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Adams Inequality of Adimurthi-Druet Type on whole $\mathbb{R}^n$

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INSTITUTO DE MATEMÁTICA E ESTATÍSTICA

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FÁBIO SODRÉ ROCHA

Desigualdade de Adams do tipo Adimurthi-Druet em todo o espaço $\mathbb{R}^n$

Tese apresentada ao Programa de Pós–Graduação do Instituto de Matemática e Estatística da Universidade Federal de Goiás , como requisito parcial para obtenção do título de Doutor em Programa de Pós-Graduação em Matemática .

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Ata Nº **29** da sessão de Defesa de Tese de **Fábio Sodré Rocha** que confere o título de Doutor em **Matemática**, na área de concentração em **Análise.**

Ao **vigésimo sétimo dia do mês de novembro do ano de dois mil e vinte e cinco**, a partir da 09h00, de forma **Videoconferência**, realizou-se a sessão pública de Defesa de Tese intitulada **“Adams Inequality of Adimurthi-Druet Type on whole Rn”**. Os trabalhos foram instalados pelo Orientador, Professor Doutor **Abiel Costa Macedo - IME/UFG,** com a participação dos demais membros da Banca Examinadora: Professor Doutor **José Francisco Alves de Oliveira - DM/UFPI**, membro titular externo; Professor Doutor **José Carlos de Albuquerque Melo Júnior - DMAT/UFPE,** membro titular externo; Professor Doutor **Uberlandio Batista Severo DM/UFPB,** membro titular externo e Professor Doutor **Pedro Eduardo Ubilla López - Departamento de Matemática y Ciencia de la Computación/USACH**. Durante a arguição os membros da banca **não fizeram** sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Tese tendo sido o candidato **aprovado** pelos seus membros. Proclamados os resultados pelo Professor Doutor **Abiel Costa Macedo**, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos Membros da Banca Examinadora, ao **vigésimo sétimo dia do mês de novembro do ano de dois mil e vinte e cinco.**

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Autor(a): Fábio Sodré Rocha

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Fábio Sodré Rocha – Author

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FÁBIO SODRÉ ROCHA

Adams Inequality of Adimurthi-Druet Type on whole $\mathbb{R}^n$

Tese defendida no Programa de Pós–Graduação do Instituto de Matemática e Estatística da Universidade Federal de Goiás como requisito parcial para obtenção do título de Doutor em Programa de Pós-Graduação em Matemática , aprovada em 27 de Novembro de 2025, pela Banca Examinadora constituída pelos professores:

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Fábio Sodré Rocha

Bachelor's degree in Mathematics from the Institute of Mathematics and Statistics at the Federal University of Goiás, where he served as a teaching assistant for the subjects Linear Algebra and Number Theory. He completed his Master's degree in Analysis at the same institution, with scholarships from CAPES and CNPq. Currently, he is conducting research on generalizations of Sobolev spaces for higher-order derivatives in unbounded domains.

To my strenght, Olavo and Luiza!

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Thank you all!

David Hilbert (1930),
„Wir müssen wissen. Wir werden wissen.“

(Devemos saber. Nós saberemos).

Abstract

SODRÉ, F.. **Desigualdade de Adams do tipo Adimurthi-Druet em todo o espaço** $\mathbb{R}^n$. Goiânia , 2025. 127p. PhD. Thesis . Instituto de Matemática e Estatística, Universidade Federal de Goiás .

Seja $W^{m,\frac{n}{m}}(\mathbb{R}^n)$, com $1 \leq m < n$, o espaço de Sobolev padrão de derivadas de ordem superior no limiar crítico de crescimento exponencial. Investigamos uma nova desigualdade do tipo Adams-Adimurthi-Druet em todo o espaço $\mathbb{R}^n$, fortemente influenciada pelo fenômeno de anulação. Especificamente, demonstramos que

$$\sup_{\substack{u\in W^{m,\frac{n}{m}}(\mathbb{R}^n) \\ \|\nabla^m u\|^{\frac{n}{m}}_{\frac{n}{m}}+\|u\|^{\frac{n}{m}}_{\frac{n}{m}}\leq 1}} \int_{\mathbb{R}^n} \Phi\left(\beta\left(\frac{1+\alpha\|u\|^{\frac{n}{m}}_{\frac{n}{m}}}{1-\gamma\alpha\|u\|^{\frac{n}{m}}_{\frac{n}{m}}}\right)^{\frac{m}{n-m}} |u|^{\frac{n}{n-m}}\right) \mathrm{d}x < +\infty,$$

onde $0 \leq \alpha < 1$, $0 < \gamma < \frac{1}{\alpha} - 1$ para $\alpha > 0$, $\nabla^m u$ é o gradiente de ordem m de u, $0 \leq \beta \leq \beta_0$, sendo β_0 a constante crítica de Adams, e $\Phi(t) = \mathrm{e}^t - \sum_{j=0}^{j_{m,n}-2} \frac{t^j}{j!}$ com $j_{m,n} = \min\{j \in \mathbb{N} : j \geq n/m\}$. Além disso, provamos que a constante β_0 é ótima.

No caso subcrítico $\beta < \beta_0$, investigamos tanto a existência quanto a não-existência de funções extremais para o caso $n = 2m$. No caso crítico $\beta = \beta_0$, estabelecemos a atingibilidade para $n = 4$ e $m = 2$. Nossa abordagem baseia-se na análise de blow-up, juntamente com um método de truncamento recentemente introduzido por DelaTorre-Mancini [16], e incorpora ideias de Chen-Lu-Zhu [10], que estudaram a desigualdade crítica de Adams em $\mathbb{R}^4$. Além disso, generalizamos esses resultados de atingibilidade para dimensões pares arbitrárias, combinando técnicas de blow-up com um truncamento poliharmônico adequado e a construção de funções teste refinadas, no espírito de [11].

Keywords

Desigualdades de Trudinger-Moser, Desigualdades de Adams, Equações Poliharmônicas, Desigualdades de Hardy

Abstract

SODRÉ, F.. **Adams Inequality of Adimurthi-Druet Type on whole $\mathbb{R}^n$.** Goiânia , 2025. 127p. PhD. Thesis . Instituto de Matemática e Estatística, Universidade Federal de Goiás .

Let $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ with $1 \leq m < n$ be the standard higher order derivative Sobolev space in the critical exponential growth threshold. We investigate a new Adams-Adimurthi-Druet type inequality on the whole space $\mathbb{R}^n$ which is strongly influenced by the vanishing phenomenon. Specifically, we prove

$$\sup_{\substack{u\in W^{m,\frac{n}{m}}(\mathbb{R}^n)\\ \|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}}+\|u\|_{\frac{n}{m}}^{\frac{n}{m}}\leq 1}} \int_{\mathbb{R}^n} \Phi\left(\beta\left(\frac{1+\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{1-\gamma\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}\right)^{\frac{m}{n-m}} |u|^{\frac{n}{n-m}}\right) \mathrm{d}x < +\infty,$$

where $0 \leq \alpha < 1$, $0 < \gamma < \frac{1}{\alpha} - 1$ for $\alpha > 0$, $\nabla^m u$ is the m-th order gradient for u, $0 \leq \beta \leq \beta_0$, with β_0 being the Adams critical constant, and $\Phi(t) = \mathrm{e}^t - \sum_{j=0}^{j_{m,n}-2} \frac{t^j}{j!}$ with $j_{m,n} = \min\{j \in \mathbb{N} : j \geq n/m\}$. In addition, we prove that the constant β_0 is sharp.

In the subcritical case $\beta < \beta_0$, we investigate both the existence and non-existence of extremal functions for the case $n = 2m$. In the critical case $\beta = \beta_0$, attainability is established for $n = 4$ and $m = 2$. Our approach relies on blow-up analysis, together with a truncation method recently introduced by DelaTorre-Mancini [16], and incorporates ideas from Chen-Lu-Zhu [10], who studied the critical Adams inequality in $\mathbb{R}^4$. Moreover, we generalize these attainability results to arbitrary even dimensions by combining blow-up techniques with a suitable polyharmonic truncation and the construction of refined test functions, in the spirit of [11].

Keywords

Trudinger-Moser inequalities, Adams' inequalities, Polyharmonic equation, Hardy inequalities

Contents

List of Symbols and Notations

General Notations

$\mathbb{R}^n$	$\{(x_1, x_2, \ldots, x_n) : \ x_i \in \mathbb{R}, \ \forall \ i \in \mathbb{N}\}$.
$\Omega \subset \mathbb{R}^n$	A bounded (connected open) domain in $\mathbb{R}^n$.
$B(x, r)$	Open ball centered at x with radius $r > 0$.
B_R	The set $\{x \in \mathbb{R}^n : \lvert x \rvert < R\}$.
$\overline{\Omega}$	Closure of Ω.
$\partial\Omega$	Boundary of Ω.
$\lvert\Omega\rvert$	Lebesgue measure of the domain Ω.
$\Omega^{\#}$	Denotes the ball centered at the origin such that $\lvert\Omega\rvert = \lvert\Omega^{\#}\rvert$.
$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right)$	Gradient of the function u.
$\Delta u = \operatorname{div}(\nabla u) = \sum_{1 \le i \le n} \frac{\partial^2 u}{\partial x_i^2}$	Laplacian of the function u.
$\rightharpoonup$	Weak convergence.
$\overset{*}{\rightharpoonup}$	Weak* convergence.
$\hookrightarrow$	Continuous embedding.
a.e.	Almost everywhere.
ω_{n-1}	Surface area of the unit sphere in $\mathbb{R}^n$.
ω_n	Volume of the unit ball in $\mathbb{R}^n$.
$\Gamma(s)$	Gamma function: $\Gamma(s+1) = s!$ if $s \in \mathbb{Z}$. If $s \in \mathbb{C}$, then $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t}\, dt$.
$u^+ = \max\{u, 0\}$	Positive part of u.
$u^- = -\min\{u, 0\}$	Negative part of u.

Function Spaces and Norms

$C^\infty(\Omega)$	Space of smooth functions in Ω.
$C_0^\infty(\Omega)$	Space of smooth functions with compact support in Ω.

$(C_0^\infty(\Omega))^n$	Product space (n times) of smooth functions with compact support in Ω.
$L^p(\Omega)$	{Equivalence class of measurable functions $u : \Omega \to \mathbb{R} : \int_\Omega \|u\|^p dx < +\infty$} for $1 \leq p < +\infty$.
$L^\infty(\Omega)$	{Equivalence class of measurable functions $u : \Omega \to \mathbb{R}$; $\exists\ C > 0$ such that $\|u(x)\| \leq C$ a.e. $x \in \Omega$}.
$W^{m,p}(\Omega)$	$\{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)$ for $0 \leq \|\alpha\| \leq m\}$, where $D^\alpha u$ denotes the weak derivative of order α of u.
$W_0^{m,p}(\Omega)$	The closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.
$H_0^1(\Omega)$	$W_0^{1,2}(\Omega)$.
$\|\cdot\|_{p,\Omega}$	Norm in the space $L^p(\Omega)$.
$\|\cdot\|_{\infty,\Omega}$	Norm in the space $L^\infty(\Omega)$.

In the norms defined above, the index Ω will be omitted throughout the text whenever there is no ambiguity.

It is worth noting that symbols and notations not included in the above list will be properly introduced and rigorously defined in the course of the text, in order to preserve clarity and precision throughout the thesis.

CHAPTER 1

Introduction

1.1 Introduction

Sobolev spaces form a foundation in modern analysis, supporting diverse research in partial differential equations, functional analysis, and approximation theory. These spaces allow for broader solution classes, particularly in elliptic and parabolic PDEs, where functional inequalities offer essential constraints linking functions and their derivatives.

A key result, the Sobolev inequality, shows that certain function norms can be bounded by norms of their derivatives, crucial for establishing regularity, integrability, and uniqueness of solutions. Building on this, the Adams inequality provides the higher-order analogue of the Trudinger–Moser inequality, capturing the critical exponential integrability of functions in higher-order Sobolev spaces, crucial in higher-dimensional settings where standard embeddings fall short.

The Adams inequality is pivotal in analyzing critical phenomena in nonlinear PDEs, including blow-up behavior and asymptotic complexity. It provides rigorous control over functions in higher-order spaces, advancing techniques in nonlinear dynamics, geometric analysis and variational methods.

In this work we present an improvement for the Adams inequality on unbounded domains in the spirit of Adimurthi-Druet [3] and its far-reaching implications, highlighting its essential role in the theoretical framework needed to address critical cases in high-dimensional Sobolev spaces and underscoring its impact on contemporary research in nonlinear analysis.

Firstly, we will give an overview of the history that underpins all our work. Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be a smooth bounded domain and let $W_0^{m,p}(\Omega), p \geq 1$ be the *m-th* order derivative classical Sobolev space on Ω. Under the strict condition $n > mp$, it is well known the optimal continuous embedding of $W_0^{m,p}(\Omega)$ into Orlicz space $L_{\varphi_*}(\Omega)$ defined by the Young function $\varphi_*(t) = t^{p^*}$ $t \geq 0$, where $p^* = np/(n - mp)$ is critical Sobolev exponent. The optimality here means that the embedding is no longer holds if we replace the function φ_* with any other Young function ψ that grows strictly more rapidly than

φ_*. This type of scenario leads to a breakdown of compactness, giving rise to interesting questions that have been the subject of investigation by several authors over time, see for instance [6] and the subsequent citations thereof. For the limiting case $n = mp$, the threshold growth is not achieved by any power-type function $\varphi_p(t) = t^p$, $t \geq 0$; instead, it is determined by exponential growth. The pioneering works are due to V. Yudovich [31], S. Pohozaev [62], J. Peetre [60], N. Trudinger [68] the proved the admissibility of the exponential growth in the first-order derivative case $m = 1$. The optmality of exponential growth was obtained by Hempel-Morris-Trudinger [28] in which they also observe the influence of this growth in the loss of compactness. Finally, Moser [55] proved an improved version of these results, which is now known as the Trudinger-Moser inequality, and the extension of Moser's result for higher order derivatives $m \geq 2$ is due to Adams [2]. Adams-Moser-Trudinger inequality and its related extremal problem has a broad range of applications in partial differential equations and geometric analysis, see for instance [4, 5], and there are a lot of extensions and generalizations, among which we point out the works [3, 67, 71] for bounded domains and [1, 7, 20, 63, 43, 64, 34] for unbounded domains in the Euclidean space.

This work aims to present extension of the Adams-Trudinger-Moser inequality of the Adimurthi-Druet type [3] in unbounded domains and to investigate its extremal problem. To precisely situate our developments, we will describe below some related advances on this topic.

1.1.1 Trudinger-Moser type inequalities: The first-order derivatives

For a smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$ Moser obtained the following sharp estimate

$$\sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_n = 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}}\, dx \leq C_n |\Omega| \quad \text{for} \quad \alpha \leq \alpha_n, \tag{1-1}$$

where $\alpha_n := n\omega_{n-1}^{1/(n-1)}$ and ω_{n-1} is the area of the surface of the unit n-ball in $\mathbb{R}^n$, $|\Omega|$ denotes the n-dimensional Lebesgue measure of Ω, and $\|\cdot\|_n$ is the standard norm in the Lebesgue space $L^n(\Omega)$. Moser also showed that α_n is sharp, i.e., the supremum (1-1) is $+\infty$ if $\alpha > \alpha_n$. Nevertheless, P.-L. Lions [46] (see also [69]) was able to prove that the exponent α_n can be improved along certain sequences. In fact, if $(u_i) \subset W_0^{1,n}(\Omega)$ with $\|\nabla u_i\|_n = 1$ and $u_i \rightharpoonup u_0$ in $W_0^{1,n}(\Omega)$, then

$$\sup_i \int_\Omega e^{\gamma \alpha_n |u_i|^{\frac{n}{n-1}}}\, dx < \infty, \quad \text{for any} \quad \gamma < P := (1 - \|\nabla u_0\|_n^n)^{-\frac{1}{n-1}}.$$

Note that, unless $u_0 = 0$, we have $P > 1$. Motivated by Lions' result, Adimurthi-Druet [3] for $n = 2$ and Yang [71] for $n \geq 3$ proposed an improved version of (1-1) that provides additional information even in the case where $u_0 = 0$. Precisely,

$$\sup_{u \in W_0^{1,n}(\Omega),\, \|\nabla u\|_n = 1} \int_\Omega e^{\alpha_n (1+\alpha \|u\|_n^n)^{\frac{1}{n-1}} |u|^{\frac{n}{n-1}}} \, dx \tag{1-2}$$

is finite for any $0 \leq \alpha < \lambda_1(\Omega)$, and the supremum is infinity for any $\alpha \geq \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ represents the eigenvalue associated with n-Laplacian. For extensions of (1-2), still within the context of first-order derivatives and without any claim of completeness, we recommend [12, 50, 67, 72, 73, 42] and the references therein.

When $|\Omega| = \infty$, the inequality (1-1) is meaningless. Nevertheless, in works Cao [7], Do Ó [20], Panda [59], and Adachi-Tanaka [1], the following Trudinger-Moser type estimate in the scaling invariant form was obtained: For $\alpha \in (0, \alpha_n)$ there exists a constant $C_{\alpha,n}$ depending only on α and n such that

$$\sup_{u \in W^{1,n}(\mathbb{R}^n),\, \|\nabla u\|_n \leq 1} \frac{1}{\|u\|_n^n} \int_{\mathbb{R}^n} \phi_n\left(\alpha |u|^{\frac{n}{n-1}}\right) dx \leq C_{\alpha,n} \tag{1-3}$$

where

$$\phi_n(t) = e^t - \sum_{j=0}^{n-2} \frac{t^j}{j!}.$$

In addition, different from the usual Trudinger-Moser inequality (1-1) in bounded domains, (1-3) with $\alpha \geq \alpha_n$ is false which excludes $\alpha = \alpha_n$. For this reason, the inequality (1-3) is currently known as subcritical Trudinger-Moser inequality in $\mathbb{R}^n$. To achieve the critical case, Li-Ruf [43] and Ruf [63] replace the Dirichlet gradient norm by the full Sobolev norm $\|u\|_{W^{1,n}(\mathbb{R}^n)} = \left(\|\nabla u\|_n^n + \|u\|_n^n\right)^{1/n}$ to obtain the following sharp critical Trudinger-Moser inequality in $\mathbb{R}^n$

$$\sup_{u \in W^{1,n}(\mathbb{R}^n),\, \|u\|_{W^{1,n}(\mathbb{R}^n)} \leq 1} \int_{\mathbb{R}^N} \phi_n\left(\alpha |u|^{\frac{n}{n-1}}\right) \mathrm{d}x < \infty, \quad \text{for all } \alpha \leq \alpha_n. \tag{1-4}$$

Moreover, if $\alpha > \alpha_n$ then the supremum in (1-4) is infinite. Inequalities of types (1-3) and (1-4) have a long history of investigation by various authors. We recommend [29, 40] for a more in-depth discussion, and particularly the recent works [9, 39], which show that these inequalities are indeed equivalent.

In the context of first-order derivatives, Trudinger-Moser inequalities of the Adimurthi-Druet type on $\mathbb{R}^n$ were developed by Do Ó-de Souza [15, 19] (see also [56]), who established the following version of (1-2) for the whole space

$$\mathrm{MT}(n,\beta,\alpha)=\sup_{u\in W^{1,n}(\mathbb{R}^n),\ \|u\|_{W^{1,n}(\mathbb{R}^n)}\le 1}\int_{\mathbb{R}^n}\phi_n\left(\beta\left(1+\alpha\|u\|_n^n\right)^{\frac{1}{n-1}}|u|^{\frac{n}{n-1}}\right)dx, \tag{1-5}$$

which is finite for any $\beta\le\alpha_n$ and $0\le\alpha<1$.

In this work, we provide extensions of inequalities (1-2) and (1-5) for higher-order derivatives, which are detailed in Theorem 1.1 below.

1.1.2 Adams-Trudinger-Moser type inequalities: The case higher-order derivatives

For any positive integer $m<n$ and $u\in C_0^\infty(\Omega)$, we set

$$\nabla^m u=\begin{cases}\Delta^{m/2}u, & \text{if } m \text{ is even,}\\ \nabla\Delta^{(m-1)/2}u, & \text{if } m \text{ is odd.}\end{cases}$$

For $|\Omega|<\infty$, Adams 1988 in [2] proved that

$$\sup_{u\in W_0^{m,\frac{n}{m}}(\Omega),\ \|\nabla^m u\|_{\frac{n}{m}}\le 1}\int_\Omega e^{\beta|u|^{\frac{n}{n-m}}}dx<\infty,\ \text{ if and only if } \beta\le\beta_0, \tag{1-6}$$

where

$$\beta_0=\beta_0(m,n)=\begin{cases}\dfrac{n}{\omega_{n-1}}\left[\dfrac{\pi^{\frac{n}{2}}2^m\Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})}\right]^{\frac{n}{n-m}}, & \text{if } m \text{ is odd,}\\[2ex] \dfrac{n}{\omega_{n-1}}\left[\dfrac{\pi^{\frac{n}{2}}2^m\Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}\right]^{\frac{n}{n-m}}, & \text{if } m \text{ is even,}\end{cases} \tag{1-7}$$

in which $\Gamma(x)=\int_0^1(-\ln t)^{x-1}dt$, $x>0$ is the gamma Euler function. Inequality (1-6) is the extension for higher order derivatives of the Moser inequality (1-1) and it is currently known as Adams inequality or Adams-Moser-Trudinger inequality. By considering was the more large space given by

$$W_{\mathcal{N}}^{m,p}(\Omega):=\{u\in W^{m,p}(\Omega): u_{|\partial\Omega}=\Delta^j u_{|\partial\Omega}=0 \text{ in the sense of trace}, 1\le j<m/2\},$$

C. Tarsi [66, Theorem 4] was able to extend (1-6) to the following

$$\sup_{u\in W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega),\ \|\nabla^m u\|_{\frac{n}{m}}\le 1}\int_\Omega e^{\beta|u|^{\frac{n}{n-m}}}dx\le C_{m,n}|\Omega|,\quad \forall\, 0\le\beta\le\beta_0, \tag{1-8}$$

for some constant $C_{m,n}>0$. In addition, β_0 is also sharp, i.e., the supremum above is $+\infty$ if $\beta>\beta_0$.

Extensions of Adams inequality for the entire space $\mathbb{R}^n$, where first considered by Ozawa [58] who obtained an extension for $W^{m,\frac{n}{m}}\left(\mathbb{R}^n\right)$ by using the restriction $\left\|\Delta^{\frac{m}{2}}u\right\|_{\frac{m}{m}} \le 1$. However, with the argument in [58], one cannot obtain the best possible exponent β for this type of inequality. Recently Ruf-Sani [64] when m is an even integer number and Lam-Lu [34, 36] for m odd integer number have obtained a version of the Adams inequality (1-6) for any domain $\Omega \subset \mathbb{R}^n$ not necessarily bounded. In fact, there hold

$$\sup_{u\in W_0^{m,\frac{n}{m}}(\Omega),\ \|(-\Delta+I)^k u\|_{\frac{n}{m}}\le 1} \int_\Omega \Phi(\beta_0|u|^{\frac{n}{n-m}})dx \le C_{m,n}, \text{ if } m=2k \tag{1-9}$$

and

$$\sup_{u\in W_0^{m,\frac{n}{m}}(\Omega),\ \|\nabla(-\Delta+I)^k u\|_{\frac{n}{m}}^{\frac{n}{m}}+\|(-\Delta+I)^k u\|_{\frac{n}{m}}^{\frac{n}{m}}\le 1} \int_\Omega \Phi(\beta_0|u|^{\frac{n}{n-m}})dx \le C_{m,n}, \text{ if } m=2k+1 \tag{1-10}$$

where

$$\Phi(t) = e^t - \sum_{j=0}^{j_{m,n}-2}\frac{t^j}{j!}, \qquad j_{m,n} := \min\{j\in\mathbb{N} : j \ge \frac{n}{m}\}. \tag{1-11}$$

Moreover, the constant β_0 is sharp for (1-9) and (1-10).

On Sobolev spaces $W^{\gamma,\frac{n}{\gamma}}\left(\mathbb{R}^n\right)$ of arbitrary positive fractional order $\gamma < n$, by using a rearrangement-free argument Lam-Lu [37] established the following Adams inequality

$$\sup_{u\in W^{\gamma,p}(\mathbb{R}^n),\ \|\tau(I-\Delta^{\frac{\gamma}{2}})u\|_p\le 1} \int_{\mathbb{R}^n} \Phi(\beta_0(n,\gamma)|u(x)|^{\frac{p}{p-1}})dx < \infty,$$

with

$$\beta_0(n,\gamma) = \frac{n}{\omega_{n-1}}\left[\frac{\pi^{\frac{n}{2}}2^\gamma\Gamma(\frac{\gamma}{2})}{\Gamma\left(\frac{n-\gamma}{2}\right)}\right]^{\frac{p}{p-1}},$$

where $p=\frac{n}{\gamma}$, $\tau>0$ and Φ is the same one in (1-11) with $j_p=\min\{j\in\mathbb{N} : j\ge p\}$ instead of $j_{m,n}$. Further, if $\beta_0(n,\gamma)$ is replaced by any $\beta > \beta_0(n,\gamma)$, then the supremum is infinite. Another extension of (1-2) for whole space $\mathbb{R}^n$, is the subcritical Adams inequality established by Lam-Lu [37] for $m=2$, Fontana-Morpurgo [23] and Lam-Lu-Zhang [39] to all $m>2$ is the following

$$\sup_{u\in W^{m,\frac{n}{m}}(\mathbb{R}^n),\ \|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}}+\|u\|_{\frac{n}{m}}^{\frac{n}{m}}\le 1} \int_{\mathbb{R}^n} \Phi(\beta|u(x)|^{\frac{n}{n-m}})dx \begin{cases} \le C_{m,n}, & \text{if } \beta \le \beta_0(m,n) \\ = \infty, & \text{if } \beta > \beta_0(m,n). \end{cases}$$

Finally, we would like to mention the following sharpened Trudinger-Moser inequality in $\mathbb{R}^2$ with exact growth condition due to Ibrahim-Masmoudi-Nakanishi [29]: there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}^2} \frac{e^{4\pi u^2}-1}{(1+|u|)^2} dx \leq C\|u\|^2_{L^2(\mathbb{R}^2)}, \quad \forall u \in W^{1,2}(\mathbb{R}^2) \text{ with } \|\nabla u\|_{L^2(\mathbb{R}^2)} \leq 1. \tag{1.4}$$

Moreover, this fails if the power 2 in the denominator is replaced with any $p < 2$. In [53] Masmound-Sani extended (1.4) to the corresponding second-order Adams' inequality in $W^{2,2}(\mathbb{R}^4)$ and then for arbitrary dimensions $n \geq 2$ in [54]. After, Lu-Tang in [48] were able to provide an extension to the framework of hyperbolic space. For a more in-depth discussion and improvements of (1.4) we recommend [38, 39, 40] and the references therein.

Finally, we would like to emphasize that, despite the large number of extensions of (1-6), except for recent work in [16, 49] for $n = 2m$ and bounded domains, to our knowledge, no other Adimurthi-Druet type extension (1-2) has been established for $m \geq 2$, for either bounded or unbounded domains $\Omega \subset \mathbb{R}^n$. In this thesis, we aim to fill this gap by presenting such extensions, which differ from the usual ones (even for $m = 1$), as our proposed inequality is strongly governed by the *vanishing* phenomenon.

1.1.3 Maximizers: the extremal problem

Since the establishment of the Adams-Trudinger-Moser inequality (1-1)-(1-6), a natural question arises: *does there exist* $u \in W_0^{m,\frac{n}{m}}(\Omega)$ *with* $\|\nabla^m u\|_{\frac{n}{m}} = 1$ *such that*

$$\int_\Omega e^{\beta_0|u|^{\frac{n}{n-m}}} dx = \sup_{u\in W_0^{m,\frac{n}{m}}(\Omega),\ \|\nabla^m u\|_{\frac{n}{m}}\leq 1} \int_\Omega e^{\beta|u|^{\frac{n}{n-m}}} dx? \tag{1-12}$$

This is the famous extremal problem for the Adams-Trudinger-Moser inequality under the critical condition $\beta = \beta_0$, which has been studied by various authors over the years. For the first order derivative case $m = 1$ it was completely solved through the works [8, 65, 21, 45, 16]. In fact, Carleson-Chang [8] solved it when $\Omega = B_1(0) \subset \mathbb{R}^n, n \geq 2$ is the unit ball, M. Struwe [65] used blow-up analysis to ensure extremal functions for a class of nonsymmetric domains in $\mathbb{R}^2$, Flucher [21] applied conformal rearrangement to derived an isoperimetric inequality which implies the existence of extremal functions to any smooth bounded domain in $\mathbb{R}^2$. Finally, Lin [45] ensures extremal function to any smooth bounded domain in $\mathbb{R}^n, n \geq 2$. In contrast to the case $m = 1$, extremal problem (1-12) for $m \geq 2$ has been solved only for some particular cases. We can only mention Lu-Yang [49], which proved the existence of extremals in the case $m = 2$ with $\Omega \subset \mathbb{R}^4$ and more recently DelaTorre-Mancini [16], where the existence of extremals for the case $H_0^m(\Omega)$ with $\Omega \subset \mathbb{R}^{2m}$ is proved. We also draw attention to the work [14], where a sharp estimate for the concentration levels of the Adams-Trudinger-Moser functional is obtained.

The existence of extremal function for the Adimurthi-Druet inequality (1-2) it was obtained by Yang in [70] for $n = 2$ and [71] for $n \geq 3$. On the other hand, the unbounded situation $\Omega = \mathbb{R}^n$ for the either subcritical (1-3) or critical (1-4) the corresponding extremal problems were investigated in [30, 43, 63] and in the recent work [40]. Further, the extremal for the Adimurthi-Druet type supremum (1-5) was recently studied in [56]. It is worth mentioning that, in the unbounded situation, even for $\alpha < \alpha_n$ attainability is nontrivial due to the loss of compactness, as observed by Ishiwata [30] (see also [46, 47]). To illustrate the situation, let (u_j) with $\|u_j\|_{W^{1,n}(\mathbb{R}^n)} = 1$ and $u_j \rightharpoonup u$ weakly in $W^{1,n}(\mathbb{R}^n)$ be a maximizing sequence for the supremum $\mathrm{MT}(n, \beta, \alpha)$ in (1-5). Then the compactness of (u_j) follows if we can exclude both the *concentration* phenomenon

$$(a) \quad u = 0 \text{ and } \lim_{j\to\infty} \int_{B_\rho^c} (|\nabla u_j|^n + |u_j|^n) dx = 0 \text{ for any } \rho > 0$$

and the *vanishing* phenomenon

$$(b) \quad \lim_{j\to\infty} \|\nabla u_j\|_n = 0 \text{ and } \limsup_{j\to\infty} \|u_j\|_n > 0.$$

In most works on the existence of extremals a lower bound for the supremum is established which, after sharp estimates for both concentrating and vanishing levels, ensures the compactness of the any maximizing sequence, see for instance [8, 30, 45, 43, 56, 63]. Now we give an outline about this work. In the next chapter, we discuss some notations, definitions and preliminary results. In following chapters we give the main results.

1.1.4 Overview and mains results

This work is structured into seven chapters, in addition to this introduction. Each of them plays a crucial role in building up the main results. In Chapter 2, we gather the definitions, notations, and auxiliary results indispensable for the development of the thesis. We present tools such as symmetric rearrangements, Gagliardo–Nirenberg type estimates, regularity results for the polyharmonic operator, and fundamental lemmas concerning radial decay. We also introduce the notions of concentrating and vanishing sequences, which play a central role in the compactness analysis and in the understanding of the loss of compactness phenomena in higher-order Sobolev spaces.

The Chapter 3 constitutes the initial core of the original results. In this chapter, we prove the main result , Theorem 1.1, which establishes a new Adams inequality of Adimurthi–Druet type in the whole space $\mathbb{R}^n$. This result extends the classical Adams inequality to unbounded domains, incorporating in a decisive way the vanishing phenomenon. Furthermore, the optimality of the critical constant β_0 is established, ensuring that the statement attains its sharp form.

Theorem 1.1 *Let* $n \geq 2$, $0 \leq \alpha < 1$, $0 < \gamma < \frac{1}{\alpha} - 1$ *for* $\alpha > 0$, *and* $1 \leq m < n$. *Then*

$$\sup_{\substack{u \in W^{m,\frac{n}{m}}(\mathbb{R}^n) \\ \|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}} + \|u\|_{\frac{n}{m}}^{\frac{n}{m}} \leq 1}} \int_{\mathbb{R}^n} \Phi\left(\beta_0 \left(\frac{1+\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{1-\gamma\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}\right)^{\frac{m}{n-m}} |u|^{\frac{n}{n-m}}\right) dx < \infty. \tag{1-13}$$

In addition, the constant β_0 *is sharp.*

In order to emphasize the nature of the inequality (1-13), let us consider (u_j) be a normalized vanishing type sequence in $W^{m,\frac{n}{m}}(\mathbb{R}^n)$, that is,

$$\|\nabla^m u_j\|_{\frac{n}{m}}^{\frac{n}{m}} + \|u_j\|_{\frac{n}{m}}^{\frac{n}{m}} = 1 \text{ and } \lim_{j\to\infty} \|\nabla^m u_j\|_{\frac{n}{m}} = 0.$$

Hence, along this type of sequence, we observe that the exponent

$$\beta\left(\frac{1+\alpha\|u_j\|_{\frac{n}{m}}^{\frac{n}{m}}}{1-\gamma\alpha\|u_j\|_{\frac{n}{m}}^{\frac{n}{m}}}\right)^{\frac{m}{n-m}}$$

converges to the constant $\beta(1+\alpha)/(1-\gamma\alpha)$ as $j \to \infty$, which can be arbitrarily large if $\alpha > 0$ is close to 0 and γ is close to the upper limit $1/\alpha - 1$. It proves to be a challenge to the maximizing problem, since that in (1-13) the vanishing phenomenon implies in the lack of compactness for the associated functional even in the subcritical case $\beta < \beta_0$. From this reason, we say that our inequality is strongly governed by the vanishing phenomenon.

In Chapter 4, we analyze the structure of *compactness–concentration–vanishing* in the subcritical regime ($\beta < \beta_0$). We present estimates that distinguish the different scenarios for maximizing sequences and discuss the implications of this delicate balance for the existence of extremals. This chapter provides the necessary technical framework for the existence and non-existence results developed in the subsequent chapters.

Let us denote

$$AD(n,m,\beta,\alpha,\gamma) := \sup_{\substack{u \in W^{m,\frac{n}{m}}(\mathbb{R}^n) \\ \|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}} + \|u\|_{\frac{n}{m}}^{\frac{n}{m}} \leq 1}} \int_{\mathbb{R}^n} \Phi\left(\beta \left(\frac{1+\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{1-\gamma\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}\right)^{\frac{m}{n-m}} |u|^{\frac{n}{n-m}}\right) dx. \tag{1-14}$$

Initially, we consider the existence and non-existence of extremal function to

$AD(n,m,\beta,\alpha,\gamma)$ in the subcritical and critical case. In this case, we consider

$$\mathscr{B}_{GN} := \sup_{u\in W^{m,2}_{rad}(\mathbb{R}^{2m})} \frac{\|u\|_4^4}{\|\nabla^m u\|_2^2\|u\|_2^2}, \tag{1-15}$$

which is the Gagliardo-Nirenberg constant in $W^{m,2}(\mathbb{R}^{2m})$. We are able to prove the following existence results.

Chapter 5 is devoted to the study of the extremal problem in the subcritical case in even dimensions, that is, when $n = 2m$. We prove Theorem 1.2, which ensures the existence of functions that attain the supremum in certain intervals of β. On the other hand, Theorem 1.3 demonstrates the *non-attainability* in other regimes, highlighting the subtlety of the problem and the decisive influence of the vanishing phenomenon even in the subcritical scenario. The next two subsequent results, whose proofs are presented in Chapter 5, asserts that

Theorem 1.2 *Suppose* $0 < \alpha < 1$, $n = 2m$ *and* $0 \leq \gamma < \min\{\frac{1}{\alpha} - 1, \frac{m(1+\alpha)-1-2\alpha}{m\alpha^2+\alpha m-\alpha^2}\}$. *Then* $AD(2m,m,\beta,\gamma,\alpha)$ *is attained for any* $\beta \in \left(\frac{1+2\alpha-\gamma\alpha^2}{1+\alpha(1-\gamma)-\gamma\alpha^2}\frac{2}{\mathscr{B}_{GN}}, \beta_0\right)$.

The next result shows that the attainability of Theorem 1.2 is lost for small β, highlighting that the corresponding extremal problem in (1-13) remains challenging even in the subcritical case.

Theorem 1.3 *Let* $0 < \alpha < 1$, $0 \leq \gamma < \frac{1}{\alpha} - 1$ *and* $n = 2m$ *for* $\beta \ll \beta_0$, *then* $AD(2m,m,\beta,\alpha,\gamma)$ *is not attained.*

In Chapter 6, we address the *critical case in dimension four* ($n = 4, m = 2$). By employing *blow-up analysis* combined with biharmonic truncations, we describe the asymptotic behavior of maximizing sequences, establish sharp upper bounds for the associated functional, and construct refined test functions. This effort culminates in the proof of Theorem 1.4, which guarantees the existence of extremals in the critical regime, unveiling the depth and precision of the analysis in four dimensions.

Theorem 1.4 *Let* $0 < \alpha < 1$ *and* $0 \leq \gamma < \min\{\frac{1}{\alpha} - 1, \frac{1}{\alpha^2+2\alpha}, \gamma_0\}$ *for some* $\gamma_0 > 0$, *then there exists* $\alpha_0 \in (0,1)$ *such that* $AD(4,2,\beta_0,\alpha,\gamma)$ *is attained for any* $0 \leq \alpha < \alpha_0$.

Finally, Chapter 7 generalizes the approach of the previous chapter to arbitrary even dimensions. We introduce sophisticated polyharmonic truncations and develop appropriate test functions, which, together with the blow-up analysis, allow us to prove Theorem 1.5. This result confirms the existence of extremals in the critical case for all even dimensions, substantially broadening the scope of the results and consolidating the theoretical framework developed throughout the thesis.

Theorem 1.5 *Let* $0 < \alpha < 1$ *and* $0 \leq \gamma < \min\{\frac{1}{\alpha} - 1, \frac{1}{\alpha^2+2\alpha}\}$, *then* $AD(2m, m, \beta_0, \alpha, \gamma)$ *is attained.*

The techniques employed in the last two results are inspired by the approach of Chen-Lu-Zhu, DelaTorre and Nguyen [11, 16, 3, 33] who studied the subcritical and critical Adams inequality in $\mathbb{R}^4$ and $\mathbb{R}^n$ in both bounded and unbounded domains using a refined blow-up analysis with a sofisticated polyharmonic truncation argument and the construction of a suitable test function to get the results of attainbility and upper bounds.

CHAPTER 2

Preliminaries

In this chapter, we introduce the essential definitions, notations, and auxiliary results that will be used throughout the thesis. We begin by presenting the fundamental concepts and conventions adopted in the sequel. We then establish key auxiliary tools, including a version of the Radial Lemma, which provides decay estimates for radial functions, and an adaptation of the Gagliardo–Nirenberg constant estimation to our setting. These results play a crucial role in deriving compactness properties and *a priori* estimates.

2.1 Notations and Definitions

In this work we'll denote by

$$F_{n,m,\beta,\alpha,\gamma}(u) = \int_{\mathbb{R}^n} \Phi\left(\beta\left(\frac{1+\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{1-\gamma\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}\right)|u|^{\frac{n}{n-m}}\right)dx, \text{ for any } u \in W^{m,\frac{n}{m}}(\mathbb{R}^n) \quad (2\text{-}1)$$

and

$$AD(n,m,\beta,\alpha,\gamma) := \sup_{\substack{u\in W^{m,\frac{n}{m}}(\mathbb{R}^n)\\ \|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}}+\|u\|_{\frac{n}{m}}^{\frac{n}{m}}\leq 1}} \int_{\mathbb{R}^n} \Phi\left(\beta\left(\frac{1+\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{1-\gamma\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}\right)^{\frac{m}{n-m}}|u|^{\frac{n}{n-m}}\right)dx. \quad (2\text{-}2)$$

We give now the definition of the radially symmetric and decreasing rearrangement of a function defined on all of $\mathbb{R}^n$.

Definition 2.1.1 *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a Borel measurable function. It is said to* ***vanish at infinity*** *if, for every $t>0$, the level sets $\{|f|>t\}$ are of finite measure.*

Definition 2.1.2 *Let $f:\mathbb{R}^n\to\mathbb{R}$ be a Borel measurable function vanishing at infinity.*

*Then its **spherically symmetric and decreasing rearrangement**, f^*, is defined on $\mathbb{R}^N$ by*

$$f^*(x) = \int_0^\infty \chi_{\{|f|>t\}^*}(x)dt. \tag{2-3}$$

Since the integrand in (2-3) is in terms of characteristic functions of balls, clearly, f^* is radially symmetric. Also it is obvious that it is radially decreasing, i.e. if $x, y \in \mathbb{R}^n$ are such that $|x| \leq |y|$, then $f^*(x) \geq f^*(y)$.

Definition 2.1.3 *Given a function $u \in L^1(\mathbb{R}^n)$, we define its **Fourier Transform** by*

$$\widehat{u}(\xi) = \int_{\mathbb{R}^n} u(x)e^{-2\pi i x\cdot\xi}dx, \tag{2-4}$$

where, $x \cdot \xi = x_1\xi_1 + \ldots + x_n\xi_n$ and i stands for the imaginary unit.

Remark 2.1.1 *For more properties and details about Fourier transform, see [26], [27] and [41].*

Definition 2.1.4 *Given a function $u \in L^2(\mathbb{R}^n)$, we define its **Fourier rearrengement** to be the function given by*

$$u^\# = \mathscr{F}^{-1}[(\mathscr{F}(u))^*] \tag{2-5}$$

where $\mathscr{F}$ is the Fourier Transform of u and $(\mathscr{F}(u))^*$ is the Schwarz Symmetrization of the Fourier transform of u.

Using the properties of Fourier rearrangement (which can be found in [41]) one can obtain

$$\begin{cases} \|\nabla^m u^\#\|_2 \leq \|\nabla^m u\|_2; \\ \|u^\#\|_2 = \|u\|_2, \ \ \|u^\#\|_q \geq \|u\|_2, (q > 2). \end{cases}$$

Denoting $\mathscr{H} := \{u \in W^{m,2}(\mathbb{R}^{2m}); \|u\|_2^2 + \|\nabla^m u\|_2^2 = 1\}$. We have,

$$\sup_{u\in\mathscr{H}} F_{2m,m,\beta,\alpha,\gamma}(u) = \sup_{u\in\mathscr{H}\cap W^{m,2}_{rad}(\mathbb{R}^{2m})} F_{2m,m,\beta,\alpha,\gamma}(u). \tag{2-6}$$

Below, we present the definitions of concentrating and vanishing sequences, which play a pivotal role in this work. These concepts will be explored in detail and applied extensively in Chapter 4.

Definition 2.1.5 *Let $(u_j) \subset W^{m,\frac{n}{m}}(\mathbb{R}^n)$, $\|\nabla^m u\|^{\frac{n}{m}}_{\frac{n}{m}} + \|u_j\|^{\frac{n}{m}}_{\frac{n}{m}} \leq 1$, for $j \in \mathbb{N}$, be a **concentrating sequence** at the origin, if*

$$\lim_{j\to\infty} \int_{B_R} |\nabla^m u_j|^{\frac{n}{m}} dx = 1 \quad \forall R > 0. \tag{2-7}$$

Definition 2.1.6 *Let* $(u_j) \subset W^{m,\frac{n}{m}}(\mathbb{R}^n)$, $\|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}} + \|u_j\|_{\frac{n}{m}}^{\frac{n}{m}} \leq 1$, *for* $j \in \mathbb{N}$, *be a* ***vanishing sequence****, if*

$$\lim_{j\to\infty} \|\nabla^m u_j\|_{\frac{n}{m}} = 0. \tag{2-8}$$

2.1.1 Auxiliary Results

Let us state a lemma inspired on Lemma 2.1 from [10].

Lemma 2.1 *For any* $0 < \beta < 32\pi^2$, *there exists a radially symmetric extremal function* $u \in \mathcal{H}$ *such that*

$$F_{2m,m,\beta,\alpha,\gamma}(u) = \sup_{u\in\mathcal{H}} F_{2m,m,\beta,\alpha,\gamma}(u).$$

Now we state the following Radial Lemma that can be easily extended from [32], Lemma 1.1, Chapter 6, which will be useful in our analysis.

Lemma 2.2 *If* $u \in W^{1,\frac{n}{m}}_{rad}(\mathbb{R}^n)$, *then*

$$|u(x)| \leq \left(\frac{1}{m\sigma_n}\right)^{\frac{m}{n}} \frac{1}{|x|^{(\frac{n-1}{n})m}} \|u\|_{W^{1,\frac{n}{m}}} \qquad a.e\ x \in \mathbb{R}^n,$$

where σ_n *is the volume of unit ball in* $\mathbb{R}^n$.

The next auxiliary result is due to Tarsi and can be found in [66]. This result generalizes the classical Adams inequality to spaces with homogeneous Navier boundary conditions

Theorem 2.3 *Let* $n > 2$, $\Omega \subset \mathbb{R}^n$ *be a bounded domain. Then there exist a constant* $C_n > 0$ *such that for all* $u \in W^{m,\frac{n}{m}}_{\mathcal{N}}(\Omega)$ *with* $\|\nabla^m u\|_{\frac{n}{m}} \leq 1$

$$\int_\Omega e^{\beta|u|^{\frac{n}{n-m}}} dx \leq C_n|\Omega|, \quad \forall \beta < \beta_{n,m}. \tag{2-9}$$

And the constant $\beta_{n,m}$ *is sharp.*

We also mention the following Adams type inequality, which is analogous to the inequality proved by Adachi-Tanaka in [1],in a scale-invariant form, which will be useful for us to prove Theorem 1.3.

Let $n > m \geq 2$ be integers. Then given $\beta \in (0, \beta_0)$ there exists $C_{\beta,m,n} = C(\beta, m, n)$ depending only on β, m and n such that

$$\int_{\mathbb{R}^n} \Phi\left(\beta\left(\frac{|u|}{\|\nabla^m u\|_{n/m}}\right)^{n/(n-m)}\right) dx \leq C_{\beta,m,n}\frac{\|u\|^{n/m}_{n/m}}{\|\nabla^m u\|^{n/m}_{n/m}}, \quad \forall\, u \in W^{m,n/m}_{rad}(\mathbb{R}^n)\setminus\{0\}, \tag{2-10}$$

where Φ was defined in (1-11), β_0 was given in (1-7) and $W^{m,n/m}_{rad}(\mathbb{R}^n)$ denote the subspace of the radial $W^{m,n/m}(\mathbb{R}^n)$-functions. Moreover, for $\beta \in [\beta_0, \infty)$ inequality (2-10) fail, i.e., there exists $(u_i) \subset W^{m,n/m}_{rad}(\mathbb{R}^n)$ such that

$$\frac{\|\nabla^m u_i\|_{n/m}^{n/m}}{\|u_i\|_{n/m}^{n/m}} \int_{\mathbb{R}^n} \Phi\left(\beta\left(\frac{|u_i|}{\|\nabla^m u_i\|_{n/m}}\right)^{n/(n-m)}\right) dx \to \infty. \tag{2-11}$$

Remark 2.1.2 *It is important to mention that from the proof of* (2-10) *we can see that the constant* $C_{\beta,m,n}$ *tend exponentially to infinite when* β *tends to* β_0.

2.1.2 Gagliardo-Niremberg Constant Estimates

We'll give an estimate to the constant $\mathscr{B}_{GN}$ in (2-12) for the even case, which will be important in our existence of extremals result. Indeed, let us consider the same function introduced by D. Adams in [2].

Let $\varphi(t) \in C^\infty[0,1]$ such that,

$$\varphi(0) = \varphi'(0) = \cdots = \varphi^{(m-1)}(0) = 0,$$
$$\varphi(1) = \varphi'(1) = 1 \qquad \varphi''(1) = \varphi'''(1) = \cdots = \varphi^{(m-1)}(1) = 0.$$

For $0 < \varepsilon < \frac{1}{2}$, we define

$$H(t) = \begin{cases} \varepsilon\varphi(\frac{1}{\varepsilon}t), & \text{if} \quad t \le \varepsilon \\ t, & \text{if} \quad \varepsilon \le t \le 1-\varepsilon \\ 1-\varepsilon\varphi(\frac{1}{\varepsilon}(1-t)), & \text{se} \quad 1-\varepsilon \le t \le 1 \\ 1, & \text{if} \quad 1 \le t, \end{cases}$$

and

$$\psi_\lambda(r) = (log(\lambda))^{1/2} H\left((log(\lambda))^{-1}\log\frac{1}{r}\right).$$

For all $\lambda > 1$, $\psi_\lambda(|x|)$ is defined on $B_1(0)$ and can be extended for whole space $W^{m,2}_{0,rad}(\mathbb{R}^{2m})$. More than that, $\psi_\lambda(|x|) = (log(\lambda))^{1/2}$ for $|x| \le 1/\lambda$ and, as proved by D. Adams in [2], we have

$$\|\nabla^m \psi_\lambda\|_2^2 = (2m)^{-1}\beta_0 A_{\lambda,\varepsilon},$$

where

$$A_{\lambda,\varepsilon} \le \left[1 + 2\varepsilon\left(\|\Phi'\|_\infty + O\left((log(\lambda))^{-1}\right)\right)^2\right].$$

Now, computing explicitly $\|\psi_\lambda\|_2^2$, we obtain

$$\begin{aligned}\|\psi_\lambda\|_2^2 &\leq \int_{B_1(0)} |\psi_\lambda(x)|^2 dx \\ &\leq \int_{B_1(0)} |(\log(\lambda))^{1/2}|^2 dx \\ &\leq \int_{B_1(0)} \log(\lambda) dx \\ &= \frac{\omega_{2m-1}}{2m}\log(\lambda).\end{aligned}$$

Now, with the aim to majorate the Gagliardo-Nirenberg constant in $W^{m,2}(\mathbb{R}^4)$, we compute

$$\begin{aligned}\|\psi_\lambda\|_4^4 &\geq \int_{B_{1/\lambda}} |\psi_\lambda(|x|)|^4 dx \\ &= 2\log_\lambda \frac{\omega_{2m-1}}{2m}\frac{1}{\lambda^{2m}}\end{aligned}$$

where, $\frac{\pi^m}{m!}\left(\frac{1}{\lambda}\right)^m$ is the volume of the $2m$-ball of radius $1/\lambda$. Moreover this, notice that $\|\psi_\lambda\|_2^2 = \pi^m(m!)^{-1}\log\lambda$. Therefore,

$$\begin{aligned}\mathscr{B}_{GN} &\geq \frac{\|\psi_\lambda\|_4^4}{\|\nabla^m\psi_\lambda\|_2^2\|\psi_\lambda\|_2^2} \\ &= \frac{2log(\lambda)\frac{\omega_{2m-1}}{2m}\frac{1}{\lambda^{2m}}}{\left(\pi^m(m!)^{-1}log(\lambda)\right)\left((2m)^{-1}\beta_0 A_{\lambda,\varepsilon}\right)}\end{aligned}$$

since $\omega_{2m-1} = \frac{2\pi^m}{(m-1)!}$, we get

$$\mathscr{B}_{GN} \geq \frac{2m}{\lambda^{2m}\beta_0 A_{\lambda,\varepsilon}}.$$

Taking the limit as $\varepsilon \to 0$, we obtain $\limsup A_{\lambda,\varepsilon} \leq 1$. Next, taking $\lambda \to 1^+$, we derive the following we estimate

$$\mathscr{B}_{GN} \geq \frac{2m}{\beta_0}. \tag{2-12}$$

2.1.3 Regularity Results for for the Polyharmonic Operator $(\Delta)^m$

Lemma 2.4 *(Gazzola, [25]) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary, and take $k, m \in \mathbb{N}$, $k \geq 2m$ and $\gamma \in (0,1)$. If $u \in H^m(\Omega)$ is a weak solutions of the problem*

$$\begin{cases} (-\Delta)^m u = f & \text{in } \Omega \\ \partial_\nu^i u = h_i, & \text{on } \partial\Omega, 0 \leq i \leq m-1 \end{cases}$$

with $f \in C^{k-2m,\gamma}(\Omega)$ and $h_i \in C^{k-i,\gamma}(\partial\Omega)$ then $u \in C^{k,\gamma}(\Omega)$, and there exists a constant $C = C(\Omega, k, \gamma)$ such that

$$\|u\|_{C^{k,\gamma}(V)} \leq C\left(\|f\|_{C^{k-2m,\gamma}(\Omega)} + \sum_{i=0}^{m-1} \|h_i\|_{C^{k-i,\gamma}(\partial\Omega)}\right).$$

Similarly, if $f \in C^{k-2m,\gamma}(\Omega)$ and u is a weak solution of $(-\Delta)^m u = f$ in Ω, then $u \in C^{k,\gamma}_{loc}(\Omega)$ and for any open set $V \Subset \Omega$, then there exists a constant $C = C(k, p, V, \Omega)$ such that

$$\|u\|_{C^{k,\gamma}} \leq C(\|f\|_{C^{k-2m,\gamma}(\Omega)} + \|u\|_{L^1(\Omega)}).$$

Lemma 2.5 *(Gazzola, [25]) Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary and take $m, k \in \mathbb{N}$, $k \geq 2m$ and $p > 1$. If $f \in W^{k-2m,p}(\Omega)$ and $u \in H^m(\Omega)$ is a weak solution of of $(-\Delta)^m u = f$ in Ω, then $u \in W^{k,p}_{loc}(\Omega)$, and for any open set $V \subset\subset \Omega$, then there exists a constant $C = C(k, p, V, \Omega)$ such that*

$$\|u\|_{W^{k,p}(V)} \leq C(\|f\|_{W^{k-2m,p}(\Omega)} + \|u\|_{L^1(\Omega)}).$$

Similarly, if $f \in C^{k-2m,\gamma}(\Omega)$ and u is a weak solution of $(-\Delta)^m u = f$ in Ω, then $u \in C^{k,\gamma}_{loc}(\Omega)$ and for any open set $V \subset\subset \Omega$, then there exists a constant $C = C(k, p, V, \Omega)$ such that

$$\|u\|_{C^{k,\gamma}(V)} \leq C(\|f\|_{C^{k-2m,\gamma}(\Omega)} + \|u\|_{L^1(\Omega)}).$$

Theorem 2.6 (Martinazzi [52]) *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain and let u be a solution of $\Delta^m u = f \in L(\ln L)^\alpha$with the Dirichlet boundary condition, for some $0 \leq \alpha \leq 1$ and $n \geq 2m$. Then $\nabla^{2m-l} u \in L^{(\frac{n}{n-l}, \frac{1}{\alpha})}(\Omega), 1 \leq l \leq 2m-1$ and*

$$\|\nabla^{2m-l} u\|_{L^{(\frac{n}{n-l}, \frac{1}{\alpha})}} \leq C\|f\|_{L(\ln L)^\alpha}, \tag{2-13}$$

where $L(\ln L)^\alpha(\Omega)$ is the Zygmund space

$$L(\ln L)^\alpha(\Omega) := \left\{ f \in L^1(\Omega) : \|f\|_{L(\ln L)^\alpha} := \int_\Omega |f| \ln^\alpha(2 + |f|) dx < \infty \right\}.$$

2.1.4 Polynomial Behaviour Auxiliar Results

Lemma 2.7 *(Pizzeti, [61]) Let* $u \in C^{2m}(B_R(x_0))$, $B_R(x_0) \subset \mathbb{R}^n$, *for some* $m, n \in \mathbb{Z}_+$. *Then there exists constants* $c_i = c_i(n)$. *Then*

$$\fint_{B(x_0)} u(x)dx = \sum_{i=0}^{m-1} c_i R^{2i} \Delta^i u(x_0) + c_m R^{2m} \Delta^m u(\xi), \tag{2-14}$$

for some $\xi \in \mathbb{R}^n$

Lemma 2.8 *(Martinazzi, [52]) Let* u *satisfying the biharmonic equation* $(-\Delta)^2 u = 0$ *with* $u \leqslant (1 + |x|^l)$ *for some* $l \geq 0$. *Then* u *is a polynomial degree at most* $\max\{l, 2\}$.

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CHAPTER 3

Adams-Adimurthi-Druet type inequality for entire space

3.1 Unbounded Domains - Theorem 1.1

In this chapter, we will prove the Adams inequality of Adimurthi-Druet type for the whole space $\mathbb{R}^n$, as stated in Theorem 1.1. This result constitutes the main theorem of this thesis and extends classical Adams-type inequalities to unbounded domains.

3.1.1 Proof of Theorem 1.1

We will proceed following the scaling argument in [56]. For $\tau > 0$ and $u \in W^{m,\frac{n}{m}}(\mathbb{R}^n)$, by setting $u_\tau(x) = u(\tau^{\frac{1}{n}}x)$, we have

$$\|\nabla^m u_\tau(x)\|_{\frac{n}{m}}^{\frac{n}{m}} = \|\nabla^m u(x)\|_{\frac{n}{m}}^{\frac{n}{m}} \text{ and } \|u_\tau(x)\|_{\frac{n}{m}}^{\frac{n}{m}} = \tau^{-1}\|u(x)\|_{\frac{n}{m}}^{\frac{n}{m}}.$$

Hence

$$\begin{aligned} K_\tau &:= \sup_{\substack{u\in W^{m,\frac{n}{m}}(\mathbb{R}^n) \\ \|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}}+\tau\|u\|_{\frac{n}{m}}^{\frac{n}{m}}\le 1}} \int_{\mathbb{R}^n} \Phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \\ &= \frac{1}{\tau} \sup_{\substack{u\in W^{m,\frac{n}{m}}(\mathbb{R}^n) \\ \|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}}+\|u\|_{\frac{n}{m}}^{\frac{n}{m}}\le 1}} \int_{\mathbb{R}^n} \Phi\left(\beta_0 |u|^{\frac{n}{n-m}}\right) dx \qquad (3\text{-}1) \\ &= \frac{K_1}{\tau} < \infty, \end{aligned}$$

which is finite by a result that can be found in [22, Theorem 1-(b)]. Since $\alpha < 1$ and $\gamma < \frac{1}{\alpha} - 1$, we can chose $0 < \tau = 1 - \alpha < 1$, $0 < \mu = \tau - \gamma\alpha < 1$. Thus, for $u \in W^{m,\frac{n}{m}}(\mathbb{R}^n)$ with $\|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}} + \|u\|_{\frac{n}{m}}^{\frac{n}{m}} \le 1$, defining

$$v := \frac{u}{(\|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}} + \tau\|u\|_{\frac{n}{m}}^{\frac{n}{m}})^{\frac{m}{n}}} \quad \text{and} \quad w := \frac{v}{(\|\nabla^m v\|_{\frac{n}{m}}^{\frac{n}{m}} + \mu\|v\|_{\frac{n}{m}}^{\frac{n}{m}})^{\frac{m}{n}}}$$

we have $\|\nabla^m v\|_{\frac{n}{m}}^{\frac{n}{m}} + \tau \|v\|_{\frac{n}{m}}^{\frac{n}{m}} = 1$ and $\|\nabla^m w\|_{\frac{n}{m}}^{\frac{n}{m}} + \mu \|w\|_{\frac{n}{m}}^{\frac{n}{m}} = 1$. Hence, (3-1) yields

$$\int_{\mathbb{R}^n} \Phi\left(\beta_0 |w|^{\frac{n}{n-m}}\right) dx \leq \frac{K_1}{\mu}. \tag{3-2}$$

Notice that by the choice of τ and μ, we get

$$\begin{cases} |u|^{\frac{n}{n-m}} \leq \left(1 - \alpha \|u\|_{\frac{n}{m}}^{\frac{n}{m}}\right)^{\frac{m}{n-m}} |v|^{\frac{n}{n-m}} \\ \\ |v|^{\frac{n}{n-m}} \leq \left(1 - \gamma\alpha \|v\|_{\frac{n}{m}}^{\frac{n}{m}}\right)^{\frac{m}{n-m}} |w|^{\frac{n}{n-m}}. \end{cases}$$

Thus

$$\begin{cases} \left(\frac{1+\alpha\|u\|_{\frac{n}{m}}^{\frac{m}{m}}}{1-\gamma\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}\right)^{\frac{m}{n-m}} |u|^{\frac{n}{n-m}} & \leq \left(\frac{1-\alpha^2\|u\|_{\frac{n}{m}}^{2\frac{m}{m}}}{1-\gamma\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}\right)^{\frac{m}{n-m}} |v|^{\frac{n}{n-m}} \\ \\ \left(\frac{1-\alpha^2\|u\|_{\frac{n}{m}}^{2\frac{m}{m}}}{1-\gamma\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}\right)^{\frac{m}{n-m}} |v|^{\frac{n}{n-m}} & \leq \left(\frac{(1-\alpha^2\|u\|_{\frac{n}{m}}^{2\frac{m}{m}})(1-\gamma\alpha\|v\|_{\frac{n}{m}}^{\frac{n}{m}})}{1-\gamma\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}\right)^{\frac{m}{n-m}} |w|^{\frac{n}{n-m}} \\ & \leq \left(\frac{(1-\alpha^2\|u\|_{\frac{n}{m}}^{2\frac{m}{m}})(1-\gamma\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}})}{1-\gamma\alpha\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}\right)^{\frac{m}{n-m}} |w|^{\frac{n}{n-m}} \\ & \leq |w|^{\frac{n}{n-m}}, \end{cases} \tag{3-3}$$

where we have used that

$$\|v\|_{\frac{n}{m}}^{\frac{n}{m}} = \frac{1 - \|\nabla^m v\|_{\frac{n}{m}}^{\frac{n}{m}}}{\tau} = \frac{1 - \frac{\|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}}}{\|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}} + \tau\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}}{\tau} = \|u\|_{\frac{n}{m}}^{\frac{n}{m}}.$$

Thus, (3-3) with (3-2) give us the desired result.

Finally, we recall that the constant β_0 is sharp: if $\beta > \beta_0$, then the supremum in the inequality becomes infinite. This follows from the classical constructions introduced by Adams [2] and further developed in the unbounded domains by Ruf-Sani [64], Lam-Lu [35] and Fontana-Morpurgo [22]. The argument is that the same sequence of test functions that establishes the sharpness of Adams' inequality in bounded domains also yields divergence in the unbounded setting, as noted for instance in Proposition 6.2 in [64]. Hence, the inequality holds precisely up to β_0.

CHAPTER 4

Compactness-concentrating-vanishing alternative: Subcritical Case

In this chapter, unless otherwise mentioned, we assume $n \geq 2m$, $0 \leq \alpha < 1$, $0 < \gamma < \frac{1}{\alpha} - 1$ for $\alpha > 0$, and the subcritical condition $0 < \beta < \beta_0$. By simplicity, for (u_j) sequence in $W^{m,\frac{n}{m}}(\mathbb{R}^n)$ with $\|u_j\|_{\frac{n}{m}} \leq 1$, we also will denote

$$\zeta_j = \zeta_j(u_j, m, n, \alpha, \gamma) := \left(\frac{1 + \alpha \|u_j\|_{\frac{n}{m}}^{\frac{n}{m}}}{1 - \gamma\alpha \|u_j\|_{\frac{n}{m}}^{\frac{n}{m}}} \right)^{\frac{m}{n-m}}. \tag{4-1}$$

4.1 Concentrating sequences

Lemma 4.1 *Let $(u_j) \subset W^{m,\frac{n}{m}}_{rad}(\mathbb{R}^n)$ with $\|\nabla^m u_j\|_{\frac{n}{m}}^{\frac{n}{m}} + \|u_j\|_{\frac{n}{m}}^{\frac{n}{m}} \leq 1$ be a concentrated sequence at the origin, i.e.*

$$\lim_{j\to\infty} \int_{B_R} |\nabla^m u_j|^{\frac{n}{m}} dx = 1 \quad \forall R > 0. \tag{4-2}$$

Then

$$\limsup_{j\to\infty} \int_{\mathbb{R}^n} \Phi\left(\beta\zeta_j |u_j|^{\frac{n}{n-m}}\right) dx = 0. \tag{4-3}$$

<u>**Proof:**</u> First, observe that we have $\lim_{j\to\infty} \|u_j\|_{\frac{n}{m}} \to 0$. Thus, $\zeta_j \to 1 \quad$ as $j \to \infty$. By Lemma 2.2 we can chose R such that $u_j(x) < 1$ for all $x \in \mathbb{R}^n \setminus B_R$ and any $j \in \mathbb{N}$. For $j \geq 1$, we set

$$I_{1_j} := \int_{B_R} \Phi\left(\beta\zeta_j |u_j|^{\frac{n}{n-m}}\right) dx$$

and

$$I_{2_j} := \int_{\mathbb{R}^n\setminus B_R} \Phi\left(\beta\zeta_j |u_j|^{\frac{n}{n-m}}\right) dx.$$

Then,

$$I_{2_j} = \int_{\mathbb{R}^n\setminus B_R} \Phi\left(\beta\zeta_j|u_j|^{\frac{n}{n-m}}\right)dx \le e^{\beta\zeta_j}\int_{\mathbb{R}^n\setminus B_R}|u_j|^{\frac{n}{n-m}\left(j\frac{n}{m}-1\right)}dx$$
$$\le e^{\beta\zeta_j}\int_{\mathbb{R}^n\setminus B_R}|u_j|^{\frac{n}{m}}\,dx \;\to 0,$$

as $j \to +\infty$. Now, to estimate I_{1_j}, we'll proceed as Sani, Ruf in [64] and Lam, Lu in [34]. Let us define

$$g_i(|x|) := \begin{cases} |x|^{m-2i}, & m = 2k, k\in\mathbb{N} \\ |x|^{m-1-2i}, & m = 2k+1, k\in\mathbb{N} \end{cases} \qquad \forall x \in B_R,$$

such that $g_i \in W^{m,\frac{n}{m}}_{rad}(B_R)$ and

$$\Delta^j g_i(|x|) = \begin{cases} c_i^j|x|^{m-2(i+j)}, & \text{for } j\in\{1,2,...,k-i\}, \text{ and } m \text{ even} \\ c_i^j|x|^{m-1-2(i+j)}, & \text{for } j\in\{1,2,...,k-i\}, \text{ and } m \text{ odd} \\ 0 & \text{for } j\in\{k-i+1,...,k\}, \end{cases}, \quad \forall x\in B_R,$$

where, for $j \in \{1,2,...,k-i\}$

$$c_i^j := \begin{cases} \prod_{h=1}^{j}[n+2k-2(h+i)][2h-2(i+h-1)], & \text{when } m \text{ is even} \\ \prod_{h=1}^{j}[n+2k-2-2(h+i-1)][2k-2(i+h-1)], & \text{when } m \text{ is odd.} \end{cases}$$

Now, we also define

$$v_j(|x|) := u_j(|x|) - \sum_{i=1}^{k-1} a_i g_i(|x|) - a_k, \quad \forall x\in B_R, \tag{4-4}$$

where

$$a_0 := \frac{\Delta u_j(R)}{\Delta^{k-i}g_0(R)},$$
$$a_i := \frac{\Delta^{k-i}u_j(R) - \sum_{l=1}^{i-1}a_l\Delta^{k-i}g_l(R)}{\Delta^{k-i}g_i(R)}, \quad \forall i\in\{1,2,...,k-1\},$$
$$a_k := u_j(R) - \sum_{i=1}^{k-1}a_i g_i(R).$$

Notice that by construction $v_j \in W^{m,\frac{n}{m}}_{\mathcal{N}}(B_R) \cap W^{m,\frac{n}{m}}_{rad}(B_R)$ and $\Delta^k v_j = \Delta^k u_j$ in B_R or equivalently $\nabla^m v_j = \nabla^m u_j$ in B_R. To simplify notation, we'll write

$$\tilde{u}_j(|x|) := \sum_{i=1}^{k-1} a_i g_i(|x|) + a_k,\ \forall x \in B_R$$

and $p' = \frac{n}{n-m}$. So, we can notice that $u_j(|x|) = v_j(|x|) + \tilde{u}_j(|x|)$. For sake of simplicity, let us consider $\tilde{u}_j(|x|) := \tilde{u}_j(R)$. We also know that

$$I_{1_j} := \int_{B_R} \Phi(\beta\zeta_j |u_j|^{\frac{n}{n-m}})\,dx \le \int_{B_R} (e^{\beta\zeta_j |u_j|^{\frac{n}{n-m}}} - 1)\,dx.$$

Then, the Adams Functional along u_j can be rewritten as

$$\int_{B_R} (e^{\beta\zeta_j |u_j|^{p'}} - 1)\,dx = \int_{B_R} (e^{\beta\zeta_j |v_j + \tilde{u}_j(R)|^{p'}} - 1)\,dx. \tag{4-5}$$

By construction, $v_j \in W_0^{1,\frac{n}{m}}(B_R)$ and $\|\Delta^k v_j\| = \|\Delta^k u_j\|$. Now, by the elementary inequality

$$(a+b)^p \le (1+\delta)^p a^p + \left(1 + \frac{1}{\delta}\right)^p b^p, \quad \text{for } p \ge 1,\ a,b > 0 \text{ and } \delta > 0 \tag{4-6}$$

and (4-5), it follows that

$$\begin{aligned}
\int_{B_R} (e^{\beta\zeta_j |v_j + \tilde{u}_j(R)|^{p'}} - 1)\,dx &\le \int_{B_R} \left(e^{\beta\zeta_j \left| (1+\delta)^{p'} (v_j)^{p'} + (1+\frac{1}{\delta})^{p'} (\tilde{u}_j(R))^{p'} \right|} - 1 \right) dx \\
&\le \int_{B_R} \left(e^{\beta\zeta_j \left(\left| (1+\delta)^{p'} (v_j)^{p'} \right| + \left| (1+\frac{1}{\delta})^{p'} (\tilde{u}_j(R))^{p'} \right| \right)} - 1 \right) dx \\
&\le \int_{B_R} e^{\beta\zeta_j \left| (1+\delta)^{p'} (v_j)^{p'} \right|} e^{\beta\zeta_j \left| (1+\frac{1}{\delta})^{p'} (\tilde{u}_j(R))^{p'} \right|}\,dx \\
&\le \int_{B_R} e^{\beta\zeta_j \left| (1+\delta)^{p'} (v_j)^{p'} \right|} e^{C(n,m,R)\frac{1}{R}\|u_j\|_{n/2}}\,dx \\
&\le e^{C(n,m,R)\frac{1}{R}\zeta_j \|u_j\|_{n/m}} \int_{B_R} e^{\beta\zeta_j \left| (1+\delta)^{p'} (v_j)^{p'} \right|}\,dx.
\end{aligned}$$

Then

$$\|\Delta^k v_j\|^{p'}_{L^{p'}(B_R)} \le \|\Delta^k u_j\|^{p'}_{L^{p'}(B_{\bar{R}})}.$$

Noticing that $e^{C(n,m,R)\frac{1}{R}\zeta_j \|u_j\|_{n/m}} \to 1$, and by Vitali's Convergence Theorem and by Tarsi [66] we got the result (4-3) taking limit as $j \to \infty$. ■

4.2 Compactness and vanish level estimates

Lemma 4.2 *Let* $(u_j) \subset W^{m,\frac{n}{m}}_{rad}(\mathbb{R}^n)$ *be a sequence satisfying* $\|u_j\|^{\frac{n}{m}}_{\frac{n}{m}} + \|\nabla^m u_j\|^{\frac{n}{m}}_{\frac{n}{m}} \leq 1$ *and* $\|u_j\|^{\frac{n}{m}}_{\frac{n}{m}} \to \theta$ *for some* $\theta \in [0,1)$. *Assume, up to a subsequence,* $u_j \rightharpoonup u$ *in* $W^{m,\frac{n}{m}}_{rad}(\mathbb{R}^n)$. *Then,*

1) *if* $\frac{n}{m}$ *is not integer,*

$$\lim_{j\to+\infty}\int_{\mathbb{R}^n}\Phi\left(\beta\zeta_j|u_j|^{\frac{n}{n-m}}\right)dx = \int_{\mathbb{R}^n}\Phi\left(\beta\left(\frac{1+\theta\alpha}{1-\theta\gamma\alpha}\right)^{\frac{n}{n-m}}|u|^{\frac{n}{n-m}}\right)dx. \tag{4-7}$$

2) *if* $\frac{n}{m}$ *is integer*

$$\lim_{j\to+\infty}\int_{\mathbb{R}^n}\Phi\left(\beta\zeta_j|u_j|^{\frac{m}{n-m}}\right)dx = \int_{\mathbb{R}^n}\Phi\left(\beta\left(\frac{1+\theta\alpha}{1-\theta\gamma\alpha}\right)^{\frac{m}{n-m}}|u|^{\frac{n}{n-m}}\right)dx \tag{4-8}$$
$$+\frac{\beta^{\frac{n-m}{m}}}{(\frac{n}{m}-1)!}\left(\frac{1+\theta\alpha}{1-\theta\gamma\alpha}\right)\left(\theta-\|u\|^{\frac{n}{m}}_{\frac{n}{m}}\right).$$

Proof: We divide the proof into two cases.

Case $\frac{n}{m}\notin\mathbb{Z}$: Note that

$$\zeta_j = \left(\frac{1+\alpha\|u_j\|^{\frac{n}{m}}_{\frac{n}{m}}}{1-\gamma\alpha\|u_j\|^{\frac{n}{m}}_{\frac{n}{m}}}\right)^{\frac{m}{n-m}} \to \zeta := \left(\frac{1+\alpha\theta}{1-\gamma\alpha\theta}\right)^{\frac{m}{n-m}}. \tag{4-9}$$

By Lemma 2.2 we can estimate as follows

$$\begin{aligned}\Phi(\beta\zeta_j|u_j|^{\frac{n}{n-m}}) &\leq e^{\beta\zeta_j}|u_j|^{(\frac{n}{m})j_{\frac{n}{m}}}\\ &\leq e^{\beta\zeta_j}\left(\left(\frac{1}{m\sigma_n}\right)^{\frac{m}{n}}\frac{1}{|x|^{(\frac{n-1}{n})m}}\|u_j\|_{W^{1,\frac{n}{m}}}\right)^{(\frac{n}{m})j_{\frac{n}{m}}}.\end{aligned}$$

Since that $j_{\frac{n}{m}}\geq\frac{n}{m}$,

$$\Phi(\beta\zeta_j|u_j|^{\frac{n}{n-m}}) \leq e^{\beta\zeta_j}\left(\frac{\left(\frac{1}{m\sigma_n}\right)^{\frac{n}{m}}\|u_j\|^{\frac{n^2}{m^2}}_{W^{1,\frac{n}{m}}}}{|x|^{\frac{n(n-1)}{m}}}\right).$$

Now, by integrating both sides outside the ball centered at origin with radius R and since $\zeta_j \geq 2$ as $j \to \infty$, we got

$$\int_{\mathbb{R}^n \setminus B_R} \Phi(\beta \zeta_j |u_j|^{\frac{n}{n-m}}) \mathrm{d}x \leq C(n,m,\beta) \int_{\mathbb{R}^n \setminus B_R} \frac{1}{|x|^{\frac{n(n-1)}{m}}} \mathrm{d}x,$$

using polar coordinates, there exists $\epsilon > 0$ such that, for a sufficiently large R and $n > m$ the integral in right side of the above inequality becomes

$$\begin{aligned}\int_{\mathbb{R}^n \setminus B_R} \frac{1}{|x|^{\frac{n(n-1)}{m}}} \mathrm{d}x &= \omega_{n-1} \int_R^{+\infty} \frac{r^{n-1}}{r^{\frac{n(n-1)}{m}}} \mathrm{d}r \\ &= \int_R^{+\infty} r^{\frac{-n^2+n-m+mn}{m}} \mathrm{d}r \\ &= \left[\frac{m}{-n^2+nm+1} \right] \frac{1}{R^{\frac{n^2-nm-1}{m}}} \\ &< \epsilon\end{aligned}$$

where ω_{n-1} denotes the surface area of the $(n-1)$-dimensional unit ball. Thus, we just proved that,

$$\int_{\mathbb{R}^n \setminus B_R} \Phi(\beta \zeta_j |u_j|^{\frac{n}{n-m}}) \mathrm{d}x \to 0.$$

In other words, the sequence (u_j) is tight on whole $\mathbb{R}^n$. Thus, using additionally the fact of the sequence be a uniformly integrable sequence over $\mathbb{R}^n$, applying again the Vitali's Convergence Theorem, we have proven the result.

Case $\frac{n}{m} \in \mathbb{Z}$ **:** Notice that we need to separate in two cases (integer and non-integer), because the Radial Lemma can't be efficently applied by the influence of the first term on the summation in integer case. We observe that

$$\begin{aligned}\Phi(\beta \zeta_j |u_j|^{\frac{n}{n-m}}) &= \exp(\beta \zeta_j |u_j|^{\frac{n}{n-m}}) - \sum_{i=0}^{j_{n,m}-2} \frac{(\beta \zeta_j |u_j|^{\frac{n}{n-m}})^i}{i!} \\ &= \frac{(\beta \zeta_j |u_j|^{\frac{n}{n-m}})^{j_{n,m}-1}}{(j_{n,m}-1)!} + \sum_{i=j_{n,m}}^{\infty} \frac{(\beta \zeta_j |u_j|^{\frac{n}{n-m}})^i}{i!}.\end{aligned}$$

Since $\frac{n}{m} \in \mathbb{Z}$, this implies $j_{n,m} = \frac{n}{m}$. So we obtain

$$\Phi(\beta \zeta_j |u_j|^{\frac{n}{n-m}}) = \frac{\beta^{\frac{n}{n-m}} \zeta_j^{\frac{n}{n-m}} |u_j|^{\frac{n}{m}}}{(\frac{n}{m}-1)!} + \sum_{i=\frac{n}{m}}^{\infty} \frac{(\beta \zeta_j |u_j|^{\frac{n}{n-m}})^i}{i!}.$$

Integrating both sides

$$\int_{\mathbb{R}^n} \Phi(\beta\zeta_j|u_j|^{\frac{n}{n-m}})dx = \int_{\mathbb{R}^n} \frac{\beta^{\frac{n}{n-m}}\zeta_j^{\frac{n}{n-m}}|u_j|^{\frac{n}{m}}}{(\frac{n}{m}-1)!}dx + \int_{\mathbb{R}^n}\sum_{i=\frac{n}{m}}^{\infty}\frac{(\beta\zeta_j|u_j|^{\frac{n}{n-m}})^i}{i!}dx$$
$$= \frac{\beta^{\frac{n}{n-m}}\zeta_j^{\frac{n}{n-m}}}{(\frac{n}{m}-1)!}\int_{\mathbb{R}^n}|u_j|^{\frac{n}{m}}dx + \int_{\mathbb{R}^n}\sum_{i=\frac{n}{m}}^{\infty}\frac{(\beta\zeta_j|u_j|^{\frac{n}{n-m}})^i}{i!}dx$$
$$= \frac{\beta^{\frac{n}{n-m}}\zeta_j^{\frac{n}{n-m}}}{(\frac{n}{m}-1)!}\|u_j\|_{\frac{n}{m}}^{\frac{n}{m}} + \int_{\mathbb{R}^n}\sum_{i=\frac{n}{m}}^{\infty}\frac{(\beta\zeta_j|u_j|^{\frac{n}{n-m}})^i}{i!}dx$$

Since $\|u_j\|_{\frac{n}{m}}^{\frac{n}{m}} \to \theta$ and by (4-9), taking limit as $j \to \infty$.

$$= \frac{\beta^{\frac{n}{n-m}}}{(\frac{n}{m}-1)!}\left(\frac{1+\alpha\theta}{1-\gamma\alpha\theta}\right)\theta + \lim_{j\to\infty}\int_{\mathbb{R}^n}\sum_{i=\frac{n}{m}}^{\infty}\frac{(\beta\zeta_j|u_j|^{\frac{n}{n-m}})^i}{i!}dx.$$

Using the same argument as in non-integer case, we can prove that the sequence $(u_j) \subset W_{rad}^{n,\frac{n}{m}}(\mathbb{R}^n)$ is uniformly integrable and tight on whole $\mathbb{R}^n$, by the Vitali's convergence Theorem, we can change the sign of the limit with the integral on the right side, and one can get

$$= \frac{\beta^{\frac{n}{n-m}}}{(\frac{n}{m}-1)!}\left(\frac{1+\alpha\theta}{1-\alpha\gamma\theta}\right)\theta + \int_{\mathbb{R}^n}\sum_{i=\frac{n}{m}}^{\infty}\frac{(\beta\zeta|u|^{\frac{n}{n-m}})^i}{i!}dx$$
$$= \frac{\beta^{\frac{n}{n-m}}}{(\frac{n}{m}-1)!}\left(\frac{1+\alpha\theta}{1-\alpha\gamma\theta}\right)\theta + \int_{\mathbb{R}^n}\left[\sum_{i=\frac{n}{m}}^{\infty}\frac{(\beta\zeta|u|^{\frac{n}{n-m}})^i}{i!} + \frac{\beta^{\frac{n}{n-m}}\zeta^{\frac{n}{n-m}}|u|^{\frac{n}{m}}}{(\frac{n}{m}-1)!} - \frac{\beta^{\frac{n}{n-m}}\zeta^{\frac{n}{n-m}}|u|^{\frac{n}{m}}}{(\frac{n}{m}-1)!}\right]dx$$
$$= \frac{\beta^{\frac{n}{n-m}}}{(\frac{n}{m}-1)!}\left(\frac{1+\alpha\theta}{1-\alpha\gamma\theta}\right)\theta + \int_{\mathbb{R}^n}\left[\sum_{i=\frac{n}{m}-1}^{\infty}\frac{(\beta\zeta|u|^{\frac{n}{n-m}})^i}{i!} - \frac{\beta^{\frac{n}{n-m}}\zeta^{\frac{n}{n-m}}|u|^{\frac{n}{m}}}{(\frac{n}{m}-1)!}\right]dx$$
$$= \frac{\beta^{\frac{n}{n-m}}}{(\frac{n}{m}-1)!}\left(\frac{1+\alpha\theta}{1-\alpha\gamma\theta}\right)\theta + \int_{\mathbb{R}^n}\Phi(\beta\zeta|u_j|^{\frac{n}{n-m}})dx - \int_{\mathbb{R}^n}\frac{\beta^{\frac{n}{n-m}}\zeta^{\frac{n}{n-m}}|u|^{\frac{n}{m}}}{(\frac{n}{m}-1)!}dx$$
$$= \frac{\beta^{\frac{n}{n-m}}}{(\frac{n}{m}-1)!}\left(\frac{1+\alpha\theta}{1-\alpha\gamma\theta}\right)\theta + \int_{\mathbb{R}^n}\Phi(\beta\zeta|u|^{\frac{n}{n-m}})dx - \frac{\beta^{\frac{n}{n-m}}}{(\frac{n}{m}-1)!}\left(\frac{1+\alpha\theta}{1-\alpha\gamma\theta}\right)\int_{\mathbb{R}^n}|u|^{\frac{n}{m}}dx$$
$$= \frac{\beta^{\frac{n}{n-m}}}{(\frac{n}{m}-1)!}\left(\frac{1+\alpha\theta}{1-\alpha\gamma\theta}\right)(\theta - \|u\|_{\frac{n}{m}}^{\frac{n}{m}}) + \int_{\mathbb{R}^n}\Phi(\beta\zeta|u|^{\frac{n}{n-m}})dx$$

we got the desired result. ■

Remark 4.2.1 *The Lemma 4.2 works for* $\beta = \beta_0$ *if* $\alpha, \theta > 0$.

Proposition 4.3 *Let* $0 < \alpha < 1$, $0 \le \gamma < \frac{1}{\alpha} - 1$ *and* $AD(n,m,\beta,\alpha,\gamma)$ *as in* (1-14). *For* $n > m$*, then*

$$AD(n,m,\beta,\alpha,\gamma) > \frac{\beta^{\frac{n}{m}-1}}{\left(\frac{n}{m}-1\right)!}\left(\frac{1+\alpha}{1-\gamma\alpha}\right) \tag{4-10}$$

for $\beta \in (0,\beta_0]$ *when* $\frac{n}{m} \ge 3$ *and for* $\beta \in \left(\frac{1+2\alpha-\gamma\alpha^2}{1+\alpha(1-\gamma)-\gamma\alpha^2}\frac{2}{\mathscr{B}_{GN}},\beta_0\right]$ *with* $0 \le \gamma < \frac{m(1+\alpha)-1-2\alpha}{m\alpha^2+\alpha m-\alpha^2}$ *when* $\frac{n}{m} = 2$.

Proof: First, we introduce the following operator in $W^{m,n/m}(\mathbb{R}^n)$. Let $t > 0$, we define $H_t : W^{m,n/m}(\mathbb{R}^n) \to W^{m,n/m}(\mathbb{R}^n)$ by

$$H_t(u)(x) := t^{\frac{m}{n}} u(t^{\frac{1}{n}}x). \tag{4-11}$$

So, we can compute

$$\|H_t(u)\|_{\frac{n}{m}}^{\frac{n}{m}} + \|\nabla^m H_t(u)\|_{\frac{n}{m}}^{\frac{n}{m}} = \left\|t^{\frac{m}{n}} u\left(t^{\frac{1}{n}}x\right)\right\|_{\frac{n}{m}}^{\frac{n}{m}} + \left\|t^{\frac{m}{n}}\nabla^m\left(u\left(t^{\frac{1}{n}}x\right)\right)\right\|_{\frac{n}{m}}^{\frac{n}{m}}$$
$$= \|u(x)\|_{\frac{n}{m}}^{\frac{n}{m}} + t\|\nabla^m(u(x))\|_{\frac{n}{m}}^{\frac{n}{m}},$$

and

$$\frac{\|H_t(u)\|_{\frac{n}{m}}^{\frac{n}{m}}}{\|H_t(u)\|_{\frac{n}{m}}^{\frac{n}{m}} + \|\nabla^m H_t(u)\|_{\frac{n}{m}}^{\frac{n}{m}}} = \frac{\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{\|u(x)\|_{\frac{n}{m}}^{\frac{n}{m}} + t\|\nabla^m(u(x))\|_{\frac{n}{m}}^{\frac{n}{m}}}.$$

By simplicity, we denote

$$\eta_u(t) = \|u\|_{\frac{n}{m}}^{\frac{n}{m}} + t\|\nabla^m u\|_{\frac{n}{m}}^{\frac{n}{m}} \tag{4-12}$$

and

$$\rho(t) := \rho(t,\alpha,\gamma) = \frac{1+\alpha t}{1-\gamma\alpha t}, \quad \text{for all } t \in [0,1]. \tag{4-13}$$

Note that $\rho(t)$ is an increasing function with $1 \le \rho(t) \le (1+\alpha)/(1-\gamma\alpha)$ for $t \in [0,1]$.

Now, since

$$\Phi(\beta t) \ge \frac{\beta^{j_{\frac{n}{m}}-1}}{\left(j_{\frac{n}{m}}-1\right)!}t^{j_{\frac{n}{m}}-1} + \frac{\beta^{j_{\frac{n}{m}}}}{\left(j_{\frac{n}{m}}\right)!}t^{j_{\frac{n}{m}}} \text{ and } j_{\frac{n}{m}} \ge \frac{n}{m}$$

we can obtain

$$\begin{aligned}
&\Phi\left(\beta\left(\frac{1+\alpha\frac{\|H_t(u)\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}}{1-\gamma\alpha\frac{\|H_t(u)\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}}\right)^{\frac{m}{n-m}}\frac{|H_t(u)|^{\frac{n}{n-m}}}{(\eta_u(t))^{\frac{m}{n-m}}}\right)=\Phi\left(\beta\left(\rho\left(\frac{\|H_t(u)\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}\right)\right)^{\frac{m}{n-m}}\frac{|H_t(u)|^{\frac{n}{n-m}}}{(\eta_u(t))^{\frac{m}{n-m}}}\right)\\
&\geq\frac{\beta^{\frac{n}{m}-1}}{(\frac{n}{m}-1)!}\left[\left(\rho\left(\frac{\|H_t(u)\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}\right)\right)^{\frac{m}{n-m}}\frac{|H_t(u)|^{\frac{n}{n-m}}}{(\eta_u(t))^{\frac{m}{n-m}}}\right]^{\frac{n-m}{m}}\\
&+\frac{\beta^{\frac{n}{m}}}{(\frac{n}{m})!}\left[\left(\rho\left(\frac{\|H_t(u)\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}\right)\right)^{\frac{m}{n-m}}\frac{|H_t(u)|^{\frac{n}{n-m}}}{(\eta_u(t))^{\frac{m}{n-m}}}\right]^{\frac{n}{m}}.
\end{aligned}\tag{4-14}$$

Since $\|H_t(u)\|_q = t^{\frac{qm-n}{qn}}\|u\|_q$, we have

$$\rho\left(\frac{\|H_t(u)\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}\right)=\rho\left(\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}\right).$$

Hence, by integrating in (4-14) and taking the supremum

$$\begin{aligned}
AD(n,m,\beta,\alpha,\gamma)&\geq\frac{\beta^{\frac{n}{m}-1}}{(\frac{n}{m}-1)!}\rho\left(\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}\right)\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta(t)}+\frac{\beta^{\frac{n}{m}}}{(\frac{n}{m})!}\left[\rho\left(\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}\right)\right]^{\frac{m}{n-m}}\frac{t^{\frac{m}{n-m}}\|u\|_{\frac{n^2}{(n-m)m}}^{\frac{n^2}{(n-m)m}}}{\eta(t)^{\frac{n}{n-m}}}\\
&\geq\frac{\beta^{\frac{n}{m}-1}}{(\frac{n}{m}-1)!}\left[\rho\left(\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}\right)\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}+\frac{\beta}{\frac{n}{m}}\left[\rho\left(\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}\right)\right]^{\frac{m}{n-m}}\frac{t^{\frac{m}{n-m}}\|u\|_{\frac{n^2}{(n-m)m}}^{\frac{n^2}{(n-m)m}}}{\eta_u(t)^{\frac{n}{n-m}}}\right].
\end{aligned}$$

Let us define

$$h(t):=\rho\left(\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}\right)\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}+\frac{\beta}{\frac{n}{m}}\left[\rho\left(\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}\right)\right]^{\frac{m}{n-m}}\frac{t^{\frac{m}{n-m}}\|u\|_{\frac{n^2}{(n-m)m}}^{\frac{n^2}{(n-m)m}}}{\eta_u(t)^{\frac{n}{n-m}}},$$

which we will also denote in terms of its components as

$$h(t)=f(t)+\frac{\beta}{\frac{n}{m}}t^{\frac{m}{n-m}}g(t),$$

where

$$g(t) := \left[\rho\left(\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{m}}}{\eta_u(t)}\right)\right]^{\frac{m}{n-m}} \frac{\|u\|_{\frac{n^2}{(n-m)m}}^{\frac{n^2}{(n-m)m}}}{\eta_u(t)^{\frac{n}{n-m}}}.$$

Thus, we write

$$AD(n,m,\beta,\alpha,\gamma) \geq \frac{\beta^{\frac{n}{m}-1}}{(\frac{n}{m}-1)!}h(t). \tag{4-15}$$

Note that

$$\eta_u(0) = \|u\|_{\frac{n}{m}}^{\frac{n}{m}}, \ \ \rho(1) = \frac{1+\alpha}{1-\gamma\alpha} \ \text{ and } \ h(0) = \rho(1). \tag{4-16}$$

Thus, by taking into account (4-15), for $\frac{n}{m} \geq 3$ we need to prove that $h(t)$ is a increasing function for t near to 0. Indeed, we have

$$h'(t) = f'(t) + \frac{\beta}{\frac{n}{m}}\left(\frac{m}{n-m}\right)t^{\frac{2m-n}{n-m}}g(t) + \frac{\beta}{\frac{n}{m}}t^{\frac{m}{n-m}}g'(t).$$

A direct calculation shows that both f' and g' are bounded for t near 0. Thus, since $g(0) > 0$ and $t^{\frac{2m-n}{n-m}} \to +\infty$ as $t \to 0$, we obtain $h'(t) > 0$ for t near 0 which gives the desired result.

Now, suppose $\frac{n}{m} = 2$. In this case, we have

$$\rho\left(\frac{\|u\|_2^2}{\eta_u(t)}\right) = \frac{1 + \frac{\alpha\|u\|_2^2}{\eta_u(t)}}{1 - \frac{\gamma\alpha\|u\|_2^2}{\eta_u(t)}}$$

$$f(t) = \rho\left(\frac{\|u\|_2^2}{\eta_u(t)}\right)\frac{\|u\|_2^2}{\eta_u(t)}$$

$$g(t) = \rho\left(\frac{\|u\|_2^2}{\eta_u(t)}\right)\frac{\|u\|_4^4}{\eta_u(t)^2}.$$

By noticing that

$$\rho_0 := \frac{d}{dt}\left[\rho\left(\frac{\|u\|_2^2}{\eta_u(t)}\right)\right]\Bigg|_{t=0} = -\frac{\alpha+\gamma\alpha}{(1-\gamma\alpha)^2}\frac{\|\nabla^m u\|_2^2}{\|u\|_2^2} \tag{4-17}$$

and taking into account (4-16), we obtain

$$\begin{aligned}
f(0) &= \frac{1+\alpha}{1-\gamma\alpha}\\
f'(0) &= \rho_0 - \frac{1+\alpha}{1-\gamma\alpha}\frac{\|\nabla^m u\|_2^2}{\|u\|_2^2} = -\frac{1+2\alpha-\gamma\alpha^2}{(1-\gamma\alpha)^2}\frac{\|\nabla^m u\|_2^2}{\|u\|_2^2}\\
g(0) &= \frac{1+\alpha}{1-\gamma\alpha}\frac{\|u\|_4^4}{\|u\|_2^4} = \frac{1-\gamma\alpha+\alpha-\gamma\alpha^2}{(1-\gamma\alpha)^2}\frac{\|u\|_4^4}{\|u\|_2^4}\\
g'(0) &= \rho_0\frac{\|u\|_4^4}{\|u\|_2^4} - \frac{1+\alpha}{1-\gamma\alpha}\frac{\|u\|_4^4}{\|u\|_2^4}\frac{\|\nabla^m u\|_2^2}{\|u\|_2^2} = -\frac{1+2\alpha-\gamma\alpha^2}{(1-\gamma\alpha)^2}\frac{\|u\|_4^4}{\|u\|_2^4}\frac{\|\nabla^m u\|_2^2}{\|u\|_2^2}.
\end{aligned} \tag{4-18}$$

If $n = 2m$, we have $h(t) = f(t) + \frac{\beta}{2}tg(t)$ and, thus $h'(0) = f'(0) + \frac{\beta}{2}g(0)$ and $h(0) = f(0)$. So, the Taylor expansion and (4-18) yield

$$\begin{aligned}
h(t) &= f(0) + (f'(0) + \frac{\beta}{2}g(0))t + O(t^2)\\
&= \frac{1+\alpha}{1-\gamma\alpha} + O(t^2)\\
&+ \frac{1}{(1-\gamma\alpha)^2}\frac{\|u\|_4^4}{\|u\|_2^4}\left[-(1+2\alpha-\gamma\alpha^2)\frac{\|\nabla^m u\|_2^2\|u\|_2^2}{\|u\|_4^4} + \frac{\beta}{2}(1-\gamma\alpha+\alpha-\gamma\alpha^2)\right]t\\
&= \frac{1+\alpha}{1-\gamma\alpha} + O(t^2)\\
&+ \frac{(1+\alpha)(1-\gamma\alpha)}{2(1-\gamma\alpha)^2}\frac{\|u\|_4^4}{\|u\|_2^4}\left(\beta - \frac{\|\nabla^m u\|_2^2\|u\|_2^2}{\|u\|_4^4}\frac{2(1+2\alpha-\gamma\alpha^2)}{1+\alpha-\gamma\alpha-\gamma\alpha^2}\right)t.
\end{aligned}$$

Therefore, by estimate (4-15) and expanding h in MacLaurin series, we obtain

$$\begin{aligned}
AD(2m,m,\beta,\alpha,\gamma) &\geq \beta\left(\frac{1+\alpha}{1-\gamma\alpha}\right)\\
&+ \frac{\beta^2}{2}\frac{1+\alpha}{1-\gamma\alpha}\frac{\|u\|_4^4}{\|u\|_2^4}\left(1 - \frac{2}{\beta}\frac{\|\nabla^m u\|_2^2\|u\|_2^2}{\|u\|_4^4}\frac{1+2\alpha-\gamma\alpha^2}{1+\alpha-\gamma\alpha-\gamma\alpha^2}\right)t + O(t^2).
\end{aligned}$$

By the arbitrariness of u, taking into account (1-15) and (2-12), we conclude the result.
■

CHAPTER 5

Existence and non-existence of extremals for the subcritical Adams functional of Adimurthi-Druet type in even dimension

In this chapter we concerned to prove both Theorem 1.2 and Theorem 1.3. In fact, we will study the attainability and the non-attainability of $AD(2m, m, \beta, \alpha, \gamma)$ in the subcritical case $\beta < \beta_0$. Let us recall the Adams functional

$$F_{2m,m,\beta,\alpha,\gamma}(u) = \int_{\mathbb{R}^{2m}} \left(e^{\beta\rho(\|u\|_2^2)u^2} - 1 \right) dx. \tag{5-1}$$

where $\rho(\|u\|_2^2) = \rho(\|u\|_2^2, \alpha, \gamma)$ is given by (4-13).

5.1 Proof of Theorem 1.2: Attainability in $n = 2m$ case

Taking into account (2-6), we can choose $(u_j) \in W^{m,2}_{rad}(\mathbb{R}^{2m})$ with $\|u_j\|_{W^{m,2}(\mathbb{R}^{2m})} = 1$ radially symmetric maximizing sequence for $AD(2m, m, \beta, \alpha, \gamma)$, that is,

$$\|\nabla^m u_j\|_2^2 + \|u_j\|_2^2 = 1, \quad \lim_{j\to\infty} F_{2m,m,\beta,\alpha,\gamma}(u_j) = \sup_{\substack{u\in W^{m,2}(\mathbb{R}^{2m}) \\ \|\nabla^m u\|_2^2+\|u\|_2^2\leq 1}} F_{2m,m,\beta,\alpha,\gamma}(u). \tag{5-2}$$

Up to a subsequence, we can assume that $u_j \rightharpoonup u_0$ weakly in $W^{m,2}_{rad}(\mathbb{R}^{2m})$, $\|u_j\|_2^2 \to \theta_0$ and $\|\nabla^m u_j\|_2^2 \to \theta_1$ for some $\theta_0, \theta_1 \in [0,1]$ such that $\theta_0 + \theta_1 := \overline{\theta} \leq 1$. From Lemma 4.2-(4-8), we derive

$$\lim_{j\to+\infty} \int_{\mathbb{R}^{2m}} \left(e^{\beta\rho(\|u_j\|_2^2)u_j^2} - 1 \right) dx = \int_{\mathbb{R}^{2m}} \left(e^{\beta\rho(\theta_0)u_0^2} - 1 \right) dx + \beta\rho(\theta_0)\left(\theta_0 - \|u_0\|_2^2\right). \tag{5-3}$$

If $u_0 \equiv 0$, combing (5-2) with (5-3),

$$AD(2m, m, \beta, \alpha, \gamma) = \beta\rho(\theta_0)\theta_0 \leq \beta\left(\frac{1+\alpha}{1-\gamma\alpha}\right),$$

which is impossible due to Proposition 4.3. Then, we can suppose that $u_0 \neq 0$. By the lower semicontinuity of the norm $\|u_0\|_2^2 \leq \liminf_{j\to\infty} \|u_j\|_2^2 = \theta_0$. So, let us define $\tau \geq 1$ and v given by

$$\tau^{2m} = \frac{\theta_0}{\|u_0\|_2^2} \quad \text{and} \quad v(x) = u_0\left(\frac{x}{\tau}\right). \tag{5-4}$$

It follows that

$$\|v\|_2^2 = \tau^{2m}\|u_0\|_2^2 = \theta_0 \quad \text{and} \quad \|\nabla^m v\|_2^2 = \|\nabla^m u_0\|_2^2. \tag{5-5}$$

Then, by lower semicontinuity again

$$\|\nabla^m v\|_2^2 + \|v\|_2^2 = \|\nabla^m u_0\|_2^2 + \tau^{2m}\|u_0\|_2^2 \leq \liminf_{j\to+\infty} \|\nabla^m u_j\|_2^2 + \theta_0 = \theta_1 + \theta_0 \leq 1.$$

Hence, from (5-5)

$$\begin{aligned}
AD(2m,m,\beta,\alpha,\gamma) &\geq \int_{\mathbb{R}^{2m}} \Phi\left(\beta\rho(\|v\|_2^2)v^2\right) dx = \tau^{2m}\int_{\mathbb{R}^{2m}} \left(e^{\beta\rho(\theta_0)u_0^2} - 1\right) dx \\
&= \int_{\mathbb{R}^{2m}} \left(e^{\beta\rho(\theta_0)u_0^2} - 1\right) dx + (\tau^{2m} - 1)\beta\rho(\theta_0)\|u_0\|_2^2 \\
&+ (\tau^{2m} - 1)\int_{\mathbb{R}^{2m}} \left[e^{\beta\rho(\theta_0)u_0^2} - 1 - \beta\rho(\theta_0)u_0^2\right] dx \\
&= \int_{\mathbb{R}^{2m}} \left(e^{\beta\rho(\theta_0)u_0^2} - 1\right) dx + \beta\rho(\theta_0)(\theta_0 - \|u_0\|_2^2) \\
&+ (\tau^{2m} - 1)\int_{\mathbb{R}^{2m}} \left[e^{\beta\rho(\theta_0)u_0^2} - 1 - \beta\rho(\theta_0)u_0^2\right] dx.
\end{aligned}$$

By (5-2) and (5-3), we obtain

$$AD(2m,m,\beta,\alpha,\gamma) \geq AD(2m,m,\beta,\alpha,\gamma) + (\tau^{2m} - 1)\int_{\mathbb{R}^{2m}} \left[e^{\beta\rho(\theta_0)u_0^2} - 1 - \beta\rho(\theta_0)u_0^2\right] dx.$$

Since $u_0 \neq 0$ we need to have $\tau^{2m} = 1$ or equivalently $\theta_0 = \|u_0\|_2^2$. Therefore, by (5-2) and (5-3) again we have

$$AD(2m,m,\beta,\alpha,\gamma) = \int_{\mathbb{R}^{2m}} \left(e^{\beta\rho(\|u_0\|_2^2)u_0^2} - 1\right) dx.$$

Since $\|\nabla^m u_0\|_2^2 + \|u_0\|_2^2 \leq \liminf_{j\to\infty}(\|\nabla^m u_j\|_2^2 + \|u_j\|_2^2) \leq 1$, we get that u_0 maximizes $AD(2m,m,\beta,\alpha,\gamma)$.

5.2 Proof of Theorem 1.3: Non-attainability in $\frac{n}{m}=2$ case

This section is devoted to prove that $AD(2m,m,\beta,\alpha,\gamma)$ is not attained for β sufficiently small. We will proceed analogously Ishiwata and Nguyen in [56], [30]. Let $\beta<(4\pi)^m m!\left(\frac{1-\alpha}{2+2\alpha}\right)$. Firstly, by (2-10) with $n=2m$, we derive

$$\int_{\mathbb{R}^{2m}}\left[\exp\left(\beta\frac{|u|^2}{\|\nabla^m u\|_2^2}\right)-1\right]dx\leq C_{\beta,m,2m}\frac{\|u\|_2^2}{\|\nabla^m u\|_2^2},\quad \forall\, u\in W^{m,2}_{rad}(\mathbb{R}^{2m})\setminus\{0\}. \tag{5-6}$$

From this we can observe that for any fixed $\tilde{\beta}<(4\pi)^m m!$,

$$\frac{\tilde{\beta}^j}{j!}\frac{\|u\|_{2j}^{2j}}{\|\nabla^m u\|_2^{2j}}\leq C_{2m,m,\tilde{\beta}}\frac{\|u\|_2^2}{\|\nabla^m u\|_2^2},\ \text{ for any } j\geq 1. \tag{5-7}$$

Let $\mathscr{H}=\{u\in W^{m,2}(\mathbb{R}^{2m})\ :\ \|u\|_2^2+\|\nabla^m u\|_2^2=1\}$ and let $v\in\mathscr{H}$. By using the same notation in (4-11) and (4-12), for $t>0$ we define the family of functions

$$v_t:=H_t(v)(x)=t^{\frac{1}{2}}v(t^{\frac{1}{2m}}x)\quad\text{and } w_t=\frac{v_t}{\|v_t\|_2^2+\|\nabla^m v_t\|_2^2}=\frac{v_t}{\eta_v(t)}$$

where we have used that $\|v_t\|_2^2=\|v\|_2^2$ and $\|\nabla^m v_t\|_2^2=t\|\nabla^m v\|_2^2$. If $v\in W^{m,2}_{rad}(\mathbb{R}^{2m})$ is a maximizer for $AD(2m,m,\beta,\alpha,\gamma)$, then $v\in\mathscr{H}\cap W^{m,2}_{rad}(\mathbb{R}^{2m})$ and each w_t is a curve in $W^{m,2}_{rad}(\mathbb{R}^{2m})\cap\mathscr{H}$ such that $w_1=v$, and

$$\frac{d}{dt}F_{2m,m,\beta,\alpha,\gamma}(w_t)\Big|_{t=1}=0. \tag{5-8}$$

The idea is to show that the identity (5-8) does not occur for any $v\in W^{m,2}_{rad}(\mathbb{R}^{2m})\cap\mathscr{H}$, if β is sufficiently small. This leads to the non-existence of a maximizer v. Using the relations $\|v_t\|_{2k}^{2k}=t^{k-1}\|v\|_{2k}^{2k}$ and $\|\nabla^m v_t\|_2^2=t\|\nabla^m v\|_2^2$, we obtain

$$F_{2m,m,\beta,\alpha,\gamma}(w_t)=\sum_{k=1}^{\infty}\frac{\beta^k}{k!}\|v\|_{2k}^{2k}\left[\rho\left(\frac{\|v\|_2^2}{\eta_v(t)}\right)\right]^k\frac{t^{k-1}}{\eta_v^k(t)}$$

where ρ is given by (4-13). Now, we note that

$$\begin{aligned}
&\eta_v(1)=\|\nabla^m v\|_2^2+\|v\|_2^2=1\\
&\rho\left(\frac{\|v\|_2^2}{\eta_v(1)}\right)=\rho(\|v\|_2^2)=\frac{1+\alpha\|v\|_2^2}{1-\gamma\alpha\|v\|_2^2}\\
&\rho_1:=\frac{d}{dt}\left[\rho\left(\frac{\|v\|_2^2}{\eta_v(t)}\right)\right]\Bigg|_{t=1}=-\frac{\|v\|_2^2\|\nabla^m v\|_2^2}{(1-\gamma\alpha\|v\|_2^2)^2}\left[\alpha(1-\gamma\alpha\|v\|_2^2)+\gamma\alpha(1+\alpha\|v\|_2^2)\right]\\
&\nu_k:=\frac{d}{dt}\left[\frac{t^{k-1}}{\eta_v^k(t)}\right]\Bigg|_{t=1}=(k-1)-k\|\nabla^m v\|_2^2.
\end{aligned} \tag{5-9}$$

Hence, the identities in (5-9) yield

$$\frac{d}{dt}\left[F_{2m,m,\beta,\alpha,\gamma}(w_t)\right]|_{t=1} = \sum_{k=1}^{\infty}\frac{\beta^k}{k!}\|v\|_{2k}^{2k}\left[\rho(\|v\|_2^2)\right]^k\left[\frac{k\rho_1}{\rho(\|v\|_2^2)}+\nu_k\right]$$
$$\leq \sum_{k=1}^{\infty}\frac{\beta^k}{k!}\|v\|_{2k}^{2k}\left[\rho(\|v\|_2^2)\right]^k\nu_k,$$

where we have used that $k\rho_1/\rho(\|v\|_2^2)<0$, which is also a consequence of (5-9). Now, since $\nu_1=-\|\nabla^m v\|_2^2$ and $\nu_k\leq k$ we can derive

$$\sum_{k=1}^{\infty}\frac{\beta^k}{k!}\|v\|_{2k}^{2k}\left[\rho(\|v\|_2^2)\right]^k\nu_k = -\beta\|v\|_2^2\|\nabla^m v\|_2^2\rho(\|v\|_2^2)+\sum_{k=2}^{\infty}\frac{\beta^k}{k!}\|v\|_{2k}^{2k}\left[\rho(\|v\|_2^2)\right]^k\nu_k$$
$$\leq -\beta\|v\|_2^2\|\nabla^m v\|_2^2\rho(\|v\|_2^2)+\sum_{k=2}^{\infty}\frac{\beta^k}{k!}\|v\|_{2k}^{2k}\left[\rho(\|v\|_2^2)\right]^k k$$
$$= \beta\|v\|_2^2\|\nabla^m v\|_2^2\rho(\|v\|_2^2)\left[\sum_{k=2}^{\infty}\frac{\beta^{k-1}}{(k-1)!}\left[\rho(\|v\|_2^2)\right]^{k-1}\frac{\|v\|_{2k}^{2k}}{\|v\|_2^2\|\nabla^m v\|_2^2}-1\right].$$

By using $\rho(\|v\|_2^2)\leq(1+\alpha)/(1-\gamma\alpha)$ we obtain

$$\frac{d}{dt}\left[F_{2m,m,\beta,\alpha,\gamma}(w_t)\right]|_{t=1}$$
$$\leq \beta\|v\|_2^2\|\nabla^m v\|_2^2\rho(\|v\|_2^2)\left[\sum_{k=2}^{\infty}\frac{\beta^{k-1}}{(k-1)!}\left(\frac{1+\alpha}{1-\gamma\alpha}\right)^{k-1}\frac{\|v\|_{2k}^{2k}}{\|v\|_2^2\|\nabla^m v\|_2^2}-1\right].$$

From inequality (5-7) and since $\|\nabla^m v\|_2^2\leq 1$, for some fixed $\tilde{\beta}<(4\pi)^m m!$ such that $\beta/\tilde{\beta}<\frac{1-\alpha}{2+2\alpha}$ we obtain

$$\frac{\beta^{k-1}}{(k-1)!}\frac{\|v\|_{2k}^{2k}}{\|v\|_2^2\|\nabla^m v\|_2^2} = \frac{k}{\tilde{\beta}}\left(\frac{\beta}{\tilde{\beta}}\right)^{k-1}\left[\frac{\tilde{\beta}^k}{k!}\frac{\|v\|_{2k}^{2k}}{\|\nabla^m v\|_2^{2k}}\right]\frac{\|\nabla^m v\|_2^{2k-2}}{\|v\|_2^2}$$
$$\leq \frac{k}{\tilde{\beta}}\left(\frac{\beta}{\tilde{\beta}}\right)^{k-1}\|\nabla^m v\|_2^{2k-4}C_{2m,m,\tilde{\beta}}$$
$$\leq \frac{k}{\tilde{\beta}}\left(\frac{\beta}{\tilde{\beta}}\right)^{k-1}C_{2m,m,\tilde{\beta}},$$

for all $k\geq 2$. Consequently, we derive

$$\frac{d}{dt}\left[F_{2m,m,\beta,\alpha,\gamma}(w_t)\right]|_{t=1}\leq \beta\|v\|_2^2\|\nabla^m v\|_2^2\rho(\|v\|_2^2)\left[\sum_{k=2}^{\infty}\frac{k}{\tilde{\beta}}\left(\left(\frac{1+\alpha}{1-\gamma\alpha}\right)\frac{\beta}{\tilde{\beta}}\right)^{k-1}C_{2m,m,\tilde{\beta}}-1\right]$$
$$\leq \beta\|v\|_2^2\|\nabla^m v\|_2^2\rho(\|v\|_2^2)$$
$$\cdot\left[\beta\left(\frac{1+\alpha}{1-\gamma\alpha}\right)\frac{1}{\tilde{\beta}^2}\sum_{k=2}^{\infty}k\left(\frac{1}{2}\right)^{k-2}C_{2m,m,\tilde{\beta}}-1\right].$$

Thus, for $\beta < \min\left\{\left(\left(\frac{1+\alpha}{1-\gamma\alpha}\right)\frac{1}{\tilde{\beta}^2}\sum_{k=2}^{\infty}k\left(\frac{1}{2}\right)^{k-2}C_{2m,m,\tilde{\beta}}\right)^{-1}, \tilde{\beta}/2\right\}$, we have

$$\frac{d}{dt}\left[F_{2m,m,\beta,\alpha,\gamma}(w_t)\right]|_{t=1} < 0, \text{ for any } v \in W^{m,2}_{rad}(\mathbb{R}^{2m}) \cap \mathscr{H}.$$

From (5-8), we conclude that $AD(2m, m, \beta, \alpha, \gamma)$ does not admit maximizers if β is chosen as above.

CHAPTER 6

Critical Case on Fourth Dimension

In this chapter, we apply blow-up analysis together together a truncation argument by DelaTorre-Mancini [16] and some ideas by Chen-Lu-Zhu [10], to obtain the attainability stated in Theorem 1.4 under the critical regime $\beta = 32\pi^2$. Let (u_j) be a sequence in $W^{2,2}(\mathbb{R}^4)$ formed by maximizers for the subcritical supremum of $AD(4,2,\beta_j,\alpha,\gamma)$ with $\beta_j = 32\pi^2 - 1/j$, ensured by Theorem 1.2, i.e., $\|u_j\|_{W^{2,2}(\mathbb{R}^4)} = 1$ and

$$AD(4,2,\beta_j,\alpha,\gamma) = \int_{\mathbb{R}^4} \left(e^{\beta_j \rho(\|u_j\|_2^2) u_j^2} - 1 \right) dx = \sup_{\substack{u \in W^{2,2}(\mathbb{R}^4) \\ \|\Delta u\|_2^2 + \|u\|_2^2 \le 1}} \int_{\mathbb{R}^4} \left(e^{\beta_j \rho(\|u\|_2^2) u^2} - 1 \right) dx, \quad (6\text{-}1)$$

with ρ as in (4-13) and $\frac{1+2\alpha-\gamma\alpha^2}{1+\alpha(1-\gamma)-\gamma\alpha^2}\frac{2}{\mathscr{B}_{GN}} < \beta_j < 32\pi^2$. From (2-6), we can assume that each u_j is a positive radially symmetric function and also suppose that $u_j \rightharpoonup u$ in $W^{2,2}_{rad}(\mathbb{R}^4)$.

Lemma 6.1 *We have*

$$AD(4,2,32\pi^2,\alpha,\gamma) = \lim_{j\to\infty} AD(4,2,\beta_j,\alpha,\gamma) = \lim_{j\to\infty} \int_{\mathbb{R}^4} \left(e^{\beta_j \rho(\|u_j\|_2^2) u_j^2} - 1 \right) dx.$$

Proof: We first note that

$$\limsup_{j\to\infty} AD(4,2,\beta_j,\alpha,\gamma) \le AD(4,2,32\pi^2,\alpha,\gamma). \quad (6\text{-}2)$$

The reverse inequality is a consequence of the Fatou Lemma. Indeed, for any $u \in W^{2,2}(\mathbb{R}^4)$ with $\|u\|_{W^{2,2}(\mathbb{R}^4)} \le 1$ we have

$$\liminf_{j\to\infty} \int_{\mathbb{R}^4} \left(e^{\beta_j \rho(\|u\|_2^2) u^2} - 1 \right) dx \ge \int_{\mathbb{R}^4} \left(e^{32\pi^2 \rho(\|u\|_2^2) u^2} - 1 \right) dx$$

which implies that

$$\liminf_{j\to\infty} AD(4,2,\beta_j,\alpha,\gamma) \ge \int_{\mathbb{R}^4} \left(e^{32\pi^2 \rho(\|u\|_2^2) u^2} - 1 \right) dx.$$

Taking the supremum over $u \in W^{2,2}(\mathbb{R}^4)$ with $\|u\|_{W^{2,2}(\mathbb{R}^4)} \leq 1$ we obtain

$$\liminf_{j\to\infty} AD(4,2,\beta_j,\alpha,\gamma) \geq AD(4,2,32\pi^2,\alpha,\gamma). \tag{6-3}$$

From (6-2) and (6-3) we conclude the result. ∎

A straightforward computation shows that the Euler-Lagrange equation of u_j is given by

$$\begin{cases} \Delta^2 u_j + u_j = \frac{u_j}{\lambda_j}\tilde{\zeta}_j e^{\beta_j \tilde{\zeta}_j u_j^2} + \mu_j u_j \quad \text{in } \mathbb{R}^4, \\ \|\Delta u_j\|_2^2 + \|u_j\|_2^2 = 1 \\ \beta_j = 32\pi^2 - \frac{1}{j}, \\ \tilde{\zeta}_j = \rho(\|u_j\|_2^2) = \frac{1+\alpha\|u_j\|_2^2}{1-\gamma\alpha\|u_j\|_2^2}, \\ \mu_j = \frac{\alpha(1+\gamma)}{(1-\gamma\alpha\|u_j\|_2^2)^2}, \\ \lambda_j = \frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2}\int_{\mathbb{R}^4} u_j^2 e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx. \end{cases} \tag{6-4}$$

By using Lemma 2.5 together with (6-4), we can see that $u_j \in C^\infty(\mathbb{R}^4)$. From Lemma 2.2, we can always take a point $x_j \in \mathbb{R}^4$ such that

$$c_j = u_j(x_j) = \max_{\mathbb{R}^4} |u_j|. \tag{6-5}$$

We divide our argument into two cases:

(a) $\sup_j c_j < \infty$,

(b) $c_j \to +\infty$, as $j \to \infty$.

Definition 6.0.1 *Let (u_j) be a sequence in $W^{2,2}(\mathbb{R}^4)$ such that $u_j \rightharpoonup u$ in $W^{2,2}(\mathbb{R}^4)$. We say that (u_j) is a normalized vanishing sequence [(NVS) in short] if $\|u_j\|_{W^{2,2}(\mathbb{R}^4)} = 1$, $u = 0$ and*

$$\lim_{\rho\to\infty}\lim_{j\to\infty}\int_{B_\rho}(e^{\beta_j\rho(\|u_j\|_2^2)u_j^2} - 1)dx = 0.$$

Firstly, we deal with the bounded case.

Lemma 6.2 *If $\sup_j c_j < +\infty$, then only one of the following alternatives is satisfied:*

(i) $u \neq 0$ and $AD(4,2,32\pi^2,\alpha,\gamma)$ is achieved by some function in $W^{2,2}_{rad}(\mathbb{R}^4)$;

(ii) (u_j) is a NVS and $AD(4,2,32\pi^2,\alpha,\gamma) \leq 32\pi^2\rho(1)$.

Proof: If $\sup_j c_j < +\infty$, then by standard elliptic estimates we conclude that $u_j \to u$ in $C^3_{loc}(\mathbb{R}^4)$, see Lemma 2.5. Then, for any $\rho > 1$, from Lemma 2.2

$$\begin{aligned} e^{\beta_j\rho(\|u_j\|_2^2)u_j^2} - 1 - \beta_j\rho(\|u_j\|_2^2)u_j^2 &= \sum_{i=2}^{\infty}\frac{(\beta_j\rho(\|u_j\|_2^2)u_j^2)^i}{i!} \le \sum_{i=2}^{\infty}\frac{(\rho(1)\beta_j)^i}{i!}u_j^{2i} \\ &\le \sum_{i=2}^{\infty}\frac{(\rho(1)\beta_j)^i}{i!}\left(\frac{C}{\rho^3}\right)^{i-1}u_j^2 \le \frac{u_j^2}{C\rho}\sum_{i=2}^{\infty}\frac{(32\pi^2 C\rho(1))^i}{i!} \\ &= C'\frac{u_j^2}{\rho}, \end{aligned} \tag{6-6}$$

on $\mathbb{R}^4 \setminus B_\rho$, where C' is independent of j and ρ. Thus, by (6-6)

$$\lim_{\rho\to\infty}\lim_{j\to\infty}\int_{\mathbb{R}^4\setminus B_\rho}\left[e^{\beta_j\rho(\|u_j\|_2^2)u_j^2} - 1 - \beta_j\rho(\|u_j\|_2^2)u_j^2\right]dx = 0. \tag{6-7}$$

Up to a subsequence, we can assume that $\|u_j\|_2^2 \to \theta$ with $\theta \in [0,1]$, as $j \to \infty$. Thus, the convergence in $C^3_{loc}(\mathbb{R}^4)$ yield

$$\lim_{\rho\to\infty}\lim_{j\to\infty}\int_{B_\rho}\left[e^{\beta_j\rho(\|u_j\|_2^2)u_j^2} - 1 - \beta_j\rho(\|u_j\|_2^2)u_j^2\right]dx = \int_{\mathbb{R}^4}\left[e^{32\pi^2\rho(\theta)u^2} - 1 - 32\pi^2\rho(\theta)u^2\right]dx. \tag{6-8}$$

Since

$$\lim_{j\to\infty}\int_{\mathbb{R}^4}\beta_j\rho(\|u_j\|_2^2)u_j^2 dx = 32\pi^2\rho(\theta)\theta,$$

it follows from (6-1), (6-7) and (6-8) that

$$\begin{aligned} \lim_{j\to\infty} AD(4,2,\beta_j,\alpha,\gamma) &= \lim_{j\to\infty}\int_{\mathbb{R}^4}\left(e^{\beta_j\rho(\|u_j\|_2^2)u_j^2} - 1\right)dx \\ &= \int_{\mathbb{R}^4}\left(e^{32\pi^2\rho(\theta)u^2} - 1\right)dx + 32\pi^2\rho(\theta)(\theta - \|u\|_2^2). \end{aligned} \tag{6-9}$$

From (6-9) and Lemma 6.1, we can write

$$AD(4,2,32\pi^2,\alpha,\gamma) = \int_{\mathbb{R}^4}\left(e^{32\pi^2\rho(\theta)u^2} - 1\right)dx + 32\pi^2\rho(\theta)(\theta - \|u\|_2^2). \tag{6-10}$$

If $u \neq 0$, as in (5-4) we define $\tau \ge 1$ and v given by

$$\tau^4 = \frac{\theta}{\|u\|_2^2} \quad \text{and} \quad v(x) = u\left(\frac{x}{\tau}\right). \tag{6-11}$$

Then, by (6-10)

$$
\begin{aligned}
AD(4,2,32\pi^2,\alpha,\gamma) &\geq \int_{\mathbb{R}^4} \Phi\left(32\pi^2\rho(\|v\|_2^2)v^2\right) dx = \tau^4 \int_{\mathbb{R}^4} \left(e^{32\pi^2\rho(\theta)u^2} - 1\right) dx \\
&= \int_{\mathbb{R}^4} \left(e^{32\pi^2\rho(\theta)u^2} - 1\right) dx + (\tau^4 - 1)32\pi^2\rho(\theta)\|u\|_2^2 \\
&+ (\tau^4 - 1)\int_{\mathbb{R}^4} \left[e^{32\pi^2\rho(\theta)u^2} - 1 - 32\pi^2\rho(\theta)u^2\right] dx \\
&= AD(4,2,32\pi^2,\alpha,\gamma) + (\tau^4 - 1)\int_{\mathbb{R}^4} \left[e^{32\pi^2\rho(\theta)u^2} - 1 - 32\pi^2\rho(\theta)u^2\right] dx.
\end{aligned}
$$

This forces $\tau = 1$. Consequently, we conclude that u is a maximizer for $AD(4,2,32\pi^2,\alpha,\gamma)$.

Now, suppose $u = 0$. From the convergence $u_j \to 0$ in $C^3_{loc}(\mathbb{R}^4)$ we get

$$
\lim_{j\to\infty} \int_{B_\rho} (e^{\beta_j\rho(\|u_j\|_2^2)u_j^2} - 1)dx = \int_{B_\rho} (e^{32\pi^2\rho(\theta)u^2} - 1)dx = 0, \text{ for any } \rho > 0.
$$

Thus, (u_j) is a NVS according with the Definition 6.0.1. Finally, directly from (6-10) with $u = 0$, we obtain

$$
AD(4,2,32\pi^2,\alpha,\gamma) = 32\pi^2\rho(\theta)\theta \leq 32\pi^2\rho(1).
$$

■

Next, we perform the blow-up analysis to deal with the case when $c_j \to +\infty$ as $j \to \infty$.

6.1 Blow-up Analysis

Throughout this section we assume that $c_j \to +\infty$ as $j \to \infty$. Our analysis is based on the works [10], [56].

Firstly, we note that, with the notation in (6-4), the condition $1 - \mu_j\|u_j\|_2^2 = 0$ with $0 < \|u_j\|_2^2 < 1$, which is undesirable in the definition of λ_j, occurs if and only if

$$
1 = \sqrt{\mu_j}\|u_j\|_2 = \frac{\sqrt{\alpha(1+\gamma)}}{1 - \gamma\alpha\|u_j\|_2^2}\|u_j\|_2
$$

or equivalently $\gamma\alpha\|u_j\|_2^2 + \sqrt{\alpha(1+\gamma)}\|u_j\|_2 - 1 = 0$ with $0 < \|u_j\|_2 < 1$. Hence, the undesirable condition occurs if and only if

$$
\|u_j\|_2 = \frac{-\sqrt{\alpha(1+\gamma)} + \sqrt{\alpha(\gamma+1) + 4\gamma\alpha}}{2\gamma\alpha}. \tag{6-12}
$$

Although there exist values of j , α, and γ such that (6-12) holds, this is not possible if j is sufficiently large. In fact, we will see below that $\|u_j\|_2 \to 0$ as $j \to \infty$.

Lemma 6.3 *Let δ_0 be the Dirac Measure supported at 0. Then, $|\Delta u_j|^2 \mathrm{d}x \overset{*}{\rightharpoonup} \delta_0$ weakly in the sense of measure. Further, $u_j \to 0$ in $L^2(\mathbb{R}^2)$, $\tilde{\zeta}_j \to 1$ and $\mu_j \to \alpha(1+\gamma)$, as $j \to \infty$.*

Proof: By contradiction, suppose that $|\Delta u_j|^2 \mathrm{d}x \overset{*}{\not\rightharpoonup} \delta_0$. Then, there are $R > 0$ and $\mu < 1$ such that

$$\lim_{j\to\infty} \int_{B_R} |\Delta u_j|^2 \mathrm{d}x = \mu < 1. \tag{6-13}$$

Set $\hat{u}_{j,R}(r) = u_j(r) - u_j(R)$ for $x \in B_R$ with $r = |x|$. Then $\hat{u}_{j,R} \in W^{2,2}_{\mathcal{N}}(B_R)$, and

$$\int_{B_R} |\Delta \hat{u}_{j,R}|^2 \mathrm{d}x = \int_{B_R} |\Delta u_j|^2 \mathrm{d}x. \tag{6-14}$$

By Lemma 2.2, for any $\delta > 0$ we derive

$$u_j^2 \leq (1+\delta)\hat{u}_{j,R}^2 + c_\delta u_j^2(R) \leq (1+\delta)\hat{u}_{j,R}^2 + c_\delta \frac{C}{R^3},$$

with $c_\delta = (1-(1+\delta)^{-1})^{-1}$. Now, set $\hat{v}_{j,R} := \frac{\hat{u}_{j,R}}{\|\Delta \hat{u}_{j,R}\|_2}$. We have

$$\begin{aligned}
\exp(\beta_j \tilde{\zeta}_j u_j^2) &\leq \exp\left(\beta_j \tilde{\zeta}_j (1+\delta)\hat{u}_{j,R}^2 + c_\delta \beta_j \tilde{\zeta}_j \frac{C}{R^3}\right) \\
&\leq e^{c_\delta 32\pi^2 \rho(1) \frac{C}{R^3}} \exp\left(\beta_j \tilde{\zeta}_j (1+\delta)\hat{u}_{j,R}^2\right) \\
&= C(R,\delta) \exp\left(\beta_j \tilde{\zeta}_j (1+\delta)\hat{u}_{j,R}^2\right) \\
&= C(R,\delta) \exp\left(\beta_j (1+\delta)\tilde{\zeta}_j \|\Delta \hat{u}_{j,R}\|_2^2 \hat{v}_{j,R}^2\right),
\end{aligned}$$

on the ball B_R. Recalling $\|\Delta u_j\|_2^2 + \|u_j\|_2^2 = 1$ and $\|\Delta \hat{u}_{j,R}\|_2 = \|\Delta u_j\|_{L^2(B_R)} \leq \|\Delta u_j\|_2$, from (6-13) and (6-14) we can write

$$\begin{aligned}
\tilde{\zeta}_j \|\Delta \hat{u}_{j,R}\|_2^2 &= \frac{(1+\alpha\|u_j\|_2^2)\|\Delta \hat{u}_{j,R}\|_2^2}{(1-\gamma\alpha\|u_j\|_2^2)} \\
&= \frac{\left(1+\alpha-\alpha\|\Delta u_j\|_2^2\right)\|\Delta \hat{u}_{j,R}\|_2^2}{(1-\gamma\alpha\|u_j\|_2^2)} \\
&\leq \frac{\left(1+\alpha-\alpha\|\Delta \hat{u}_{j,R}\|_2^2\right)\|\Delta \hat{u}_{j,R}\|_2^2}{(1-\gamma\alpha+\gamma\alpha\|\Delta \hat{u}_{j,R}\|_2^2)}.
\end{aligned}$$

Letting $j \to \infty$, we get

$$\lim_{j\to\infty} \left(\beta_j (1+\delta)\tilde{\zeta}_j \|\Delta \hat{u}_{j,R}\|_2^2\right) \leq 32\pi^2 (1+\delta) \frac{\mu + \alpha(1-\mu)\mu}{1-\gamma\alpha(1-\mu)}. \tag{6-15}$$

Since the positive numbers γ, α and $1-\mu$ are less than 1, we obtain $1-\gamma\alpha(1-\mu) > 0$, thus

$$\frac{\mu+\alpha(1-\mu)\mu}{1-\gamma\alpha(1-\mu)}<1 \text{ iff } \alpha(1-\mu)\mu<(1-\mu)-\gamma\alpha(1-\mu)$$
$$\text{iff } \alpha\mu<1-\gamma\alpha$$
$$\text{iff } \mu+\gamma<\frac{1}{\alpha}. \tag{6-16}$$

We are assuming the condition $0<\gamma<\frac{1}{\alpha}-1$. Thus, for any $0<\mu<1$ we obtain $\mu+\gamma<(\mu-1)+\frac{1}{\alpha}<\frac{1}{\alpha}$. So, (6-16) holds and from (6-17), for $\delta>0$ small enough it follows that

$$\lim_{j\to\infty}\left(\beta_j(1+\delta)\tilde{\zeta}_j\|\Delta\hat{u}_{j,R}\|_2^2\right)<32\pi^2. \tag{6-17}$$

Therefore, we can apply the Adams-Trudinger-Moser type inequality (2.3) by C. Tarsi [66, Theorem 4] to conclude that $\exp\left(\beta_j(1+\delta)\tilde{\zeta}_j\|\Delta\hat{u}_{j,R}\|_2^2\hat{v}_{j,R}^2\right)$ is bounded in $L^p(B_R)$ for some $p>1$. Consequently, $\exp(\beta_j\tilde{\zeta}_ju_j^2)$ is also bounded in $L^p(B_R)$. Since, (u_j) is bounded in $L^q(B_R)$ for any $q<\infty$, by Hölder inequality together with Lemma 6.4 below, it holds that

$$\frac{u_j}{\lambda_j}\exp\left(\beta_j\tilde{\zeta}_ju_j^2\right)$$

is bounded in $L^s(B_R)$ for any $s>1$. Hence, by apply standard elliptic estimates we have that (u_j) is uniformly bounded in $B_{R/2}$ which contradicts the hypothesis that $c_j\to\infty$. Thus, we need to have $|\Delta u_j|^2dx\overset{*}{\rightharpoonup}\delta_0$. As direct consequence, since $\|\Delta u_j\|_2^2+\|u_j\|_2^2=1$, we have $\|u_j\|_2^2\to 0$ as $j\to\infty$. Thus, $u_j\to 0$ and $u\equiv 0$ in $L^2(\mathbb{R}^4)$, which also implies that $\tilde{\zeta}_j\to 1$, $\mu_j\to\alpha(1+\gamma)$ as $j\to\infty$. ■

Next, we show that (λ_j) is bounded away from zero.

Lemma 6.4 *There holds* $\inf_{j\to\infty}\lambda_j>0$.

Proof: By contradiction, suppose that $\inf_{j\to\infty}\lambda_j=0$. By applying the elementary inequality $(e^t-1)\le te^t$ for $t\ge 0$ and noticing that $(1-\mu_j\|u_j\|_2^2)^{-1}\ge 1$, we obtain

$$\begin{aligned}
0&=\liminf_{j\to\infty}\beta_j\lambda_j\\
&=\liminf_{j\to\infty}\frac{\rho(\|u_j\|_2^2)}{1-\mu_j\|u_j\|_2^2}\int_{\mathbb{R}^4}\beta_ju_j^2\,e^{\beta_j\rho(\|u_j\|_2^2)u_j^2}\,dx\\
&\ge\liminf_{j\to\infty}\int_{\mathbb{R}^4}\beta_j\rho(\|u_j\|_2^2)u_j^2\,e^{\beta_j\rho(\|u_j\|_2^2)u_j^2}\,dx\\
&\ge\liminf_{j\to\infty}\int_{\mathbb{R}^4}\left(e^{\beta_j\rho(\|u_j\|_2^2)u_j^2}-1\right)dx\\
&=AD(4,2,32\pi^2,\alpha,\gamma)
\end{aligned}$$

which is the desired contradiction. ■

6.1.1 Assymptotic behavior

Here, we shall proceed exactly the same way as was done in [10]. Since (u_j) is a bounded sequence in $W^{2,2}_{rad}(\mathbb{R}^4)$, we have

$$\begin{cases} u_j \rightharpoonup u \text{ in } W^{2,2}_{rad}(\mathbb{R}^4); \\ u_j \to u \text{ in } L^p(\mathbb{R}^4), \quad \forall p > 2; \\ \beta_j \nearrow 32\pi^2. \end{cases} \tag{6-18}$$

In the case $c_j \to \infty$, from Lemma 2.2 we can assume that the sequence $(x_j) \subset \mathbb{R}^4$ in (6-5) satisfies

$$x_j \to 0, \text{ as } j \to \infty.$$

With aim to study the assymptotic behavior of (u_j) near to the blow-up point, let us define

$$r_j^4 := \frac{\lambda_j}{c_j^2 e^{\beta_j \tilde{\zeta}_j c_j^2}}. \tag{6-19}$$

Lemma 6.5 *For any $\xi < 32\pi^2$, we have $\limsup_{j\to\infty} r_j^4 c_j^2 e^{\xi\tilde{\zeta}_j c_j^2} = 0$. In particular, $\lim_{j\to+\infty} r_j^4 = 0$*

Proof: Let $\xi < 32\pi^2$, then

$$\begin{aligned} r_j^4 c_j^2 e^{\xi\tilde{\zeta}_j c_j^2} &= e^{(\xi-\beta_j)\tilde{\zeta}_j c_j^2} \frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2} \int_{\mathbb{R}^4} u_j^2 e^{\beta_j\tilde{\zeta}_j u_j^2} dx \\ &\le \frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2} \int_{\mathbb{R}^4} u_j^2 e^{\beta_j\tilde{\zeta}_j u_j^2} e^{(\xi-\beta_j)\tilde{\zeta}_j u_j^2} dx \\ &\le \frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2} \int_{\mathbb{R}^4} u_j^2 e^{\xi\tilde{\zeta}_j u_j^2} dx. \end{aligned}$$

From Lemma 6.3 we have $\frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2} \to 1$, as $j \to +\infty$. In particular, $\frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2}$ is bounded and we also have

$$\begin{aligned} r_j^4 c_j^2 e^{\xi\tilde{\zeta}_j c_j^2} &\le C\Big(\int_{\mathbb{R}^4} u_j^2(e^{\xi\tilde{\zeta}_j u_j^2}-1)dx + \int_{\mathbb{R}^4} u_j^2 dx\Big) \\ &\le C\left(\left(\int_{\mathbb{R}^4} |u_j|^p dx\right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^4} (e^{\xi\tilde{\zeta}_j u_j^2}-1)^{\frac{p}{p-2}} dx\right)^{\frac{p-2}{p}} + \int_{\mathbb{R}^4} u_j^2 dx\right) \\ &\le C(p)\left(\int_{\mathbb{R}^4} |u_j|^p dx\right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^4} \left(e^{\frac{\xi\tilde{\zeta}_j p}{p-2} u_j^2}-1\right)^{\frac{p}{p-2}} dx\right)^{\frac{p-2}{p}} + C(p)\int_{\mathbb{R}^4} u_j^2 dx, \end{aligned}$$

for some constants C and $C(p)$, which are independent of j. Taking into account the Adams-Adimurthi-Druet inequality in (1-13), Lemma 6.3 and (6-18) we obtain the result.

■

Now, we need to define some auxiliary sequences to understand the assymptotic behavior of u_j near to the blow-up point. Namely,

$$\begin{cases} w_j(x) = \dfrac{u_j(x_j + r_j x)}{c_j} \\ z_j(x) = c_j(u_j(x_j + r_j x) - c_j) \\ v_j(x) = u_j(x_j + r_j x) - c_j \end{cases} \tag{6-20}$$

where the all sequences are defined on the sequence of set $\Omega_j = \{x \in \mathbb{R}^4 \ : \ x_j + r_j x \in B_1\}$.

Lemma 6.6 *$w_j(x) \to 1$ in $C^3_{loc}(\mathbb{R}^4)$.*

Proof: By the Euler-Lagrange equation (6-4), the definition of r_j and the fact of $w_j \leq 1$, we know that for any $R > 0$ and $x \in B_R(0)$, $w_j(x)$ satisfies

$$\begin{aligned} |\Delta^2(w_j(x))| &= \left| \frac{r_j^4}{c_j}(\Delta^2 u_j)(x_j + r_j x) \right| \\ &= \left| \frac{r_j^4}{c_j}\left(\lambda_j^{-1}\tilde{\zeta}_j u_j(x_j + r_j x)\, e^{\beta_j \tilde{\zeta}_j u_j^2(x_j + r_j x)} + (\mu_j - 1)u_j(x_j + r_j x)\right) \right| \\ &\leq \left| r_j^4 \left(\lambda_j^{-1}\tilde{\zeta}_j w_j(x)\, e^{\beta_j \tilde{\zeta}_j c_j^2} + (\mu_j - 1) w_j(x)\right) \right| \\ &= \left| \frac{\tilde{\zeta}_j w_j(x)}{c_j^2} + (\mu_j - 1)\frac{w_j(x)\lambda_j}{c_j^2\, e^{\beta_j \tilde{\zeta}_j c_j^2}} \right| \\ &= \left| \frac{w_j(x)}{c_j^2}\left(\tilde{\zeta}_j + (\mu_j - 1)\frac{\lambda_j}{e^{\beta_j \tilde{\zeta}_j c_j^2}}\right) \right| \\ &\leq \frac{1}{c_j^2}\left| \left(\tilde{\zeta}_j + (\mu_j - 1)\frac{\lambda_j}{e^{\beta_j \tilde{\zeta}_j c_j^2}}\right) \right| \to 0, \end{aligned}$$

where we have used that $\mu_j \to \alpha(1 + \gamma)$, $\tilde{\zeta}_j \to 1$ and

$$\frac{\lambda_j}{e^{\beta_j \tilde{\zeta}_j c_j^2}} \leq \frac{\tilde{\zeta}_j}{1 - \mu_j \|u_j\|_2^2} \int_{\mathbb{R}^4} u_j^2\, e^{\beta_j \tilde{\zeta}_j (u_j^2 - c_j^2)}\, dx \leq M \|u_j\|_2^2 \leq C.$$

Furthermore, $w_j(x)$ is bounded in $L^1_{loc}(\mathbb{R}^4)$. By, the standard regularity theory, we got that for any $R > 0$ and $0 < \kappa < 1$, the sequence of $\|w_j(x)\|_{C^{3,\kappa}(B_R(0))}$ is uniformly bounded for every j. Finally, by the Ascoli-Arzela Theorem, there exists a function $w \in C^3(\mathbb{R}^4)$ such that the sequence $w_j(x)$ converges to w in $C^3(\mathbb{R}^4)$ having the property of $\Delta^2 w(x) = 0$ for all $x \in \mathbb{R}^4$. Now, since $w_j(0) = 1$, by the Liouville's Theorem for harmonic functions we obtain w is constant and equal to 1 in $\mathbb{R}^4$. ■

Now, we are in a position to prove the following:

Lemma 6.7 *It holds* $v_j(x) = u_j(x_j + r_j x) - c_j \to 0$ *in* $C^3_{loc}(\mathbb{R}^4)$*. Therefore,*

$$|\nabla^i u_j(x)| = o\left(\frac{1}{r_j^i}\right) \text{ in } B_{Rr_j}, \quad i = 1,2,3,$$

for any $R > 0$.

Proof: We can notice that v_j satisfies the equation

$$\begin{aligned}(-\Delta)^2 v_j &= \frac{r_j^4}{\lambda_j}\tilde{\zeta}_j u_j(x_j + r_j x)\, e^{\beta_j \tilde{\zeta}_j u_j^2(x_j + r_j x)} + r_j^4(\mu_j - 1)u_j(x_j + r_j x)\\ &= \frac{\tilde{\zeta}_j u_j(x_j + r_j x)}{c_j^2}\, e^{\beta_j \tilde{\zeta}_j (u_j^2(x_j + r_j x) - c_j^2)} + r_j^4(\mu_j - 1)u_j(x_j + r_j x).\end{aligned}$$

By setting $\Delta v_j = g_j$ and then $\Delta g_j = f_j$, where

$$f_j = \frac{\tilde{\zeta}_j u_j(x_j + r_j x)}{c_j^2}\, e^{\beta_j \tilde{\zeta}_j (u_j^2(x_j + r_j x) - c_j^2)} + r_j^4(\mu_j - 1)u_j(x_j + r_j x).$$

Since (u_j) is bounded in $H^2(\mathbb{R}^4)$, it is clear that $\int_{\mathbb{R}^4} |g_j|^2 dx = \int_{\mathbb{R}^4} |\Delta u_j|^2 dx < c$. By the fact of (f_j) is bounded in $L^p_{loc}(\mathbb{R}^4)$ for any $p \geq 1$, by Lemma 2.5 joint with Morrey's inequality, we got that for some $0 < \kappa < 1$,

$$\|g_j\|_{C^{1,\kappa}(B_R)} \leq c, \tag{6-21}$$

for any $R > 0$. Therefore by Pizzetti's formula (2-14), we obtain

$$\int_{B_R} v_j(x) dx = c_0 R^8 \Delta^2 v_j(t) + c_1 R^6 \Delta v_j(0) + c_2 R^4 v_j(0),$$

for some $t \in B_R$, where B_R is a ball centered at origin with radius R.

Now, we note that $v_j \leq 0$, $v_j(0) = 0$ and by (6-21), one can conclude that $v_j(x)$ is bounded in $L^1_{loc}(\mathbb{R}^4)$. Thus, by Lemma 2.5 again, there exists a $v \in C^3(\mathbb{R}^4)$ to which the sequennce $v_j(x)$ converges in $C^3(\mathbb{R}^4)$, satisfying $(-\Delta)^2 v = 0$. Finally, by Lemma 2.8 and knowing that $v \leq 0$, we guarantee that v is a polynomial of degree at most 6. Therefore, by

$$\int_{\mathbb{R}^4} |\Delta v|^2 dx \leq \lim_{j\to\infty} \int_{\mathbb{R}^4} |\Delta v_j|^2 dx \leq C,$$

then v must be a constant. By the above estimate together with the fact of $v(0) = 0$, we conclude the result. ■

In the next lemma we will establish a gradient estimates on B_{Rr_j}, and it will be of utmost importance for our aim in studying and determining the limit behavior of $z_j(x)$. The proof is basically the same procedure did in [10]. Let us state first a theorem which can be found in [52], which involves uniform estimates for $\nabla^{2m-l}u$, with $1 \leq l \leq 2m-1$ of a solution $\Delta^m u = f$ in the Lorentz space $L^{(\frac{n}{n-l},\frac{1}{\alpha})}(\Omega)$, $0 \leq \alpha \leq 1$.

Lemma 6.8 *It holds that for any $R > 0$,*

$$c_j \int_{B_{Rr_j}} |\Delta u_j| dx \leq c(Rr_j)^2.$$

Furthermore,

$$\int_{B_R} |\Delta z_j| dx = \frac{c_j}{r_j^2} \int_{B_{Rr_j}} |\Delta u_j| dx \leq cR^2, \tag{6-22}$$

where z_j is given by (6-20).

Proof: For any $R_0 > 0$, let $u_j^{R_0}$ be the biharmonic functions that solve the problem

$$\begin{cases} \Delta^2 u_j^{R_0} = 0, & \text{in } \overline{B}_{R_0}(x_j) \\ \partial_\nu^i u_j^{R_0} = \partial_\nu^i u_j, & \text{on } \partial\overline{B}_{R_0}(x_j);\ \ i = 0, 1. \end{cases}$$

By the radial lemma and the elliptic estimates (Lemma 2.5), we obtain

$$\|u_j^{R_0}\|_{C^4(B_{R_0})} < \frac{c}{R_0^\tau}, \text{ for some } \tau > 0. \tag{6-23}$$

Note that $u_j - u_j^{R_0}$ satisfies the equation

$$\begin{cases} \Delta^2(u_j - u_j^{R_0}) = \lambda_j^{-1} u_j \zeta_j e^{\beta_j \zeta_j u_j^2} + (\mu_j - 1)u_j, & \text{in } \overline{B}_{R_0}(0) \\ \partial_\nu^i (u_j - u^{R_0}) = 0, & \text{on } \partial\overline{B}_{R_0}(0), i = 0, 1. \end{cases}$$

Set $f_j := \lambda_j^{-1} u_j \zeta_j \exp(\beta_j \zeta_j u_j^2) + (\mu_j - 1)u_j$. Then (f_j) is bounded in $L(\ln L)^\alpha(B_{R_0})$. So, as consequence of the definitions above and joining the result in Theorem 7.8, we got

$$\|\nabla^i(u_j - u_j^{R_0})\|_{L^{(\frac{4}{i},2)}} \leq C, \ \ i = 1, 2, 3, \tag{6-24}$$

where $\|\cdot\|_{L^{(\frac{4}{i},2)}}$ is the Lorentz norm.

We have the estimate

$$|\Delta^2((u_j - u_j^{R_0})^2)| \leq |2(u_j - u_j^{R_0})\Delta^2(u_j - u_j^{R_0})| + C\sum_{i=1}^{3} |\nabla^i(u_j - u_j^{R_0})||\nabla^{4-i}(u_j - u_j^{R_0})|.$$

From (6-24), the Hölder type inequality and by O'Neil in [57], the term $\sum_{i=1}^{3} |\nabla^i(u_j - u_j^{R_0})||\nabla^{4-i}(u_j - u_j^{R_0})|$ is bounded in $L^1(B_{R_0})$. Now, we must prove that the term $|2(u_j -$

$u_j^{R_0})\Delta^2(u_j - u_j^{R_0})|$ is also bounded in $L^1(B_{R_0})$. Indeed, we separate into two integrals as follows

$$\int_{B_{R_0}} |2(u_j - u_j^{R_0})\Delta^2(u_j - u_j^{R_0})|dx \leq 2\left(\int_{B_{R_0}} |u_j\Delta^2 u_j|dx + \int_{B_{R_0}} |u_j^{R_0}\Delta^2 u_j|dx\right) = 2(I_1 + I_2).$$

Firstly, by the Euler-Lagrange equation (6-4) and applying integration by parts, we derive

$$\begin{aligned} I_1 = \int_{B_{R_0}} |u_j\Delta^2 u_j|dx &\leq \left|\int_{\mathbb{R}^4} \left(\tilde{\zeta}_j \frac{u_j^2}{\lambda_j} e^{\beta_j\tilde{\zeta}_j u_j^2} + \mu_j u_j^2\right)dx\right| + \int_{\mathbb{R}^4} |u_j|^2 dx \\ &= \left|\int_{\mathbb{R}^4} u_j(\Delta^2 u_j + u_j)dx\right| + \int_{\mathbb{R}^4} |u_j|^2 dx \\ &\leq \int_{\mathbb{R}^4} |\Delta u_j|^2 dx + 2\int_{\mathbb{R}^4} |u_j|^2 dx \leq c. \end{aligned}$$

On the other hand, we have

$$I_2 = \int_{B_{R_0}} |u_j^{R_0}\Delta^2 u_j|dx \leq c\int_{B_{R_0}} |u_j\Delta^2 u_j|dx + c\int_{B_{R_0}\cap\{|u_j|\leq 1\}} |\Delta^2 u_j|dx \leq c(R_0). \tag{6-25}$$

Hence, $\int_{B_{R_0}} |\Delta^2(u_j - u_j^{R_0})^2|dx \leq c$. Now, we want to prove that for any $R > 0$,

$$\int_{B_{Rr_j}} \Delta((u_j - u_j^{R_0})^2)dx \leq c(Rr_j)^2. \tag{6-26}$$

Indeed, we will proceed as [52]. Firstly, we claim that

$$\|\Delta u_j^2\|_{L^1(B_{Rr_j})} \leq C. \tag{6-27}$$

Notice that

$$|\Delta u_j^2| \leq 2|u_j\Delta u_j| + 2|\nabla u_j|^2.$$

Firstly, by the Hölder inequality and we get

$$\int_{B_{Rr_j}} |u_j\Delta u_j|dx \leq \|u_j\|_{L^2(B_{Rr_j})}\|\Delta u_j\|_{L^2(B_{Rr_j})} \leq c_1.$$

For the second term $2|\nabla u_j|^2$ it suffices to see that by Lemma 6.7, $|\nabla u_j| = o(r_j^{-1})$ for any R and j sufficiency large, and the claim is proved. Now, from (6-25) and (6-27), we have that (6-26) holds. By (6-26), (6-23), and Lemma 6.7,

$$\int_{B_{Rr_j}} |\Delta(u_j)^2| dx \le c \int_{B_{Rr_j}} (\Delta(u_j - u_j^{R_0})^2) dx + o(r_j^2). \tag{6-28}$$

In other hand,

$$c_j|\Delta u_j| \le cu_j|\Delta u_j| \le c(\Delta(u_j^2) + |\nabla u_j|^2) \le c\Delta(u_j^2) + o\left(\frac{1}{r_j^2}\right). \tag{6-29}$$

Finally, by (6-28) and (6-29), we conclude

$$c_j \int_{B_{Rr_j}} |\Delta u_j| dx \le c(Rr_j)^2.$$

Thus, for any $R > 0$,

$$\int_{B_R} |\Delta z_j| dx = \frac{c_j}{r_j^2} \int_{B_{Rr_j}} |\Delta u_j| dx \le cR^2.$$

■

Let us analyze the limit behavior of $z_j(x)$ in (6-20).

Lemma 6.9 *It holds $z_j(x) \to z$ in $C^3_{loc}(\mathbb{R}^4)$ with z satisfying the equation $(-\Delta)^2 z = \exp(64\pi^2 z)$. Moreover,*

$$z(x) = -\frac{1}{16\pi^2} \ln\left(1 + \frac{\pi}{\sqrt{6}}|x|^2\right) \tag{6-30}$$

and

$$\int_{\mathbb{R}^4} e^{64\pi^2 z(x)} dx = 1.$$

Proof: By the Euler-Lagrange equation (6-4), we can notice that z satisfies

$$\begin{aligned}(-\Delta)^2 z_j + c_j r_j^4 (1-\mu_j) u_j(x_j + r_j x) &= \frac{c_j r_j^4}{\lambda_j} \zeta_j u_j(x_j + r_j x) e^{\beta_j \zeta_j u_j^2(x_j + r_j x)} \\ &= \frac{\zeta_j u_j(x_j + r_j x)}{c_j} e^{\beta_j \zeta_j (u_j^2(x_j + r_j x) - c_j^2)} \\ &= \frac{\zeta_j u_j(x_j + r_j x)}{c_j} e^{\beta_j \zeta_j c_j (u_j(x_j + r_j x) - c_j)\left(\frac{u_j(x_j + r_j x)}{c_j} + 1\right)}.\end{aligned}$$

By Lemma 6.8, we know that $\int_{B_R} |\Delta z_j| dx \le cR^2$. Hence, by elliptic estimates in [25], we obtain $\|\Delta z_j\|_{C^{1,\alpha}_{loc}} \le c$. By Lemma 6.7, there exists $z \in C^3(\mathbb{R}^4)$ such that $z_j \to z$ in $C^3_{loc}(\mathbb{R}^4)$ with z satisfying the equation

$$(-\Delta)^2 z = e^{64\pi^2 z}.$$

By Fatou's Lemma, we have

$$\liminf_{j\to\infty}\int_{\mathbb{R}^4} e^{64\pi^2 z}\,dx \le \lambda_j^{-1}\int_{\mathbb{R}^4}\tilde{\zeta}_j u_j^2 e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx \le \lambda_j^{-1}\frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2}\int_{\mathbb{R}^4} u_j^2 e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx \le 1.$$

Now, let us suppose by contradiction that $z(x)$ is not of the form in (6-30). Then, by [44], there exist a negative number N such that $\lim_{|x|\to+\infty}(-\Delta)z(x) = N$. Consequently, we would have that

$$\lim_{j\to+\infty}\int_{B_R}|\Delta z_j(x)|dx = |N|\mathrm{vol}(B_1)R^4 + o(R^4)$$

as $R\to\infty$, which contradicts (6-22). Thus, we have (6-30). By computations as done in [10], ones can see that

$$\int_{\mathbb{R}^4} e^{64\pi^2 z(x)}\,dx = \int_{\mathbb{R}^4}\left(\frac{1}{1+\frac{\pi}{\sqrt{6}}|x|^2}\right)^4 = 1.$$

■

6.1.2 Bi-harmonic Truncations

In this subsection, we will use the bi-harmonic truncation proposed in [10] which are inspired by [16], which in turn are generalizations of the truncation argument introduced in [3]. Basically, for any $A > 1$, we will introduce a new function u_j^A valued close to c_j/A in a small ball centered at x_j and coinciding with u_j outside this ball. The main objective of this section is to study the properties of u_j^A.

Lemma 6.10 (DelaTorre, Lemma 4.20 [16]) *For any $A > 1$ and $j \in \mathbb{N}$, $\Omega\subset\mathbb{R}^4$ smooth bounded domain, there exists a radius $0 < \rho_j^A < \mathrm{dist}(x_j,\partial\Omega)$ and a constant C_A depending on A, such that*

1. $u_j \ge \frac{c_j}{A}$ *in* $B_{\rho_j^A}(x_j)$;
2. $|u_j - \frac{c_j}{A}| \le \frac{C_A}{c_j}$ *on* $\partial B_{\rho_j^A}(x_j)$;
3. $|\nabla^l u_j| \le \frac{C_A}{c_j(\rho_j^A)^l}$ *on* $\partial B_{\rho_j^A}(x_j)$ *for any* $1\le l\le 3$;
4. $\lim_{j\to\infty}\rho_j^A = 0$ *and, if r_j is defined as in (6-19), then* $\lim_{j\to\infty}\frac{\rho_j^A}{r_j} = \infty$.

Let $\rho_j^A > 0$ and $v_j^A \in C^4(\overline{B}_{\rho_j^A}(x_j))$ be the unique solution of the problem

$$\begin{cases}\Delta^2 v_j^A = 0, & \text{in } B_{\rho_j^A}(x_j)\\ \partial_\nu^i v_j^A = \partial_\nu^i u_j, & \text{on } \partial B_{\rho_j^A}(x_j),\quad i=0,1.\end{cases}$$

Let us consider the function

$$u_j^A = \begin{cases} v_j^A, & \text{in } B_{\rho_j^A}(x_j) \\ u_j, & \text{in } \mathbb{R}^4 \backslash B_{\rho_j^A}(x_j). \end{cases}$$

Lemma 6.11 *For $A > 1$, we*

$$u_j^A = \frac{c_j}{A} + O\Big(\frac{1}{c_j}\Big)$$

uniformly on $\overline{B}_{\rho_j^A}(x_j)$.

Proof: Set $\tilde{v}_j(x) = v_j^A(x_j + \rho_j^A x) - \frac{c_j}{A}$ for $x \in B_1$. By elliptic estimates [16, Proposition A.2], we have

$$\begin{aligned} \|v_j^A - \frac{c_j}{A}\|_{L^\infty(B_{\rho_j^A}(x_j))} = \|\tilde{v}_j\|_{L^\infty(B_1)} &\le C\big[\|\tilde{v}_j\|_{L^\infty(\partial B_1)} + \|\nabla \tilde{v}_j\|_{L^\infty(\partial B_1)}\big] \\ &= C\big[\|v_j^A - \frac{c_j}{A}\|_{L^\infty(\partial B_{\rho_j^A}(x_j))} + \rho_j^A \|\nabla v_j^A\|_{L^\infty(\partial B_{\rho_j^A}(x_j))}\big] \\ &= C\big[\|u_j - \frac{c_j}{A}\|_{L^\infty(\partial B_{\rho_j^A}(x_j))} + \rho_j^A \|\nabla u_j\|_{L^\infty(\partial B_{\rho_j^A}(x_j))}\big]. \end{aligned}$$

This together with Lemma 6.10 yields the result. ■

Lemma 6.12 *For any $A > 1$, there holds*

$$\limsup_{j\to+\infty} \int_{\mathbb{R}^4} \big(|\Delta u_j^A|^2 + |u_j^A|^2\big)\, dx \le \frac{1}{A}.$$

Proof: By combining (6-4) with Lemma 6.10, we obtain $(-\Delta)^2 u_j \ge 0$ in $B_{\rho_j^A}(x_j)$ for j large enough. So, the maximum principle yields $u_j \ge u_j^A$ in $B_{\rho_j^A}(x_j)$ and, from Lemma 6.11, $u_j^A \ge 0$ on $B_{\rho_j^A}(x_j)$ for j large enough. In addition, since $u_j^A \equiv u_j$ in $\mathbb{R}^n \setminus B_{\rho_j^A}(x_j)$ and in view of Lemma 6.11, by using $u_j - u_j^A$ as test function in (6-4) and recalling that $\rho_j^A / r_j \to \infty$, for any $R > 0$ and j sufficiently large, we obtain

$$\begin{aligned} &\int_{B_{\rho_j^A}(x_j)} \big[\Delta u_j \Delta(u_j - u_j^A) + u_j(u_j - u_j^A)\big] dx = \int_{B_{\rho_j^A}(x_j)} \frac{u_j \tilde{\zeta}_j}{\lambda_j} \big[e^{\beta_j \tilde{\zeta}_j u_j^2} + \mu_j u_j\big](u_j - u_j^A) dx \\ &\ge \frac{\tilde{\zeta}_j}{\lambda_j} \int_{B_{Rr_j}(x_j)} u_j e^{\beta_j \tilde{\zeta}_j u_j^2}(u_j - u_j^A) dx \\ &= \frac{\tilde{\zeta}_j}{\lambda_j} r_j^4 \int_{B_R(0)} \Big(c_j + \frac{z_j}{c_j}\Big) e^{\beta_j \tilde{\zeta}_j (c_j^2 + 2z_j + \frac{z_j^2}{c_j^2})} \Big(c_j + \frac{z_j}{c_j} - \frac{c_j}{A} + O(c_j^{-1})\Big) dx \\ &= \tilde{\zeta}_j \int_{B_R(0)} \Big(1 + \frac{z_j}{c_j^2}\Big) e^{\beta_j \tilde{\zeta}_j (2z_j + \frac{z_j^2}{c_j^2})} \Big(1 - \frac{1}{A} + \frac{z_j}{c_j^2} + O(c_j^{-2})\Big) dx. \end{aligned}$$

From Lemma 6.7 and Lemma 6.6, we have $z_j/c_j = v_j \to 0$ and $z_j/c_j^2 = w_j - 1 \to 0$ in $C^3_{loc}(\mathbb{R}^2)$. So, by applying Lemma 6.9, we can write

$$\int_{B_{\rho_j^A}(x_j)} \left[\Delta u_j \Delta(u_j - u_j^A) + u_j(u_j - u_j^A)\right] dx \geq \left(1 - \frac{1}{A}\right) \int_{B_R} e^{64\pi^2 z} dx + o_j(1),$$

and letting $R \to +\infty$, we obtain

$$\int_{B_{\rho_j^A}(x_j)} \left[\Delta u_j \Delta(u_j - u_j^A) + u_j(u_j - u_j^A)\right] dx \geq 1 - \frac{1}{A} + o_j(1). \tag{6-31}$$

Recalling $\|\Delta u_j\|_2^2 + \|u_j\|_2^2 = 1$, we have

$$\begin{aligned}
\int_{\mathbb{R}^4} \left(|\Delta u_j^A|^2 + |u_j^A|^2\right) dx &= \int_{B_{\rho_j^A}(x_j)} \left(|\Delta u_j^A|^2 + |u_j^A|^2\right) dx + \int_{\mathbb{R}^4 \setminus B_{\rho_j^A}(x_j)} \left(|\Delta u_j|^2 + |u_j|^2\right) dx \\
&= \int_{B_{\rho_j^A}(x_j)} \left(|\Delta u_j^A|^2 + |u_j^A|^2\right) dx + 1 - \int_{B_{\rho_j^A}(x_j)} \left(|\Delta u_j|^2 + |u_j|^2\right) dx \\
&= 1 - \int_{B_{\rho_j^A}(x_j)} \left[\Delta u_j \Delta(u_j - u_j^A) + u_j(u_j - u_j^A)\right] dx \\
&\quad + \int_{B_{\rho_j^A}(x_j)} \Delta u_j^A \Delta\left(u_j^A - u_j\right) dx + \int_{B_{\rho_j^A}(x_j)} u_j^A(u_j^A - u_j) dx.
\end{aligned}$$

From (6-31) and recalling $u_j^A(u_j^A - u_j) \leq 0$ in $B_{\rho_j^A}(x_j)$, we derive

$$\begin{aligned}
\int_{\mathbb{R}^4} \left(|\Delta u_j^A|^2 + |u_j^A|^2\right) dx &\leq \frac{1}{A} + \int_{B_{\rho_j^A}(x_j)} \Delta u_j^A \Delta(u_j^A - u_j) dx + o_j(1) \\
&= \frac{1}{A} + o_j(1),
\end{aligned}$$

where we used integration by parts and $\Delta^2 u_j^A = 0$ in $B_{\rho_j^A}(x_j)$ to obtain the last identity. ∎

Lemma 6.13 *We have*

$$AD(4, 2, 32\pi^2, \alpha, \gamma) = \lim_{j\to\infty} \int_{\mathbb{R}^4} (e^{\beta_j \tilde{\zeta}_j u_j^2} - 1) dx = \lim_{\hat{R}\to\infty} \lim_{j\to\infty} \int_{B_{\hat{R}r_j}} (e^{\beta_j \tilde{\zeta}_j u_j^2} - 1) dx = \lim_{j\to\infty} \frac{\lambda_j}{c_j^2}$$

and consequently

$$\frac{\lambda_j}{c_j} \to \infty \ \textit{and} \ \sup_j \frac{c_j^2}{\lambda_j} < \infty$$

Proof: The first identity has already been proved in Lemma 6.1. Now, let us write

$$\begin{aligned}\int_{\mathbb{R}^4}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx &= \int_{B_{\rho_j^A}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx+\int_{\mathbb{R}^4\setminus B_{\rho_j^A}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx\\ &\leq \int_{B_{\rho_j^A}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx+\int_{\mathbb{R}^4}(e^{\beta_j\tilde{\zeta}_j (u_j^A)^2}-1)dx.\end{aligned} \tag{6-32}$$

By Radial Lemma 2.2, for some radius $\hat{R}$ such that $u_j \leq 1$ on $\mathbb{R}^4\setminus B_{\hat{R}}$, we get

$$\int_{\mathbb{R}^4\setminus B_{\hat{R}}}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx \leq C\int_{\mathbb{R}^4}u_j^2dx \to 0, \text{ as } j\to\infty. \tag{6-33}$$

Note that $\beta_j\tilde{\zeta}_j \to 32\pi^2$, which together with Lemma 6.12 and by Tarsi's Adams inequality (2.3), implies

$$\sup_{j\to\infty}\int_{B_{\hat{R}}}(e^{\beta_j\tilde{\zeta}_j q'(u_j^A-u_j(\hat{R}))^2}-1)dx<\infty$$

for any $q' < A^2$ and j sufficiently large. In addition, we have

$$q(u_j^A)^2 \leq q'(u_j^A-u_j(\hat{R}))^2+c(q,q'), \text{ for } q<q',$$

and so

$$\limsup_{j\to\infty}\int_{B_{\hat{R}}}(e^{\beta_j\tilde{\zeta}_j q(u_j^A)^2}-1)dx<\infty$$

for any $q < A^2$. Now, recalling $A > 1$, then $\exp(\beta_j\tilde{\zeta}_j(u_j^A)^2)$ is uniformly integrable and $u_j \to u = 0$ a.e. Then, the Vitali's Convergence Theorem provides

$$\lim_{j\to\infty}\int_{B_{\hat{R}}}(e^{\beta_j\tilde{\zeta}_j(u_j^A)^2}-1)dx=0. \tag{6-34}$$

Hence, from (6-32), (6-33) and recalling that Lemma 6.10-(4) yields $\rho_j^A \to 0$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^4}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx &= \int_{B_{\rho_j^A}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx+o_j(1)\\ &= \int_{B_{\rho_j^A}(x_j)}e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx+o_j(1)\end{aligned} \tag{6-35}$$

and from Lemma 6.10-(1)

$$\begin{aligned}\lim_{j\to\infty}\int_{\mathbb{R}^4}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx &= \lim_{j\to\infty}\int_{B_{\rho_j^A}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\\ &\le \lim_{j\to\infty}\frac{A^2}{c_j^2}\int_{B_{\rho_j^A}(x_j)} u_j^2\, e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\\ &\le \lim_{j\to\infty}\frac{A^2}{c_j^2}\int_{\mathbb{R}^4} u_j^2\, e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\\ &= A^2\lim_{j\to\infty}\frac{\lambda_j}{c_j^2}\frac{1-\mu_j\|u_j\|_2^2}{\tilde{\zeta}_j}\\ &= A^2\lim_{j\to\infty}\frac{\lambda_j}{c_j^2}.\end{aligned} \tag{6-36}$$

By taking $A\to 1^+$ in (6-36), we obtain

$$\lim_{j\to\infty}\int_{\mathbb{R}^4}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx \le \lim_{j\to\infty}\frac{\lambda_j}{c_j^2}.$$

To reverse inequality, we note that

$$\begin{aligned}\lambda_j &= \frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2}\left[\int_{\mathbb{R}^4} u_j^2\left(e^{\beta_j\tilde{\zeta}_j u_j^2}-1\right)dx+\int_{\mathbb{R}^2} u_j^2 dx\right]\\ &\le \frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2}\left[\int_{\mathbb{R}^4} c_j^2\left(e^{\beta_j\tilde{\zeta}_j u_j^2}-1\right)dx+o_j(1)\right]\end{aligned}$$

and it follows that

$$\lim_{j\to\infty}\frac{\lambda_j}{c_j^2}\le \lim_{j\to\infty}\int_{\mathbb{R}^4}\left(e^{\beta_j\tilde{\zeta}_j u_j^2}-1\right)dx.$$

Now, note that

$$\begin{aligned}\lim_{j\to\infty}\int_{B_{\hat{R}r_j}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx &= \lim_{j\to\infty}\left[\int_{B_{\hat{R}r_j}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx+|B_{\hat{R}r_j}(x_j)|\right]\\ &= \lim_{j\to\infty}\int_{B_{\hat{R}r_j}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\end{aligned} \tag{6-37}$$

for any $\hat{R}>0$. By using the definition of r_j and $u_j^2(x_j+r_jx)-c_j^2 = z_j(w_j+1)$ on $B_{\hat{R}r_j}(x_j)$, we can write

$$\int_{B_{\hat{R}r_j}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx = \frac{\lambda_j}{c_j^2}\int_{B_{\hat{R}}(0)} e^{\beta_j\tilde{\zeta}_j[u_j^2(x_j+r_jx)-c_j^2]}\,dx = \frac{\lambda_j}{c_j^2}\int_{B_{\hat{R}}(0)} e^{\beta_j\tilde{\zeta}_j z_j(w_j+1)}\,dx. \tag{6-38}$$

Taking into account (6-37), (6-38), Lemma 6.6 and Lemma 6.9, we get

$$\lim_{j\to\infty}\int_{B_{\hat{R}r_j}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx=\lim_{j\to\infty}\frac{\lambda_j}{c_j^2}\Big(\int_{B_{\hat{R}}(0)}e^{64\pi^2 z}\,dx\Big).$$

Then, from Lemma 6.9 again

$$\lim_{\hat{R}\to\infty}\lim_{j\to\infty}\int_{B_{\hat{R}r_j}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx=\lim_{j\to\infty}\frac{\lambda_j}{c_j^2}.$$

■

Let us define

$$\xi_{R,j}=\frac{\lambda_j}{\int_{B_R(x_j)}|u_j|\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx},$$

$$\tau=\lim_{j\to\infty}\frac{\xi_j}{c_j},\quad\text{with}\quad \xi_j=\xi_{\rho_j^A,j},$$

$$\varphi=\lim_{R\to\infty}\lim_{j\to\infty}\frac{\int_{B_R(x_j)}u_j\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx}{\int_{B_R(x_j)}|u_j|\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx}.$$

Lemma 6.14 *We have* $\varphi=1$.

Proof: For all $A>1$ and $R>0$, it holds

$$\int_{B_R(x_j)}u_j\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx=\int_{B_{\rho_j^A}(x_j)}u_j\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx+\int_{B_R(x_j)\setminus B_{\rho_j^A}(x_j)}u_j^A\,e^{\beta_j\tilde{\zeta}_j (u_j^A)^2}\,dx.$$

From Lemma 6.12, we can conclude that $\exp(\beta_j\tilde{\zeta}_j(u_j^A)^2)$ is bounded in $L^p(B_R(x_j)\setminus B_{\rho_j^A}(x_j))$ for some $p>1$. Thus,

$$\int_{B_R(x_j)\setminus B_{\rho_j^A}(x_j)}u_j^A\,e^{\beta_j\tilde{\zeta}_j (u_j^A)^2}\,dx=o_j(1). \tag{6-39}$$

which implies

$$\int_{B_R(x_j)}u_j\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx=\int_{B_{\rho_j^A}(x_j)}|u_j|\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx+o_j(1), \tag{6-40}$$

where we have used that u_j is positive in $B_{\rho_j^A}(x_j)$. Analogously,

$$\int_{B_R(x_j)}|u_j|\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx=\int_{B_{\rho_j^A}(x_j)}|u_j|\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx+o_j(1). \tag{6-41}$$

Note that

$$\begin{aligned}\int_{B_{\rho_j^A}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx &= \int_{B_{\rho_j^A}(x_j)} \left(\left(e^{\beta_j\tilde{\zeta}_j u_j^2}-1\right)+1\right)dx\\ &= \int_{B_{\rho_j^A}(x_j)} \left(e^{\beta_j\tilde{\zeta}_j u_j^2}-1\right)dx + o_j(1)\end{aligned}$$

Hence, from Lemma 6.10-(4) and Lemma 6.13, we have

$$\begin{aligned}\lim_{j\to\infty}\int_{B_{\rho_j^A}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx &= \lim_{j\to\infty}\int_{B_{\rho_j^A}(x_j)} \left(e^{\beta_j\tilde{\zeta}_j u_j^2}-1\right)dx\\ &\geq \lim_{\hat{R}\to\infty}\lim_{j\to\infty}\int_{B_{\hat{R}r_j}} (e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx\\ &= AD(4,2,32\pi^2,\alpha,\gamma) > 0.\end{aligned} \tag{6-42}$$

On the other hand, from Lemma 6.10-(1) it is easy to see that

$$\begin{aligned}c_j\int_{B_{\rho_j^A}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx &\geq \int_{B_{\rho_j^A}(x_j)} |u_j|\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\\ &\geq \frac{c_j}{A}\int_{B_{\rho_j^A}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx.\end{aligned} \tag{6-43}$$

Combining (6-40), (6-41), (6-42) and (6-43) we get

$$\frac{1}{A}+o_j(1) \leq \frac{\int_{B_R(x_j)} u_j\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx}{\int_{B_R(x_j)} |u_j|\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx} \leq 1.$$

Letting $j\to\infty$, $R\to\infty$ and $A\to 1$, we obtain the result. ■

Lemma 6.15 $\tau = 1$.

Proof: Fix $R>0$. From Lemma 6.10-(4), for j large enough, we have

$$\begin{aligned}\int_{B_{\rho_j^A}(x_j)} u_j^2\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx &\geq \int_{B_{Rr_j}(x_j)} u_j^2\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\\ &= \lambda_j\int_{B_R(0)} w_j^2\,e^{\beta_j\tilde{\zeta}_j z_j(w_j+1)}\,dx.\end{aligned} \tag{6-44}$$

From (6-44), we get

$$\lim_{j\to\infty}\frac{\xi_j}{c_j}=\lim_{j\to\infty}\frac{1}{c_j}\frac{\lambda_j}{\int_{B_{\rho_j^A}(x_j)}|u_j|\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx}\leq\lim_{j\to\infty}\frac{\lambda_j}{\int_{B_{\rho_j^A}(x_j)}u_j^2\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx}$$
$$\leq\lim_{j\to\infty}\frac{1}{\int_{B_R(0)}w_j^2\,e^{\beta_j\tilde{\zeta}_j z_j(w_j+1)}\,dx}=\frac{1}{\int_{B_R(0)}e^{64\pi^2 z}\,dx}.$$

Letting $R\to\infty$ and using Lemma 6.9, we obtain $\lim_{j\to\infty}\xi_j/c_j\leq 1$, which yields $\tau\leq 1$. Now, analogous to (6-33) and (6-40) we can write

$$\begin{aligned}\int_{\mathbb{R}^4}u_j^2\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx &\geq \int_{B_{\rho_j^A}(x_j)}u_j^2\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx+o_j(1)\\ &\geq\frac{c_j^2}{A^2}\int_{B_{\rho_j^A}(x_j)}e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx+o_j(1)\\ &=\frac{c_j^2}{A^2}\left[\int_{B_{\rho_j^A}(x_j)}e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx+o_j(1)\right].\end{aligned}\tag{6-45}$$

In addition,

$$\int_{B_{\rho_j^A}(x_j)}|u_j|\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\leq c_j\int_{B_{\rho_j^A}(x_j)}e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx.\tag{6-46}$$

By combining (6-46) and (6-45), we can write

$$\frac{\xi_j}{c_j}=\frac{1}{c_j}\frac{\frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2}\int_{\mathbb{R}^4}u_j^2\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx}{\int_{B_{\rho_j^A}(x_j)}|u_j|\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx}$$
$$\geq\frac{1}{A^2}\frac{\frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2}\left[\int_{B_{\rho_j^A}(x_j)}e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx+o_j(1)\right]}{\int_{B_{\rho_j^A}(x_j)}e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx}.$$

Thus,

$$\tau=\lim_{j\to\infty}\frac{\xi_j}{c_j}\geq\frac{1}{A^2}.$$

Letting $A\to 1$ we get $\tau\geq 1$ which yields the desired result. ■

6.1.3 Assymptotic behavior of u_j away from the blow-up point

We recall the fundamental solution of the biharmonic operator $(\Delta^2+\kappa^2)$, $\kappa>0$ in $\mathbb{R}^4$ whose properties below can be found on [18]. The fundamental solution $\Phi_\kappa(x,y)$ is the solution of the equation

$$(\Delta^2+\kappa^2)\Phi_\kappa(x,y)=\delta_x(y), \text{ in } \mathbb{R}^4.$$

We recall (cf. [18, Theorem 2.4]) that for every solution $u\in H^2(\mathbb{R}^4)\cap C^4(\mathbb{R}^4)$ of the equation $(\Delta^2+\kappa^2)u=f$ we can write

$$u(x)=\int_{\mathbb{R}^4}\Phi_\kappa(x,y)f(y)\mathrm{d}y,\ \ x\in\mathbb{R}^4. \tag{6-47}$$

Let us now provide a few estimates for Φ_κ, which will play a key role in what follows.

$$|\Phi_\kappa(x,y)|\le c\ln\left(1+|x-y|^{-1}\right), \tag{6-48}$$

$$|\nabla^i\Phi_\kappa(x,y)|\le c\left(|x-y|^{-1}\right), \text{ for } i\ge 1,\ \ \forall x,y\in\mathbb{R}^4, x\neq y \text{ with } |x-y|\to 0, \tag{6-49}$$

$$|\nabla^i\Phi_\kappa(x,y)|=o\left(\exp\left(-\frac{\sqrt{\kappa}}{\sqrt{2}}|x-y|\right)\right), \text{ for } i=0,1,2,$$
$$\forall x,y\in\mathbb{R}^4, x\neq y \text{ with } |x-y|\to\infty. \tag{6-50}$$

Lemma 6.16 *For any* $1<p<2$*, we have* (c_ju_j) *is bounded in* $W^{2,p}(\mathbb{R}^4)$.

Proof: Let v_j be the solution for the equation

$$\Delta^2 v_j+\kappa_j^2 v_j=\frac{\xi_j}{\lambda_j}u_j\tilde{\zeta}_j\,\mathrm{e}^{\beta_j\tilde{\zeta}_ju_j^2},\ \text{ in } \mathbb{R}^4, \tag{6-51}$$

where $\kappa_j=(1-\mu_j)^{1/2}\to(1-\alpha(1+\gamma))^{1/2}>0$ because we are assuming $\gamma<\frac{1}{\alpha}-1$. Hence, for j large enough, the representation formula (6-47) yields

$$v_j(x)=\frac{\xi_j}{\lambda_j}\int_{\mathbb{R}^4}\Phi_{\kappa_j}(x,y)u_j(y)\tilde{\zeta}_j\,\mathrm{e}^{\beta_j\tilde{\zeta}_ju_j^2(y)}\,\mathrm{d}y,\ \ x\in\mathbb{R}^4.$$

Computing the i-th gradient

$$|\nabla^i v_j|=\left|\frac{\xi_j}{\lambda_j}\int_{\mathbb{R}^4}\nabla^i\Phi_{\kappa_j}(x,y)u_j(y)\tilde{\zeta}_j\,\mathrm{e}^{\beta_j\tilde{\zeta}_ju_j^2(y)}\,\mathrm{d}y\right|.$$

By the definition of ξ_j/λ_j, we have

$$|\nabla^i v_j|=\left|\int_{\mathbb{R}^4}\nabla^i\Phi_{\kappa_j}(x,y)\frac{u_j(y)\,\mathrm{e}^{\beta_j\tilde{\zeta}_ju_j^2(y)}}{\int_{B_{\rho_j^A}(x_j)}|u_j(z)|\,\mathrm{e}^{\beta_j\tilde{\zeta}_ju_j^2(z)}\,\mathrm{d}z}\mathrm{d}y\right|.$$

Letting $R \to \infty$ in (6-41), we have

$$\int_{\mathbb{R}^4} |u_j(z)|\, e^{\beta_j \tilde{\zeta}_j u_j^2(z)}\, dz = \int_{B_{\rho_j^A}(x_j)} |u_j(z)|\, e^{\beta_j \tilde{\zeta}_j u_j^2(z)}\, dz + o_j^+(1).$$

Then

$$|\nabla^i v_j| \le \frac{1}{\int_{\mathbb{R}^4} g_j(z)dz} \int_{\mathbb{R}^4} \left|\nabla^i \Phi_{\kappa_j}(x,y)\right| g_j(y)dy,$$

where $g_j = |u_j|\, e^{\beta_j \tilde{\zeta}_j u_j^2}$. So, by Hölder's inequality for $1 < p < 2$, we get

$$\begin{aligned}\int_{\mathbb{R}^4} \left|\nabla^i \Phi_{\kappa_j}(x,y)\right| g_j(y)dy &= \int_{\mathbb{R}^4} \left|\nabla^i \Phi_{\kappa_j}(x,y)\right| |g_j(y)|^{\frac{1}{p}} |g_j(y)|^{\frac{1}{p'}} dy \\ &\le \left(\int_{\mathbb{R}^4} \left|\nabla^i \Phi_{\kappa_j}(x,y)\right|^p |g_j(y)| dy\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^4} |g_j(y)| dy\right)^{\frac{1}{p'}}.\end{aligned}$$

It follows that

$$|\nabla^i v_j|^p \le \int_{\mathbb{R}^4} \left|\nabla^i \Phi_{\kappa_j}(x,y)\right|^p \frac{|g_j(y)|}{\int_{\mathbb{R}^4} |g_j(z)| dz} dy,$$

for $i = 0,1,2$. Applying Fubini's theorem and using (6-48), (6-49) and (6-50)

$$\begin{aligned}\int_{\mathbb{R}^4} |\nabla^i v_j(x)|^p dx &\le \int_{\mathbb{R}^4} \left(\int_{\mathbb{R}^4} \left|\nabla^i \Phi_{\kappa_j}(x,y)\right|^p \frac{|g_j(y)|}{\int_{\mathbb{R}^4} |g_j(z)| dz} dy\right) dx \\ &\le c, \text{ for } \ i = 0,1,2.\end{aligned}$$

Therefore, $\|v_j\|_{W^{2,p}(\mathbb{R}^4)} \le c$. By noticing that $v_j = \xi_j u_j$ satisfies (6-51), we have

$$\|\xi_j u_j\|_{W^{2,p}(\mathbb{R}^4)} \le c. \tag{6-52}$$

From Lemma 6.14, we have $\xi_j / c_j \to 1$. Then the proof follows from (6-52). ■

The following result will be important to demonstrate the convergence of $c_j u_j$ to a Green function.

Lemma 6.17 *Let $\phi \in C_0^1(\mathbb{R}^4)$, then we have*

$$\lim_{j \to \infty} \frac{c_j}{\lambda_j} \int_{\mathbb{R}^4} \phi u_j e^{\beta_j \tilde{\zeta}_j u_j^2} dx = \phi(0). \tag{6-53}$$

Proof: Let $\phi \in C_0^\infty(\mathbb{R}^4)$ with $\text{supp}(\phi) \subset B_\rho$, for some $\rho > 0$. We can separate the integral as follows

$$\frac{c_j}{\lambda_j}\int_{\mathbb{R}^4}\phi u_j e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx = I_j^1 + I_j^2 + I_j^3. \tag{6-54}$$

where

$$I_j^1 = \frac{c_j}{\lambda_j}\int_{B_{\rho_j^A}(x_j)\setminus B_{\hat{R}r_j}(x_j)}\phi u_j e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx$$
$$I_j^2 = \frac{c_j}{\lambda_j}\int_{B_{\hat{R}r_j}(x_j)}\phi u_j e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx$$
$$I_j^3 = \frac{c_j}{\lambda_j}\int_{B_\rho\setminus B_{\rho_j^A}(x_j)}\phi u_j e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx$$

We will show that $I_j^i \to 0$ for $i = 1,3$ and $I_j^2 \to \phi(0)$, as $j \to \infty$. From Lemma 6.4 and Lemma 6.10-(1)

$$\begin{aligned}|I_j^1| &\le A\|\phi\|_{C^0}\frac{1}{\lambda_j}\int_{B_{\rho_j^A}(x_j)\setminus B_{\hat{R}r_j}} u_j^2 e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\\ &= A\|\phi\|_{C^0}\frac{1}{\lambda_j}\Big(\int_{\mathbb{R}^4} - \int_{B_{\hat{R}r_j}(x_j)} - \int_{\mathbb{R}^4\setminus B_{\rho_j^A}(x_j)}\Big) u_j^2 e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\\ &= A\|\phi\|_{C^0}\Big[\frac{1-\mu_j\|u_j\|_2^2}{\tilde{\zeta}_j} - \frac{1}{\lambda_j}\Big(\int_{B_{\hat{R}r_j}(x_j)} + \int_{\mathbb{R}^4\setminus B_{\rho_j^A}(x_j)}\Big) u_j^2 e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\Big].\end{aligned}$$

Hence,

$$|I_j^1| \le A\|\phi\|_{C^0}\Big[1 - \frac{1}{\lambda_j}\int_{B_{\hat{R}r_j}(x_j)} u_j^2 e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx + o_j(1)\Big]. \tag{6-55}$$

As in (6-38), we have

$$\int_{B_{\hat{R}r_j}(x_j)} u_j^2 e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx = \lambda_j\int_{B_{\hat{R}}(0)} w_j^2 e^{\beta_j\tilde{\zeta}_j z_j(w_j+1)}\,dx. \tag{6-56}$$

Taking into account (6-56), letting $j \to \infty$ and $\hat{R} \to \infty$ in (6-55), by Lemma 6.6 and Lemma 6.9, we conclude that $I_j^1 \to 0$. In addition,

$$I_j^2 = \int_{B_{\hat{R}}(0)}\phi(x_j + r_j x) w_j e^{\beta_j\tilde{\zeta}_j z_j(w_j+1)}\,dx$$

Thus, by Lemmas 6.6 and 6.9, letting $j \to \infty$ and $\hat{R} \to \infty$, we obtain $I_j^2 \to \phi(0)$. Finally, note that $\exp(\beta_j\tilde{\zeta}_j|u_j^A|^2)$ is bounded in $L^q(B_\rho)$ for some $q > 1$, and so by choosing q close to 1 and applying Hölder's inequality

$$|I_j^3| \leq \frac{c_j}{\lambda_j}\|\phi\|_{C^0}\left(\int_{B_\rho}|u_j|^{q'}dx\right)^{\frac{1}{q'}}\left(\int_{B_\rho}e^{\beta_j\tilde{\zeta}_j q|u_j^A|^2}dx\right)^{\frac{1}{q}}.$$

From Lemma 6.13, we have $c_j/\lambda_j \to 0$. Then, $I_j^3 \to 0$ as $j \to \infty$. ■

Before presenting the next result, we recall the following auxiliary proposition.

Proposition 6.18 *Let $\Omega \subset \mathbb{R}^4$ be a bounded open domain with Lipschitz boundary. Then for any $u \in H^2(\Omega)$, $\omega \in H^4(\Omega)$, we have*

$$\int_\Omega \Delta u \Delta\omega dx = \int_\Omega u\Delta^2\omega dx - \int_{\partial\Omega}\nu\cdot u\nabla\Delta\omega dS + \int_{\partial\Omega}\nu\cdot\nabla u\Delta\omega dS$$

where ν denotes the outer normal to $\partial\Omega$.

Lemma 6.19 *For any $1 < p < 2$, we have $c_ju_j \rightharpoonup G \in C^3(\mathbb{R}^4\backslash\{0\})$ weakly in $W^{2,p}(\mathbb{R}^4)$, where G is a Green function satisfying $\Delta^2 G + \kappa_0 G = \delta_0$ in $\mathbb{R}^4$, where $\kappa_0 = 1-\alpha(\gamma+1)$. Moreover*

$$G = -\frac{1}{8\pi^2}\ln|x| + K_0 + h$$

where K_0 is a constant depending on 0, $h \in C^3(\mathbb{R}^4)$ with $h(0) = 0$. Further,

$$\lim_{j\to\infty}\int_{\mathbb{R}^4\backslash B_\epsilon}\left(|\Delta(c_ju_j)|^2 + |c_ju_j|^2\right)dx = -\frac{1}{8\pi^2}\ln\epsilon - \frac{1}{16\pi^2} + K_0 + \alpha(\gamma+1)\|G\|_2^2 + O(\epsilon),$$

as $\epsilon \to 0$, where $B_\epsilon = B_\epsilon(0)$, for $\epsilon > 0$.

Proof: By Lemma 6.16, we obtain some $G \in W^{2,p}(\mathbb{R}^4)$ such that $c_ju_j \rightharpoonup G$ weakly in $W^{2,p}(\mathbb{R}^4)$, for any $1 < p < 2$. Since $\|\Delta u_j\|^2 dx \overset{*}{\rightharpoonup} \delta_0$ in the sense of measure, we have that $\exp(\beta_j\tilde{\zeta}_j u_j^2)$ is bounded in $L^p(B_R\backslash B_S)$, for any radius $0 < S < R$. Notice that c_ju_j satisfies the Euler-Lagrange equation

$$\Delta^2(c_ju_j) + c_ju_j = \frac{c_ju_j}{\lambda_j}\tilde{\zeta}_j e^{\beta_j\tilde{\zeta}_j u_j^2} + \mu_j c_ju_j \text{ in } \mathbb{R}^4. \tag{6-57}$$

Therefore, by the standard regularity theory, we got $c_ju_j \to G$ in $C^3_{loc}(\mathbb{R}^4\backslash\{0\})$. By Lemma 6.17, for any $\phi \in C_0^3(\mathbb{R}^4)$, we can write

$$\lim_{j\to\infty}\int_{\mathbb{R}^4}\phi\left(\frac{c_ju_j}{\lambda_j}\tilde{\zeta}_j e^{\beta_j\tilde{\zeta}_j u_j^2} + (\mu_j - 1)c_ju_j\right)dx = \phi(0) - \kappa_0\int_{\mathbb{R}^4}\phi G dx.$$

This implies that

$$\Delta^2 G + \kappa_0 G = \delta_0 \text{ in } \mathbb{R}^4.$$

Fix $\rho > 0$. Let $\psi \in C_0^\infty(B_{2\rho}(0))$ be a cutoff function such that $\psi = 1$ in $B_\rho(0)$ and

$$g = G + \frac{1}{8\pi^2}\psi \ln|x|.$$

By computing directly the biharmonic of g we have

$$\Delta^2 g = f \ \text{ in } \mathbb{R}^4,$$

where the function f is given by

$$\begin{aligned} f = & -\frac{1}{8\pi^2}\left(\Delta^2\psi \ln|x| + 2\nabla\Delta\psi\cdot\nabla\ln|x| + 2\Delta(\nabla\psi\cdot\nabla\ln|x|) + 2\nabla\psi\cdot\nabla\Delta\ln|x|\right) \\ & - \frac{2}{8\pi^2}\Delta\psi\Delta\ln|x| - \kappa_0 G \end{aligned}$$

where we have used that $\psi\Delta^2\ln|x| = 8\pi^2\delta_0$ in $\mathbb{R}^4$. Since $G \in W^{2,p}(\mathbb{R}^4)$ for any $1 < p < 2$, we have $f \in L^q_{loc}(\mathbb{R}^4)$ for any $q > 2$. Again, by the standard regularity theory, we get $g \in C^3_{loc}(\mathbb{R}^4)$. Let $K_0 = g(0)$ and

$$h = g - g(0) + \frac{1}{8\pi^2}(1-\psi)\ln|x|.$$

Then, we have

$$G = -\frac{1}{8\pi^2}\ln|x| + K_0 + h. \tag{6-58}$$

Finally, set $U_j = c_j u_j$. From (6-57), we get

$$\Delta^2 U_j + (1-\mu_j)U_j = \frac{\tilde{\zeta}_j}{\lambda_j} U_j e^{\beta_j \tilde{\zeta}_j u_j^2} \ \text{ in } \mathbb{R}^4. \tag{6-59}$$

Let $0 < \epsilon < R$. For any $\varphi \in C_c^\infty(B_R)$, by applying Proposition 6.18 on $B_R \setminus B_\epsilon$, the equation (6-59) yields

$$\begin{aligned} \int_{B_R\setminus B_\epsilon} \Delta U_j \Delta\varphi dx &= \int_{B_R\setminus B_\epsilon} \varphi\Delta^2 U_j dx - \int_{\partial B_\epsilon} \eta\cdot\varphi\nabla\Delta U_j dS + \int_{\partial B_\epsilon} \eta\cdot\nabla\varphi\Delta U_j dS \\ &= \int_{B_R\setminus B_\epsilon} (\mu_j - 1)U_j\varphi dx + \frac{\tilde{\zeta}_j}{\lambda_j}\int_{B_R\setminus B_\epsilon} U_j e^{\beta_j\tilde{\zeta}_j u_j^2} dx \\ &+ \int_{\partial B_\epsilon} \nu\left(\varphi\nabla\Delta U_j - \nabla\varphi\Delta U_j\right) dS \end{aligned}$$

where $\nu = -\eta$ is the outer normal vector of ∂B_ϵ. By density, we can choose $\varphi = U_j$ to obtain

$$\int_{B_R\setminus B_\epsilon}\left(|\Delta U_j|^2+(1-\mu_j)|U_j|^2\right)dx=\frac{\tilde{\zeta}_j c_j^2}{\lambda_j}\int_{B_R\setminus B_\epsilon}u_j^2 e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx + \int_{\partial B_\epsilon}\nu\left(U_j\nabla\Delta U_j-\nabla U_j\Delta U_j\right)dS. \tag{6-60}$$

Letting $R\to\infty$, we have

$$\int_{\mathbb{R}^4\setminus B_\epsilon}\left(|\Delta U_j|^2+(1-\mu_j)|U_j|^2\right)dx=\frac{\tilde{\zeta}_j c_j^2}{\lambda_j}\int_{\mathbb{R}^4\setminus B_\epsilon}u_j^2 e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx + \int_{\partial B_\epsilon}\nu\left(U_j\nabla\Delta U_j-\nabla U_j\Delta U_j\right)dS. \tag{6-61}$$

By taking into account the Lemmas 2.2, 6.3 and 6.13, we can write

$$\lim_{j\to\infty}\int_{\mathbb{R}^4\setminus B_\epsilon}\left(|\Delta U_j|^2+(1-\mu_j)|U_j|^2\right)dx=\int_{\partial B_\epsilon}\nu\left(G\nabla\Delta G-\nabla G\Delta G\right)dS. \tag{6-62}$$

From (6-62), by employing Fatou's Lemma we also can write

$$\int_{\mathbb{R}^4\setminus B_\epsilon}\left(|\Delta G|^2+\kappa_0|G|^2\right)dx\le\int_{\partial B_\epsilon}\nu(G\nabla\Delta G-\nabla G\Delta G)dS.$$

Since $\kappa_0>0$, we conclude that $G\in W^{2,2}(\mathbb{R}^4\setminus B_\epsilon(0))$ for any $\epsilon>0$. With this information in hand, returning to (6-60) and letting $j\to\infty$, and then $R\to\infty$ we obtain

$$\int_{\mathbb{R}^4\setminus B_\epsilon}\left(|\Delta G|^2+\kappa_0|G|^2\right)dx=\int_{\partial B_\epsilon}\nu(G\nabla\Delta G-\nabla G\Delta G)dS.$$

In particular,

$$\lim_{\epsilon\to\infty}\int_{\partial B_\epsilon}\nu(G\nabla\Delta G-\nabla G\Delta G)dS=0.$$

So, from (6-62),

$$\lim_{\epsilon\to\infty}\lim_{j\to\infty}\int_{\mathbb{R}^4\setminus B_\epsilon}|U_j|^2dx=0.$$

From this, since we already know that $U_j\to G$ in $C^3_{\mathrm{loc}}(\mathbb{R}^4\setminus\{0\})$, we obtain $U_j\to G$ in $L^2(\mathbb{R}^4)$. Now, from (6-62) we can write

$$\lim_{j\to\infty}\int_{\mathbb{R}^4\setminus B_\epsilon}\left(|\Delta U_j|^2+|U_j|^2\right)dx=\int_{\partial B_\epsilon}\nu\left(G\nabla\Delta G-\nabla G\Delta G\right)dS+\alpha(\gamma+1)\|G\|_2^2. \tag{6-63}$$

It is well known that the fundamental solution of Δ^2 in $\mathbb{R}^4$ is $-\frac{1}{8\pi^2}\ln|x|$, and it satisfies

$$\nabla \ln|x| = \frac{x}{|x|^2}, \quad \Delta(\ln|x|) = \frac{2}{|x|^2}, \quad \nabla\Delta(\ln|x|) = -\frac{4x}{|x|^4}. \tag{6-64}$$

So, we have

$$\begin{aligned} \nu G(\epsilon)(\nabla\Delta G)(\epsilon) &= \left(-\frac{1}{8\pi^2}\ln\epsilon + K_0 + O(\epsilon)\right)\left(\frac{1}{2\pi^2}\frac{1}{\epsilon^3} + O(1)\right) \\ &= \frac{1}{2\pi^2}\frac{1}{\epsilon^3}\left(-\frac{1}{8\pi^2}\ln\epsilon + K_0 + O(\epsilon)\right) \end{aligned} \tag{6-65}$$

and

$$-\nu(\nabla G)(\epsilon)(\Delta G)(\epsilon) = -\left(-\frac{1}{8\pi^2}\frac{1}{\epsilon} + O(1)\right)\left(-\frac{1}{8\pi^2}\frac{2}{\epsilon^2} + O(1)\right) = -\frac{1}{32\pi^4}\frac{1}{\epsilon^3}(1 + O(\epsilon)). \tag{6-66}$$

By combining (6-63), (6-65) and (6-66), we obtain the desired result. ■

6.1.4 The upper bound for the Adimurthi-Druet-Adams functional acting on concentrating sequences

To obtain an upper bound for the Adimurthi-Druet-Adams functional acting on concentrating sequences, we follow an approach similar to that in [43]. We start by establishing an upper bound for any blow up function sequences in $H_0^2(B_R)$.

Lemma 6.20 *Let* (u_j) *be a bounded sequence in* $H_0^2(B_R)$ *such that* $\|\Delta u_j\|_2^2 = 1$*, where* $B_R \subset \mathbb{R}^4$*. If* $u_j \rightharpoonup 0$ *in* $H_0^2(B_R)$*, then*

$$\limsup_{j\to\infty}\int_{B_R}\left(e^{\beta_j\tilde{\zeta}_j u_j^2} - 1\right)dx \le \frac{|B_R|\,e^{-\frac{1}{3}}}{3}$$

Proof: By [49, Ineq. (5.23)] we have the estimate

$$\limsup_{j\to\infty}\int_{B_R}\left(e^{\beta_j\tilde{\zeta}_j u_j^2} - 1\right)dx \le \frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2 A_0}$$

where A_0 is the value at 0 of the trace of the part regular of the Green Function G_{B_R} for the bi-Laplacian operator Δ^2 on the ball B_R. It is well known that

$$G_{B_R} = \frac{1}{8\pi^2}\ln|x| + \frac{1}{16\pi^2}\frac{|x|^2}{R^2} + \frac{1}{8\pi^2}\ln R - \frac{1}{16\pi^2}$$

and the term $\frac{1}{8\pi^2}\ln R - \frac{1}{16\pi^2}$ is the value at 0 of the trace of the regular part of G_{B_R}. So, by simple computation, we obtain the desired estimate. ■

Having established this upper bound, we proceed to derive a corresponding upper bound for the Adimurthi-Druet-Adams inequality in the entire space $\mathbb{R}^4$.

Lemma 6.21 *If $AD(4,2,32\pi^2,\alpha,\gamma)$ is not attained, then*

$$AD(4,2,32\pi^2,\alpha,\gamma) \leq \frac{\pi^2}{6} e^{\frac{5}{3}+32\pi^2 K_0},$$

where K_0 is the value at 0 of the regular part of the Green function G for the operator $\Delta^2 + \kappa_0$.

Proof: By assumption, we can assume $c_j \to \infty$ as $j \to \infty$. Let

$$\tilde{u}_j = \frac{u_j - u_j^\epsilon}{\|\Delta(u_j - u_j^\epsilon)\|_{L^2(B_\epsilon(x_j))}},$$

where

$$u_j^\epsilon(r) = u_j(\epsilon) + \frac{u_j'(\epsilon) r^2}{2\epsilon} - \frac{u_j'(\epsilon)\epsilon}{2}, \text{ with } r = |x|.$$

Since $c_j u_j(\epsilon) \to G(\epsilon)$ and $c_j u_j'(\epsilon) \to G'(\epsilon)$, as $j \to \infty$, we have

$$\begin{cases} \lim\limits_{j\to\infty} \left[c_j u_j(\epsilon) - \frac{c_j u_j'(\epsilon)\epsilon}{2} \right] = G(\epsilon) - \frac{\epsilon G'(\epsilon)}{2} = -\frac{1}{8\pi^2}\ln\epsilon + K_0 + \frac{1}{16\pi^2} + o_\epsilon(1) \\ \lim\limits_{j\to\infty} \frac{c_j u_j'(\epsilon)}{2\epsilon} = \frac{G'(\epsilon)}{2\epsilon} = -\frac{1}{16\pi^2\epsilon^2} + \frac{h'(\epsilon)}{2\epsilon}. \end{cases} \tag{6-67}$$

Let us compute $\|\Delta u_j - \Delta u_j^\epsilon)\|^2_{L^2(B_\epsilon)}$. We have

$$\|\Delta(u_j - u_j^\epsilon)\|^2_{L^2(B_\epsilon)} = \int_{B_\epsilon} |\Delta u_j|^2 dx - 2\int_{B_\epsilon} \Delta u_j \Delta u_j^\epsilon dx + \int_{B_\epsilon} |\Delta u_j^\epsilon|^2 dx = I_1 - 2I_2 + I_3.$$

Firstly,

$$\begin{aligned} I_1 = \int_{B_\epsilon} |\Delta u_j|^2 dx &= 1 - \int_{\mathbb{R}^4\setminus B_\epsilon} \left(|\Delta u_j|^2 + |u_j|^2\right) dx - \int_{B_\epsilon} |u_j|^2 dx \\ &= 1 - \frac{1}{c_j^2}\left(\int_{\mathbb{R}^4\setminus B_\epsilon} \left(|\Delta U_j|^2 + |U_j|^2\right) dx - \int_{B_\epsilon} |U_j|^2 dx \right) \end{aligned}$$

Then, from Lemma 6.19 we get

$$I_1 = 1 - \frac{1}{c_j^2}\left[-\frac{1}{8\pi^2}\ln\epsilon - \frac{1}{16\pi^2} + K_0 + \alpha(\gamma+1)\|G\|_2^2 + o_j(\epsilon) + o_\epsilon(1) \right], \tag{6-68}$$

where $o_j(\epsilon)$ means that $\lim_{j\to\infty} o_j(\epsilon) = 0$, if ϵ is fixed. By Proposition 6.18, we can decompose I_2 as follows

$$I_2 = \int_{B_\epsilon} u_j^\epsilon \Delta^2 u_j \mathrm{d}x - \int_{\partial B_\epsilon} \nu \cdot u_j^\epsilon \nabla\Delta u_j \mathrm{d}S + \int_{\partial B_\epsilon} \nu \cdot \nabla u_j^\epsilon \Delta u_j \mathrm{d}S = I_2^1 - I_2^2 + I_2^3.$$

By (6-57), we get

$$\begin{aligned} I_2^1 &= \int_{B_\epsilon} u_j^\epsilon \Delta^2 u_j \mathrm{d}x = \frac{1}{c_j}\int_{B_\epsilon} u_j^\epsilon \left(\frac{c_j u_j}{\lambda_j}\tilde{\zeta}_j \mathrm{e}^{\beta_j\tilde{\zeta}_j u_j^2} + (\mu_j - 1)c_j u_j\right)\mathrm{d}x \\ &= \frac{1}{c_j^2}\left[\left(c_j u_j(\epsilon) - \frac{c_j u_j'(\epsilon)\epsilon}{2}\right)\frac{c_j\tilde{\zeta}_j}{\lambda_j}\int_{B_\epsilon} u_j \mathrm{e}^{\beta_j\tilde{\zeta}_j u_j^2}\mathrm{d}x + \frac{c_j u_j'(\epsilon)}{2\epsilon}\frac{c_j\tilde{\zeta}_j}{\lambda_j}\int_{B_\epsilon}|x|^2 u_j \mathrm{e}^{\beta_j\tilde{\zeta}_j u_j^2}\mathrm{d}x\right] \\ &+ \frac{1}{c_j^2}\left[\left(c_j u_j(\epsilon) - \frac{c_j u_j'(\epsilon)\epsilon}{2}\right)(\mu_j - 1)\int_{B_\epsilon} c_j u_j \mathrm{d}x + \frac{c_j u_j'(\epsilon)}{2\epsilon}(\mu_j - 1)\int_{B_\epsilon}|x|^2 c_j u_j \mathrm{d}x\right]. \end{aligned} \tag{6-69}$$

By Hölder inequality and using that $c_j u_j \to G$ in $L^2(\mathbb{R}^4)$, we have

$$\begin{cases} \left|\int_{B_\epsilon} c_j u_j \mathrm{d}x\right| \le |B_\epsilon|^{\frac{1}{2}}\|c_j u_j\|_{L^2(\mathbb{R}^4)} = O(\epsilon^2) \\ \left|\int_{B_\epsilon}|x|^2 c_j u_j \mathrm{d}x\right| \le \epsilon^2|B_\epsilon|^{\frac{1}{2}}\|c_j u_j\|_{L^2(\mathbb{R}^4)} = O(\epsilon^4) \\ \lim_{j\to\infty}\left[c_j u_j(\epsilon) - \frac{c_j u_j'(\epsilon)\epsilon}{2}\right] = G(\epsilon) - \frac{\epsilon G'(\epsilon)}{2} = O(\ln\epsilon) \\ \lim_{j\to\infty}\frac{c_j u_j'(\epsilon)}{2\epsilon} = \frac{G'(\epsilon)}{2\epsilon} = O\left(\frac{1}{\epsilon^2}\right). \end{cases} \tag{6-70}$$

Taking account the Lemma 6.17, and using (6-67), (6-69) and (6-70), we also can write

$$I_2^1 = \frac{1}{c_j^2}\left[-\frac{1}{8\pi^2}\ln\epsilon + K_0 + \frac{1}{16\pi^2} + o_j(\epsilon) + o_\epsilon(1)\right].$$

Now, note that $u_j^\epsilon(x) = u_j(\epsilon)$ on ∂B_ϵ, then

$$\begin{aligned} I_2^2 = \int_{\partial B_\epsilon} \nu \cdot u_j^\epsilon \nabla\Delta u_j \mathrm{d}S &= \frac{u_j(\epsilon)}{c_j}\int_{\partial B_\epsilon} \nu \cdot \nabla\Delta(c_j u_j)\mathrm{d}S \\ &= \frac{u_j(\epsilon)}{c_j}\left[\int_{\partial B_\epsilon} \nu \cdot \nabla\Delta G \mathrm{d}S + o_j(\epsilon)\right] \\ &= \frac{1}{c_j^2}\left[c_j u_j(\epsilon)\int_{\partial B_\epsilon} \nu \cdot \nabla\Delta G \mathrm{d}S + o_j(\epsilon)\right] \end{aligned}$$

and by the identities in (6-64)

$$\int_{\partial B_\epsilon} \nu \cdot \nabla\Delta G \mathrm{d}S = \frac{1}{2\pi^2}\int_{\partial B_\epsilon}\frac{1}{|x|^3}\mathrm{d}S + \int_{\partial B_\epsilon} \nu \cdot \nabla\Delta h \mathrm{d}S = 1 + O(\epsilon^3).$$

It follows that

$$I_2^2 = \frac{1}{c_j^2}\left[G(\epsilon) + O(\epsilon^3 G(\epsilon)) + o_j(\epsilon)\right] = \frac{1}{c_j^2}\left[G(\epsilon) + o_j(\epsilon) + o_\epsilon(1)\right].$$

Analogous, from (6-64) we can write

$$\int_{\partial B_\epsilon} \Delta G \mathrm{d}S = -\frac{1}{4\pi^2}\int_{\partial B_\epsilon} \frac{1}{|x|^2}\mathrm{d}S + \int_{\partial B_\epsilon} \Delta h \mathrm{d}S = -\frac{\epsilon}{2} + O(\epsilon^3)$$

and then we get

$$\begin{aligned}
I_2^3 &= \int_{\partial B_\epsilon} \nu \cdot \nabla u_j^\epsilon \Delta u_j \mathrm{d}S = \frac{u_j'(\epsilon)}{c_j}\left[\int_{\partial B_\epsilon} \Delta G \mathrm{d}S + o_j(\epsilon)\right] \\
&= \frac{1}{c_j^2}\left[c_j u_j'(\epsilon)\int_{\partial B_\epsilon} \Delta G \mathrm{d}S + o_j(\epsilon)\right] \\
&= \frac{1}{c_j^2}\left[G'(\epsilon)\int_{\partial B_\epsilon} \Delta G \mathrm{d}S + o_j(\epsilon)\right] \\
&= \frac{1}{c_j^2}\left[-\frac{1}{8\pi^2\epsilon}\big(1 + o_\epsilon(1)\big)\int_{\partial B_\epsilon} \Delta G \mathrm{d}S + o_j(\epsilon)\right] \\
&= \frac{1}{c_j^2}\left[\frac{1}{16\pi^2} + o_j(\epsilon) + o_\epsilon(1)\right].
\end{aligned}$$

Hence,

$$\begin{aligned}
I_2 &= I_2^1 - I_2^2 + I_2^3 \\
&= \frac{1}{c_j^2}\left[-\frac{1}{8\pi^2}\ln\epsilon + K_0 + \frac{1}{16\pi^2} + o_j(\epsilon) + o_\epsilon(1)\right] \\
&+ \frac{1}{c_j^2}\left[\frac{1}{16\pi^2} - G(\epsilon) + o_j(\epsilon) + o_\epsilon(1)\right] \\
&= \frac{1}{c_j^2}\left[\frac{1}{8\pi^2} + o_j(\epsilon) + o_\epsilon(1)\right].
\end{aligned} \tag{6-71}$$

Finally, by definition of u_j^ϵ,

$$\begin{aligned}
I_3 &= \int_{B_\epsilon} |\Delta u_j^\epsilon|^2 \mathrm{d}x = \int_{B_\epsilon}\left(\frac{4u_j'(\epsilon)}{\epsilon}\right)^2 \mathrm{d}x \\
&= 8\pi^2 (u_j'(\epsilon))^2\epsilon^2 = \frac{1}{c_j^2}\left[8\pi^2(\epsilon G'(\epsilon))^2 + o_j(\epsilon)\right] \\
&= \frac{1}{c_j^2}\left[\frac{1}{8\pi^2} + o_j(\epsilon) + o_\epsilon(1)\right].
\end{aligned} \tag{6-72}$$

By (6-68), (6-71) and (6-72), we have

$$\|\Delta(u_j - u_j^\epsilon)\|^2_{L^2(B_\epsilon)} = 1 - \frac{1}{c_j^2}\Big[-\frac{1}{8\pi^2}\ln\epsilon + \frac{1}{16\pi^2} + K_0 + \alpha(\gamma+1)\|G\|_2^2 + o_j(\epsilon) + o_\epsilon(1)\Big]. \tag{6-73}$$

Thus,

$$\begin{aligned}
\tilde{u}_j^2 &= \frac{(u_j - u_j^\epsilon)^2}{1 - \frac{1}{c_j^2}\Big[-\frac{1}{8\pi^2}\ln\epsilon + \frac{1}{16\pi^2} + K_0 + \alpha(\gamma+1)\|G\|_2^2 + o_j(\epsilon) + o_\epsilon(1)\Big]} \\
&= u_j^2\Big(1 + \frac{1}{c_j^2}\Big[-\frac{1}{8\pi^2}\ln\epsilon + \frac{1}{16\pi^2} + K_0 + \alpha(\gamma+1)\|G\|_2^2 + o_j(\epsilon) + o_\epsilon(1)\Big]\Big) \\
&\quad - \big(2u_j u_j^\epsilon + (u_j^\epsilon)^2\big)\big(1 + o_j(\epsilon)\big) \\
&= u_j^2 - c\ln\epsilon^4 + o_j(\epsilon).
\end{aligned}$$

By Lemma 6.13,

$$\lim_{\hat{R}\to\infty}\lim_{j\to\infty}\int_{B_\rho\setminus B_{\hat{R}r_j}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx = |B_\rho|, \quad \text{for any } \rho < \epsilon.$$

Therefore,

$$\begin{aligned}
\lim_{\hat{R}\to\infty}\lim_{j\to\infty}\int_{B_\rho\setminus B_{\hat{R}r_j}(x_j)} e^{\beta_j\tilde{\zeta}_j \tilde{u}_j^2}\,dx &\le O\Big(\frac{1}{\epsilon^{4c}}\Big)\lim_{\hat{R}\to\infty}\lim_{j\to\infty}\int_{B_\rho\setminus B_{\hat{R}r_j}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx \\
&= O\Big(\frac{|B_\rho|}{\epsilon^{4c}}\Big) \to 0 \text{ as } \rho\to 0.
\end{aligned}$$

So, by Lemma 6.3, we have

$$\lim_{j\to\infty}\int_{B_\epsilon\setminus B_\rho}\Big(e^{\beta_j\tilde{\zeta}_j\tilde{u}_j^2(x)} - 1\Big)\,dx = 0.$$

Thus, with Lemma 6.20

$$\lim_{\hat{R}\to\infty}\lim_{j\to\infty}\int_{B_{\hat{R}r_j}}\Big(e^{\beta_j\tilde{\zeta}_j\tilde{u}_j^2(x)} - 1\Big)\,dx = \lim_{j\to\infty}\int_{B_\epsilon}\Big(e^{\beta_j\tilde{\zeta}_j\tilde{u}_j^2(x)}\Big)\,dx \le \frac{1}{3}|B_\epsilon|e^{-\frac{1}{3}}.$$

Fix any $\hat{R} > 0$ and any $x \in B_{\hat{R}r_j}(x_j)$,

$$\begin{aligned}
\beta_j u_j^2(x) &= \beta_j\left(\frac{u_j(x)}{\|\Delta(u_j - u_j^\epsilon)\|_{L^2(B_\epsilon)}}\right)^2\int_{B_\epsilon}|\Delta(u_j - u_j^\epsilon)|^2 dx \\
&= \beta_j\left(\tilde{u}_j + \frac{u_j^\epsilon(x)}{\|\Delta(u_j - u_j^\epsilon)\|_{L^2(B_\epsilon)}}\right)^2\int_{B_\epsilon}|\Delta(u_j - u_j^\epsilon)|^2 dx.
\end{aligned}$$

By the estimate (6-73), we have

$$\begin{aligned}
\beta_j u_j^2 &= \beta_j \left(\tilde{u}_j + u_j^{\epsilon} + O(c_j^{-2})\right)^2 \\
&\times \left(1 - \frac{1}{c_j^2}\left[-\frac{1}{8\pi^2}\ln\epsilon + \frac{1}{16\pi^2} + K_0 + \alpha(\gamma+1)\|G\|_2^2 + o_j(\epsilon) + o_\epsilon(1)\right]\right) \\
&= \beta_j \tilde{u}_j^2 \left(1 + \frac{u_j^{\epsilon}}{c_j} + O(c_j^{-3})\right)^2 \\
&\times \left(1 - \frac{1}{c_j^2}\left[G(\epsilon) + \frac{1}{16\pi^2} + \alpha(\gamma+1)\|G\|_2^2 + o_j(\epsilon) + o_\epsilon(1)\right]\right).
\end{aligned}$$

Notice that

$$\lim_{j\to\infty} \frac{\tilde{u}_j(x_j + r_j x)}{c_j} = 1,$$

and since

$$\tilde{u}_j(x_j + r_j x) u_j(\epsilon) \to G(\epsilon)$$

we get

$$\begin{aligned}
\beta_j u_j^2 = \beta_j \tilde{u}_j^2 &\left(1 + \frac{1}{c_j^2}\left(G(\epsilon) + \frac{1}{16\pi^2}\right) + O\left(\frac{1}{c_j^3}\right)\right)^2 \\
&\times \left(1 - \frac{1}{c_j^2}\left[G(\epsilon) + \frac{1}{16\pi^2} + \alpha(\gamma+1)\|G\|_2^2 + o_j(\epsilon) + o_\epsilon(1)\right]\right) \\
&= \beta_j \tilde{u}_j^2 \left(1 + \frac{2G(\epsilon)}{c_j^2} + \frac{1}{8\pi^2 c_j^2} - \frac{G(\epsilon) + \frac{1}{16\pi^2} + O_j(\epsilon)}{c_j^2}\right) \\
&= \beta_j \tilde{u}_j^2 + \beta_j G(\epsilon) + \frac{\beta_j}{16\pi^2} + O_j(\epsilon).
\end{aligned}$$

Hence,

$$
\begin{aligned}
&\lim_{\hat{R}\to\infty}\lim_{j\to\infty}\int_{B_{\hat{R}r_j}(x_j)}\left(e^{\beta_j\tilde{\zeta}_j u_j^2}-1\right)dx\\
&\leq \lim_{\hat{R}\to\infty}\lim_{j\to\infty}\int_{B_{\hat{R}r_j}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\\
&\leq \lim_{\hat{R}\to\infty}\lim_{j\to\infty} e^{32\pi^2 G(\epsilon)+2+o_\epsilon(1)}\int_{B_{\hat{R}r_j}} e^{\beta_j\tilde{\zeta}_j \tilde{u}_j^2}\,dx\\
&= e^{32\pi^2 G(\epsilon)+2+o_\epsilon(1)}\,|B_\epsilon|\,\frac{1}{3}\,e^{-\frac{1}{3}}\\
&= e^{-4\ln\epsilon+32\pi^2K_0+32\pi^2h(\epsilon)+2+o_\epsilon(1)}\,|B_\epsilon|\,\frac{1}{3}\,e^{-\frac{1}{3}}\\
&= \frac{\pi^2}{6}\,e^{\frac{5}{3}+32\pi^2K_0}+o_\epsilon(1).
\end{aligned}
$$

Letting $\epsilon\to 0$, we have

$$
AD(4,2,32\pi^2,\alpha,\gamma)\leq \frac{\pi^2}{6}\,e^{\frac{5}{3}+32\pi^2K_0}.
$$

■

6.2 The test function computation

In this section, we complete the proof of Theorem 1.4 by showing that

$$
AD(4,2,32\pi^2,\alpha,\gamma) > \frac{\pi^2}{6}\,e^{\frac{5}{3}+32\pi^2K_0}, \tag{6-74}
$$

for $\alpha>0$ small enough and $\gamma\leq\gamma_0$, which together with Lemma 6.21 completes the proof. In order to get (6-74), following some ideas in [10, 56], we use some test function computations. Let us define

$$
\phi_\varepsilon=\begin{cases} C+\frac{1}{C}\left[a-\frac{1}{16\pi^2}\psi\left(\frac{|x|}{\epsilon}\right)+K_0+h(x)+b|x|^2\right], & \text{if } |x|\leq L\epsilon,\\ \frac{G(x)}{C}, & \text{if } |x|\geq L\epsilon \end{cases}
$$

where $\psi(s)=\ln(1+\frac{\pi}{\sqrt{6}}s^2)$ and L, C, a, b are functions of ε such that

(i) $\epsilon=\exp(-L)$, $\frac{1}{C^2}=O\left(\frac{1}{L}\right)$ as $\epsilon\to 0$

(ii) $a=-\frac{1}{8\pi^2}\ln(L\epsilon)-C^2+\frac{1}{16\pi^2}\psi(L)-bL^2\epsilon^2$;

(iii) $b=-\frac{1}{16\pi^2L^2\epsilon^2\left(1+\frac{\pi}{\sqrt{6}}L^2\right)}$.

Note that, by (i)

$$\psi(L) - 2\ln(L\epsilon) = \ln\left(\frac{\frac{\pi}{\sqrt{6}}}{\epsilon^2}\right) + \ln\left(1 + \frac{\sqrt{6}}{\pi}\frac{1}{L^2}\right)$$
$$= \ln\frac{\pi}{\sqrt{6}\epsilon^2} + O\left(\frac{1}{\ln^2\epsilon}\right). \tag{6-75}$$

Hence, from (i)-(ii)

$$a = -C^2 + \frac{1}{16\pi^2}\ln\frac{\pi}{\sqrt{6}\epsilon^2} + O\left(\frac{1}{\ln^2\epsilon}\right). \tag{6-76}$$

It was shown in [10] that $\phi_\epsilon \in H^2\left(\mathbb{R}^4\right)$. By Lemma 6.19, we have

$$\int_{\mathbb{R}^4\setminus B_{L\epsilon}(0)} \left(|\Delta\phi_\epsilon|^2 + |\phi_\epsilon|^2\right) dx = \frac{1}{C^2}\int_{\mathbb{R}^4\setminus B_{L\epsilon}(0)} \left(|\Delta G|^2 + |G|^2\right) dx$$
$$= \frac{1}{C^2}\left[-\frac{1}{8\pi^2}\ln L\epsilon - \frac{1}{16\pi^2} + K_0 + \alpha(\gamma+1)\|G\|_2^2 + O(L\epsilon)\right] \tag{6-77}$$

and by [10, 49], it follows

$$\int_{B_{L\epsilon}(0)} |\Delta\phi_\epsilon|^2 dx = \frac{1}{96\pi^2 C^2}\left[6\psi(L) + 1 + O\left(\frac{1}{\ln^2\epsilon}\right)\right].$$

Notice that

$$\int_{B_{L\epsilon}(0)} |\phi_\epsilon|^2 dx = \frac{1}{C^2} O\left((L\epsilon)^4 C^4\right)$$
$$\int_{\mathbb{R}^4\setminus B_{L\epsilon}(0)} |\phi_\epsilon|^2 dx = \frac{1}{C^2}\int_{\mathbb{R}^4\setminus B_{L\epsilon}(0)} |G|^2 dx = \frac{1}{C^2}\left[\|G\|_2^2 + O\Big((-L\epsilon\ln(L\epsilon))^4\Big)\right] \tag{6-78}$$

where we have used in the last identity the expression of G given in Lemma 6.19. Therefore

$$\int_{\mathbb{R}^4} \left(|\Delta\phi_\epsilon|^2 + |\phi_\epsilon|^2\right) dx = \frac{1}{32\pi^2 C^2}\left[2\psi(L) - 4\ln(L\epsilon) + O\left(\frac{1}{\ln^2\epsilon}\right)\right]$$
$$+ \frac{1}{32\pi^2 C^2}\left[-\frac{5}{3} + 32\pi^2 K_0 + 32\pi^2\alpha(\gamma+1)\|G\|_2^2\right]$$
$$= \frac{1}{32\pi^2 C^2}\left[2\ln\frac{\pi}{\sqrt{6}\epsilon^2} - \frac{5}{3} + 32\pi^2 K_0 + 32\pi^2\alpha(\gamma+1)\|G\|_2^2 + O\left(\frac{1}{\ln^2\epsilon}\right)\right].$$

By setting $\int_{\mathbb{R}^4}\left(|\Delta\phi_\varepsilon|^2 + |\phi_\varepsilon|^2\right) dx = 1$ we get

$$32\pi^2 C^2 = 2\ln\frac{\pi}{\sqrt{6}\epsilon^2} - \frac{5}{3} + 32\pi^2 K_0 + 32\pi^2\alpha(\gamma+1)\|G\|_2^2 + O\left(\frac{1}{\ln^2\epsilon}\right) \tag{6-79}$$

and so

$$C^2 = \alpha(\gamma+1)\|G\|_2^2 + \frac{1}{16\pi^2}\ln\frac{\pi}{\sqrt{6}\epsilon^2} - \frac{5}{96\pi^2} + K_0 + O\left(\frac{1}{\ln^2\epsilon}\right) = O(-\ln\epsilon). \tag{6-80}$$

In addition, since $\frac{1+\alpha t^2}{1-\alpha\gamma t^2} = 1+\alpha(\gamma+1)t^2+O(t^4)$, as $t\to 0$ and, from (6-78) we have $\|\phi_\epsilon\|_2^2 = O\left(\frac{1}{C^2}\right) \to 0$ as $\epsilon\to 0$. So, by using (6-78) again we can write

$$\begin{aligned}\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2} &= 1+\alpha(\gamma+1)\|\phi_\epsilon\|_2^2 + O(\|\phi_\epsilon\|_2^4)\\ &= 1+\frac{\alpha(\gamma+1)\|G\|_2^2}{C^2} + O\left(C^{-2}(-L\epsilon\ln(L\epsilon))^4\right) + O\left(C^{-4}\right).\end{aligned} \tag{6-81}$$

Firstly, since $e^t - 1 \geq t$, for $t \geq 0$, from (6-78) and (6-81), we can write

$$\begin{aligned}&\int_{\mathbb{R}^4\setminus B_{L\epsilon}}\left(e^{32\pi^2\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2}-1\right)dx\\ &\geq 32\pi^2\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\int_{\mathbb{R}^4\setminus B_{L\epsilon}}\phi_\epsilon^2 dx\\ &= \frac{32\pi^2}{C^2}\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\left[\|G\|_2^2 + O\left((-L\epsilon\ln(L\epsilon))^4\right)\right]\\ &= \frac{32\pi^2\|G\|_2^2}{C^2} + O\left(C^{-4}\right).\end{aligned} \tag{6-82}$$

Now, from (6-81), we also have

$$\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2 = \left(1+\frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}\right)\phi_\epsilon^2 + \left[O\left((-L\epsilon\ln(L\epsilon))^4\right)+O\left(C^{-2}\right)\right]\frac{\phi_\epsilon^2}{C^2}. \tag{6-83}$$

Next, we shall estimate each term on the right hand side of (6-83) on $B_{L\varepsilon}$. Firstly, by (*ii*), (6-76) and (6-80), for any $x\in B_{L\varepsilon}$,

$$\begin{aligned}\phi_\epsilon^2 &\geq C^2 + 2\left(a - \frac{1}{16\pi^2}\psi\left(\frac{|x|}{\epsilon}\right) + K_0 + h(x) + b|x|^2\right)\\ &= C^2 + 2a - \frac{1}{8\pi^2}\psi\left(\frac{|x|}{\epsilon}\right) + 2(K_0+h(x)+b|x|^2)\\ &= -\alpha(\gamma+1)\|G\|_2^2 + \frac{1}{16\pi^2}\ln\frac{\pi}{\sqrt{6}\epsilon^2} - \frac{5}{96\pi^2} + K_0 + O\left(\frac{1}{\ln^2\varepsilon}\right)\\ &+ \frac{10}{96\pi^2} - \frac{1}{8\pi^2}\psi\left(\frac{|x|}{\epsilon}\right) + 2(h(x)+b|x|^2)\\ &= C^2 - 2\alpha(\gamma+1)\|G\|_2^2 + \frac{10}{96\pi^2} + 2z\left(\frac{x}{\epsilon}\right) + O\left(\frac{1}{\ln^2\epsilon}\right).\end{aligned} \tag{6-84}$$

where $z(x)$ is given by (6-30) and we have used in the last identity (6-84) the fact that $h \in C^3$ with $h(0) = 0$, which implies that $2(h(x) + b|x|^2) = O(L\epsilon) = O\left(\frac{1}{\ln^2 \epsilon}\right)$ for $x \in B_{L\varepsilon}$. From (6-84), we obtain

$$\left(1 + \frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}\right)\phi_\epsilon^2 \geq \left(1 + \frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}\right)2z\left(\frac{x}{\epsilon}\right) + C^2 - \alpha(\gamma+1)\|G\|_2^2 + \frac{10}{96\pi^2} - \frac{2\alpha^2(\gamma+1)^2\|G\|_2^4 + \frac{10}{96\pi^2}}{C^2} + O\left(\frac{1}{\ln^2 \epsilon}\right)$$

and from (6-80)

$$\left(1 + \frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}\right)\phi_\epsilon^2 \geq \left(1 + \frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}\right)2z\left(\frac{x}{\epsilon}\right) + \frac{5}{96\pi^2} + K_0 + \frac{1}{16\pi^2}\ln\frac{\pi}{\sqrt{6}\epsilon^2} + O\left(\frac{1}{\ln^2 \epsilon}\right) \tag{6-85}$$

Now, for $x \in B_{L\epsilon}$

$$\left|\psi\left(\frac{|x|}{\epsilon}\right)\right| \leq \ln\left(1 + \frac{\pi}{\sqrt{6}}L^2\right) = 2\ln L + \ln\left(\frac{\pi}{\sqrt{6}} + \frac{1}{L^2}\right) = O\Big(\ln(-\ln\epsilon)\Big).$$

So, by definition of ϕ_ϵ, (iii), (6-76) and (6-80), on $B_{L\epsilon}$ we get

$$\left|\frac{\phi_\epsilon}{C}\right| = \left|1 + \frac{1}{C^2}\left[a - \frac{1}{16\pi^2}\psi\left(\frac{|x|}{\epsilon}\right) + K_0 + h(x) + b|x|^2\right]\right| \leq O\Big(C^{-2}\ln(-\ln\epsilon)\Big) + O\Big(C^{-2}\Big) \to 0, \text{ as } \epsilon \to 0.$$

It follows that

$$\left[O\Big((-L\epsilon\ln(L\epsilon))^4\Big) + O\Big(C^{-2}\Big)\right]\frac{\phi_\varepsilon^2}{C^2} = O\left(\frac{1}{\ln^2\epsilon}\right).$$

So, by combining (6-83) with (6-85), for $x \in B_{L\epsilon}$ we can write

$$32\pi^2\left(\frac{1 + \alpha\|\phi_\epsilon\|_2^2}{1 - \alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2 \geq \frac{5}{3} + 32\pi^2 K_0 + \ln\frac{\pi^2}{6} + \ln\frac{1}{\epsilon^4} + \left(1 + \frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}\right)64\pi^2 z\left(\frac{x}{\epsilon}\right) + O\left(\frac{1}{\ln^2\epsilon}\right). \tag{6-86}$$

Hence,

$$\begin{aligned}&\int_{B_{L\epsilon}}\left(e^{32\pi^2\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2}-1\right)dx=\int_{B_{L\epsilon}}e^{32\pi^2\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2}dx+O((\epsilon L)^4)\\&\geq\frac{\pi^2}{6\epsilon^4}e^{\frac{5}{3}+32\pi^2K_0}\int_{B_{L\epsilon}}\left(1+\frac{\pi}{\sqrt{6}}\frac{|x|^2}{\epsilon^2}\right)^{-4\left(1+\frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}\right)}dx+O\Big(\frac{1}{\ln^2\epsilon}\Big)\\&=\frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2K_0}\int_{B_L}\left(1+\frac{\pi}{\sqrt{6}}|x|^2\right)^{-4\left(1+\frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}\right)}dx+O\Big(\frac{1}{\ln^2\epsilon}\Big).\end{aligned}\tag{6-87}$$

By Taylor expansion (cf. [13, Lemma 19]) we have

$$\frac{6\Gamma(2)\Gamma(2+y)}{\Gamma(4+y)}=1-\frac{5y}{6}+O(y^2),\ \text{ as }\ y\to 0.\tag{6-88}$$

Thus, by setting $y=y_\epsilon:=\frac{4\alpha(\gamma+1)\|G\|_2^2}{C^2}$ we obtain

$$\begin{aligned}\int_{B_L}\left(1+\frac{\pi}{\sqrt{6}}|x|^2\right)^{-4\left(1+\frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}\right)}dx&=\int_{B_L}\frac{1}{(1+\frac{\pi}{\sqrt{6}}|x|^2)^{4+y_\epsilon}}dx\\&=6\int_0^{\frac{\pi}{\sqrt{6}}L^2}\frac{s}{(1+s)^{4+y_\epsilon}}ds\\&=6\int_0^{\infty}\frac{s}{(1+s)^{4+y_\epsilon}}ds+O(L^{-2})\\&=\frac{6\Gamma(2)\Gamma(2+y_\epsilon)}{\Gamma(4+y_\epsilon)}+O(L^{-2})\\&=1-\frac{10}{3}\frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}+O(L^{-2})+O(C^{-4})\end{aligned}\tag{6-89}$$

This together with (6-87) yields

$$\begin{aligned}&\int_{B_{L\epsilon}}\left(e^{32\pi^2\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2}-1\right)dx\geq\frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2K_0}\\&+\frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2K_0}\frac{1}{C^2}\left[-\frac{10}{3}\alpha(\gamma+1)\|G\|_2^2+O(C^2L^{-2})+O(C^{-2})+O\Big(\frac{C^2}{\ln^2\epsilon}\Big)\right].\end{aligned}\tag{6-90}$$

By combining (6-90) with (6-82), it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^4}\left(e^{32\pi^2\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2}-1\right)dx \geq \frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2K_0}\\
&+\frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2K_0}\frac{1}{C^2}\left[-\frac{10}{3}\alpha(\gamma+1)\|G\|_2^2+O(C^2L^{-2})+O(C^{-2})+O\Big(\frac{C^2}{\ln^2\epsilon}\Big)\right]\\
&+\frac{32\pi^2\|G\|_2^2}{C^2}+O\Big(C^{-4}\Big)\\
&=\frac{\pi^2}{6}e^{\frac{5}{3}+32\pi^2K_0}\\
&+\frac{\pi^2}{6C^2}e^{\frac{5}{3}+32\pi^2K_0}\left[\Big(192\,e^{-\frac{5}{3}-32\pi^2K_0}-\frac{10}{3}\alpha(\gamma+1)\Big)\|G\|_2^2+O(C^2L^{-2})+O(C^{-2})+O\Big(\frac{C^2}{\ln^2\epsilon}\Big)\right]
\end{aligned} \tag{6-91}$$

Since

$$O(C^2L^{-2})+O(C^{-2})+O\Big(\frac{C^2}{\ln^2\epsilon}\Big)\to 0 \text{ as } \epsilon\to 0$$

and $\gamma \leq \gamma_0$ is bounded, by choosing $\alpha_0 > 0$ small such that

$$192\,e^{-\frac{5}{3}-32\pi^2K_0}-\frac{10}{3}\alpha_0(\gamma+1)>0$$

we conclude that (6-74) holds for all $0 \leq \alpha < \alpha_0$.

CHAPTER 7

Critical Case on Even Dimensions

7.1 Critical Case

In this chapter, we generalize the results obtained in the previous chapter, following the approach outlined in [11]. Most of the results presented here are similar to those in the previous chapter.

Remark 7.1.1 *We would like to point out to the reader that the texts and computations of the proofs presented in this chapter, although for the most part identical to those carried out for the corresponding results in the particular case $n = 4$, have been deliberately included here. The reason is to ensure that the present chapter can be read independently of the previous one, thus avoiding the need for frequent consultation of the special case while reading the general case developed here.*

Let $\{\beta_j\}$ be an increasing sequence converging to $\beta_{2m,m}$. Let us assume from now that $u_k \rightharpoonup u$ in $W^{m,2}(\mathbb{R}^{2m})$ formed by maximizers for the subcritical supremum of $AD(2m, m, \beta_j, \alpha, \gamma)$ ensured by Theorem 1.2, i.e. $\|u_j\|_{W^{m,2}(\mathbb{R}^{2m})}$ and

$$AD(2m, m, \beta_j, \alpha, \gamma) = \int_{\mathbb{R}^{2m}} \left(e^{\beta_j \rho(\|u_j\|_2^2) u_j^2} - 1\right) dx = \sup_{\substack{u \in W^{m,2}(\mathbb{R}^{2m}) \\ \|\nabla^m u_j\|_2^2 + \|u_j\|_2^2 \leq 1}} \int_{\mathbb{R}^{2m}} \left(e^{\beta_j \rho(\|u\|_2^2) u^2} - 1\right) dx, \tag{7-1}$$

with ρ as in (4-13) and $\frac{1+2\alpha-\gamma\alpha^2}{1+\alpha(1-\gamma)-\gamma\alpha^2}\frac{2}{\mathscr{B}_{GN}} < \beta_j < \beta_{2m,m}$. From (2-6), we can assume that each u_j is a positive radially symmetric function and also suppose that $u_j \rightharpoonup u$ in $W^{m,2}_{rad}(\mathbb{R}^{2m})$.

Let $\{\beta_j\}$ be an increasing sequence converging to $\beta_{2m,m}$. Let us assume from now that $u_j \rightharpoonup u$ in $W^{m,2}(\mathbb{R}^{2m})$. In the aim to analyze the assymptotic behavior of maximizers u_j.

Lemma 7.1 *We have*

$$AD(2m, m, \beta_{2m,m}, \alpha, \gamma) = \lim_{j\to\infty} AD(2m, m, \beta_j, \alpha, \gamma) = \lim_{j\to\infty} \int_{\mathbb{R}^{2m}} \left(e^{\beta_j \rho(\|u_j\|_2^2) u_j^2} - 1\right) dx.$$

Proof: We first note that

$$\limsup_{j\to\infty} AD(2m, m, \beta_j, \alpha, \gamma) \leq AD(2m, m, \beta_{2m,m}, \alpha, \gamma). \tag{7-2}$$

The reverse inequality is a consequence of the Fatou Lemma. Indeed, for any $u \in W^{m,2}(\mathbb{R}^{2m})$ with $\|u\|_{W^{m,2}(\mathbb{R}^{2m})} \leq 1$ we have

$$\liminf_{j} \int_{\mathbb{R}^{2m}} \left(e^{\beta_j \rho(\|u\|_2^2) u^2} - 1\right) dx \geq \int_{\mathbb{R}^{2m}} \left(e^{\beta_{2m,m} \rho(\|u\|_2^2) u^2} - 1\right) dx$$

which implies that

$$\liminf_{j\to\infty} AD(2m, m, \beta_j, \alpha, \gamma) \geq \int_{\mathbb{R}^{2m}} \left(e^{\beta_{2m,m} \rho(\|u\|_2^2) u^2} - 1\right) dx.$$

Taking the supremum over $u \in W^{m,2}(\mathbb{R}^{2m})$ with $\|u\|_{W^{m,2}(\mathbb{R}^{2m})} \leq 1$ we obtain

$$\liminf_{j\to\infty} AD(2m, m, \beta_j, \alpha, \gamma) \geq AD(2m, 2, \beta_{2m,m}, \alpha, \gamma). \tag{7-3}$$

From (7-2) and (7-3) we conclude the result. ■

Analogously, a straightforward computation shows that the Euler-Lagrange equation of u_j is given by

$$\begin{cases} (-\Delta)^m u_j + u_j = \frac{u_j}{\lambda_j} \tilde{\zeta}_j e^{\beta_j \tilde{\zeta}_j u_j^2} + \mu_j u_k \quad \text{in } \mathbb{R}^{2m}, \\ \|\nabla^m u_j\|_2^2 + \|u_j\|_2^2 = 1, \\ \beta_j \nearrow \beta_{2m,m}; \\ \tilde{\zeta}_j = \rho(\|u_j\|_2^2) = \frac{1+\alpha\|u_j\|_2^2}{1-\gamma\alpha\|u_j\|_2^2}, \\ \mu_j = \frac{\alpha(1+\gamma)}{(1-\gamma\alpha\|u_j\|_2^2)^2}, \\ \lambda_j = \frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2} \int_{\mathbb{R}^{2m}} u_j^2 e^{\beta_j \tilde{\zeta}_j u_j^2} dx. \end{cases} \tag{7-4}$$

By using Lemma 2.5 together with (7-4), we can see that $u_j \in C^\infty(\mathbb{R}^{2m})$. From Lemma 2.2, we can always take a point $x_j \in \mathbb{R}^{2m}$ such that

$$c_j = u_j(x_j) = \max_{\mathbb{R}^{2m}} |u_j|. \tag{7-5}$$

We divide our argument into two cases:

(a) $\sup_j c_j < \infty$,

(b) $c_j \to +\infty$, as $j \to \infty$.

Definition 7.1.1 *Let (u_j) be a sequence in $W^{m,2}(\mathbb{R}^{2m})$ such that $u_j \rightharpoonup u$ in $W^{m,2}(\mathbb{R}^{2m})$. We say that (u_j) is a normalized vanishing sequence [(NVS) in short] if $\|u_j\|_{W^{m,2}(\mathbb{R}^{2m})} = 1$, $u = 0$ and*

$$\lim_{\rho\to\infty}\lim_{j\to\infty}\int_{B_\rho}(e^{\beta_j\rho(\|u_j\|_2^2)u_j^2}-1)dx = 0.$$

Firstly, we deal with the bounded case.

Lemma 7.2 *If $\sup_j c_j < +\infty$, then only one of the following alternatives is satisfied:*

(*i*) *$u \neq 0$ and $AD(2m, m, \beta_{2m,m}, \alpha, \gamma)$ is achieved by some function in $W^{m,2}_{rad}(\mathbb{R}^{2m})$;*
(*ii*) *(u_j) is a NVS and $AD(2m, m, \beta_{2m,m}, \alpha, \gamma) \leq \beta_{2m,m}\rho(1)$.*

<u>**Proof:**</u> If $\sup_j c_j < +\infty$, then by standard elliptic estimates we conclude that $u_j \to u$ in $C^{2m-1}_{loc}(\mathbb{R}^{2m})$, see Lemma 2.5. Then, for any $\rho > 1$, from Lemma 2.2

$$\begin{aligned} e^{\beta_j\rho(\|u_j\|_2^2)u_j^2}-1-\beta_j\rho(\|u_j\|_2^2)u_j^2 &= \sum_{i=2}^{\infty}\frac{(\beta_j\rho(\|u_j\|_2^2)u_j^2)^i}{i!} \leq \sum_{i=2}^{\infty}\frac{(\rho(1)\beta_j)^i}{i!}u_j^{2i} \\ &\leq \sum_{i=2}^{\infty}\frac{(\rho(1)\beta_j)^i}{i!}\left(\frac{C_m}{\rho^{2m-1}}\right)^{i-1}u_j^2 \leq \frac{u_j^2}{C_m\rho}\sum_{i=2}^{\infty}\frac{(\beta_{2m,m}C_m\rho(1))^i}{i!} \\ &= C'_m\frac{u_j^2}{\rho}, \text{ for any } m \end{aligned} \tag{7-6}$$

on $\mathbb{R}^{2m}\setminus B_\rho$, where C'_m is independent of j and ρ. Thus, by (7-6)

$$\lim_{\rho\to\infty}\lim_{j\to\infty}\int_{\mathbb{R}^{2m}\setminus B_\rho}\left[e^{\beta_j\rho(\|u_j\|_2^2)u_j^2}-1-\beta_j\rho(\|u_j\|_2^2)u_j^2\right]dx = 0. \tag{7-7}$$

Up to a subsequence, we can assume that $\|u_j\|_2^2 \to \theta$ with $\theta \in [0,1]$, as $j \to \infty$. Thus, the convergence in $C^{2m-1}_{loc}(\mathbb{R}^{2m})$ yield

$$\lim_{\rho\to\infty}\lim_{j\to\infty}\int_{B_\rho}\left[e^{\beta_j\rho(\|u_j\|_2^2)u_j^2}-1-\beta_j\rho(\|u_j\|_2^2)u_j^2\right]dx = \int_{\mathbb{R}^{2m}}\left[e^{\beta_{2m,m}\rho(\theta)u^2}-1-\beta_{2m,m}\rho(\theta)u^2\right]dx. \tag{7-8}$$

Since

$$\lim_{j\to\infty}\int_{\mathbb{R}^{2m}}\beta_j\rho(\|u_j\|_2^2)u_j^2dx = \beta_{2m,m}\rho(\theta)\theta$$

it follows from (7-1), (7-7) and (7-8) that

$$\begin{aligned} \lim_{j\to\infty}AD(2m,m,\beta_j,\alpha,\gamma) &= \lim_{j\to\infty}\int_{\mathbb{R}^{2m}}\left(e^{\beta_j\rho(\|u_j\|_2^2)u_j^2}-1\right)dx \\ &= \int_{\mathbb{R}^{2m}}\left(e^{\beta_{2m,m}\rho(\theta)u^2}-1\right)dx + \beta_{2m,m}\rho(\theta)(\theta-\|u\|_2^2). \end{aligned} \tag{7-9}$$

From (7-9) and Lemma 7.1, we can write

$$AD(2m, m, \beta_{2m,m}, \alpha, \gamma) = \int_{\mathbb{R}^{2m}} \left(e^{\beta_{2m,m}\rho(\theta)u^2} - 1\right) dx + \beta_{2m,m}\rho(\theta)(\theta - \|u\|_2^2). \qquad (7\text{-}10)$$

If $u \neq 0$, we define for $\tau \geq 1$ and v given by

$$\tau^{2m} = \frac{\theta}{\|u\|_2^2} \text{ and } v(x) = u\left(\frac{x}{\tau}\right). \qquad (7\text{-}11)$$

Then, by (7-10)

$$\begin{aligned} AD(2m, m, \beta_{2m,m}, \alpha, \gamma) &\geq \int_{\mathbb{R}^{2m}} \Phi\left(\beta_{2m,m}\rho(\|v\|_2^2)v^2\right) dx = \tau^{2m} \int_{\mathbb{R}^{2m}} \left(e^{\beta_{2m,m}\rho(\theta)u^2} - 1\right) dx \\ &= \int_{\mathbb{R}^{2m}} \left(e^{\beta_{2m,m}\rho(\theta)u^2} - 1\right) dx + (\tau^{2m} - 1)\beta_{2m,m}\rho(\theta)\|u\|_2^2 \\ &+ (\tau^{2m} - 1)\int_{\mathbb{R}^{2m}} \left[e^{\beta_{2m,m}\rho(\theta)u^2} - 1 - \beta_{2m,m}\rho(\theta)u^2\right] dx \\ &= AD(2m, 2, \beta_{2m,m}, \alpha, \gamma) + (\tau^{2m} - 1)\int_{\mathbb{R}^{2m}} \left[e^{\beta_{2m,m}\rho(\theta)u^2} - 1 - \beta_{2m,m}\rho(\theta)u^2\right] dx. \end{aligned}$$

This forces $\tau = 1$. Consequently, we conclude that u is a maximizer for $AD(2m, m, \beta_{2m,m}, \alpha, \gamma)$.

Now, suppose $u = 0$. From the convergence $u_j \to 0$ in $C^{2m-1}_{loc}(\mathbb{R}^{2m})$ we get

$$\lim_{j\to\infty} \int_{B_\rho} (e^{\beta_j\rho(\|u_j\|_2^2)u_j^2} - 1)dx = \int_{B_\rho} (e^{\beta_{2m,m}\rho(\theta)u^2} - 1)dx = 0, \text{ for any } \rho > 0.$$

Thus, (u_j) is a NVS according with the Definition 7.1.1. Finally, directly from (7-10) with $u = 0$, we obtain

$$AD(2m, m, \beta_{2m,m}, \alpha, \gamma) = \beta_{2m,m}\rho(\theta)\theta \leq \beta_{2m,m}\rho(1).$$

■

7.1.1 Blow-up Analysis

In what follows, we'll perform a detailed non-standard blow-up analysis to consider the scenario where c_j approaches $+\infty$. The proof is based on the works [11], [56], [16].

Now, we'll prove again the important result about the behaviour of the norms in $\|\nabla^m u_j\|_2^2 + \|u_j\|_2^2$.

Lemma 7.3 *It holds*

$$|\nabla^m u_j|^2 dx \overset{*}{\rightharpoonup} \delta_0, \tag{7-12}$$

in the sense of measure, where δ_0 *is the Dirac Measure supported at 0. Furthermore, we have* $u \equiv 0$, $\tilde{\zeta}_j \to 1$ *and* $\mu_j \to \alpha(1+\gamma)$ *as* $j \to \infty$.

Proof: By contradiction, suppose that $|\nabla^m u_j|^2 \mathrm{d}x \overset{*}{\not\rightharpoonup} \delta_0$. Then, there are $R > 0$ and $\mu < 1$ such that

$$\lim_{j\to\infty} \int_{B_R} |\nabla^m u_j|^2 \mathrm{d}x = \mu < 1. \tag{7-13}$$

We now use the Sani-Ruf cut function defined in (4-4) $v_{j,R}(r) = u_j(r) - \tilde{u}_j(R)$ for $x \in B_R$ with $r = |x|$. Then $v_{j,R} \in W^{m,2}_{\mathcal{N}}(B_R)$, and $\nabla^m v_{j,R} = \nabla^m u_j$ in B_R

$$\int_{B_R} |\nabla^m v_{j,R}|^2 \mathrm{d}x = \int_{B_R} |\nabla^m u_j|^2 \mathrm{d}x. \tag{7-14}$$

By Lemma 2.2, for any $\delta > 0$ we derive

$$u_j^2 \leq (1+\delta) v_{j,R}^2 + c_\delta u_j^2(R) \leq (1+\delta) v_{j,R}^2 + c_\delta \frac{C}{R^{2m-1}},$$

with $c_\delta = (1-(1+\delta)^{-1})^{-1}$. Now, set $\hat{v}_{j,R} := \frac{v_{j,R}}{\|\nabla^m v_{j,R}\|_2}$. We have

$$\begin{aligned}
\mathrm{e}^{(\beta_j \tilde{\zeta}_j u_j^2)} &\leq \mathrm{e}^{\left(\beta_j \tilde{\zeta}_j (1+\delta) \hat{u}_{j,R}^2 + c_\delta \beta_j \tilde{\zeta}_j \frac{C}{R^{2m-1}}\right)} \\
&\leq \mathrm{e}^{c_\delta \beta_{2m,m} \rho(1) \frac{C}{R^{2m-1}}} \mathrm{e}^{\left(\beta_j \tilde{\zeta}_j (1+\delta) \hat{v}_{j,R}^2\right)} \\
&= C(R,\delta) \mathrm{e}^{\left(\beta_j \tilde{\zeta}_j (1+\delta) \hat{v}_{j,R}^2\right)} \\
&= C(R,\delta) \mathrm{e}^{\left(\beta_j (1+\delta) \tilde{\zeta}_j \|\nabla^m v_{j,R}\|_2^2 \hat{v}_{j,R}^2\right)},
\end{aligned}$$

on the ball B_R. Recalling $\|\nabla^m u_j\|_2^2 + \|u_j\|_2^2 = 1$ and $\|\nabla^m \hat{v}_{j,R}\|_2 = \|\nabla^m u_j\|_{L^2(B_R)} \leq \|\nabla^m u_j\|_2$, from (7-13) and (7-14) we can write

$$\begin{aligned}
\tilde{\zeta}_j \|\nabla^m v_{j,R}\|_2^2 &= \frac{(1+\alpha\|u_j\|_2^2)\|\nabla^m v_{j,R}\|_2^2}{(1-\gamma\alpha\|u_j\|_2^2)} \\
&= \frac{\left(1+\alpha-\alpha\|\nabla^m u_j\|_2^2\right)\|\nabla^m v_{j,R}\|_2^2}{(1-\gamma\alpha\|u_j\|_2^2)} \\
&\leq \frac{\left(1+\alpha-\alpha\|\nabla^m v_{j,R}\|_2^2\right)\|\nabla^m v_{j,R}\|_2^2}{(1-\gamma\alpha+\gamma\alpha\|\nabla^m \hat{v}_{j,R}\|_2^2)}.
\end{aligned}$$

Letting $j \to \infty$, we get

$$\lim_{j\to\infty} \left(\beta_j (1+\delta) \tilde{\zeta}_j \|\nabla^m v_{j,R}\|_2^2\right) \leq \beta_{2m,m} (1+\delta) \frac{\mu + \alpha(1-\mu)\mu}{1-\gamma\alpha(1-\mu)}. \tag{7-15}$$

Since the positive numbers γ, α and $1-\mu$ are less than 1, we obtain $1-\gamma\alpha(1-\mu) > 0$, thus

$$\begin{aligned}\frac{\mu+\alpha(1-\mu)\mu}{1-\gamma\alpha(1-\mu)} < 1 \text{ iff } & \alpha(1-\mu)\mu < (1-\mu)-\gamma\alpha(1-\mu) \\ \text{iff } & \alpha\mu < 1-\gamma\alpha \\ \text{iff } & \mu+\gamma < \frac{1}{\alpha}. \end{aligned} \tag{7-16}$$

We are assuming the condition $0 < \gamma < \frac{1}{\alpha} - 1$. Thus, for any $0 < \mu < 1$ we obtain $\mu + \gamma < (\mu - 1) + \frac{1}{\alpha} < \frac{1}{\alpha}$. So, (7-16) holds and from (7-17), for $\delta > 0$ small enough it follows that

$$\lim_{j\to\infty} \left(\beta_j(1+\delta)\tilde{\zeta}_j\|\nabla^m v_{j,R}\|_2^2\right) < \beta_{2m,m}. \tag{7-17}$$

Therefore, we can apply the Adams-Trudinger-Moser type inequality (2.3) by C. Tarsi [66, Theorem 4] to conclude that $e^{\left(\beta_j(1+\delta)\tilde{\zeta}_j\|\nabla^m v_{j,R}\|_2^2 \hat{v}_{j,R}^2\right)}$ is bounded in $L^p(B_R)$ for some $p > 1$. Consequently, $e^{(\beta_j\tilde{\zeta}_j u_j^2)}$ is also bounded in $L^p(B_R)$. Since, (u_j) is bounded in $L^q(B_R)$ for any $q < \infty$, by Hölder inequality together with Lemma 7.4 below, it holds that

$$\frac{u_j}{\lambda_j} e^{\beta_j \tilde{\zeta}_j u_j^2}$$

is bounded in $L^s(B_R)$ for any $s > 1$. Hence, by apply standard elliptic estimates we have that (u_j) is uniformly bounded in $B_{R/2}$ which contradicts the hypothesis that $c_j \to \infty$. Thus, we need to have $|\nabla^m u_j|^2 dx \overset{*}{\rightharpoonup} \delta_0$. As direct consequence, since $\|\nabla^m u_j\|_2^2 + \|u_j\|_2^2 = 1$, we have $\|u_j\|_2^2 \to 0$ as $j \to \infty$. Thus, $u_j \to 0$ and $u \equiv 0$ in $L^2(\mathbb{R}^{2m})$, which also implies that $\tilde{\zeta}_j \to 1$, $\mu_j \to \alpha(1+\gamma)$ as $j \to \infty$.

■

Next, we state the result without proof, as the argument is identical to that of the corresponding result in the previous chapter.

Lemma 7.4 *There holds* $\inf_{j\to\infty} \lambda_j > 0$.

7.1.2 Assymptotic Behavior

Here, we'll proceed exactly the same way as was done in [11]. Since u_j is a bounded sequence in $W^{m,2}_{rad}(\mathbb{R}^{2m})$, we have

$$\begin{cases} u_j \rightharpoonup u \text{ in } W^{m,2}_{rad}(\mathbb{R}^{2m}); \\ u_j \to u \text{ in } L^p(\mathbb{R}^{2m}), \ \forall p > 2; \\ \beta_j \nearrow \beta_{2m,m}. \end{cases} \tag{7-18}$$

In the case $c_j \to \infty$, from Lemma 2.2 we can assume that the sequence $(x_j) \subset \mathbb{R}^{2m}$ in (6-5) satisfies

$$x_j \to 0, \text{ as } j \to \infty.$$

With aim to study the assymptotic behavior of $\{u_j\}$ near to the blow-up point, let us define

$$r_j^{2m} := \frac{\lambda_j}{c_j^2 \, e^{\beta_j \tilde{\zeta}_j c_j^2}} \tag{7-19}$$

Lemma 7.5 *For any $\xi < \beta_{2m,m}$, we have $\limsup_{j\to\infty} r_j^{2m} c_j^2 \, e^{\xi \tilde{\zeta}_j c_j^2} = 0$. In particular,*

$$\lim_{j\to+\infty} r_j^{2m} = 0$$

Proof: Let $\xi < \beta_{2m,m}$, then

$$\begin{aligned}
r_j^{2m} c_j^2 \, e^{\xi \tilde{\zeta}_j c_j^2} &= e^{(\xi-\beta_j)\tilde{\zeta}_j c_j^2} \frac{\tilde{\zeta}_j}{1-\mu_j \|u_j\|_2^2} \int_{\mathbb{R}^{2m}} u_j^2 \, e^{\beta_j \tilde{\zeta}_j u_j^2} \, dx \\
&\le \frac{\tilde{\zeta}_j}{1-\mu_j \|u_j\|_2^2} \int_{\mathbb{R}^{2m}} u_j^2 \, e^{\beta_j \tilde{\zeta}_j u_j^2} \, e^{(\xi-\beta_j)\tilde{\zeta}_j u_j^2} \, dx \\
&\le \frac{\tilde{\zeta}_j}{1-\mu_j \|u_j\|_2^2} \int_{\mathbb{R}^{2m}} u_j^2 \, e^{\xi \tilde{\zeta}_j u_j^2} \, dx.
\end{aligned}$$

From Lemma 6.3 we have $\frac{\tilde{\zeta}_j}{1-\mu_j \|u_j\|_2^2} \to 1$, as $j \to +\infty$. In particular, $\frac{\tilde{\zeta}_j}{1-\mu_j \|u_j\|_2^2}$ is bounded and we also have

$$\begin{aligned}
r_j^{2m} c_j^2 \, e^{\xi \tilde{\zeta}_j c_j^2} &\le C\Big(\int_{\mathbb{R}^{2m}} u_j^2 (e^{\xi \tilde{\zeta}_j u_j^2} - 1) dx + \int_{\mathbb{R}^{2m}} u_j^2 dx\Big) \\
&\le C\left(\left(\int_{\mathbb{R}^{2m}} |u_j|^p dx\right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^{2m}} (e^{\xi \tilde{\zeta}_j u_j^2} - 1)^{\frac{p}{p-2}} dx\right)^{\frac{p-2}{p}} + \int_{\mathbb{R}^{2m}} u_j^2 dx\right) \\
&\le C(p)\left(\int_{\mathbb{R}^{2m}} |u_j|^p dx\right)^{\frac{2}{p}} \left(\int_{\mathbb{R}^{2m}} \left(e^{\frac{\xi \tilde{\zeta}_j p}{p-2} u_j^2} - 1\right)^{\frac{p}{p-2}} dx\right)^{\frac{p-2}{p}} + C(p) \int_{\mathbb{R}^{2m}} u_j^2 dx,
\end{aligned}$$

for some constants C and $C(p)$, which are independent of j. Taking into account the Adams-Adimurthi-Druet inequality in (1-13), Lemma 6.3 and (6-18) we obtain the result. ■

Now, we need to define again the same auxiliary sequences to understand the assymptotic behavior of u_j near to the blow-up point. Namely,

$$\begin{cases} w_j(x) = \dfrac{u_j(x_j + r_j x)}{c_j} \\ z_j(x) = c_j(u_j(x_j + r_j x) - c_j) \\ v_j(x) = u_j(x_j + r_j x) - c_j \end{cases} \tag{7-20}$$

where the sequences are all defined on the sequence of sets $\Omega_j = \{x \in \mathbb{R}^{2m}; x_j + r_j x \in B_1(0)\}$. So we state

Lemma 7.6 $w_j(x) \to 1$ *in* $C^{2m-1}_{Loc}(\mathbb{R}^{2m})$.

Proof: By the Euler-Lagrange equation (7-4), the definition of r_j and the fact of $w_j \leq 1$, we know that for any $R > 0$ and $x \in B_R(0)$, $w_j(x)$ satisfies

$$\begin{aligned} |(-\Delta)^m(w_j(x))| &= \left| \frac{r_j^{2m}}{c_j}((-\Delta)^m u_j)(x_j + r_j x) \right| \\ &= \left| \frac{r_j^{2m}}{c_j} \left(\lambda_j^{-1} \tilde{\zeta}_j u_j(x_j + r_j x)\, e^{\beta_j \tilde{\zeta}_j u_j^2(x_j + r_j x)} + (\mu_j - 1) u_j(x_j + r_j x) \right) \right| \\ &\leq \left| r_j^{2m} \left(\lambda_j^{-1} \tilde{\zeta}_j w_j(x)\, e^{\beta_j \tilde{\zeta}_j c_j^2} + (\mu_j - 1) w_j(x) \right) \right| \\ &= \left| \frac{\tilde{\zeta}_j w_j(x)}{c_j^2} + (\mu_j - 1) \frac{w_j(x) \lambda_j}{c_j^2\, e^{\beta_j \tilde{\zeta}_j c_j^2}} \right| \\ &= \left| \frac{w_j(x)}{c_j^2} \left(\tilde{\zeta}_j + (\mu_j - 1) \frac{\lambda_j}{e^{\beta_j \tilde{\zeta}_j c_j^2}} \right) \right| \\ &\leq \frac{1}{c_j^2} \left| \left(\tilde{\zeta}_j + (\mu_j - 1) \frac{\lambda_j}{e^{\beta_j \tilde{\zeta}_j c_j^2}} \right) \right| \to 0, \end{aligned}$$

where we have used that $\mu_j \to \alpha(1 + \gamma)$, $\tilde{\zeta}_j \to 1$ and

$$\frac{\lambda_j}{e^{\beta_j \tilde{\zeta}_j c_j^2}} \leq \frac{\tilde{\zeta}_j}{1 - \mu_j \|u_j\|_2^2} \int_{\mathbb{R}^{2m}} u_j^2\, e^{\beta_j \tilde{\zeta}_j (u_j^2 - c_j^2)}\, dx \leq M \|u_j\|_2^2 \leq C.$$

Furthermore, $w_j(x)$ is bounded in $L^1_{loc}(\mathbb{R}^{2m})$. By, the standard regularity theory, we got that for any $R > 0$ and $0 < \kappa < 1$, the sequence of $\|w_j(x)\|_{C^{2m-1,\kappa}(B_R(0))}$ is uniformly bounded for every j. Finally, by the Ascoli-Arzela Theorem, there exists a function $w \in C^{2m-1}(\mathbb{R}^{2m})$ such that the sequence $w_j(x)$ converges to w in $C^{2m-1}(\mathbb{R}^{2m})$ having the property of $(-\Delta)^m w(x) = 0$ for all $x \in \mathbb{R}^{2m}$. Now, since $w_j(0) = 1$, by the Liouville's Theorem for harmonic functions we obtain w is constant and equal to 1 in $\mathbb{R}^{2m}$. ■

Lemma 7.7 *It holds* $v_j(x) = u_j(x_j + r_j x) - u_j(x_j) \to 0$ *in* $C^{2m-1}_{Loc}(\mathbb{R}^{2m})$ *as* $j \to 0$. *Therefore,*

$$|\nabla^i u_j(x)| = o\left(\frac{1}{r_j^i}\right) \text{ in } B_{Rr_j}, \quad i = 1,2,3,...,2m-1,$$

for any $R > 0$.

Proof: We can notice that v_j satisfies the equation

$$\begin{aligned}(-\Delta)^m(v_j) &= r_j^{2m}\lambda_j^{-1}\tilde{\zeta}_j u_j(x_j + r_j x)\, e^{(\beta_j\tilde{\zeta}_j u_j^2(x_j+r_jx))} + r_j^{2m}(\mu_j - 1)u_j(x_j + r_j x)\\ &= \frac{\tilde{\zeta}_j u_j(x_j + r_j x)}{c_j^2}\, e^{(\beta_j\tilde{\zeta}_j(u_j^2(x_j+r_jx)-c_j^2))} - r_j^{2m}(\mu_j - 1)u_j(x_j + r_j x).\end{aligned}$$

By Sobolev embedding theorem $H^{m-2}(\mathbb{R}^{2m}) \hookrightarrow L^m(\mathbb{R}^{2m})$, it follows that

$$\int_{\mathbb{R}^{2m}} |\Delta v_j|^m dx = \int_{\mathbb{R}^{2m}} |\Delta u_j|^m dx \le c\|u_j\|_{H^m(\mathbb{R}^{2m})}. \tag{7-21}$$

By setting $\Delta v_j = g_j$ and then $\Delta^{(m-1)}g_j = f_k$, where

$$f_j := \frac{\tilde{\zeta}_j u_j(x_j + r_j x)}{c_j^2}\, e^{(\beta_j\tilde{\zeta}_j(u_j^2(x_j+r_jx)-c_j^2))} - r_j^{2m}(\mu_k - 1)u_j(x_j + r_j x).$$

Since $f_k \in L^p_{loc}(\mathbb{R}^{2m})$ for any $p > 1$ and $g_j \in L^1_{loc}(\mathbb{R}^{2m})$ and using the standard elliptic estimates, Lemma 2.5 and Morrey's inequality, we get for some $\alpha > 0$ that $\|g_j\|_{C^{2m-3,\alpha}(B_R)} \le c$, for any $R > 0$. Therefore by Pizzetti's formula (2-14), we obtain

$$\int_{B_R} v_j(x)dx = \sum_{i=0}^{m-1} c_i R^{2(m+i)}\Delta^i v_j(0) + c_m R^{4m}\Delta^m v_j(t), \quad t \in B_R$$

where B_R is a ball centered at origin with radius R.
Notice now, that $v_j(0) = 0$, one can conclude that $v_k(x)$ is bounded in $L^1_{loc}(\mathbb{R}^{2m})$. Again, using standard elliptic estimates, we derive that there exists some $v \in C^{2m-1}(\mathbb{R}^{2m})$ such that $v_j(x) \to v$ in $C^{2m-1}_{loc}(\mathbb{R}^{2m})$, with v satisfying $(-\Delta)^m v = 0$.
By Lemma 2.8 , it follows that v has degree at most $2(m-1)$. Since

$$\int_{\mathbb{R}^{2m}} |\Delta v|^m dx \le \lim_{j\to\infty} \int_{\mathbb{R}^{2m}} |\Delta v_j|^m dx \le c,$$

and $v \le 0$, then v must be a constant. Joining this with the fact that $v(0) = 0$, we conclude $v \equiv 0$. ■

In the next lemma we'll establish a gradient estimates on B_{Rr_j}, and it will be of utmost importance for our aim in studying and determining the limit behavior of $z_j(x)$.

The proof is basically the same procedure did in [10]. Let us state first a theorem which can be found on [52].

Theorem 7.8 *(Martinazzi, Thm. 10) Let* u *solve* $\Delta^m u = f \in L(logL)^\alpha$ *in smooth bounded* Ω *with the Dirichlet boundary condition for some* $0 \leq \alpha \leq 1$ *and* $n \geq 2m$*. Then* $\nabla^{2m-l} u \in L^{(\frac{n}{n-l},\frac{1}{\alpha})}(\Omega)$, $1 \leq l \leq 2m-1$ *and*

$$\|\nabla^{2m-l} u\|_{(\frac{n}{n-l},\frac{1}{\alpha})} \leq C\|f\|_{L(logL)^\alpha}. \tag{7-22}$$

Lemma 7.9 *It holds that for any* $R > 0$ *and* $1 \leq k \leq 2m-1$,

$$c_j \int_{B_{Rr_j}} |\nabla^k u_j| dx \leq c(Rr_j)^{2m-k}.$$

And furthermore, we have

$$\int_{B_R} |\nabla^k z_j| dx = \frac{c_j}{r_j^{2m-k}} \int_{B_{Rr_j}} |\nabla^k u_j| dx \leq cR^{2m-k}. \tag{7-23}$$

<u>Proof:</u> For any $R_0 > 0$, let $u_j^{R_0}$ poly-harmonic functions wich are solutions for the following problem

$$\begin{cases} (-\Delta)^m u_j^{R_0} = 0, & \text{in } \overline{B_{R_0}(x_j)} \\ \partial_\nu^i u_j^{R_0} = \partial_\nu^i u_j, & \text{on } \partial\overline{B_{R_0}(x_j)};\ \ i = 0,1,...,m-1 \end{cases}$$

By the Radial lemma and the elliptic estimates (Lemma 2.5), we obtain

$$\|u_j^{R_0}\|_{C^{2m}(B_{R_0})} < \frac{c}{R_0^\tau}, \text{ for some } \tau > 0. \tag{7-24}$$

And notice that $u_j - u_j^{R_0}$ satisfies the equation

$$\begin{cases} (-\Delta)^m (u_j - u_j^{R_0}) = \lambda_j^{-1} u_j \tilde{\zeta}_j e^{(\beta_j \tilde{\zeta}_j u_j^2)} + (\mu_j - 1)u_j, & \text{in } \overline{B_{R_0}(0)} \\ \partial_\nu^i (u_j - u^{R_0}) = 0, & \text{on } \partial\overline{B_{R_0}(0)}, i = 0,1,...,m-1. \end{cases}$$

Define the sequence $f_j := \lambda_j^{-1} u_j \tilde{\zeta}_j e^{(\beta_j \tilde{\zeta}_j u_j^2)} + (\mu_k - 1)u_j$ and the Zygmund space $L(logL)^\alpha(B_{R_0}) := \{f \in L^1(B_{R_0}) : \|f\|_{L(logL)^\alpha} < \infty\}$, where $\|f\|_{L(logL)^\alpha}$ denotes the norm defined as

$$\|f\|_{L(logL)^\alpha} := \int_{B_{R_0}} |f|(log^\alpha(2+|f|))dx.$$

We have f_j is bounded in $L(logL)^{\frac{1}{2}}(B_{R_0})$. So, as consequence of the definitions above and joining the result in Theorem (7.8), we got

$$\|\nabla^k (u_j - u_j^{R_0})\|_{L\left(\frac{2m}{j},2\right)} \leq C, \quad k = 1,2,...,2m-1. \tag{7-25}$$

onde $\|\cdot\|_{L\left(\frac{2m}{j},2\right)}$ is the Lorentz norm.
We estimate,

$$|(-\Delta)^m (u_j - u_j^{R_0})^2| \leq |2(u_k - u_j^{R_0})(-\Delta)^m (u_j - u_j^{R_0})| + C\sum_{j=1}^{2m-1} |\nabla^j (u_j - u_j^{R_0})||\nabla^{2m-j}(u_j - u_j^{R_0})|$$

Due to (7-25) and a Hölder type inequality by O'Neil in [57], the term $\sum_{j=1}^{2m-1} |\nabla^j (u_k - u_j^{R_0})||\nabla^{2m-j}(u_k - u_j^{R_0})|$ is bounded in $L^1(B_{R_0})$. Now we must prove that the term $|2(u_k - u_j^{R_0})(-\Delta)^m (u_k - u_j^{R_0})|$ is also bounded in $L^1(B_{R_0})$. Indeed, we separate in two integrals as follows

$$\int_{B_{R_0}} |2(u_k - u_j^{R_0})(-\Delta)^m (u_k - u_j^{R_0})| dx \leq 2\left(\int_{B_{R_0}} |u_k(-\Delta)^m (u_k)| dx + \int_{B_{R_0}} |u_k^{R_0}(-\Delta)^m (u_k)| dx\right) := I_1 + I_2.$$

To majorate I_1, we can notice by the Euler-Lagrange equation (7-4) and applying integration by parts, we derive

$$\begin{aligned} I_1 &\leq 2\int_{\mathbb{R}^{2m}} \left(\tilde{\zeta}\frac{u_j^2}{\lambda_j} e^{(\beta_j \tilde{\zeta}_j u_j^2)} + (\mu_k - 1)u_j^2\right) dx \\ &= 2\int_{\mathbb{R}^{2m}} |\nabla^m u_j|^2 dx + (\mu_k + 1)\int_{\mathbb{R}^{2m}} |u_j|^2 dx \leq c. \end{aligned}$$

To majorate I_2, we have

$$I_2 = 2\int_{B_{R_0}} |u_j^{R_0}(-\Delta)^m (u_j)| dx \leq c\int_{B_{R_0}} |u_j(-\Delta)^m (u_k)| dx + c\int_{B_{R_0}\cap\{|u_k|\leq 1\}} |(-\Delta)^m (u_k)| dx \leq c(R_0). \tag{7-26}$$

Hence, $\int_{B_{R_0}} |(-\Delta)^m ((u_j - u_j^{R_0})^2)| dx \leq c$.
Now, by the same procedure as in proof of Lemma 6 in [52], we have for $R > 0$ and any $1 \leq j \leq 2m-1$,

$$\int_{B_{Rr_j}} \nabla^j ((u_j - u_j^{R_0})^2) dx \leq c(Rr_j)^{2m-j}. \tag{7-27}$$

Combining (7-27) and (7-24), we get by Lemma 7.7, that

$$\int_{B_{Rr_j}} |\nabla^j u_j^2| dx \leq c \Big\{ \int_{B_{Rr_j}} \nabla^j ((u_j - u_j^{R_0})^2) dx + \int_{B_{Rr_j}} \left| \nabla^j \left(u_j^{R_0} \right)^2 \right| dx$$
$$+ c \sum_{i=0}^{j} \int_{B_{Rr_j}} |\nabla^i u_j^{R_0} \nabla^{j-i} u_j| dx \Big\} \tag{7-28}$$
$$\leq c \int_{B_{Rr_j}} \nabla^j ((u_j - u_j^{R_0})^2) dx + o\left(r_j^{2m-j} \right)$$

On the other hand,

$$c_j |\nabla^j u_j| \leq c u_j |\nabla^j u_j| \leq \left(\nabla^j (u_j^2) + \sum_{i=1}^{j} |\nabla^i u_j| |\nabla^{j-i} u_j| \right) \leq c \nabla^j (u_j^2) + o\left(\frac{1}{r_j^2} \right) \tag{7-29}$$

. Finally, by (7-28) and (7-29), we conclude, for any $1 \leq j \leq 2m-1$, that

$$c_j \int_{B_{Rr_j}} |\nabla^j (u_j)| dx \leq c (R_{r_j})^{2m-j}.$$

Thus, for any $R > 0$,

$$\int_{B_R} |\nabla^j z_j| dx = \frac{c_j}{r_j^2} \int_{B_{Rr_j}} |\nabla^{2m-j} u_j| dx \leq c R^{2m-j}.$$

■

Let us analyze the limit behavior of $z_j(x)$.

Lemma 7.10 *It holds $z_j(x) \to z$ in $C^{2m-1}_{loc}(\mathbb{R}^{2m})$ with z satisfying the equation $(-\Delta)^m z = e^{(2\beta_{2m,m} z)}$. Moreover,*

$$z(x) = \frac{m}{\beta_{2m,m}} log \left(\frac{1}{1 + \frac{\omega_{2m}^{1\backslash m}}{4} |x|^2} \right) \tag{7-30}$$

where, $\omega_{2m} = \frac{2^{m+1}\pi^m}{(2m-1)!!}$, *and*

$$\int_{\mathbb{R}^{2m}} e^{\{2\beta_{2m,m} z(x)\}} dx = 1.$$

Proof: By the Euler-Lagrange equation (7-4), we can notice that z satisfies

$$
\begin{aligned}
(-\Delta)^m(z_j) + c_k r_j^{2m}(1-\mu_k)u_j(x_j+r_jx) &= \frac{c_k r_j^{2m}}{\lambda_j}\tilde{\zeta}_j u_j(x_j+r_jx)\, e^{(\beta_j\tilde{\zeta}_j u_j^2(x_j+r_jx))} \\
&= \frac{\tilde{\zeta}_j u_j(x_j+r_jx)}{c_j}\, e^{(\beta_j\tilde{\zeta}_j(u_j^2(x_j+r_jx)-c_j^2))} \\
&= \frac{\tilde{\zeta}_j u_j(x_j+r_jx)}{c_j}\, e^{\left(\beta_j\tilde{\zeta}_j c_j(u_j(x_j+r_jx)-c_j)\left(\frac{u_j(x_j+r_jx)}{c_k}+1\right)\right)}
\end{aligned}
$$

By Lemma 7.9, we know that $\int_{B_R}|\nabla^j z_j|dx \le cR^2$ for $1 \le j \le 2m-1$. Joining this result with the Elliptic Estimates in [25], we obtain $\|\nabla^j z_k\|_{C^{2m-3,\alpha}_{loc}} \le c$. Since, by Lemma 7.7, there exists $z \in C^{2m-1}(\mathbb{R}^{2m})$ such that

$$z_j(x) \to z \text{ in } C^{2m-1}_{loc}(\mathbb{R}^{2m})$$

with z satisfying the equation

$$(-\Delta)^m z = e^{(2\beta_{2m,m}z)}.$$

By Fatou's Lemma, we have

$$\liminf_{k\to\infty}\int_{\mathbb{R}^{2m}} e^{(2\beta_{2m,m}z)}\,dx \le \lambda_j^{-1}\int_{\mathbb{R}^{2m}}\tilde{\zeta}u_j^2\, e^{(\beta_k\tilde{\zeta}u_j^2)}\,dx \le \lambda_j^{-1}\frac{\tilde{\zeta}_j}{1-\mu_k\|u_k\|_2^2}\int_{\mathbb{R}^{2m}} u_j^2\, e^{\left(\beta_j\tilde{\zeta}_j u_j^2\right)} \le 1.$$

Now, let us suppose by contradiction that $z(x)$ isn't the form in (7-30). Then, by [44] (see also [51]), there exist $l \in \mathbb{N}$ satisfying $1 \le l \le m-1$ and some negative number N such that $\lim_{|x|\to+\infty}(-\Delta)^l z(x) = N$. And as consequence, we would have that

$$\lim_{k\to+\infty}\int_{B_R}|\Delta^l z_j(x)|dx = |N|vol(B_1(0))R^{2m} + o(R^{2m})$$

with $R \to \infty$, which contradicts (7-23). Thus, we have that the function z is of the form as (7-30). And to conclude, by computations as done in [10], ones can prove

$$\int_{\mathbb{R}^{2m}} e^{2\beta_{2m,m}z(x)} = \int_{\mathbb{R}^{2m}}\left(\frac{1}{1+\frac{\omega_{2m}^{1\backslash m}}{4}|x|^2}\right)^{2m} = 1, \tag{7-31}$$

and we get the last equality through a simple change of variable, namely, by setting $t = \frac{\omega_{2m}^{1\backslash m}}{4}|x|^2$. ■

7.1.3 Poly-harmonic Truncations

In this subsection, we shall employ poly-harmonic truncations, inspired by [16] and ultimately rooted in the truncation argument of [3]. This construction is entirely analogous to that used in the previous chapter for the particular case $n = 4$, and we now extend it to the general even-dimensional setting. Concretely, for any $A > 1$, we introduce a new function u_j^A, which takes values close to $\frac{c_j}{A}$ in a small ball centered at x_j and coincides with u_j outside this ball. The purpose of this section is to investigate the properties of u_j^A. To that end, we first state the following lemma:

Lemma 7.11 *(DelaTorre, Lemma 4.20 [16]) For any $A > 1$ and $k \in \mathbb{N}$, $\Omega \subset \mathbb{R}^{2m}$ a smooth bounded domain, there exists a radius $0 < \rho_j^A < dist(x_n, \partial\Omega)$ and a constant C_A depending on A, such that*

1. $u_j \geq \frac{c_j}{A}$ *in* $B_{\rho_j^A}(x_j)$*;*
2. $|u_j - \frac{c_j}{A}| \leq \frac{C_A}{c_j}$ *on* $\partial B_{\rho_j^A}(x_j)$*;*
3. $|\nabla^l u_j| \leq \frac{C_A}{c_j(\rho_j^A)^l}$ *on* $\partial B_{\rho_j^A}(x_j)$ *for any* $1 \leq l \leq 2m-1$*;*
4. $\lim\limits_{j\to\infty} \rho_j^A = 0$ *and if r_j is defined as in (7-19)* $\lim\limits_{j\to\infty} \frac{\rho_j^A}{r_j} \to \infty$.

Let $\rho_j^A > 0$ and $u_j^{\rho_j^A} \in C^{2m}(\overline{B_{\rho_j^A}(x_j)})$ be the unique solution of:

$$\begin{cases} (-\Delta)^m u_j^{\rho_j^A} = 0, & \text{in } B_{\rho_j^A}(x_j) \\ \partial_\nu^i u_j^{\rho_j^A} = \partial_\nu^i u_j, & \text{on } \partial B_{\rho_j^A}(x_j), \quad i = 0, 1, \ldots, m-1. \end{cases}$$

. Let us consider the function

$$u_j^A = \begin{cases} u_j^{\rho_j^A} & \text{in } B_{\rho_j^A}(x_j) \\ u_j, & \text{in } \mathbb{R}^{2m} \backslash B_{\rho_j^A}(x_j). \end{cases}$$

Lemma 7.12 *For any $A > 1$, we have $u_j^A = \frac{c_j}{A} + O(\frac{1}{c_j})$, uniformly on $\overline{B_{\rho_j^A}(x_j)}$.*

Proof: Set $\tilde{v}_j(x) = v_j^A(x_j + \rho_j^A x) - \frac{c_j}{A}$ for $x \in B_1$. By elliptic estimates [16, Proposition A.2], we have

$$\begin{aligned} \|v_j^A - \frac{c_j}{A}\|_{L^\infty(B_{\rho_j^A}(x_j))} = \|\tilde{v}_j\|_{L^\infty(B_1)} &\leq C\big[\|\tilde{v}_j\|_{L^\infty(\partial B_1)} + \sum_{l=1}^{m-1} \|\nabla^l \tilde{v}_j\|_{L^\infty(\partial B_1)}\big] \\ &= C\big[\|v_j^A - \frac{c_j}{A}\|_{L^\infty(\partial B_{\rho_j^A}(x_j))} + \sum_{l=1}^{m-1} (\rho_j^A)^l \|\nabla^l v_j^A\|_{L^\infty(\partial B_{\rho_j^A}(x_j))}\big] \\ &= C\big[\|u_j - \frac{c_j}{A}\|_{L^\infty(\partial B_{\rho_j^A}(x_j))} + \sum_{l=1}^{m-1} (\rho_j^A)^l \|\nabla^l u_j\|_{L^\infty(\partial B_{\rho_j^A}(x_j))}\big]. \end{aligned}$$

This together with Lemma 7.11 yields the result. ■

Lemma 7.13 *For any* $A > 1$*, there holds*

$$\limsup_{j\to+\infty}\int_{\mathbb{R}^{2m}}\left(|\nabla^m u_j^A|^2+|u_j^A|^2\right)dx\le\frac{1}{A}$$

Proof: By combining (7-4) with Lemma 7.11, we obtain $(-\Delta)^m u_j \geq 0$ in $B_{\rho_j^A}(x_j)$ for j large enough. So, the maximum principle yields $u_j \geq u_j^A$ in $B_{\rho_j^A}(x_j)$ and, from Lemma 7.12, $u_j^A \geq 0$ on $B_{\rho_j^A}(x_j)$ for j large enough. In addition, since $u_j^A \equiv u_j$ in $\mathbb{R}^n \setminus B_{\rho_j^A}(x_j)$ and in view of Lemma 7.12, by using $u_j - u_j^A$ as test function in (7-4) and recalling that $\rho_j^A/r_j \to \infty$, for any $R > 0$ and j sufficiently large, we obtain

$$\begin{aligned}
&\int_{B_{\rho_j^A}(x_j)}\left[\nabla^m u_j\nabla^m(u_j-u_j^A)+u_j(u_j-u_j^A)\right]\mathrm{d}x=\int_{B_{\rho_j^A}(x_j)}\frac{u_j\tilde{\zeta}_j}{\lambda_j}\left[\mathrm{e}^{\beta_j\tilde{\zeta}_j u_j^2}+\mu_j u_j\right](u_j-u_j^A)\mathrm{d}x\\
&\ge\frac{\tilde{\zeta}_j}{\lambda_j}\int_{B_{Rr_j}(x_j)}u_j\,\mathrm{e}^{\beta_j\tilde{\zeta}_j u_j^2}(u_j-u_j^A)\mathrm{d}x\\
&=\frac{\tilde{\zeta}_j}{\lambda_j}r_j^{2m}\int_{B_R(0)}\left(c_j+\frac{z_j}{c_j}\right)\mathrm{e}^{\beta_j\tilde{\zeta}_j(c_j^2+2z_j+\frac{z_j^2}{c_j^2})}\left(c_j+\frac{z_j}{c_j}-\frac{c_j}{A}+O(c_j^{-1})\right)\mathrm{d}x\\
&=\tilde{\zeta}_j\int_{B_R(0)}\left(1+\frac{z_j}{c_j^2}\right)\mathrm{e}^{\beta_j\tilde{\zeta}_j(2z_j+\frac{z_j^2}{c_j^2})}\left(1-\frac{1}{A}+\frac{z_j}{c_j^2}+O(c_j^{-2})\right)\mathrm{d}x.
\end{aligned}$$

From Lemma 7.7 and Lemma 7.6, we have $z_j/c_j = v_j \to 0$ and $z_j/c_j^2 = w_j - 1 \to 0$ in $C^{2m-1}_{loc}(\mathbb{R}^{2m})$. So, by applying Lemma 7.10, we can write

$$\int_{B_{\rho_j^A}(x_j)}\left[\nabla^m u_j\nabla^m(u_j-u_j^A)+u_j(u_j-u_j^A)\right]\mathrm{d}x\ge\left(1-\frac{1}{A}\right)\int_{B_R}\mathrm{e}^{2\beta_{2m,m}z}\,\mathrm{d}x+o_j(1),$$

and letting $R \to +\infty$, we obtain

$$\int_{B_{\rho_j^A}(x_j)}\left[\nabla^m u_j\nabla^m(u_j-u_j^A)+u_j(u_j-u_j^A)\right]\mathrm{d}x\ge 1-\frac{1}{A}+o_j(1). \tag{7-32}$$

Recalling $\|\nabla^m u_j\|_2^2 + \|u_j\|_2^2 = 1$, we have

$$\begin{aligned}
\int_{\mathbb{R}^{2m}} \left(|\nabla^m u_j^A|^2 + |u_j^A|^2\right) \mathrm{d}x &= \int_{B_{\rho_j^A}(x_j)} \left(|\nabla^m u_j^A|^2 + |u_j^A|^2\right) \mathrm{d}x + \int_{\mathbb{R}^{2m}\setminus B_{\rho_j^A}(x_j)} \left(|\nabla^m u_j|^2 + |u_j|^2\right) \mathrm{d}x \\
&= \int_{B_{\rho_j^A}(x_j)} \left(|\nabla^m u_j^A|^2 + |u_j^A|^2\right) \mathrm{d}x + 1 - \int_{B_{\rho_j^A}(x_j)} \left(|\nabla^m u_j|^2 + |u_j|^2\right) \mathrm{d}x \\
&= 1 - \int_{B_{\rho_j^A}(x_j)} \left[\nabla^m u_j \Delta(u_j - u_j^A) + u_j(u_j - u_j^A)\right] \mathrm{d}x \\
&\quad + \int_{B_{\rho_j^A}(x_j)} \nabla^m u_j^A \nabla^m \left(u_j^A - u_j\right) \mathrm{d}x + \int_{B_{\rho_j^A}(x_j)} u_j^A (u_j^A - u_j) \mathrm{d}x.
\end{aligned}$$

From (7-32) and recalling $u_j^A(u_j^A - u_j) \leq 0$ in $B_{\rho_j^A}(x_j)$, we derive

$$\begin{aligned}
\int_{\mathbb{R}^{2m}} \left(|\nabla^m u_j^A|^2 + |u_j^A|^2\right) \mathrm{d}x &\leq \frac{1}{A} + \int_{B_{\rho_j^A}(x_j)} \nabla^m u_j^A \nabla^m (u_j^A - u_j) \mathrm{d}x + o_j(1) \\
&= \frac{1}{A} + o_j(1),
\end{aligned}$$

where we used integration by parts and $(-\Delta)^m u_j^A = 0$ in $B_{\rho_j^A}(x_j)$ to obtain the last identity. ■

As a corollary of the previous result, we have

Corollary 7.14 *For any $\epsilon > 0$, it holds*

$$\limsup_{j\to+\infty} \int_{\mathbb{R}^{2m}\setminus B_\epsilon} \left(|\nabla^m u_j^A|^2 + |u_j^A|^2\right) dx = 0.$$

Let us state

Lemma 7.15 *We have*

$$\begin{aligned}
AD(2m, m, \beta_{2m,m}, \alpha, \gamma) &= \lim_{j\to+\infty} \int_{\mathbb{R}^{2m}} (\mathrm{e}^{\beta_j \tilde{\zeta}_j u_j^2} - 1) dx \\
&= \lim_{\hat{R}\to+\infty} \lim_{j\to+\infty} \int_{B_{\hat{R}r_j}} (\mathrm{e}^{\beta_j \tilde{\zeta}_j u_j^2} - 1) dx = \lim_{j\to\infty} \lambda_j c_j^{-2}
\end{aligned}$$

and consequently

$$\frac{\lambda_j}{c_j} \to \infty \text{ and } \sup_j \frac{c_j^2}{\lambda_j} < \infty$$

Proof: We recall that the first identity has already been proven in 7.1. So now, we'll write

$$\begin{aligned}\int_{\mathbb{R}^{2m}}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx &= \int_{B_{\rho_j^A}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx+\int_{\mathbb{R}^{2m}\setminus B_{\rho_j^A}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx\\ &\leq \int_{B_{\rho_j^A}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx+\int_{\mathbb{R}^{2m}}(e^{\beta_j\tilde{\zeta}_j (u_j^A)^2}-1)dx.\end{aligned} \tag{7-33}$$

By Radial Lemma 2.2, for some radius, $\hat{R}$ such that $u_j \leq 1$ on $\mathbb{R}^{2m}\setminus B_{\hat{R}}$, we get

$$\lim_{j\to+\infty}\int_{\mathbb{R}^{2m}\setminus B_{\hat{R}}}(e^{(\beta_j\tilde{\zeta}_j u_j^2)}-1)dx \leq \lim_{j\to\infty} c\int_{\mathbb{R}^{2m}} u_j^2 dx = 0, \quad c \text{ constant.} \tag{7-34}$$

Observe that $\beta_j\tilde{\zeta}_j \to \beta_{2m,m}$, with Lemma 7.13 and by the Tarsi's Adams inequality with Navier Boundary condition 2.3, we obtain

$$\sup_{j\to\infty}\int_{B_{\hat{R}}}(e^{\beta_j\tilde{\zeta}_j q'(u_j^A-u_j(\hat{R}))^2}-1)dx < \infty$$

for any $q' < A^2$ and j sufficiently large. Moreover this, we can observe that

$$q(u_j^A)^2 \leq q'(u_j^A-u_j(\hat{R}))^2 + c(q,q'), \text{ for } q < q',$$

therefore

$$\limsup_{j\to\infty}\int_{B_{\hat{R}}}(e^{\beta_j\tilde{\zeta}_j q(u_j^A)^2}-1)dx < \infty$$

for any $q < A^2$. Now, recalling that $A > 1$, then $e^{\beta_j\tilde{\zeta}_j(u_j^A)^2}$ is uniformly integrable and $u_j \to u$ a.e., we apply the Vitali's Convergence Theorem, and we get

$$\lim_{j\to+\infty}\int_{B_{\hat{R}}}(e^{\beta_j\tilde{\zeta}_j(u_j^A)^2}-1)dx = 0.$$

Hence, from (7-33) and (7-34) and using the fact in Lemma 7.11-(4) which gives $\rho_j^A \to 0$, we get

$$\begin{aligned}\int_{\mathbb{R}^{2m}}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx &= \int_{B_{\rho_j^A}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx + o_j(1)\\ &= \int_{B_{\rho_j^A}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx + o_j(1)\end{aligned} \tag{7-35}$$

and from Lemma 7.11-(1), we obtain

$$\begin{aligned}\lim_{j\to+\infty}\int_{\mathbb{R}^{2m}}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx &= \lim_{j\to+\infty}\int_{B_{\rho_j^A}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\\ &\le \lim_{j\to+\infty}\frac{A^2}{c_j^2}\int_{B_{\rho_j^A}(x_j)} u_j^2\, e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\\ &\le \lim_{j\to+\infty}\frac{A^2}{c_j^2}\int_{\mathbb{R}^{2m}} u_j^2\, e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx \qquad (7\text{-}36)\\ &= \lim_{j\to+\infty}\frac{\lambda_j}{c_j^2}\frac{1-\mu_j\|u_j\|_2^2}{\tilde{\zeta}_j}\\ &= A^2\lim_{k\to+\infty}\frac{\lambda_j}{c_j^2}\end{aligned}$$

By taking $A\to 1^+$ in (7-36), we derive

$$\lim_{j\to\infty}\int_{\mathbb{R}^{2m}}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx \le \lim_{j\to\infty}\frac{\lambda_j}{c_j^2}.$$

Conversely, notice that

$$\begin{aligned}\lambda_j &= \frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2}\left[\int_{\mathbb{R}^{2m}} u_j^2\left(e^{\beta_j\tilde{\zeta}_j u_j^2}-1\right)dx+\int_{\mathbb{R}^2} u_j^2 dx\right]\\ &\le \frac{\tilde{\zeta}_j}{1-\mu_j\|u_j\|_2^2}\left[\int_{\mathbb{R}^{2m}} c_j^2\left(e^{\beta_j\tilde{\zeta}_j u_j^2}-1\right)dx+o_j(1)\right]\end{aligned}$$

and it follows

$$\lim_{j\to\infty}\frac{\lambda_j}{c_j^2}\le\lim_{j\to\infty}\int_{\mathbb{R}^{2m}}\left(e^{\beta_j\tilde{\zeta}_j u_j^2}-1\right)dx.$$

Now, note that

$$\begin{aligned}\lim_{j\to\infty}\int_{B_{\hat{R}r_j}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx &= \lim_{j\to\infty}\left[\int_{B_{\hat{R}r_j}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx+|B_{\hat{R}r_j}(x_j)|\right]\\ &= \lim_{j\to\infty}\int_{B_{\hat{R}r_j}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx\end{aligned}\qquad (7\text{-}37)$$

for any $\hat{R}>0$. By using the definition of r_j and $u_j^2(x_j+r_jx)-c_j^2=z_j(w_j+1)$ on $B_{\hat{R}r_j}(x_j)$, we can write

$$\int_{B_{\hat{R}r_j}(x_j)} e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx=\frac{\lambda_j}{c_j^2}\int_{B_{\hat{R}}(0)} e^{\beta_j\tilde{\zeta}_j[u_j^2(x_j+r_jx)-c_j^2]}\,dx = \frac{\lambda_j}{c_j^2}\int_{B_{\hat{R}}(0)} e^{\beta_j\tilde{\zeta}_j z_j(w_j+1)}\,dx. \qquad (7\text{-}38)$$

Taking into account (7-37), (7-38), Lemma 7.6 and Lemma 7.10, we get

$$\lim_{j\to\infty}\int_{B_{\hat{R}r_j}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx=\lim_{j\to\infty}\frac{\lambda_j}{c_j^2}\Big(\int_{B_{\hat{R}}(0)}e^{64\pi^2 z}\,dx\Big).$$

Then, from Lemma 7.10 again

$$\lim_{\hat{R}\to\infty}\lim_{j\to\infty}\int_{B_{\hat{R}r_j}(x_j)}(e^{\beta_j\tilde{\zeta}_j u_j^2}-1)dx=\lim_{j\to\infty}\frac{\lambda_j}{c_j^2}.$$

■

Let us define

$$\xi_{R,j}=\frac{\lambda_j}{\int_{B_R(x_j)}|u_j|\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx},$$

$$\tau=\lim_{k\to\infty}\frac{\xi_j}{c_j},\quad \text{with } \xi_j=\xi_{\rho_j^A,j},$$

$$\varphi=\lim_{R\to\infty}\lim_{k\to\infty}\frac{\int_{B_R(x_j)}u_j\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx}{\int_{B_R(x_j)}|u_j|\,e^{\beta_j\tilde{\zeta}_j u_j^2}\,dx},$$

where $\xi_j=\xi_{\rho_j^A,k}$. Following exactly the same argument as in previous chapter for 6.14, 6.15, we derive both results:

Lemma 7.16 $\varphi=1$.

Lemma 7.17 $\tau=1$.

7.1.4 Assymptotic behavior of u_k away from the blow-up point

We recall the fundamental solution of the biharmonic operator $(\Delta^m+\kappa^2)$ for $\kappa>0$ in $\mathbb{R}^{2m}$ whose properties below can be found on [18] and [17]. The fundamental solution $\Phi_\kappa(x,y)$ is the solution of the equation

$$((-\Delta)^m+\kappa^2)\Phi_\kappa(x,y)=\delta_x(y),\ \text{ in } \mathbb{R}^{2m}$$

and also every function $u\in H^m(\mathbb{R}^{2m})\cap C^{2m}(\mathbb{R}^{2m})$ that can be expressed as

$$u(x)=\int_{\mathbb{R}^{2m}}\Phi_\kappa(x,y)f(y)dy, \tag{7-39}$$

satisfying $((-\Delta)^m + \kappa^2)u = f$.

Let us state some estimates for Φ_κ which will play a key role in what follows

$$|\Phi_\kappa(x,y)| \le c \ln\left(1 + \frac{1}{|x-y|}\right), \tag{7-40}$$

$$|\nabla^l \Phi_\kappa(x,y)| \le c\left(\frac{1}{|x-y|}\right), \text{ for } l \ge 1, \ \forall x,y \in \mathbb{R}^{2m}, x \neq y \text{ with } |x-y| \to 0. \tag{7-41}$$

$$|\nabla^l \Phi_\kappa(x,y)| = o\left(\exp\left(-\frac{\sqrt{\kappa}}{\sqrt{2}}|x-y|\right)\right), \text{ for } l = 0,1,2,...,m. \ \forall x,y \in \mathbb{R}^{2m},$$
$$x \neq y \text{ with } |x-y| \to +\infty. \tag{7-42}$$

Lemma 7.18 *For any* $1 < p < 2$*,* $c_k u_k$ *is bounded in* $W^{2,p}(\mathbb{R}^{2m})$.

Proof: Let v_j be the solution for the equation

$$(-\Delta)^m v_j + \kappa_j^2 v_j = \frac{\xi_j}{\lambda_j} u_j \tilde{\zeta}_j e^{\beta_j \tilde{\zeta}_j u_j^2}, \text{ in } \mathbb{R}^{2m}, \tag{7-43}$$

where $\kappa_j = (1-\mu_j)^{1/2} \to (1-\alpha(1+\gamma))^{1/2} > 0$ because we are assuming $\gamma < \frac{1}{\alpha} - 1$. Hence, for j large enough, the representation formula (7-39) yields

$$v_j(x) = \frac{\xi_j}{\lambda_j} \int_{\mathbb{R}^{2m}} \Phi_{\kappa_j}(x,y) u_j(y) \tilde{\zeta}_j e^{\beta_j \tilde{\zeta}_j u_j^2(y)} \mathrm{d}y, \ \ x \in \mathbb{R}^{2m}.$$

Computing the i-th gradient

$$|\nabla^i v_j| = \left| \frac{\xi_j}{\lambda_j} \int_{\mathbb{R}^{2m}} \nabla^i \Phi_{\kappa_j}(x,y) u_j(y) \tilde{\zeta}_j e^{\beta_j \tilde{\zeta}_j u_j^2(y)} \mathrm{d}y \right|.$$

By the definition of ξ_j/λ_j, we have

$$|\nabla^i v_j| = \left| \int_{\mathbb{R}^{2m}} \nabla^i \Phi_{\kappa_j}(x,y) \frac{u_j(y) e^{\beta_j \tilde{\zeta}_j u_j^2(y)}}{\int_{B_{\rho_j^A}(x_j)} |u_j(z)| e^{\beta_j \tilde{\zeta}_j u_j^2(z)} \mathrm{d}z} \mathrm{d}y \right|.$$

Letting $R \to \infty$ we have

$$\int_{\mathbb{R}^{2m}} |u_j(z)| e^{\beta_j \tilde{\zeta}_j u_j^2(z)} \mathrm{d}z = \int_{B_{\rho_j^A}(x_j)} |u_j(z)| e^{\beta_j \tilde{\zeta}_j u_j^2(z)} \mathrm{d}z + o_j^+(1).$$

Then

$$|\nabla^i v_j| \le \frac{1}{\int_{\mathbb{R}^{2m}} g_j(z) \mathrm{d}z} \int_{\mathbb{R}^{2m}} \left| \nabla^i \Phi_{\kappa_j}(x,y) \right| g_j(y) \mathrm{d}y,$$

where $g_j = |u_j| e^{\beta_j \tilde{\zeta}_j u_j^2}$. So, by Hölder's inequality for $1 < p < 2$, we get

$$\int_{\mathbb{R}^{2m}} \left|\nabla^i \Phi_{\kappa_j}(x,y)\right| g_j(y) dy = \int_{\mathbb{R}^{2m}} \left|\nabla^i \Phi_{\kappa_j}(x,y)\right| |g_j(y)|^{\frac{1}{p}} |g_j(y)|^{\frac{1}{p'}} dy$$
$$\leq \left(\int_{\mathbb{R}^{2m}} \left|\nabla^i \Phi_{\kappa_j}(x,y)\right|^p |g_j(y)| dy\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^{2m}} |g_j(y)| dy\right)^{\frac{1}{p'}}.$$

It follows that

$$|\nabla^i v_j|^p \leq \int_{\mathbb{R}^{2m}} \left|\nabla^i \Phi_{\kappa_j}(x,y)\right|^p \frac{|g_j(y)|}{\int_{\mathbb{R}^{2m}} |g_j(z)| dz} dy,$$

for $i = 0,1,2,...,m$. Applying Fubini's theorem and using (7-40), (7-41) and (7-42)

$$\int_{\mathbb{R}^{2m}} |\nabla^i v_j(x)|^p dx \leq \int_{\mathbb{R}^{2m}} \left(\int_{\mathbb{R}^{2m}} \left|\nabla^i \Phi_{\kappa_j}(x,y)\right|^p \frac{|g_j(y)|}{\int_{\mathbb{R}^{2m}} |g_j(z)| dz} dy\right) dx$$
$$\leq c, \text{ for } i = 0,1,2,...,m.$$

Therefore, $\|v_j\|_{W^{2,p}(\mathbb{R}^{2m})} \leq c$. By noticing that $v_j = \xi_j u_j$ satisfies (7-43), we have

$$\|\xi_j u_j\|_{W^{2,p}(\mathbb{R}^{2m})} \leq c. \tag{7-44}$$

From Lemma 7.16, we have $\xi_j / c_j \to 1$. Then the proof follows from (7-44). ■

The following result will be important to demonstrate the convergence of $c_j u_j$ to a Green function.

Lemma 7.19 *Let $\phi \in C_0^\infty(\mathbb{R}^{2m})$, then we have*

$$\lim_{k\to\infty} \int_{\mathbb{R}^{2m}} \phi(x) \left(\frac{c_k u_j}{\lambda_j} \tilde{\zeta}_j e^{(\beta_j \tilde{\zeta}_j u_j^2)} + \mu_j c_j u_j\right) dx = \phi(0). \tag{7-45}$$

<u>**Proof:**</u> The proof follows exactly as in 6.17. ■

Before we state the next result, let us consider the following auxiliary proposition

Proposition 7.20 *Let $\Omega \subseteq \mathbb{R}^{2m}$ be a bounded open domain with Lipschitz boundary. Then, for any $u \in H^m(\Omega), v \in H^{2m}(\Omega)$, we have*

$$\int_\Omega \nabla^m u \cdot \nabla^m v \, dx = \int_\Omega u(-\Delta)^m v \, dx - \sum_{j=0}^{m-1} \int_{\partial\Omega} (-1)^{m+j} \nu \cdot \nabla^j u \cdot \nabla^{2m-j-1} v \, d\sigma$$

where ν denotes the outer normal to $\partial\Omega$.

Now, we'll prove the convergence of $c_j u_j$ to a Green Function.

Lemma 7.21 *Let G be a Green Function such that* $c_j u_j \rightharpoonup G$ *in* $C^{2m-1}(\mathbb{R}^{2m})\backslash\{0\}$*, weakly in* $W^{m,p}(\mathbb{R}^{2m})$*, where G is a Green function satisfying the equation* $(-\Delta)^m G + \kappa_0 G = \delta_0$ *in* $\mathbb{R}^{2m}$*, where* $\kappa_0 = 1 - \alpha(\gamma+1)$ *Moreover this, we have:*

$$G = -\frac{1}{\gamma_m} In|x| + K_0 + \phi(x)$$

where K_0 *is a constant depending on* 0*,* $\phi(x) \in C^{2m-1}(\mathbb{R}^{2m})$ *and* $\phi(0) = 0$*. And*

$$\lim_{k\to\infty}\left(\int_{\mathbb{R}^{2m}\backslash B_\delta(0)} |\nabla^m(c_j u_j)|^2 + |c_j u_j|^2 dx\right) = -\frac{1}{\gamma_m} In|\delta| + H_m + K_0 + \alpha(1+\gamma)\|G\|_2^2 + O(\epsilon).$$

as $\epsilon \to 0$ *and where* $\gamma_m = \frac{\beta_{2m,m}}{2m}$ *and* $H_m = \frac{1}{2\gamma_m}\sum_{j=1}^{m-1}\frac{(-1)^{\left[\frac{2j}{m}\right]}}{j}$.

Proof: Since, by Lemma 7.18 we can afford that there exists $G \in W^{m,p}(\mathbb{R}^{2m})$ such that $c_j u_j \rightharpoonup G$ weakly in $W^{m,p}(\mathbb{R}^{2m})$, for any $1 < p < 2$. Since $|\nabla^m u_j|^2 dx \overset{*}{\rightharpoonup} \delta_0$ in the sense of measure, we have that for any radius $0 < S < R$, $e^{\beta_j \tilde{\zeta}_j u_j^2)}$ is bounded in $L^p(B_R\backslash B_S)$. Notice that $c_j u_j$ satisfies the Euler-Lagrange equation

$$(-\Delta)^m(c_j u_j) + c_j u_j = c_j u_j \lambda_j^{-1} e^{\beta_j \tilde{\zeta}_j u_j^2} + c_j \mu_j u_j \text{ in } \mathbb{R}^{2m}. \tag{7-46}$$

Therefore, by the standard regularity theory, we got $c_j u_j \to G$ in $C^{2m-1}_{loc}(\mathbb{R}^{2m}\backslash 0)$. By Lemma 7.19, for any $\phi(x) \in C_0^\infty(\mathbb{R}^{2m})$, we have

$$\phi(0) - \kappa_0 \int_{\mathbb{R}^{2m}} G(x)\phi(x)dx = \lim_{j\to\infty}\int_{\mathbb{R}^{2m}} \phi(x)\left(c_j u_j^2 \lambda_j^{-1} e^{\beta_j \tilde{\zeta}_j u_j^2} - (\mu_j - 1)c_j u_j\right) dx.$$

This implies that

$$(-\Delta)^m G + \kappa_0 G = \delta_0 \text{ in } \mathbb{R}^{2m}.$$

Now, as long as we known that the fundamental solution of $(-\Delta)^m$ in $\mathbb{R}^{2m}$ is

$$-\frac{1}{\gamma_m}\log|x|,$$

where

$$\gamma_m = \omega_{2m-1} 2^{2(m-1)}[(m-1)!]^2 = \frac{\beta_{2m,m}}{2m}.$$

Fixing a radius $\rho > 0$ and taking some cutoff function $\psi \in C_0^\infty(B_{2\rho}(0))$ such that $\psi = 1$ in $B_\rho(0)$ and

$$g(x) = G(x) + \frac{1}{\gamma_m}\psi(x)\ln|x|.$$

By computing directly the polyharmonic of g, we have

$$(-\Delta)^m g(x) = \tilde{f}(x), \quad \text{in } \mathbb{R}^{2m},$$

where the function $\tilde{f}$ is given as follows

$$\tilde{f}(x) = -\frac{1}{\gamma_m}\left(\sum_{i=0}^{2m-1}\binom{2m}{i}\nabla^{2m-i}\psi\cdot\nabla^i \ln|x| + \psi\cdot(-\Delta)^m \ln|x|\right) + \delta_0 - \kappa_0 G.$$

Since, $-\frac{1}{\gamma_m}\psi\cdot(-\Delta)^m \ln|x| = \delta(x)$ in $\mathbb{R}^{2m}$, we can rewrite the $\tilde{f}$ expression as

$$\tilde{f}(x) = -\frac{1}{\gamma_m}\left(\sum_{i=0}^{2m-1}\binom{2m}{i}\nabla^{2m-i}\psi\cdot\nabla^i \ln|x|\right) - \kappa_0 G.$$

Since $G \in W^{m,p}(\mathbb{R}^{2m})$ for any $1 < p < 2$, we have $\tilde{f}(x) \in L^p_{loc}(\mathbb{R}^{2m}$ for any $p > 2$. Again, by the standard regularity theory, we get $g \in C^{2m-1}_{loc}(\mathbb{R}^{2m})$. Let $K_0 = g(0)$ and

$$h(x) = g(x) - g(0) + \frac{1}{\gamma_m}(1-\psi)\ln|x|.$$

Then, we have

$$G = -\frac{1}{\gamma_m}\ln|x| + K_0 + h(x). \tag{7-47}$$

with K_0 is a constant depending on 0, $h(x) \in C^{2m-1}(\mathbb{R}^{2m})$ and $h(0) = 0$. as we want. Finally, define

$$U_j = c_j u_j,$$

clearly, U_j satisfies the Euler-Lagrange equation (7-46), and we get the following

$$(-\Delta)^m U_j + (1-\mu_j)U_j = \frac{\tilde{\zeta}_j}{\lambda_j} U_j e^{\beta_j \zeta_j u_j^2} \quad \text{in } \mathbb{R}^{2m}. \tag{7-48}$$

. Let $0 < \epsilon < R$. Then, using computations similar to those for the analogous result in the previous chapter and arguing by density, we can test with U_j and apply Proposition (7.20), we obtain

$$\begin{aligned}\int_{B_R\setminus B_\epsilon} |\nabla^m U_j|^2\,dx &= \int_{B_R\setminus B_\epsilon} U_j(-\Delta)^m U_j\,dx \\ &- \sum_{i=0}^{m-1}(-1)^{m+i}\left[\int_{\partial B_R} \nu\,\nabla^i U_j\,\nabla^{2m-i-1}U_j\,d\sigma - \int_{\partial B_\epsilon} \nu\,\nabla^i U_j\,\nabla^{2m-i-1}U_j\,d\sigma\right].\end{aligned}$$

Plugging equation (7-48) into the first term of the above equation, we get

$$\int_{B_R\setminus B_\epsilon} |\nabla^m U_j|^2\,dx + (1-\mu_j)\int_{B_R\setminus B_\epsilon} |U_j|^2\,dx \tag{7-49}$$

$$= \int_{B_R\setminus B_\epsilon} \frac{\tilde{\zeta}_j}{\lambda_j} U_j^2\, e^{\beta_j \tilde{\zeta}_j u_j^2}\,dx \tag{7-50}$$

$$- \sum_{i=0}^{m-1} (-1)^{m+i} \left[\int_{\partial B_R} \nu\,\nabla^i U_j\,\nabla^{2m-i-1} U_j\,d\sigma - \int_{\partial B_\epsilon} \nu\,\nabla^i U_j\,\nabla^{2m-i-1} U_j\,d\sigma\right], \tag{7-51}$$

Letting $R \to \infty$, we derive the following

$$\int_{\mathbb{R}^{2m}\setminus B_\epsilon} |\nabla^m U_j|^2\,dx + (1-\mu_j)\int_{\mathbb{R}^{2m}\setminus B_\epsilon} |U_j|^2\,dx = \int_{B_R\setminus B_\epsilon} \frac{\tilde{\zeta}_j}{\lambda_j} U_j^2\, e^{\beta_j \tilde{\zeta}_j u_j^2}\,dx$$
$$- \sum_{i=0}^{m-1} (-1)^{m+i} \int_{\partial B_\epsilon} \nu\,\nabla^i U_j\,\nabla^{2m-i-1} U_j\,d\sigma,$$

where ν denotes the unit outward normal vector to $\mathbb{R}^{2m}\setminus B_\epsilon(0)$, i.e., pointing into $B_\epsilon(0)$. By taking into account the Lemmas 2.2, 7.3 and 7.15, we can write

$$\lim_{j\to\infty} \int_{\mathbb{R}^{2m}\setminus B_\epsilon} \left(|\nabla^m U_j|^2 + (1-\mu_j)|U_j|^2\right)\mathrm{d}x = - \sum_{i=0}^{m-1} (-1)^{m+i} \int_{\partial B_\epsilon} \nu\,\nabla^i G\,\nabla^{2m-i-1} G\,d\sigma. \tag{7-52}$$

From (7-52), by employing Fatou's Lemma we also can write

$$\int_{\mathbb{R}^{2m}\setminus B_\epsilon} \left(|\nabla^m G|^2 + \kappa_0|G|^2\right)\mathrm{d}x \le \sum_{i=0}^{m-1} (-1)^{m+i} \int_{\partial B_\epsilon} \nu\,\nabla^i G\,\nabla^{2m-i-1} G\,d\sigma,$$

Since $\kappa_0 > 0$, we conclude that $G \in W^{m,2}(\mathbb{R}^{2m}\setminus B_\epsilon(0))$ for any $\epsilon > 0$. With this information in hand, returning to (7-49) and letting $j \to \infty$, and then $R \to \infty$ we obtain

$$\int_{\mathbb{R}^{2m}\setminus B_\epsilon} \left(|\nabla^m G|^2 + \kappa_0|G|^2\right)\mathrm{d}x = \sum_{i=0}^{m-1} (-1)^{m+i} \int_{\partial B_\epsilon} \nu\,\nabla^i G\,\nabla^{2m-i-1} G\,d\sigma.$$

Similar to the approach in the previous chapter, we can write

$$\lim_{j\to\infty} \int_{\mathbb{R}^{2m}\setminus B_\epsilon} \left(|\nabla^m U_j|^2 + |U_j|^2\right)\mathrm{d}x = \sum_{i=0}^{m-1} (-1)^{m+i} \int_{\partial B_\epsilon} \nu\,\nabla^i G\,\nabla^{2m-i-1} G\,d\sigma + \alpha(\gamma+1)\|G\|_2^2. \tag{7-53}$$

Since

$$\nabla^{2l}\log|x| = \frac{\tilde{K}_{m,l}}{|x|^{2l}},\quad \nabla^{2l+1}(\log|x|) = -\frac{2l\tilde{K}_{m,l}x}{|x|^{2l+2}},\quad \nabla^{l}(\log|x|) = K_{m,\frac{l}{2}}\frac{e_l(x)}{|x|^l} \tag{7-54}$$

where, $\tilde{K}_{m,l} = (-1)^{l+1}2^{2l-1}\frac{(l-1)!(m-1)!}{(m-l-1)!}$ and

$$K_{m,\frac{l}{2}} = \begin{cases} \tilde{K}_{m,\frac{l}{2}}, & \text{for } l \text{ even },\\ -(l-1)\tilde{K}_{m,\frac{l-1}{2}}, & \text{for } l \text{ odd }, l \geq 3\\ 1, & \text{for } l = 1 \end{cases}$$

and

$$e_l(x) = \begin{cases} 1, & l \text{ even},\\ \frac{x}{|x|}, & l \text{ odd}. \end{cases}$$

we have

$$\nu G(\epsilon)\nabla^{2m-1}G(\epsilon) = \left(-\frac{1}{\gamma_m}\ln|\epsilon| + K + O(\epsilon)\right)\left(\frac{2(m-1)}{\gamma_m}\tilde{K}_{m,m-1}\frac{1}{\epsilon^{2m-1}} + O(1)\right) \tag{7-55}$$

$$= -\frac{1}{\gamma_m}K_{m,\frac{2m-1}{2}}\frac{1}{\epsilon^{2m-1}}\left(-\frac{1}{\gamma_m}\ln\epsilon + K + O(\epsilon)\right). \tag{7-56}$$

and

$$\nu\nabla G\nabla^{2m-i-1}G = \left(\frac{1}{\gamma_m}\right)^2 K_{m,\frac{i}{2}}K_{m,\frac{2m-i-1}{2}}\frac{1}{\epsilon^{2m-1}} + o\left(\frac{1}{\epsilon^{2m-1}}\right). \tag{7-57}$$

Joining the equations (7-55) and (7-57) into (7-52), and by the identity

$$\frac{1}{\gamma_m}K_{m,\frac{2m-1}{2}} = \frac{(-1)^{m-1}}{\omega_{2m-1}}$$

and by Remark 2.4 in [16], we obtain

$$\lim_{j\to\infty}\left(\int_{\mathbb{R}^{2m}\setminus B_\epsilon(0)}|\nabla^m U_j|^2\,dx + \int_{\mathbb{R}^{2m}\setminus B_\epsilon(0)}|U_j|^2\,dx\right)$$
$$= (-1)^m\left(\frac{1}{\gamma_m}\right)^2 K_{m,\frac{2m-1}{2}}\cdot\omega_{2m-1}\ln\epsilon + H_m + A + \alpha(\gamma+1)\|G\|_2^2 + O(\epsilon)$$
$$= -\frac{1}{\gamma_m}\ln\epsilon + H_m + K_0 + \alpha(\gamma+1)\|G\|_2^2 + O(\epsilon)$$

where

$$H_m = \frac{1}{2\gamma_m}\sum_{j=1}^{m-1}\frac{(-1)^{\frac{2j}{m}}}{j}.$$

■

7.1.5 The Upper Bound for the Adimurthi-Druet-Adams Inequality acting on concentrating Sequences

To derive an upper bound for the Adams inequality of Adimurthi-Druet type in $\mathbb{R}^{2m}$, we follow a strategy similar to that employed in [11] and [43]. We begin by establishing a preliminary result concerning the upper bound for arbitrary sequences of functions in $H_0^m(B_R)$.

Lemma 7.22 *Let u_j be a bounded sequence in $H_0^m(B_R)$ such that $\|\nabla^m u_j\|_2^2 = 1$, where $B_R \subset \mathbb{R}^{2m}$. If $u_j \rightharpoonup 0$, then*

$$\limsup_{j\to+\infty} \int_{B_R} \left[e^{\beta_j \tilde{\zeta}_j u_j^2} - 1 \right] dx \leq \frac{\omega_{2m}}{2^{2m}} R^{2m} e^{\left(m \sum_{j=1}^{m-1} \frac{(-1)^j \binom{m-1}{j}}{j} - \beta_{2m,m} \mathscr{I}_m \right)}$$

where $\mathscr{I}_m$ is defined as in [16]:

$$\mathscr{I}_m := -\frac{m 4^{2m}}{\beta_{2m,m}\omega_{2m}} \int_{\mathbb{R}^{2m}} \frac{\log\left(1+\frac{y^2}{4}\right)}{(4+y^2)^{2m}} dy$$

<u>Proof:</u> By Proposition 4.2 in [16], we have the estimate

$$\limsup_{k\to+\infty} \int_{B_R} \left[e^{\beta_j \tilde{\zeta}_j u_j^2} - 1 \right] dx \leq \frac{\omega_{2m}}{2^{2m}} e^{\beta_{2m,m}(A_0 - \mathscr{I}_m)}$$

where A_0 is the value at 0 of the trace of the part regular of the Green Function G for the poly-harmonic operator $(-\Delta)^m$. Moreover this, we also known that when the domain is the unitary ball centered at origin, by the Boggios's Formula [equation (4.30) in [24])]

$$\begin{aligned} G_1 &= \frac{1}{2\gamma_m} \int_0^{1-|x|^2} s^{m-1}(1-s)^{-1} ds \\ &= -\frac{1}{2\gamma_m} \int_{|x|^2}^1 \frac{(1-t)^{m-1}}{t} dt \\ &= -\frac{1}{2\gamma_m} \left(\ln|x|^2 + \sum_{j=1}^{m-1} (-1)^j \binom{m-1}{j} \frac{(1-|x|^{2j})}{j} \right) \end{aligned} \tag{7-58}$$

Considering the ball B_R, it's easy to demonstrate that $G_1\left(\frac{x}{R}\right)$ is the Green function G_R of $(-\Delta)^m$ on B_R, thus, the value at 0 of the trace of the regular part of G_R is

$$-\frac{1}{\gamma_m}\ln R+\frac{1}{2\gamma_m}\sum_{j=1}^{m-1}\frac{\binom{m-1}{j}(-1)^j}{j}.$$

Hence, we get

$$\begin{aligned}\int_{B_R}\left[e^{\beta_j\tilde{\zeta}_j u_j^2}-1\right]dx &\leq \frac{\omega_{2m}}{2^{2m}}e^{\left(2m\log(R)+m\sum_{j=1}^{m-1}\frac{(-1)^j\binom{m-1}{j}}{j}-\beta_{2m,m}\mathscr{I}_m\right)}\\ &\leq \frac{\omega_{2m}}{2^{2m}}R^{2m}e^{\left(m\sum_{j=1}^{m-1}\frac{(-1)^j\binom{m-1}{j}}{j}-\beta_{2m,m}\mathscr{I}_m\right)}.\end{aligned}$$

■

Still following the same procedure in [11], we introduce the polyharmonic truncate function

$$u_j^{\epsilon}(r)=\sum_{k=1}^{m-1}\frac{(\binom{m-1}{k}+d_k(\epsilon))u_j^{(k)}(\epsilon)(r^{2k}-\epsilon^{2k})}{2\cdot k!\epsilon^k}+u_j(\epsilon), \tag{7-59}$$

where $d_k(\epsilon)$, for $j=1,2,...,m-1$ are constants such that

$$(u_j^{\epsilon})^{(i)}=u_j^{(i)}(\epsilon),\quad i=0,1,2,...,m-1. \tag{7-60}$$

It's clear that $(-\Delta)^m u_j^{\epsilon}=0$. We now prove that $d_j(\epsilon)\to 0$ as $\epsilon\to 0$, $j=1,2,...,m-1$.

Lemma 7.23 *For any $k=1,2,...,m-1$, we have $d_k(\epsilon)=o_\epsilon(1)$.*

Proof: From the convergence $c_ju_j\to G$ in $C^{m-1}_{\text{loc}}(\mathbb{R}^{2m}\setminus\{0\})$ and the expansion of G near the origin, we have

$$u_j^{(i)}(\epsilon)=\frac{G^{(i)}(\epsilon)+o_j(1)}{c_j}=\frac{1}{c_j}\left(-\frac{1}{\gamma_m}\frac{(-1)^{i-1}(i-1)!}{\epsilon^i}+O_\epsilon(1)+o_j(1)\right).$$

On the other hand, differentiating (7-59) and evaluating at $r=\epsilon$ gives

$$(u_j^{\epsilon})^{(i)}(\epsilon)=\sum_{k=\lceil i/2\rceil}^{m-1}\frac{\binom{m-1}{k}+d_k(\epsilon)}{2(k)!\,\epsilon^k}u_j^{(k)}(\epsilon)\,(2k)(2k-1)\dots(2k-i+1)\,\epsilon^{2k-i}.$$

Substituting the expansion of $u_j^{(k)}(\epsilon)$ into the above and using (7-60) yields

$$\begin{aligned}
&\frac{1}{c_j}\left(-\frac{1}{\gamma_m}\frac{(-1)^{i-1}(i-1)!}{\epsilon^i}+O_\epsilon(1)+o_j(1)\right)\\
&=\frac{1}{c_j}\left(-\frac{1}{\gamma_m\epsilon^i}\sum_{k=\lceil i/2\rceil}^{m-1}\left(\binom{m-1}{k}+d_k(\epsilon)\right)\right.\\
&\left.\cdot(-1)^{k-1}(k-1)!(2k)(2k-1)\ldots(2k-i+1)+O_\epsilon(1)+o_j(1)\right).
\end{aligned}$$

The terms with $\binom{m-1}{k}$ reproduce $(i-1)!$ by the combinatorial identity

$$\sum_{k=\lceil i/2\rceil}^{m-1}\binom{m-1}{k}(-1)^{k-i}(2k)(2k-1)\ldots(2k-i+1)=(i-1)!.$$

What remains is a linear system in the variables $d_k(\epsilon)$ with a constant coefficient matrix (independent of ϵ) which is non-degenerate. This forces

$$d_k(\epsilon)=o_\epsilon(1),\quad k=1,2,\ldots,m-1.$$

■

Now, we state an auxiliary result

Lemma 7.24 *For any* $\epsilon>0$, *we have*

$$\int_{B_\epsilon(x_j)}|\nabla^m(u_j(x)-u_j^\epsilon(x))|^2dx=\int_{B_\epsilon(x_j)}|\nabla^m u_j(x)|^2dx-\int_{B_\epsilon(x_j)}\nabla^m u_j^\epsilon(x)\nabla^m u_j(x)dx.$$

Proof: Let $\epsilon>0$, then by definition of the truncation u_j^ϵ, we have

$$\begin{aligned}
\int_{B_\epsilon(x_j)}\left|\nabla^m\left(u_j(x)-u_j^\epsilon(x)\right)\right|^2\,\mathrm{d}x&=\int_{B_\epsilon(x_j)}\left(u_j(x)-u_j^\epsilon(x)\right)(-\Delta)^m u_j(x)\mathrm{d}x\\
&=\int_{B_\epsilon(x_j)}u_j(x)(-\Delta)^m u_j(x)\mathrm{d}x-\int_{B_\epsilon(x_j)}u_j^\epsilon(x)(-\Delta)^m u_j(x)\mathrm{d}x\\
&=I-II
\end{aligned}$$

Computing I and II and by applying 7.20, we get

$$\begin{aligned}
I&=\int_{B_\epsilon(x_j)}u_j(x)(-\Delta)^m u_j(x)\mathrm{d}x\\
&=\int_{B_\epsilon(x_j)}|\nabla^m u_j(x)|^2\mathrm{d}x+\sum_{j=0}^{m-1}\int_{\partial B_\epsilon(x_j)}(-1)^{m+j}\nu\cdot\nabla^j u_j\nabla^{2m-j-1}u_j\,\mathrm{d}\sigma(x)
\end{aligned}$$

and

$$II = \int_{B_\epsilon(x_j)} u_j^\epsilon(x)(-\Delta)^m u_j(x)\mathrm{d}x$$
$$= \int_{B_\epsilon(x_j)} \nabla^m u_j^\epsilon(x)\nabla^m u_j(x)\mathrm{d}x + \sum_{j=0}^{m-1}\int_{\partial B_\epsilon(x_j)} (-1)^{m+j} v\cdot\nabla^j u_j^\epsilon \nabla^{2m-j-1}u_j \,\mathrm{d}\sigma(x)$$

Thus,

$$\int_{B_\epsilon(x_j)} \left|\nabla^m\left(u_j(x)-u_j^\epsilon(x)\right)\right|^2 \mathrm{d}x$$
$$= \int_{B_\epsilon(x_j)} \nabla^m u_j(x)\nabla^m u_j(x)\mathrm{d}x + \sum_{j=0}^{m-1}\int_{\partial B_\epsilon(x_j)} (-1)^{m+j} v\cdot\Delta^{\frac{j}{2}} u_j \nabla^{2m-j-1}u_j \,\mathrm{d}\sigma(x)$$
$$- \int_{B_\epsilon(x_j)} \nabla^m u_j^\epsilon(x)\nabla^m u_j(x)\mathrm{d}x - \sum_{j=0}^{m-1}\int_{\partial B_\epsilon(x_j)} (-1)^{m+j} v\cdot\nabla^j u_j^\epsilon \nabla^{2m-j-1}u_j \,\mathrm{d}\sigma(x)$$
$$= \int_{B_\epsilon(x_j)} |\nabla^m u_j(x)|^2 dx - \int_{B_\epsilon(x_j)} \nabla^m u_j^\epsilon(x)\nabla^m u_j(x)dx$$

■

Now we can establish an upper bound for the Adams-Adimurthi-Druet inequality on whole $\mathbb{R}^{2m}$ as follows.

Theorem 7.25 *If $AD(\alpha,\gamma)$ is not attained, then*

$$AD(\alpha,\gamma) \le \frac{\omega_{2m}}{2^{2m}} e^{\beta_{2m,m}(K_0-\mathscr{I}_m)},$$

where K_0 is the value at 0 of the regular part of the Green function G for the operator $(-\Delta)^m+\kappa^2$.

Proof: we define

$$\tilde{u}_j(x) = \frac{u_j(x)-u_j^\epsilon}{\|\nabla^m(u_j(x)-u_j^\epsilon)\|_{L^2(B_\epsilon(x_j))}},$$

We have by lemma 7.21

$$\lim_{j\to\infty}\left(\int_{\mathbb{R}^{2m}\setminus B_\epsilon} |c_j\nabla^m u_j|^2 dx + \int_{\mathbb{R}^{2m}\setminus B_\epsilon} |c_j u_j|^2 dx\right) = -\frac{1}{\gamma_m} In|\epsilon| + H_m + K_0 + \alpha(1+\gamma)\|G\|_2^2 + O_j(\epsilon).$$

Let us compute $\|\nabla^m(u_j(x)-u_j^\epsilon)\|_{L^2(B_\epsilon(x_j))}$.
By Lemma 7.24, we have

$$\|\nabla^m(u_j(x)-u_j^\epsilon)\|_{L^2(B_\epsilon(x_j))} = \int_{B_\epsilon(x_j)} |\nabla^m u_j(x)|^2 dx - \int_{B_\epsilon(x_j)} \nabla^m u_j^\epsilon(x)\nabla^m u_j(x)dx$$
$$= I_1 - I_2.$$

For the first,

$$I_1 = \int_{B_\epsilon} |\nabla^m u_j|^2 \, dx = 1 - \int_{\mathbb{R}^{2m}\setminus B_\epsilon} |\nabla^m u_j|^2 \, dx - \int_{\mathbb{R}^{2m}\setminus B_\epsilon} |u_j|^2 \, dx - \int_{B_\epsilon} u_j^2 \, dx$$
$$= 1 - \frac{-\frac{1}{\gamma_m}\ln\epsilon + H_m + K_0 + \alpha(1+\gamma)\|G\|_2^2 + O_j(\epsilon)}{c_j^2} \tag{7-61}$$

On the other hand, by the definition of u_j^ϵ we have

$$\int_{B_\epsilon} \nabla^m\left(c_j u_j^\epsilon\right)\nabla^m\left(c_j u_j\right) dx = \int_{B_\epsilon} \left(c_j u_j^\epsilon\right)(-\Delta)^m\left(c_j u_j\right) dx$$
$$- \sum_{j=0}^{m-1}\int_{\partial B_\epsilon} (-1)^{m+j} v\cdot\nabla^j\left(c_j u_j^\epsilon\right)\nabla^{2m-j-1}\left(c_j u_j\right) d\sigma(x)$$
$$= \int_{B_\epsilon} \left(c_j u_j^\epsilon\right)(-\Delta)^m\left(c_j u_j\right) dx \tag{7-62}$$
$$- \sum_{j=0}^{m-1}\int_{\partial B_\epsilon} (-1)^{m+j} v\cdot\nabla^j\left(c_j u_j\right)\nabla^{2m-j-1}\left(c_j u_j\right) d\sigma(x)$$
$$= \int_{B_\epsilon} \left(c_j u_j^\epsilon\right)\left(\delta(x) - G + o_j(1)\right) dx$$
$$- \left(-\frac{1}{\gamma_m}\log\epsilon + K_0 + H_m + O_j(\epsilon)\right)$$
$$= G(\epsilon) + c_0 - \left(-\frac{1}{\gamma_m}\log\epsilon + K_0 + H_m + O_j(\epsilon)\right) = c_0 - H_m + O_j(\epsilon) \tag{7-63}$$

where $c_0 = \lim_{\epsilon\to 0}\lim_{j\to\infty} c_j\left(u_j^\epsilon - u_j(\epsilon)\right)(0)$. Using Lemma 7.23, we can show that

$$c_0 = -\frac{m}{\beta_{2m,m}}\sum_{k=1}^{m-1}\frac{\binom{m-1}{k}(-1)^k}{k}, \tag{7-64}$$

With (7-64), the equation (7-61) and Lemma 7.24 yields

$$
\begin{aligned}
\int_{B_\epsilon(x_j)} \left|\nabla^m \left(u_j(x) - u_j^\epsilon(x)\right)\right|^2 \, dx &= 1 - \frac{-\frac{1}{\gamma_m}\ln\epsilon + H_m + K_0 + \alpha(1+\gamma)\|G\|_2^2 + O_j(\epsilon)}{c_j^2} \\
&\quad - \frac{c_0 - H_m + O_j(\epsilon)}{c_j^2} \\
&= 1 - \frac{-\frac{1}{\gamma_m}\ln\epsilon + K_0 + \alpha(1+\gamma)\|G\|_2^2 + c_0 + O_j(\epsilon)}{c_j^2}
\end{aligned}
\tag{7-65}
$$

Hence,

$$
\begin{aligned}
\tilde{u}_j^2(x) &= \frac{\left(u_j(x) - u_j^\epsilon(x)\right)^2}{1 - \frac{-\frac{1}{\gamma_m}\ln\epsilon + K_0 + \alpha(1+\gamma)\|G\|_2^2 + c_0 + O_j(\epsilon)}{c_j^2}} \\
&= u_j^2(x)\left(1 - \frac{-\frac{1}{\gamma_m}\ln\epsilon + K_0 + \alpha(1+\gamma)\|G\|_2^2 + c_0 + O_j(\epsilon)}{c_j^2}\right) \\
&\quad - \left(2u_j^\epsilon u_j - \left(u_j^\epsilon\right)^2\right)\left(1 + o_j(\epsilon)\right) \\
&= u_j^2(x) - c\ln\epsilon^{2m} + o_j(\epsilon)
\end{aligned}
$$

On the other hand, we have by lemma 7.15,

$$
\lim_{R\to\infty}\lim_{j\to\infty}\int_{B_\rho\setminus B_{Rr_j}(x_j)} e^{\beta_j\tilde{\zeta}u_j^2}\, dx = |B_\rho|, \text{ for any } \rho < \epsilon.
$$

Therefore,

$$
\begin{aligned}
\lim_{R\to\infty}\lim_{j\to\infty}\int_{B_\rho\setminus B_{Rr_j}(x_j)} e^{\beta_j\tilde{\zeta}u_j^2}\, dx &\le O(\epsilon^{-2mc})\lim_{R\to\infty}\lim_{j\to\infty}\int_{B_\rho\setminus B_{Rr_j}(x_j)} e^{\beta_j\tilde{\zeta}u_j^2}\, dx \\
&= O(\epsilon^{-2mc}|B_\rho|) \to 0, \text{ as } \rho \to 0.
\end{aligned}
$$

We also have, by (7-12)

$$
\lim_{j\to\infty}\int_{B_\epsilon\setminus B_\rho}\left[e^{\beta_j\tilde{\zeta}_j\tilde{u}_j^2} - 1\right] dx = 0.
$$

Thus, from Lemma 7.22 that

$$\lim_{R\to\infty}\lim_{j\to\infty}\int_{B_{Rr_j}}\left(e^{\beta_j\tilde{\zeta}\tilde{u}_j^2}-1\right)dx$$
$$=\lim_{j\to\infty}\int_{B_\delta}\left(e^{\beta_j\tilde{\zeta}\tilde{u}_j^2}-1\right)dx\le\frac{\omega_{2m}}{2^{2m}}\epsilon^{2m}e^{\left(m\sum_{j=1}^{m-1}\frac{\binom{m-1}{j}(-1)^j}{j}-\beta_{2m,m}I_m\right)}. \tag{7-66}$$

Fixing $\bar{R}>0$, then for any $x\in B_{\bar{R}r_j(x_j)}$, we have

$$\beta_j u_j^2=\beta_j\left(\frac{u_j}{\left\|\nabla^m\left(u_j(x)-u_j^\epsilon(x)\right)\right\|_{L^2(B_\epsilon)}}\right)^2\int_{B_\epsilon}\left|\nabla^m\left(u_j(x)-u_j^\epsilon(x)\right)\right|^2dx$$
$$=\beta_j\left(\tilde{u}_j+\frac{u_j^\epsilon(x)}{\left\|\nabla^m\left(u_j(x)-u_j^\epsilon(x)\right)\right\|_{L^2(B_\epsilon)}}\right)^2\int_{B_\epsilon}\left|\nabla^m\left(u_j(x)-u_j^\epsilon(x)\right)\right|^2dx$$

By the identity (7-65) and some direct computations yields

$$\beta_j u_j^2=\beta_j\left(\tilde{u}_j+u_j^\epsilon+O\left(\frac{1}{c_j^2}\right)\right)^2\cdot\left(1-\frac{-\frac{1}{\gamma_m}\ln\epsilon+K_0+\alpha(1+\gamma)\|G\|_2^2+c_0+O_j(\epsilon)}{c_j^2}\right)$$
$$=\beta_j\tilde{u}_j^2\left(1+\frac{u_j^\epsilon}{c_j}+O\left(\frac{1}{c_j^3}\right)\right)^2\left(1-\frac{-\frac{1}{\gamma_m}\ln\epsilon+K_0+\alpha(1+\gamma)\|G\|_2^2+c_0+O_j(\epsilon)}{c_j^2}\right)$$

Observe that

$$\lim_{j\to\infty}\frac{\tilde{u}_j\left(x_j+r_jx\right)}{c_j}=1,\ \text{and}\ \tilde{u}_j\left(x_j+r_jx\right)u_j(\epsilon)\to G(\epsilon).$$

Thus, we have

$$\beta_j u_j^2=\beta_j\tilde{u}_j^2\left(1+\frac{u_j(\epsilon)}{c_j}+\frac{c_0+O_j(\epsilon)}{c_j}+O\left(\frac{1}{c_j^3}\right)\right)^2$$
$$\cdot\left(1-\frac{-\frac{1}{\gamma_m}\ln\epsilon+K_0+\alpha(1+\gamma)\|G\|_2^2+c_0+O_j(\epsilon)}{c_j^2}\right)$$
$$=\beta_j(\tilde{u}_j^2+G(\epsilon)+c_0)+O_j(\epsilon) \tag{7-67}$$

Combining (7-66) and (7-67), we have

$$
\begin{aligned}
&\lim_{\bar{R}\to\infty}\lim_{j\to\infty}\int_{B_{\bar{R}r_j}(x_j)}\left(e^{\beta_j\tilde{\zeta}_j u_j^2}-1\right)dx\\
&\leq \lim_{\bar{R}\to\infty}\lim_{j\to\infty}\int_{B_{\bar{R}r_j}(x_j)}\left(e^{\beta_j\tilde{\zeta}_j u_j^2}\right)dx\\
&\leq \lim_{\bar{R}\to\infty}\lim_{j\to\infty} e^{\beta_{2m,m}(G(\epsilon)+c_0)+o_\epsilon(1)}\int_{B_{\bar{R}r_j}} e^{\beta_j\tilde{\zeta}_j\tilde{u}_j^2}dx\\
&= e^{\beta_{2m,m}(G(\epsilon)+c_0)+o_\epsilon(1)}\frac{\omega_{2m}}{2^{2m}}\epsilon^{2m}e^{\left(m\sum_{j=1}^{m-1}\frac{\binom{m-1}{j}(-1)^j}{j}-\beta_{2m,m}I_m\right)}\\
&= e^{\left(-2m\ln\epsilon+\beta_{2m,m}K_0+\varphi(\epsilon)+o_\epsilon(1)\right)}\frac{\omega_{2m}}{2^{2m}}\epsilon^{2m}e^{\left(-\beta_{2m,m}\mathscr{I}_m\right)}
\end{aligned}
$$

where in the last inequality we used (7-64).Therefore, taking limit as $\epsilon\to 0$, we get

$$AD(\alpha,\gamma)\leq\frac{\omega_{2m}}{2^{2m}}e^{\left(\beta_{2m,m}K_0-\beta_{2m,m}\mathscr{I}_m\right)}.$$

■

7.2 The test function computation

In this section, we construct the test function as introduced in [10] and [16], following a procedure inspired by the computations carried out in the previous chapter. We set

$$\phi_\varepsilon=\begin{cases} C+\frac{1}{C}\left[-\frac{m}{\beta_{2m,m}}\psi\left(\frac{|x|}{\varepsilon}\right)+K_0+\phi(x)+p_\epsilon(x)\right] & \text{if } |x|\leq L\varepsilon,\\ \frac{G(x)}{C} & \text{if } |x|\geq L\varepsilon\end{cases}.$$

where, $\psi(s)=\ln(1+c_n s^{2m})$, where c_n is a dimensional constant. L, C are functions of ε such that

(i) $\varepsilon=\exp(-L)$, $\frac{1}{C^2}=O\left(\frac{1}{L}\right)$ as $\varepsilon\to 0$,
(ii) $p_\varepsilon(x)=-C^2+\sum_{j=0}^{m-1}c_i(\varepsilon,L)|x|^{2j}$, where the coefficients $c_i(\varepsilon,L)$ has the following form:

$$c_0(\varepsilon,L)=-\frac{2m}{\beta_{2m,m}}\ln(2\varepsilon)+d_0(L)\text{ and }c_j(\varepsilon,L)=\varepsilon^{-2j}L^{-2j}d_j(L)$$

where $d_j(L)=O\left(L^{-2}\right)$ as $L\to+\infty$, for $0\leq j\leq m-1$. We remark that this form of $p_\varepsilon(x)$ can make sure that $\partial_\nu^i\phi_\varepsilon(L\varepsilon)=0$, for any $0\leq i\leq m-1$,(see [16], Lemma 5.1). By Lemma 7.21, we have

$$\int_{\mathbb{R}^{2m}\setminus B_{L\varepsilon}(0)} \left(|\nabla^m\phi_\varepsilon|^2+|\phi_\varepsilon|^2\right) \mathrm{d}x = \frac{1}{C^2}\int_{\mathbb{R}^{2m}\setminus B_{L\varepsilon}(0)} \left(|\nabla^m G|^2+|G|^2\right) \mathrm{d}x$$
$$= \frac{1}{C^2}\left(-\frac{1}{\gamma_m}\ln L\varepsilon + H_m + K_0 + \alpha(1+\gamma)\|G\|_2^2 + O_j(L\varepsilon)\right)$$

and by ([16], Proposition 5.3), one has

$$\int_{B_{L\varepsilon}(0)} |\nabla^m\phi_\varepsilon|^2 \ \mathrm{d}x = \frac{1}{C^2}\left(\frac{1}{\gamma_m}\ln\frac{L}{2} + I_m - H_m + O\left(L^{-2}\ln L\right)\right).$$

Notice that the integral over $B_{L\varepsilon}(0)$ satisfies

$$\int_{B_{L\varepsilon}(0)} |\phi_\varepsilon|^2 \mathrm{d}x = O\left((L\varepsilon)^{2m}C^{2m}\right). \tag{7-68}$$

On the other hand, on its complement, we have

$$\int_{\mathbb{R}^n\setminus B_{L\varepsilon}(0)} |\phi_\varepsilon|^2 \mathrm{d}x = \frac{1}{C^2}\int_{\mathbb{R}^n\setminus B_{L\varepsilon}(0)} |G|^2 \mathrm{d}x = \frac{1}{C^2}\left[\|G\|_2^2 + O\left((-L\varepsilon\ln(L\varepsilon))^{2m}\right)\right]. \tag{7-69}$$

thus,

$$\int_{\mathbb{R}^{2m}} \left(|\nabla^m\phi_\varepsilon|^2+|\phi_\varepsilon|^2\right) dx = \frac{1}{C^2}\left[-\frac{1}{\gamma_m}\ln(L\varepsilon) + \frac{1}{\gamma_m}\ln\frac{L}{2} + \mathscr{I}_m\right.$$
$$\left. + K_0 + \alpha(1+\gamma)\|G\|_2^2 + O_j(L\varepsilon) + O\left(L^{-2}\ln L\right)\right] + O\left(\frac{(L\varepsilon)^{2m}}{C^2}\right)$$
$$= \frac{1}{C^2}\left[-\frac{1}{\gamma_m}(\ln 2\varepsilon) + \mathscr{I}_m + K_0 + \alpha(1+\gamma)\|G\|_2^2\right.$$
$$\left. + O(L\varepsilon) + O\left(L^{-2}\ln L\right) + O\left((L\varepsilon)^{2m}\right)\right]. \tag{7-70}$$

By setting $\int_{\mathbb{R}^{2m}} \left(|\nabla^m\phi_\varepsilon|^2+|\phi_\varepsilon|^2\right) \mathrm{d}x = 1$, we have

$$C^2 = -\frac{1}{\gamma_m}(\ln 2\varepsilon) + \mathscr{I}_m + K_0 + \alpha(1+\gamma)\|G\|_2^2 + +O(L\varepsilon) + O\left(L^{-2}\ln L\right) + O\left((L\varepsilon)^{2m}\right) \tag{7-71}$$

then

$$C^2 \sim \frac{1}{\gamma_m}\ln\frac{1}{\varepsilon}. \tag{7-72}$$

In addition, since $\lim_{t\to 0}\frac{1+\alpha t^2}{1-\alpha\gamma t^2} = 1+\alpha(\gamma+1)t^2 + O(t^4)$ and, from (7-69) we have $\|\phi_\varepsilon\|_2^2 =$

$O\left(\frac{1}{C}\right) \to 0$ as $\varepsilon \to 0$. Therefore, once again by (7-69), we can write

$$\begin{aligned}\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2} &= 1+\alpha(\gamma+1)\|\phi_\epsilon\|_2^2+O(\|\phi_\epsilon\|_2^4)\\ &= 1+\frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}+O\Big(C^{-2}(-L\epsilon\ln(L\epsilon))^4\Big)+O\Big(C^{-4}\Big).\end{aligned} \tag{7-73}$$

Recalling $e^t-1 \geq t$, for $t \geq 0$, from (7-69) and (7-73), we get

$$\begin{aligned}&\int_{\mathbb{R}^{2m}\setminus B_{L\epsilon}}\left(e^{\beta_{2m,m}\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2}-1\right)dx\\ &\geq \beta_{2m,m}\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\int_{\mathbb{R}^{2m}\setminus B_{L\epsilon}}\phi_\epsilon^2 dx\\ &= \frac{\beta_{2m,m}}{C^2}\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\left[\|G\|_2^2+O\Big((-L\epsilon\ln(L\epsilon))^{2m}\Big)\right]\\ &= \frac{\beta_{2m,m}\|G\|_2^2}{C^2}+O\Big(C^{-2m}\Big).\end{aligned} \tag{7-74}$$

Now, from (7-73), we also have

$$\begin{aligned}\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2 &= \left(1+\frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}\right)\phi_\epsilon^2+\left[O\Big((-L\epsilon\ln(L\epsilon))^4\Big)+O\Big(C^{-2}\Big)\right]\frac{\phi_\epsilon^2}{C^2}\\ &= \mathscr{T}_1+\mathscr{T}_2.\end{aligned} \tag{7-75}$$

Now, let us estimate each term on the right side of the (7-75) on $B_{L\varepsilon}$. Firstly, we have by the definition of the test function

$$\begin{aligned}\phi_\epsilon^2 &\geq C^2+2\left(-\frac{m}{\beta_{2m,m}}\ln\left(1+\frac{x^2}{4\varepsilon^2}\right)+K_0+\varphi(x)-C^2+\sum_{j=0}^{m-1}c_j(\varepsilon,L)|x|^{2j}\right)\\ &= C^2-\frac{2m}{\beta_{2m,m}}\ln\left(1+\frac{x^2}{4\varepsilon^2}\right)+2K_0+2\varphi(x)-2C^2+2\sum_{j=0}^{m-1}c_j(\varepsilon,L)|x|^{2j}\\ &= -C^2-\frac{2m}{\beta_{2m,m}}\ln\left(1+\frac{x^2}{4\varepsilon^2}\right)+2K_0+2\varphi(x)+2\sum_{j=0}^{m-1}c_j(\varepsilon,L)|x|^{2j}.\end{aligned}$$

For the first term $\mathscr{T}_1$, we have the following estimate

$$\begin{aligned}\mathscr{T}_1 &\geq \left(1+\frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}\right)\left[-C^2-\frac{2m}{\beta_{2m,m}}\ln\left(1+\frac{x^2}{4\varepsilon^2}\right)+2K_0+2\varphi(x)+2\sum_{j=0}^{m-1}c_j(\varepsilon,L)|x|^{2j}\right]\\ &= -C^2-\alpha(\gamma+1)\|G\|_2^2+\\ &\left(1+\frac{\alpha(\gamma+1)\|G\|_2^2}{C^2}\right)\left[-\frac{1}{\gamma_m}\ln\left(1+\frac{x^2}{4\varepsilon^2}\right)+2K_0+2\varphi(x)+2\sum_{j=0}^{m-1}c_j(\varepsilon,L)|x|^{2j}\right]\end{aligned}$$

Using the expression of (7-71), we have after some careful computations

$$\begin{aligned}
\mathscr{T}_1 \geq& \frac{1}{\gamma_m}\ln(2\varepsilon) - \mathscr{I}_m - K_0 - 2\alpha(\gamma+1)\|G\|_2^2 \\
&- \frac{2m}{\beta_{2m,m}}\ln\left(1+\frac{x^2}{4\varepsilon^2}\right) + 2K_0 + 2\varphi(x) + 2\sum_{j=0}^{m-1} c_j(\varepsilon,L)|x|^{2j} + O\left(\frac{1}{\ln(1/\varepsilon)}\right) \\
=& \frac{1}{\gamma_m}\ln(2\varepsilon) - \frac{2m}{\beta_{2m,m}}\ln\left(1+\frac{x^2}{4\varepsilon^2}\right) - \mathscr{I}_m + K_0 \\
&- 2\alpha(\gamma+1)\|G\|_2^2 + O\left(L^{-2}\ln(L)\right) + O\left(\frac{1}{\ln(1/\varepsilon)}\right).
\end{aligned} \tag{7-76}$$

This follows from the fact that $\varphi \in C^{2m-1}$ and $\varphi(0)=0$. Now, for the second term $\mathscr{T}_2$ and for $x \in B_{L\epsilon}$, it follows that

$$\left|\psi\left(\frac{|x|}{\epsilon}\right)\right| \leq \ln\left(1+c_n L^{2m}\right) = m\ln L + \ln\left(c_n + \frac{1}{L^{2m}}\right) = O(\ln(-\ln\epsilon)).$$

So, by definition of ϕ_ϵ and (7-73), on $B_{L\epsilon}$ we get

$$\left|\frac{\phi_\epsilon}{C}\right| \leq O\Big(C^{-2}\ln(-\ln\epsilon)\Big) + O\Big(C^{-2}\Big) \to 0, \text{ as } \epsilon \to 0.$$

It follows that

$$\mathscr{T}_2 = \left[O\Big((-L\epsilon\ln(L\epsilon))^4\Big) + O\Big(C^{-2}\Big)\right]\frac{\phi_\varepsilon^2}{C^2} = O\Big(\frac{1}{\ln^2\epsilon}\Big).$$

So, by combining (7-75) with (7-76), for $x \in B_{L\epsilon}$ we can write

$$\begin{aligned}
&\beta_{2m,m}\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2 \\
&\geq \beta_{2m,m}\Big(-\frac{1}{\gamma_m}\ln(2\varepsilon) - \frac{2m}{\beta_{2m,m}}\ln\left(1+\frac{x^2}{4\varepsilon^2}\right) - \mathscr{I}_m \\
&+ K_0 - 2\alpha(\gamma+1)\|G\|_2^2 + O\left(L^{-2}\ln(L)\right) + O\left(\frac{1}{\ln(1/\varepsilon)}\right) + O\Big(\frac{1}{\ln^2\epsilon}\Big)\Big) \\
&= -2m\ln(2\varepsilon) - 2m\ln\left(1+\frac{x^2}{4\varepsilon^2}\right) + \beta_{2m,m}(K_0 - \mathscr{I}_m) \\
&- 2\beta_{2m,m}\alpha(\gamma+1)\|G\|_2^2 + \beta_{2m,m}\left(O\left(L^{-2}\ln(L)\right) + O\left(\frac{1}{\ln(1/\varepsilon)}\right) + O\Big(\frac{1}{\ln^2\epsilon}\Big)\right)
\end{aligned}$$

Hence,

$$\int_{B_{L\epsilon}} \left(e^{\beta_{2m,m}\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2} - 1 \right) \mathrm{d}x \geq \int_{B_{L\epsilon}} e^{\beta_{2m,m}\left(\frac{1+\alpha\|\phi_\epsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2} \mathrm{d}x + O\left((L\varepsilon)^{2m} C^2\right)$$
$$\geq (2\varepsilon)^{-2m} e^{\beta_{2m,m}(K_0 - \mathscr{I}_m)} e^{-2\beta_{2m,m}\alpha(\gamma+1)\|G\|_2^2} \int_{B_{L\varepsilon}} \left(1 + \frac{x^2}{4\varepsilon^2}\right)^{-2m} \mathrm{d}x + +O\left((L\varepsilon)^{2m} C^2\right)$$
$$= (2\varepsilon)^{-2m} e^{\beta_{2m,m}(K_0 - \mathscr{I}_m)} e^{-2\beta_{2m,m}\alpha(\gamma+1)\|G\|_2^2} \frac{1}{\epsilon^{2m}} \int_{B_L} \left(1 + \frac{z^2}{4}\right)^{-2m} dz + O\left((L\varepsilon)^{2m} C^2\right). \tag{7-77}$$

Since $\int_{B_L} \left(1 + \frac{|z|^2}{4}\right)^{-2m} dz = \omega_{2m} + O\left(L^{-2m}\right)$, we have

$$\int_{B_{L\epsilon}} \left(e^{\beta_{2m,m}\left(\frac{1+\alpha\|\phi_\varepsilon\|_2^2}{1-\alpha\gamma\|\phi_\varepsilon\|_2^2}\right)\phi_\epsilon^2} - 1 \right) \mathrm{d}x$$
$$= 2^{-2m} \omega_{2m} e^{\beta_{2m,m}(K_0 - \mathscr{I}_m)} e^{-2\beta_{2m,m}\alpha(\gamma+1)\|G\|_2^2} + O\left(L^{-2m}\right) + O\left((L\varepsilon)^{2m} C^2\right). \tag{7-78}$$

Combining (7-74) and (7-78), we get

$$\int_{B_{L\epsilon}} \left(e^{\beta_{2m,m}\left(\frac{1+\alpha\|\phi_\varepsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2} - 1 \right) \mathrm{d}x$$
$$\geq 2^{-2m} \omega_{2m} e^{\beta_{2m,m}(K_0 - \mathscr{I}_m)} e^{-2\beta_{2m,m}\alpha(\gamma+1)\|G\|_2^2}$$
$$+ \frac{\beta_{2m,m}\|G\|_2^2}{C^2} + O\left(L^{-2m}\right) + O\left((L\varepsilon)^{2m} C^2\right) + O\left(C^{-2m}\right)$$

By (7-72), we know $C^2 \sim |\ln \varepsilon|$, which implies $\frac{\beta_{2m,m}\|G\|_2^2}{C^2} \to 0$ as $\varepsilon \to 0$. Moreover this, we have by (i) that $O\left(L^{-2m}\right) + O\left((L\varepsilon)^{2m} C^2\right) + O\left(C^{-2m}\right) \to 0$ as $\varepsilon \to 0$. Therefore, by the fact $\gamma \leq \gamma_0$ is bounded, by choosing $\alpha_0 > 0$ small such that $e^{-2\beta_{2m,m}\alpha_0(\gamma+1)\|G\|_2^2} > 1$, ones can conclude that

$$\int_{\mathbb{R}^{2m}} \left(e^{\beta_{2m,m}\left(\frac{1+\alpha\|\phi_\varepsilon\|_2^2}{1-\alpha\gamma\|\phi_\epsilon\|_2^2}\right)\phi_\epsilon^2} - 1 \right) \mathrm{d}x > 2^{-2m} \omega_{2m} e^{(\beta_{2m,m} K_0 - \beta_{2m,2} \mathscr{I}_m)}$$

for ε small enough and suitable. This accomplishes the proof.

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