical bifurcations.

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