UNIVERSIDADE FEDERAL DE GOIÁS (UFG)
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA (IME)
PROGRAMA DE PÓS-GRADUAÇÃO EM MATEMÁTICA

NARA REGES FARIA DE PAIVA PEREIRA

Length Coordinates for Moduli Spaces and the Distribution of Semi-Arithmetic Hyperbolic Surfaces

Coordenadas de Comprimento para Espaços Moduli e a Distribuição de Superfícies Hiperbólicas Semi-Aritméticas

Goiânia
2026

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NARA REGES FARIA DE PAIVA PEREIRA

Length Coordinates for Moduli Spaces and the Distribution of Semi-Arithmetic Hyperbolic Surfaces

Coordenadas de Comprimento para Espaços Moduli e a Distribuição de Superfícies Hiperbólicas Semi-Aritméticas

Tese apresentada ao Programa de Pós–Graduação em Matemática, do Instituto de Matemática e Estatística (IME) da Universidade Federal de Goiás (UFG), como requisito para obtenção do título de Doutora em Matemática.
Área de concentração: Geometria.
Orientador: Dr. Cayo Rodrigo Felizardo Dória.

Goiânia
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Length Coordinates for Moduli Spaces and the Distribution of Semi-Arithmetic Hyperbolic Surfaces [Manuscrito] = Coordenadas de Comprimento para Espaços Moduli e a Distribuição de Superfícies Hiperbólicas Semi-Aritméticas / Nara Reges Faria de Paiva Pereira. - 2026.

100 f.: 2026

Orientador: Prof. Dr. Cayo Rodrigo Felizardo Dória

Tese (Doutorado) - Universidade Federal de Goiás, Instituto de Matemática e Estatística (IME), Programa de Pós-Graduação em Matemática, Goiânia, 2026.

Bibliografia.

Inclui: lista de figuras.

1. Hyperbolic Surfaces. 2. Teichmüller Space. 3. Length Spectrum. 4. Semi-arithmetic Surfaces. 5. Fuchsian Groups.
I. Dória, Cayo Rodrigo Felizardo, orient. II. Título.

CDU 51

UNIVERSIDADE FEDERAL DE GOIÁS

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ATA DE DEFESA DE TESE

Ata Nº **31** da sessão de Defesa de Tese de **Nara Reges Faria de Paiva Pereira** que confere o título de Doutora em **Matemática**, na área de concentração em **Geometria.**

Ao **vigésimo quarto dia do mês de abril do ano de dois mil e vinte e seis**, a partir da 11h00, de forma Híbrida na Sala de Aula do IME/UFG, realizou-se a sessão pública de Defesa de Tese intitulada **"Length Coordinates for Moduli Spaces and the Distribution of Semi-Arithmetic Hyperbolic Surfaces"**. Os trabalhos foram instalados pelo Orientador, Professor Doutor **Cayo Rodrigo Felizardo Dória - IME/UFG,** com a participação dos demais membros da Banca Examinadora: Professor Doutor **Ronaldo Alves Garcia - IME/UFG**, membro titular interno; Professor Doutor **André Salles de Carvalho - IME/USP**, membro titular externo; Professora Doutora **Gisele Teixeira Paula - DMAT/UFPR** membra titular externa; Professor Doutor **Mikhail Viktorovich Belolipetsky - IMPA**, membro titular externo. Durante a arguição os membros da banca **não fizeram** sugestão de **alteração** do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Tese tendo sido a candidata **aprovada** pelos seus membros. Proclamados os resultados pelo Professor Doutor **Cayo Rodrigo Felizardo Dória**, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos Membros da Banca Examinadora, ao **vigésimo quarto dia do mês de abril do ano de dois mil e vinte e seis.**

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Aos meus amados filhos, Lucas e Laura,
que dão sentido a tudo o que sou e a tudo o que faço.

Agradecimentos

Ao longo desta caminhada aprendi que uma tese nunca é construída por uma única pessoa. Ela é resultado de muitas mãos, muitos incentivos, muitas renúncias e, sobretudo, da graça de Deus.

A Deus, toda honra e toda glória. Foi Ele quem despertou em mim a curiosidade de compreender, ainda que de forma limitada, essa bela ciência que descreve a natureza que Ele próprio criou. Em cada etapa deste doutorado pude perceber Seu cuidado nos pequenos detalhes. Quando os desafios pareciam maiores do que minhas forças, foi Ele quem me sustentou, renovou minha esperança e me concedeu coragem para continuar.

Ao meu esposo, meu maior incentivador. Nós nos conhecemos ainda no ensino fundamental e, nas longas conversas por telefone daquela época, você foi responsável por ampliar meus horizontes e me fazer enxergar possibilidades que talvez eu sequer acreditasse serem para mim. Ao longo da graduação, do mestrado e do doutorado, você foi meu porto seguro. Celebrou cada conquista como se fosse sua, acreditou em mim até quando eu mesma já não conseguia acreditar e tornou este sonho possível das mais diversas formas. Quantos domingos, feriados e noites você assumiu sozinho os cuidados com nossos filhos para que eu pudesse estudar, escrever ou viajar. Este título também é seu.

Aos meus filhos, Lucas e Laura, razão maior desta conquista. O Lucas tinha apenas três anos e oito meses e a Laura um ano e cinco meses quando iniciei o doutorado. Vocês cresceram acompanhando essa jornada. Assistiram aulas comigo, dormiram em meu colo enquanto eu estudava, compreenderam pacientemente os momentos em que eu precisava trabalhar e conviveram com minha ausência mais do que eu gostaria. Se em alguns momentos este sonho me afastou de vocês, saibam que foram justamente vocês que me deram forças para realizá-lo. Espero que um dia compreendam que tudo isso também foi por vocês.

Aos meus pais, por me ensinarem, desde muito cedo, o valor da educação. Embora não tenham tido as oportunidades de estudo que eu tive, sempre acreditaram que o conhecimento poderia transformar vidas. Nunca permitiram que as limitações financeiras ou as dificuldades fossem maiores do que meus sonhos. Foram minha rede de apoio mais forte, ajudando em tudo o que esteve ao alcance deles. Hoje posso dizer, com profunda gratidão, que cheguei até aqui graças ao amor, aos sacrifícios e ao incentivo que sempre recebi.

Aos meus familiares, que também dividiram comigo o peso e a alegria desta caminhada. Ao meu irmão Diego e à minha cunhada Franciele; aos meus avós, José de Campos e Teresinha; ao meu tio Marciano; aos meus sogros, Suedis e Marlene; aos meus cunhados Filipe e Lucas, e à minha cunhada Malaine, agradeço pelo carinho, pelas orações, pelo apoio durante minhas viagens e por celebrarem comigo esta conquista. Aos meus sobrinhos Isabela, Luiza, Felipe e João Pedro, obrigada por trazerem leveza, alegria e tantos momentos felizes em meio às exigências da vida acadêmica.

Ao longo da minha formação tive o privilégio de encontrar professores extraordinários, que moldaram não apenas a pesquisadora, mas também a professora que hoje procuro ser. Ainda no ensino médio conheci o Professor Gean Godoy, um jovem matemático que tinha uma missão muito clara: fazer com que estudantes da periferia acreditassem que poderiam transformar suas histórias por meio da educação. Tenho certeza de que não fui a única profundamente marcada por seu exemplo. Na graduação, guardo com enorme carinho os ensinamentos dos Professores Romildo Pina (in memoriam), Genésio Reis (in memoriam) e Jesus Carlos, cuja paciência, dedicação e amor pela Matemática despertaram em mim o desejo de seguir nessa profissão.

Ainda na graduação, tive a felicidade de ser orientada pelo Professor Ronaldo Alves Garcia em um projeto de iniciação científica. Mesmo sendo caloura, fui acolhida por um pesquisador cuja excelência acadêmica eu só compreenderia plenamente ao longo dos anos. Durante mais de três anos de orientação, aprendi muito mais do que Matemática. Aprendi a fazer perguntas, a enfrentar problemas difíceis e a desenvolver autonomia intelectual. Sou profundamente grata pelas oportunidades, pela confiança e pelos ensinamentos que marcaram minha formação.

No mestrado, tive o privilégio de ser orientada pelo Professor Paulo Henrique. Agradeço pela dedicação, pelos ensinamentos e pela confiança depositada em meu trabalho, fundamentais para consolidar minha formação como pesquisadora e despertar em mim o desejo de continuar na vida acadêmica.

Quando procurei a UFG, seis anos após concluir o mestrado, minha intenção era apenas cursar uma disciplina como aluna especial. Foi o Professor Benedito Leandro quem me convenceu de que eu poderia ir além e me incentivou a participar do processo seletivo do doutorado. Hoje percebo que aquele incentivo mudou os rumos da minha vida acadêmica. Tive ainda o privilégio de tê-lo como professor de Geometria Riemanniana e, posteriormente, como membro da minha banca de qualificação. Seu incentivo e sua confiança foram decisivos para que esta história acontecesse.

Agradeço também ao Professor Hiuri Reis, cuja generosidade e disponibilidade marcaram profundamente minha trajetória. Além do apoio durante as disciplinas de Seminários, esteve sempre disposto a ajudar nas inúmeras dúvidas técnicas que surgiram ao longo do doutorado. Sua colaboração foi fundamental em diversos momentos desta caminhada.

Ao meu orientador, Professor Cayo Dória, minha mais sincera gratidão. Sei que aceitar orientar uma doutoranda naquele momento talvez não estivesse entre seus planos, mas sou imensamente grata por essa decisão. Foi por meio de sua orientação que conheci uma área da Matemática completamente nova para mim, que rapidamente se tornou motivo de grande entusiasmo. Além do conhecimento matemático, o professor Cayo abriu portas, apresentou pessoas, incentivou minha participação em eventos científicos e criou oportunidades que transformaram minha trajetória. Agradeço também pela paciência nos momentos mais difíceis, pela sensibilidade em compreender minha realidade como professora, esposa e mãe, e por sempre enxergar além da pesquisadora. Sua confiança e humanidade fizeram toda a diferença para que esta tese fosse concluída.

Também agradeço aos colegas da área que, embora vinculados a diferentes instituições, tornaram-se verdadeiros irmãos acadêmicos ao longo desta caminhada. Em especial, agradeço à Professora Gisele Teixeira, ao Professor Plínio Murillo e ao Gregory Cosac, que me acolheram desde o início, compartilharam conhecimentos, esclareceram dúvidas, me incluíram em grupos de estudo e incentivaram minha participação em eventos científicos, contribuindo significativamente para minha formação como pesquisadora. À Gisele, agradeço por ter sido uma referência desde o início desta jornada. Sua generosidade em compartilhar experiências e discutir Matemática foi fundamental para minha formação. Agradeço, ainda, pela leitura cuidadosa desta tese como membro da banca de defesa e pelas valiosas sugestões, que contribuíram significativamente para a versão final deste trabalho. Ao Plínio, agradeço pela disponibilidade em ajudar sempre que necessário e pelas importantes contribuições apresentadas durante minha banca de qualificação, que ajudaram a amadurecer este trabalho. Ao Gregory, agradeço pela parceria construída especialmente na etapa final do doutorado, pelas inúmeras horas de estudo em conjunto e pela satisfação de dar continuidade a essa colaboração em nosso trabalho de pesquisa em andamento.

É verdade que a caminhada acadêmica se torna muito mais leve quando compartilhada com bons amigos. Sou grata aos amigos da graduação e do mestrado, em especial Kamila Andrade, Laredo Rennan, Lucyjane de Almeida, Paula Gonçalves

e Thamara Policarpo, pelas incontáveis horas de estudo, pelas conversas sobre Matemática, carreira, fé e vida. Durante o doutorado, agradeço a Ana Paula Machado, Jefferson Luís, Eduardo Dias, Márcia Cardoso, Elaine Altino, Alejandra Muñoz, Vitória Chaves e Vítor Emanuel pela companhia, pelas discussões acadêmicas e pelo apoio mútuo.

Também registro minha gratidão às amigas Sandra, Gabriele, Paloma, Abiqueila, Abilene, Abigail, Mariele e Ione. A amizade, o incentivo constante e as palavras de encorajamento de vocês foram combustível para que eu seguisse em frente.

Agradeço aos estudantes, professores e servidores do Instituto de Matemática e Estatística da Universidade Federal de Goiás, que contribuíram para minha formação em um ambiente de constante aprendizado. Em especial, agradeço aos professores Bruno Freitas, Abiel Costa, Edcarlos Domingos e Armando Corro, pelos conhecimentos compartilhados ao longo das disciplinas cursadas e pela dedicação com que conduziram sua atividade docente.

Registro também minha gratidão aos membros da banca examinadora pela disponibilidade em ler este trabalho e pelas valiosas sugestões, que certamente contribuíram para aprimorar esta tese. Agradeço especialmente ao Professor André Salles, pelas observações e contribuições apresentadas durante a defesa, e ao Professor Mikhail Belolipetskiy, cuja leitura cuidadosa, sugestões na pesquisa e comentários ao longo deste trabalho enriqueceram significativamente os resultados aqui desenvolvidos.

Agradeço também à Universidade Federal do Oeste da Bahia (UFOB), instituição da qual tenho a honra de fazer parte como docente, pelo apoio institucional e pela concessão do afastamento para qualificação durante dois anos e meio, condição essencial para que este doutorado pudesse ser realizado. Estendo minha gratidão aos colegas de trabalho, que acompanharam esta jornada com incentivo, compreensão e palavras de encorajamento. Em especial, agradeço aos professores Bruno Trindade e Camila Oliveira, cujo apoio e amizade fizeram diferença em momentos importantes desta caminhada.

Por fim, agradeço a todos aqueles que, de alguma forma, caminharam ao meu lado durante esta jornada. Cada gesto de incentivo, cada oração, cada palavra de apoio e cada demonstração de carinho ajudaram a transformar este sonho em realidade.

"All true science is but an interpretation of the handwriting of God in the material world."

Ellen G. White,
Patriarchs and Prophets.

Resumo

Paiva, Nara. **Length Coordinates for Moduli Spaces and the Distribution of Semi-Arithmetic Hyperbolic Surfaces Coordenadas de Comprimento para Espaços Moduli e a Distribuição de Superfícies Hiperbólicas Semi-Aritméticas**. Goiânia, 2026. p100. Tese de Doutorado do Programa de Pós–Graduação em Matemática do Instituto de Matemática e Estatística da Universidade Federal de Goiás.

Esta tese investiga a interação entre geometria hiperbólica, espaço moduli e estruturas aritméticas em superfícies, com ênfase em superfícies hiperbólicas semi-aritméticas e subaritméticas.

Do ponto de vista geométrico, mostramos que qualquer superfície hiperbólica fechada de gênero $g \geq 2$, na parte espessa do espaço moduli, é determinada pelos comprimentos de no máximo $12g - 12$ geodésicas fechadas, com cotas logarítmicas explícitas em g. Esse resultado fornece uma versão quantitativa da determinação por comprimentos, ótima a menos de constantes multiplicativas.

Aplicamos esse arcabouço a problemas de natureza aritmética. Na classe de superfícies aritméticas, obtemos cotas inferiores polinomiais para a distância de Teichmüller entre recobrimentos finitos distintos de uma superfície fixa dessa classe. No contexto semi-aritmético, provamos resultados de contagem para superfícies com invariantes limitados, mostrando que sua quantidade cresce, no máximo, de forma superexponencial com o gênero.

Por fim, estabelecemos um resultado de densidade para superfícies subaritméticas de assinatura $(1, 1)$, mostrando que os Q-pedaços subaritméticos formam um subconjunto denso no espaço moduli correspondente.

Palavras-chave

Superfícies Hiperbólicas; Espaço de Teichmüller; Espectro de Comprimentos; Superfícies Semi-Aritméticas; Grupos Fuchsianos.

Abstract

Paiva, Nara. **Length Coordinates for Moduli Spaces and the Distribution of Semi-Arithmetic Hyperbolic Surfaces**. Goiânia, 2026. p100.
PhD. Thesis from the Graduate Program in Mathematics at the Institute of Mathematics and Statistics of the Federal University of Goiás.

This thesis investigates the interplay between hyperbolic geometry, moduli space, and arithmetic structures on surfaces, with a focus on semi-arithmetic and subarithmetic hyperbolic surfaces.

On the geometric side, we show that any closed hyperbolic surface of genus $g \geq 2$ in the thick part of moduli space is determined by the lengths of at most $12g - 12$ closed geodesics, with explicit logarithmic bounds in g. This provides a quantitative form of length-based determination, optimal up to multiplicative constants.

We apply this framework to arithmetic problems. Within the class of arithmetic surfaces, we obtain polynomial lower bounds for the Teichmüller distance between distinct finite coverings of a fixed surface in this class. In the semi-arithmetic setting, we prove counting results for surfaces with bounded invariants, showing that their number grows at most superexponentially with the genus.

Finally, we establish a denseness result for subarithmetic surfaces of signature $(1, 1)$, proving that subarithmetic Q–pieces form a dense subset of the corresponding moduli space.

Keywords

Hyperbolic Surfaces; Teichmüller Space; Length Spectrum; Semi-Arithmetic Surfaces; Fuchsian Groups.

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Introdution

Hyperbolic surfaces provide a natural setting for relating geometric, topological, and analytic properties of surfaces. The space of all such structures, organized up to equivalence, gives rise to the moduli space, which appears as a central object in several areas of mathematics, including low-dimensional topology, algebraic geometry, and mathematical physics (see [15, Chapter 12]). From a geometric perspective, the lengths of closed geodesics encode significant information about the surface and are closely related to spectral invariants of the Laplace operator via the Selberg trace formula (see [8, Section 9.5]). From the point of view of number theory, certain hyperbolic surfaces carry additional arithmetic structure, as in the case of arithmetic Fuchsian groups, where geometric quantities such as traces of group elements are related to algebraic data in number fields (see [17, Chapter 5]). These interactions make the study of hyperbolic surfaces a natural framework for investigating how geometric data, such as lengths of closed geodesics, reflect deeper structural properties of the surface.

Let $\mathcal{M}_g$ denote the moduli space of closed hyperbolic surfaces of genus $g \geq 2$. Points of $\mathcal{M}_g$ represent isometry classes of hyperbolic metrics on a fixed topological surface of genus g, and through the uniformization theorem every such surface can be written as $S = \Gamma\backslash\mathbb{H}$, where $\Gamma < \mathrm{PSL}(2, \mathbb{R})$ is a torsion-free cofinite Fuchsian group.

Although moduli space provides a natural setting for studying hyperbolic surfaces up to isometry, it does not retain information about how curves on different surfaces correspond to each other. In particular, when comparing geometric data such as lengths of closed geodesics, it is often necessary to keep track of the underlying topological identification between surfaces. This leads naturally to the introduction of markings and to the Teichmüller space, which records hyperbolic structures together with a reference homeomorphism from a fixed topological surface.

Formally, the Teichmüller space $\mathcal{T}_g$ consists of equivalence classes of marked hyperbolic surfaces, and it carries the Teichmüller metric defined via extremal quasiconformal maps (see [15, Chapter 11]). This metric measures the minimal quasiconformal distortion between marked surfaces and yields precise multiplicative control on the lengths of closed geodesics. In particular, small Teichmüller distance implies uniform bounds on the distortion of the length spectrum.

For every genus $g \geq 2$, Seppälä and Sorvali [30] proved that there exists a collection of $6g-4$ marked simple closed geodesics on a fixed reference surface such that the corresponding length functions determine uniquely any point of $\mathcal{T}_g$. This raises a natural quantitative problem: given a surface $S \in \mathcal{T}_g$, how short can the curves in a family of closed curves be so that their lengths still determine S?

Let $S \in \mathcal{T}_g$ be a marked hyperbolic surface with $g \geq 2$ and injectivity radius at least $s > 0$. Dória proved in [13, Proposition 4.1.7] that one can choose $9g-9$ simple closed curves on the reference surface whose length functions determine any point in $\mathcal{T}_g$ and whose lengths on S satisfy an explicit linear bound in g. More precisely, there exist constants A (universal) and $B = B(s)$ such that each of these geodesics has lengths on S at most $Ag + B$. While every hyperbolic surface of genus g admits a closed geodesic of length at most $2\log g + D$, for a universal constant D, the constructions of Brooks [7, Lemma 3.1], Buser–Sarnak [9, Section 4], and Katz–Schaps–Vishne [19, Theorem 1.10] provide sequences of surfaces for which the length of the shortest closed geodesic is bounded below by a constant multiple of $\log g$, indicating that logarithmic behavior is the best possible order.

Dória's result [13, Proposition 4.1.7] relies on pants decompositions and Fenchel–Nielsen coordinates. By contrast, Parlier [28] introduced a different approach to length coordinates, based on families of curves designed to control the geometry of a surface directly through length data.

Parlier's construction begins with a collection $\mathcal{C}$ of disjoint simple closed geodesics and a maximal family $\mathcal{A}$ of simple orthogeodesic arcs in the complement whose endpoints lie on the geodesics of $\mathcal{C}$. Each arc $a \in A$ determines an associated closed geodesic γ_a, obtained by concatenating the arc with suitable subarcs of boundary curves. The union of the curves in $\mathcal{C}$ together with the associated chains $\{\gamma_a\}$ forms what is called a *curve and chain system*. A key feature of such systems is that the lengths $\ell(\gamma)$ and twist parameters $\tau(\gamma)$ for $\gamma \in \mathcal{C}$, defined in Section 1.5, together with the lengths $\ell(\gamma_a)$ of the associated chain curves arising from orthogeodesic arcs, determine uniquely the isometry class of the surface, as shown in [28, Proposition 3.6]. Moreover, Parlier shows that every closed hyperbolic surface of genus $g \geq 2$ admits a curve and chain system in which the lengths of all curves involved grow at most of order $\log(g)$.

Building on this construction, we obtain a quantitative determination theorem for surfaces in the thick part of moduli space, which is a consequence of Theorem 3.4 in this thesis.

Theorem A *Let $s > 0$. For any closed hyperbolic surface $S \in \mathcal{M}_g$ with injectivity radius at least s, there exist closed geodesics $\gamma_1, \dots, \gamma_n \subset S$, with $n \leq 12g - 12$, such that:*

(1) The length of each curve satisfies

$$\ell(\gamma_i) \leq 16\log(4g) + 8\,\mathrm{arcsinh}\left(\frac{1}{\sinh\left(\frac{s}{2}\right)}\right), \quad \textit{for all } i = 1, \dots, n;$$

(2) If $S' \in \mathcal{M}_g$ also has injectivity radius at least s and there exists a homeomorphism $\varphi : S \to S'$ such that

$$\ell_{S'}(\varphi(\gamma_i)) = \ell_S(\gamma_i), \quad \textit{for all } i = 1, \dots, n,$$

then S and S' are isometric.

We remark that this bound is optimal up to multiplicative constants. Indeed, as mentioned above, there exist sequences of genera $\{g_i\}$ tending to infinity for which one can find closed hyperbolic surfaces S_i of genus g_i such that every closed geodesic $\gamma \subset S_i$ satisfies $\ell(\gamma) \geq c\log(g_i)$ for some universal constant $c > 0$. Consequently, the upper bound in Theorem A cannot, in general, be improved beyond the logarithmic order in g.

This optimality has important implications for quantitative estimates. An effective version was recently established by Qiliang Luo [23], who proved that the degrees of the coverings can be bounded above by a polynomial in ε^{-1}. More precisely, there exists a constant $k > 0$ such that coverings achieving $d_{\mathcal{T}}(M_\varepsilon, N_\varepsilon) \leq \varepsilon$ may be chosen with degree at most ε^{-k}.

Among all hyperbolic surfaces, arithmetic surfaces form a distinguished and rigid subclass. These surfaces are characterized by strong algebraic constraints on the traces of their uniformizing groups.

Let S be a fixed closed hyperbolic surface. For each genus g, only finitely many surfaces in $\mathcal{M}_g$ arise as coverings of S, since coverings correspond to finite index subgroups of the uniformizing group. The Ehrenpreis conjecture, proved by Kahn and Markovic [16], asserts that for any two closed hyperbolic surfaces M and N, both of genus $g \geq 2$, and for every $\varepsilon > 0$, there exist finite coverings $M_\varepsilon \to M$ and $N_\varepsilon \to N$ such that

$$d_{\mathcal{T}}(M_\varepsilon, N_\varepsilon) \leq \varepsilon.$$

Where $d_{\mathcal{T}}$ denotes the *Teichmüller metric* in $\mathcal{M}_g$. In particular, the minimal Teichmüller distance between distinct coverings of a fixed surface S of genus g tends to zero as $g \to \infty$.

The constructions of Kahn–Markovic already suggest that the optimal asymptotic behavior should be polynomial. Luo's result confirms that such a polynomial upper bound indeed holds. Our rigidity result below Theorem B shows that, in the arithmetic setting, this polynomial rate cannot in general be improved, thereby showing that the polynomial bound is essentially sharp in this case.

Let $\Gamma < \mathrm{PSL}(2,\mathbb{R})$ be a cofinite Fuchsian group and let

$$k\Gamma = \mathbb{Q}\big(\mathrm{tr}(\gamma) \mid \gamma \in \Gamma^{(2)}\big),$$

where $\Gamma^{(2)} = \langle \gamma^2 \mid \gamma \in \Gamma \rangle$, be its invariant trace field. A closed hyperbolic surface $S = \Gamma\backslash\mathbb{H}$ is called *arithmetic* if Γ satisfies the following characterization due to Takeuchi [31]:

1. the field $k\Gamma$ is a totally real number field;
2. the traces of elements of $\Gamma^{(2)}$ are algebraic integers; and
3. for every nontrivial embedding $\sigma : k\Gamma \to \mathbb{C}$, one has

$$\sigma(\mathrm{tr}(\gamma^2)) \in [-2, 2] \quad \text{for all } \gamma \in \Gamma.$$

Arithmetic hyperbolic surfaces satisfy a strong finiteness property. A theorem of Borel [6] asserts that there are only finitely many arithmetic surfaces of a fixed genus g. Moreover, Belolipetsky–Gelander–Lubotzky–Shalev [4] obtained precise asymptotic estimates showing that the number of arithmetic surfaces of genus g, denoted by $\mathcal{AS}_g$, satisfy

$$\lim_{g\to\infty} \frac{\log \mathcal{AS}_g}{g \log g} = 2. \tag{1}$$

In particular, the number of arithmetic surfaces of genus g grows superexponentially in g, with precise logarithmic asymptotics.

Arithmetic surfaces also exhibit rigidity reflected in the behavior of traces. Luo and Sarnak [24] established spacing properties for traces of arithmetic groups, showing that distinct traces cannot accumulate too closely. Such arithmetic separation results impose quantitative restrictions on the possible lengths of closed geodesics. When combined with the logarithmic length coordinates developed above, these constraints lead to lower bounds for the Teichmüller distance between distinct coverings of a fixed arithmetic surface. Combining Theorem A with the arithmetic rigidity described above leads to quantitative results on coverings of arithmetic hyperbolic surfaces.

We now state our quantitative rigidity result.

Theorem B *Let S be a closed arithmetic hyperbolic surface of genus $g \geq 2$ whose invariant trace field has degree d. There exist constants $A > 0$ and $C > 0$, depending only on d, such that for any two distinct finite coverings $S', S'' \to S$ of genus g, one has*

$$d_{\mathcal{T}}(S', S'') \geq Ag^{-C}.$$

In particular, arithmetic constraints impose a polynomial lower bound on how fast distinct coverings of a fixed arithmetic surface can accumulate in moduli space.

Now we discuss another application of Theorem A in the moduli space of a class of hyperbolic surfaces. In particular, we focus on surfaces arising from semi-arithmetic Fuchsian groups. The notion of *semi-arithmetic Fuchsian groups* was introduced by Schmutz Schaller and Wolfart [29] as a natural weakening of arithmeticity. A cofinite Fuchsian group Γ is *semi-arithmetic* if

1. $k\Gamma$ is a totally real number field, i.e., $\sigma(k\Gamma) \subseteq \mathbb{R}$, for every homomorphism $\sigma : k\Gamma \to \mathbb{C}$; and
2. the traces of $\Gamma^{(2)}$ lie in the ring of integers of that field.

Although the relationship is more subtle, arithmetic groups are a special case of semi-arithmetic ones. Unlike the arithmetic case, semi-arithmetic groups may occur in infinite families even with fixed invariant trace field and genus. In fact, Cosac and Dória [11] showed that there exist semi-arithmetic surfaces with arbitrarily small injectivity radius, which does not occur for arithmetic surfaces due to finiteness properties.

In [3], a new invariant called the *stretch* was introduced. The stretch of Γ is defined via Lipschitz Γ–equivariant maps from $\mathbb{H}$ to products $\mathbb{H}^r$, where r is the arithmetic dimension of the group (see subsection 2.3.2). It was proved in [*op. cit.*] that the stretch is a commensurability invariant, that it equals 1 for both arithmetic groups and a special class of groups which contains all triangle groups. It was also shown that it can be arbitrarily large in general.

A key result of [3] shows that if one bounds simultaneously the genus, the degree of the invariant trace field and the stretch, then only finitely many semi-arithmetic surfaces arise. This finiteness statement provides the arithmetic input for a counting result proved in this thesis. Combining the logarithmic length parametrization of Theorem A with the algebraic control of eigenvalues coming from bounded degree and bounded stretch, one reduces the geometric counting problem to a problem about algebraic integers.

More precisely, if Γ is semi-arithmetic with invariant trace field of degree at most d and stretch at most L, then the largest eigenvalues of its hyperbolic elements are zeros of integer polynomials of a special type, with degree and coefficients bounded in terms of d and L. Since a surface in the thick part is determined by finitely many closed geodesics whose lengths grow at most logarithmically with the genus, counting such surfaces reduces to counting algebraic integers with controlled degree and size.

We now pass to the semi-arithmetic setting. In contrast with the arithmetic case, where the number of surfaces of genus g satisfies the precise asymptotic relation (1), the semi-arithmetic situation is more flexible: even after fixing bounds on the degree of the invariant trace field and on the stretch, one still obtains families whose size grows superexponentially with g, but without a universal limiting constant. The following theorem makes this comparison precise by providing matching lower and upper bounds for the growth rate.

Theorem C *Fix integers $g \geq 2$, $d \geq 2$, and real number $L \geq 1$. Let $\mathrm{SA}(g,d,L)$ denote the number of semi-arithmetic hyperbolic surfacessurfaces of genus g whose invariant trace field has degree at most d and whose stretch is at most L. Then there exists a constant $C > 0$, depending only on d and L, such that*

$$2 \leq \limsup_{g\to\infty} \frac{\log\left(\mathrm{SA}[g,d,L]\right)}{g\log g} \leq C.$$

Moreover, the constant C satisfies the upper bound $C \leq ULd^2$, for some universal constant $U \geq \frac{1}{2}$.

Thus, although semi-arithmetic surfaces may occur in infinite families in fixed genus without additional constraints, imposing bounds on arithmetic complexity and stretch restores a super-exponential upper bound of order g^{Cg} in the genus. The argument illustrates how geometric determination by short curves and arithmetic control of traces interact to produce quantitative counting results.

Another natural enlargement of the arithmetic setting is given by subarithmetic groups, namely discrete groups contained in arithmetic lattices. These larger arithmetic classes continue to exhibit strong length rigidity phenomena, which impose quantitative constraints on effective approximation problems in moduli space, such as bounding the degree of coverings required to achieve a given Teichmüller distance.

The final part of this thesis studies a different phenomenon, namely denseness, we study subarithmetic Fuchsian groups arising as fundamental groups of Q-pieces, that is, hyperbolic surfaces of signature $(1,1)$, and analyze their trace parameters.

Using explicit parametrizations in terms of traces and the algebraic structure of the invariant trace field, we construct families of subarithmetic Q–pieces whose parameters vary in a dense subset of the admissible region. The key point is to show that the chosen trace parameters satisfy the necessary inequalities defining hyperbolic Q–pieces, while remaining totally real algebraic integers.

As a consequence, we prove the following denseness result for subarithmetic Q–pieces of signature $(1, 1)$.

Theorem D *Let $\mathcal{Q}$ denote the parameter space of Q–pieces of signature $(1, 1)$. The subset consisting of subarithmetic Q–pieces is dense in $\mathcal{Q}$.*

This result contrasts with the finiteness phenomena observed for arithmetic surfaces and highlights the flexibility that arises once arithmeticity is relaxed to subarithmeticity.

The results of this thesis lie at the intersection of hyperbolic geometry, moduli theory and arithmetic. On the geometric side, we develop logarithmic length coordinates that determine surfaces in the thick part of moduli space. These coordinates provide explicit quantitative control in terms of genus and injectivity radius.

On the arithmetic side, semi-arithmetic and subarithmetic surfaces carry algebraic constraints on traces and eigenvalues of hyperbolic elements. These constraints translate into restrictions on exponential lengths of closed geodesics.

The interaction between these two aspects leads to:

- quantitative lower bounds for distances between arithmetic coverings;
- counting results for semi-arithmetic surfaces;
- denseness phenomena for subarithmetic Q–pieces.

The general principle underlying these results is that geometric parametrizations by short closed geodesics can be combined with arithmetic restrictions on traces to obtain quantitative global information in moduli space.

The thesis is organized into four chapters. Chapter 1 develops the geometric framework. We review the necessary background on hyperbolic surfaces, Fuchsian groups and their actions on $\mathbb{H}$, and introduce Teichmüller space and moduli space from the point of view of markings. Particular emphasis is placed on length functions and curve–and–chain decompositions, which form the geometric foundation of the determination results proved later.

Chapter 2 is devoted to the arithmetic and semi-arithmetic setting. We review invariant trace fields, invariant quaternion algebras, and the algebraic structure of arithmetic Fuchsian groups. The definition of semi-arithmetic groups and their basic properties are presented, together with the necessary number-theoretic background. The notion of stretch, introduced in [3] is presented. We also establish the algebraic properties of traces and exponential lengths that are used later in the counting arguments, namely that eigenvalues of hyperbolic elements are reciprocal algebraic integers whose degree and size are controlled in terms of the invariant trace field and the stretch.

Chapter 3 contains the main geometric result of the thesis: the determination theorem by short closed geodesics with logarithmic bounds in the genus. The precise statement appears as Theorem A, and its proof occupies the central part of the chapter. The applications to arithmetic coverings and quantitative rigidity are then established, including Theorem B, where logarithmic length coordinates are combined with trace separation phenomena to obtain a polynomial lower bound for the Teichmüller distance between distinct coverings of a fixed arithmetic surface. The counting result for semi-arithmetic surfaces with bounded invariant trace field degree and bounded stretch, stated as Theorem C, is also proved in Chapter 3.

Finally, Chapter 4 is devoted to subarithmetic Q–pieces of signature $(1, 1)$. After recalling their parametrization and deriving the algebraic conditions characterizing subarithmeticity, we establish the denseness result stated as Theorem D.

CHAPTER 1

Hyperbolic Structures

1.1 Hyperbolic Geometry

In hyperbolic geometry, the classical trigonometric identities from Euclidean geometry are replaced by analogues involving hyperbolic functions. A systematic exposition of these relations can be found in Buser [8]. In this section, we recall the identities that will be used throughout this thesis.

1.1.1 The hyperbolic plane

Hyperbolic geometry is the canonical space of non-Euclidean geometry with constant negative curvature. Its most fundamental example is the *hyperbolic plane*, which can be described by several equivalent models. In this thesis, we will primarily work with the *upper half-plane model.*

Let

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$$

be the upper half-plane, equipped with the Riemannian metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

This is the *hyperbolic metric*, and it defines a complete Riemannian structure of constant curvature -1.

The distance $d(z_1, z_2)$ between two points $z_1, z_2 \in \mathbb{H}$, is given by

$$d(z_1, z_2) = \operatorname{arccosh}\left(1 + \frac{|z_1 - z_2|^2}{2\,\mathrm{Im}(z_1)\,\mathrm{Im}(z_2)}\right).$$

The hyperbolic plane $\mathbb{H}$ admits a natural boundary at infinity, given by

$$\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}.$$

The hyperbolic plane also admits another important model, namely the *Poincaré disk model.* In this representation, the hyperbolic plane is identified with the unit disk

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\},$$

equipped with the metric

$$ds^2 = \frac{4\,|dz|^2}{(1-|z|^2)^2}.$$

The upper half-plane model and the disk model are related by the Möbius transformation

$$f : \mathbb{H} \longrightarrow \mathbb{D}, \quad f(z) = i\,\frac{z-i}{z+i}.$$

This map is a conformal diffeomorphism and, moreover, an isometry with respect to the hyperbolic metric. In particular, the metric on $\mathbb{D}$ is obtained by transporting the hyperbolic metric of $\mathbb{H}$ via f, and coincides with the standard expression above.

As a consequence, geometric quantities in the disk model can be computed by pulling them back to the upper half-plane via f^{-1}. Geodesics in the disk correspond to images under f of vertical lines and semicircles orthogonal to the real axis in $\mathbb{H}$, and are therefore represented by circular arcs orthogonal to the boundary of $\mathbb{D}$.

Throughout this thesis, we will primarily use the upper half-plane model for computations, while the disk model will be preferred for geometric illustrations.

1.1.2 Trigonometry

In hyperbolic geometry, the classical trigonometric identities from Euclidean geometry are replaced by analogues involving hyperbolic functions. These identities serve to describe the geometry of geodesic polygons and are essential for computing distances and angles on hyperbolic surfaces. In this section, we record the identities that will be used throughout this thesis, particularly in the analysis of geometric structures built from hyperbolic polygons.

Example 1.1 (Triangles) *Let us first consider a hyperbolic triangle with interior angles α, β, θ and respective opposite sides a, b, c. Two fundamental relations govern the geometry of such triangles.*

The law of sines *relates the side lengths to the sine of the opposite angles:*

$$\frac{\sin(\alpha)}{\sinh(a)} = \frac{\sin(\beta)}{\sinh(b)} = \frac{\sin(\theta)}{\sinh(c)}. \tag{1}$$

The law of cosines *generalizes the familiar Euclidean formula by incorporating*

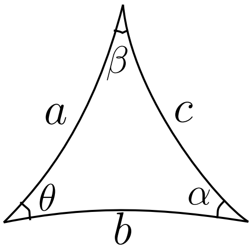

Figure 1.1: *Hyperbolic Triangle*

hyperbolic functions:

$$\cosh(c) = \cosh(a)\cosh(b) - \sinh(a)\sinh(b)\cos(\theta). \tag{2}$$

These identities will be useful in later sections for estimating distances between curves and analyzing the geometry of elementary decompositions of hyperbolic surfaces.

When the triangle has a right angle, say $\theta = \pi/2$, the trigonometric formulas simplify.

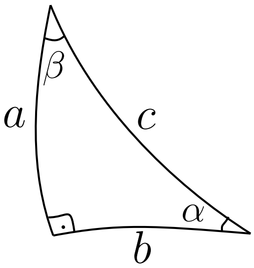

Figure 1.2: *Right-angled triangle*

The following identities are valid:

$$\cosh c = \cosh a \cosh b, \tag{3}$$
$$\sinh a = \sin\alpha \sinh c, \tag{4}$$
$$\sinh a = \cot\beta \tanh b, \tag{5}$$
$$\cos\alpha = \tanh b \coth c. \tag{6}$$

Example 1.2 (Pentagon) *Another important polygon in hyperbolic geometry is the geodesic pentagon. Consider a pentagon with consecutive sides a, β, c, α, b, and assume that all internal angles are right angles, except for the angle θ, which is opposite to the side c.*
Under these conditions, the following identity holds:

$$\frac{\cosh a}{\sinh \alpha} = \frac{\cosh b}{\sinh \beta} = \frac{\sinh c}{\sin \theta}. \tag{7}$$

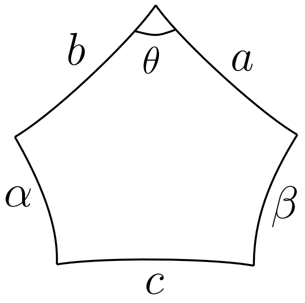

Figure 1.3: *Pentagon*

Example 1.3 (Hexagon) *Right-angled hexagons appear frequently in the geometry of hyperbolic surfaces. They arise naturally in the decomposition of surfaces into simpler pieces.*

Consider a right-angled hexagon with consecutive sides $a, \gamma, b, \alpha, c, \beta$*, where all internal angles are equal to* $\pi/2$*.*

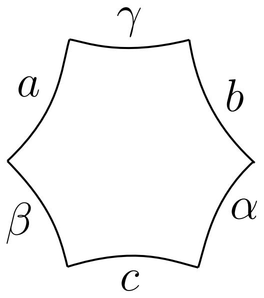

Figure 1.4: *Right-angled hexagon*

The lengths of the alternating sides are related by the following identities:

$$\cosh c = \sinh a \sinh b \cosh \gamma - \cosh a \cosh b, \tag{8}$$

$$\frac{\sinh a}{\sinh \alpha} = \frac{\sinh b}{\sinh \beta} = \frac{\sinh c}{\sinh \gamma}. \tag{9}$$

1.2 Fuchsian Groups

Fuchsian groups provide a link between hyperbolic geometry and the theory of Riemann surfaces. These groups arise as discrete subgroups of the group of orientation-preserving isometries of the hyperbolic plane and offer the algebraic structures used to describe hyperbolic surfaces. By studying their properties, geometric and topological problems can be reformulated in the language of group theory, allowing for precise classifications and constructions.

In what follows, we introduce the basic notions concerning Fuchsian groups and their connection to the geometry of the hyperbolic plane (see Katok [17]). We aim to present the algebraic and geometric foundations that will be essential for the study of hyperbolic surfaces in the subsequent chapters.

1.2.1 Automorphisms of the hyperbolic plane

The group of orientation-preserving isometries of the hyperbolic plane $\mathbb{H}$ form a Lie group, which is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$, and it is defined as the quotient

$$\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm I\},$$

where $\mathrm{SL}(2,\mathbb{R})$ denotes the group of real 2×2 matrices with determinant one. The elements of $\mathrm{PSL}(2,\mathbb{R})$ act on the upper half-plane by Möbius transformations of the form

$$z \mapsto \frac{az+b}{cz+d}, \quad \text{with} \quad ad - bc = 1.$$

This action is well-defined because the matrices A and $-A$ induce the same transformation, and it preserves the hyperbolic metric and orientation.

The full group of isometries of $\mathbb{H}$ is generated by $\mathrm{PSL}(2,\mathbb{R})$ together with the anti-holomorphic isometry

$$z \longmapsto \frac{1}{\bar{z}},$$

which reverses orientation.

Let $A \in \mathrm{SL}(2,\mathbb{R})$. The isometry of $\mathbb{H}$ defined by the Möbius transformation

$$z \mapsto \frac{az+b}{cz+d}, \quad \text{where } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

can be classified according to the value of $|\mathrm{tr}(A)|$, as follows.

Example 1.4 (Elliptic elements) *The isometry associated with A is called* elliptic *if $|tr(A)| < 2$. In this case, the eigenvalues of A are complex conjugates of unit modulus, and the corresponding isometry fixes a unique point in $\mathbb{H}$. Geometrically, it acts as a rotation about that point.*

An elliptic transformation does not preserve any geodesic globally, but it rotates angles locally. The fixed point lies strictly inside $\mathbb{H}$, and the transformation is conjugate in $\mathrm{SL}(2,\mathbb{R})$ *to a matrix of the form*

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in (0,\pi).$$

Example 1.5 (Parabolic elements) *The isometry associated with A is called* parabolic *if $|tr(A)| = 2$ and $A \neq \pm I$. In this case, the eigenvalues of A are both* *equal to* 1 *or* -1*, and the transformation fixes exactly one point on the boundary* $\partial\mathbb{H}$*.*

Parabolic elements are conjugate to matrices of the form

$$A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad t \neq 0.$$

Example 1.6 (Hyperbolic elements) *The isometry associated with* A *is called* hyperbolic *if* $|tr(A)| > 2$*. In this case,* A *has two distinct real eigenvalues* λ *and* λ^{-1}*, with* $\lambda > 1$*. The associated isometry fixes exactly two points on* $\partial\mathbb{H}$*, which lie on the boundary of a unique invariant geodesic, called its* axis*. The transformation acts as a translation along this axis.*

Every hyperbolic element is conjugate in $\mathrm{SL}(2, \mathbb{R})$ *to a diagonal matrix of the form*

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad |\lambda| > 1.$$

1.2.2 Discrete groups and properly discontinuous actions

A *topological group* is a group G equipped with a topology such that the group operations, multiplication $(g, h) \mapsto gh$ and inversion $g \mapsto g^{-1}$, are continuous. The group $\mathrm{PSL}(2, \mathbb{R})$ is of particular interest in this work. It acts on the hyperbolic plane $\mathbb{H}$ by orientation-preserving isometries. This group naturally inherits a topology from $\mathbb{R}^4$, and it has the structure of a real Lie group of dimension three.

Definition 1.7 *Let* G *be a topological group. A subgroup* $\Gamma \subset G$ *is* discrete *if it is a discrete subset in the topology inherited from* G*. Equivalently, there exists a neighborhood of the identity in* G *containing no other elements of* Γ*.*

In the case where $G = \mathrm{PSL}(2, \mathbb{R})$, discrete subgroups act on the hyperbolic plane $\mathbb{H}$ by isometries, and the corresponding quotients $\Gamma\backslash\mathbb{H}$ give rise to topological surfaces or orbifolds. To ensure that such quotients are well-defined and locally resemble the geometry of $\mathbb{H}$, one must impose additional conditions on the group action.

Definition 1.8 *Let* Γ *be a group acting on a topological space* X*. The action is said to be* properly discontinuous *if for every compact subset* $K \subset X$*, the set*

$$\{\gamma \in \Gamma \mid \gamma(K) \cap K \neq \emptyset\}$$

is finite.

When $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a discrete subgroup, its action on $\mathbb{H}$ is properly discontinuous. As a result, the quotient $\Gamma\backslash\mathbb{H}$ inherits the structure of a topological surface. These quotients will be studied in detail in the next sections.

1.2.3 Fuchsian groups and the geometry of quotients

Definition 1.9 *A* Fuchsian group *is a discrete subgroup* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$*.*

If $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a discrete subgroup, then its action on $\mathbb{H}$ is properly discontinuous (see [17, Theorem 2.2.6]).

Definition 1.10 *A Fuchsian group is called* torsion-free *if it contains no elliptic elements, that is, if the stabilizer of every point in* $\mathbb{H}$ *is trivial.*

In this work, we restrict our attention to torsion-free Fuchsian groups. In this case, the action is free, meaning that no nontrivial group element fixes any point in $\mathbb{H}$, and the quotient $S = \Gamma\backslash\mathbb{H}$ is a smooth hyperbolic surface without singularities. If Γ is cocompact, then S is a closed surface of genus $g \geq 2$. If Γ is not cocompact but has finite coarea, the surface has a finite number of cusps, each corresponding to a conjugacy class of parabolic elements in the group.

To study the global geometry of the quotient, it is useful to construct a *fundamental domain* for the action of Γ on $\mathbb{H}$.

Definition 1.11 *A fundamental domain for* Γ *is a connected, closed subset* $D \subset \mathbb{H}$ *such that:*

1. $\displaystyle\bigcup_{\gamma\in\Gamma} \gamma(D) = \mathbb{H}$,
2. *the interiors* $\mathrm{int}(\gamma(D))$ *are pairwise disjoint for* $\gamma \neq \mathrm{id}$.

A classical construction of fundamental domains for Fuchsian groups is the Dirichlet domain centered at a point $x_0 \in \mathbb{H}$, defined by

$$D(x_0) = \{x \in \mathbb{H} \mid d(x, x_0) \leq d(\gamma(x), x_0) \text{ for all } \gamma \in \Gamma\}.$$

This region is convex, and its boundary consists of finitely many geodesic segments if Γ is finitely generated.

The boundary of a fundamental domain consists of geodesic segments that appear in pairs. For each boundary side s of D, there exists an element $\gamma \in \Gamma$ such that $\gamma(s)$ is another boundary side of D. These two sides are said to be paired by the transformation γ. Points lying on paired sides are identified in the quotient space $\Gamma\backslash\mathbb{H}$. More precisely, if $x \in s$ and $y = \gamma(x)$, then x and y represent the same point in the quotient surface. In this way, the surface $\Gamma\backslash\mathbb{H}$ is obtained by gluing the sides of the fundamental domain according to the corresponding elements of Γ.

Through these side identifications, the topology of the quotient surface is encoded by the fundamental domain. In particular, when Γ is torsion-free and cocompact, the quotient is a closed surface, and the Dirichlet domain is a polygon with $4g$ sides, which are identified in pairs by elements of Γ corresponding to generators of the fundamental group $\pi_1(S)$. More generally, a fundamental domain provides a combinatorial and geometric description of the surface, allowing the analysis of geometric invariants such as lengths of closed geodesics, angles, and distances.

The identification of sides of a fundamental domain not only determines the topology of the quotient surface but also encodes the algebraic structure of the group Γ. In particular, when Γ is torsion-free and cocompact, the quotient $S = \Gamma\backslash\mathbb{H}$ is a closed surface of genus g, and the side pairings of the fundamental polygon give rise to generators of the fundamental group.

In this case, Γ is isomorphic to the fundamental group of the surface,

$$\Gamma \cong \pi_1(S),$$

which admits the classical presentation

$$\Gamma = \left\langle a_1, b_1, \ldots, a_g, b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle,$$

where $[a, b] = aba^{-1}b^{-1}$ denotes the commutator of a and b.

A key connection between geometry and group structure appears in the study of simple closed geodesics. Every closed geodesic $\beta \subset S = \Gamma\backslash\mathbb{H}$ corresponds to the projection of the axis of a hyperbolic element $T_\beta \in \Gamma$. The length of such a geodesic is determined algebraically by the trace of T_β, via the formula

$$\ell_\beta(S) = 2\cosh^{-1}\left(\frac{|\mathrm{tr}(T_\beta)|}{2}\right), \tag{10}$$

where $\mathrm{tr}(T_\beta)$ denotes the trace of a representative of T_β in $\mathrm{SL}(2, \mathbb{R})$. This relationship allows one to extract geometric information about S from the traces of elements in Γ.

Example 1.12 *Let* $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$, *the* modular group. *The quotient* $\Gamma\backslash\mathbb{H}$ *is a noncompact hyperbolic orbifold of finite area, with two elliptic points of orders* 2 *and* 3, *and one cusp corresponding to the parabolic fixed point at infinity.*

A fundamental domain for the action of $\mathrm{PSL}(2,\mathbb{Z})$ *on* $\mathbb{H}$ *is given by*

$$F = \left\{ z = x + yi \in \mathbb{H} \;\middle|\; |z| \geq 1,\ -\tfrac{1}{2} \leq x \leq \tfrac{1}{2} \right\}.$$

Its boundary consists of the vertical geodesics $x = \pm\frac{1}{2}$ *and the semicircle* $|z| = 1$ *orthogonal to the real axis.*

The side identifications are induced by the standard generators

$$S(z) = -\frac{1}{z}, \qquad T(z) = z + 1,$$

where S *pairs the two arcs of the unit semicircle and* T *identifies the vertical sides* $x = \pm\frac{1}{2}$. *These transformations generate* $\mathrm{PSL}(2,\mathbb{Z})$ *with presentation*

$$\mathrm{PSL}(2,\mathbb{Z}) = \langle S, T \mid S^2 = (ST)^3 = \mathrm{id} \rangle.$$

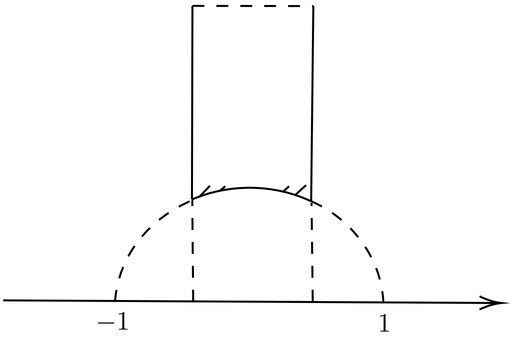

Figure 1.5: *A fundamental domain for the action of* $\mathrm{PSL}(2,\mathbb{Z})$ *on* $\mathbb{H}$.

If Γ *is a torsion-free subgroup of finite index in* $\mathrm{PSL}(2,\mathbb{Z})$, *then the quotient* $\Gamma\backslash\mathbb{H}$ *is a noncompact hyperbolic surface of finite area with finitely many cusps and no singular points. A classical example is the principal congruence subgroup of level* 2,

$$\Gamma(2) = \{ A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv I \pmod 2 \},$$

which is torsion-free and has index 6 *in* $\mathrm{PSL}(2,\mathbb{Z})$.

A fundamental domain for $\Gamma(2)$ *can be obtained as the union of the six images* F, $S(F)$, $T(F)$, $TST(F)$, $TS(F)$, *and* $ST^{-1}(F)$, *which correspond to a choice of representatives of the cosets of* $\Gamma(2)$ *in* $\mathrm{PSL}(2,\mathbb{Z})$. *These regions are disjoint up to their boundaries and together form a fundamental domain for* $\Gamma(2)$, *as illustrated in Figure 1.6.*

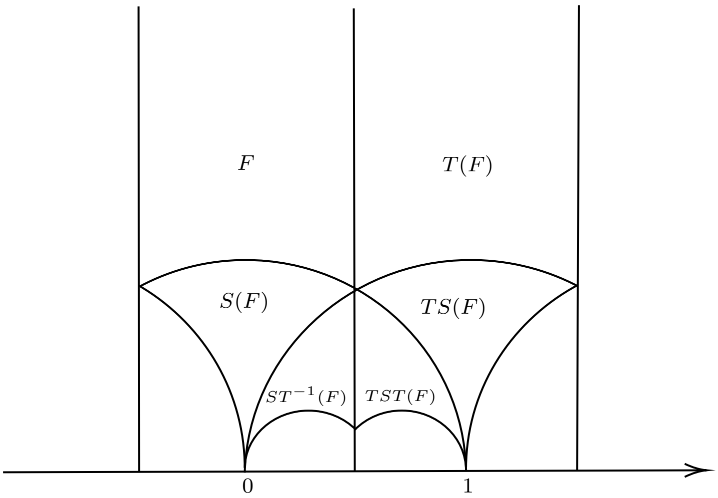

Figure 1.6: *A fundamental domain for* $\Gamma(2)$.

The resulting quotient is a complete hyperbolic surface of genus zero with three cusps. Further details on the construction of fundamental domains and the geometry of modular and congruence subgroups can be found in [17, Sections 3.2, 3.5 and 5.5].

Restricting the trace values of a Fuchsian group leads to additional arithmetic structure. For example, one may require all traces to lie in a fixed number field or to be algebraic integers. These conditions give rise to the notions of arithmetic and semi-arithmetic Fuchsian groups, which exhibit rich interactions between geometry, group theory, and number theory. These classes of groups will be central to the developments in the later chapters of this thesis.

1.3 Hyperbolic Surfaces

Hyperbolic surfaces provide the natural geometric setting in which local properties of the hyperbolic plane and algebraic structure of Fuchsian groups combine into global objects. While the hyperbolic plane serves as a universal model, the passage to quotients by discrete, torsion-free group actions produces a rich family of surfaces whose geometry reflects both the topology of the underlying surface and the algebraic features of the acting group.

The introduction of markings allows one to distinguish between different hyperbolic metrics on a fixed topological surface and leads to the Teichmüller space, which organizes these structures into a finite-dimensional parameter space. Through quasiconformal deformations, this space carries a natural metric that quantifies geometric variation and, in particular, controls the distortion of lengths of closed geodesics, a theme that recurs throughout the thesis.

By identifying hyperbolic surfaces up to isometry, the moduli space emerges as a quotient of Teichmüller space and serves as the ambient space for studying families of hyperbolic surfaces. Within this framework, surfaces of signature $(1, 1)$ appear as especially tractable and geometrically explicit examples, and they will later be used as fundamental pieces in the analysis of subarithmetic phenomena.

1.3.1 Hyperbolic structures and Teichmüller space

Definition 1.13 *A* hyperbolic surface *is a two-dimensional smooth manifold S equipped with a complete Riemannian metric of constant curvature -1. If the surface is compact and without boundary, it is called a* closed hyperbolic surface*.*

By the Killing-Hopf Theorem, every closed orientable surface of genus $g \geq 2$ admits a conformal structure compatible with a complete Riemannian metric of constant curvature -1. As a consequence, any such surface can be endowed with a hyperbolic metric and realized as a quotient

$$S = \Gamma\backslash\mathbb{H},$$

where Γ is a torsion-free, cocompact Fuchsian group acting properly discontinuously on the hyperbolic plane $\mathbb{H}$ by isometries. The metric on S is induced from the hyperbolic metric on $\mathbb{H}$, and the group Γ acts as the covering group of the universal covering map $\mathbb{H} \to S$. The group Γ is isomorphic to the fundamental group $\pi_1(S)$, and different choices of Γ yield distinct hyperbolic metrics on the same underlying topological surface.

The classification of compact orientable surfaces implies that, up to homeomorphism, there is a unique surface of genus g. However, such a surface admits infinitely many inequivalent hyperbolic metrics. To parametrize these structures, one introduces markings.

Definition 1.14 *A* marked hyperbolic surface *is a pair (X, φ), where X is a hyperbolic surface and $\varphi : F \to X$ is an orientation-preserving homeomorphism from a fixed reference surface F of genus g. The map ϕ is called a* marking*.*

Two marked hyperbolic surfaces (X_1, φ_1) and (X_2, φ_2) are considered equivalent if there exists an isometry $\Phi : X_1 \to X_2$ such that $\Phi \circ \varphi_1$ is isotopic to φ_2. This defines an equivalence relation on the set of marked hyperbolic surfaces.

Definition 1.15 *The* Teichmüller space *$\mathcal{T}_g$ is the set of equivalence classes of marked hyperbolic surfaces of genus g.*

The space $\mathcal{T}_g$ has the structure of a real analytic manifold of dimension $6g-6$. Points in $\mathcal{T}_g$ correspond to hyperbolic structures on the reference surface F, up to isotopy. Several equivalent models of $\mathcal{T}_g$ exist, including quasiconformal deformations, representations into $\mathrm{PSL}(2,\mathbb{R})$, and Fenchel–Nielsen coordinates.

To define the Teichmüller metric, it is necessary to introduce the concept of quasiconformal maps between hyperbolic surfaces. Let $\varphi : S \to S'$ be a homeomorphism between two hyperbolic surfaces. The map φ is called *K-quasiconformal* if for almost every point $x \in S$,

$$\limsup_{t\to 0} \left\{ \frac{\max\{\mathrm{dist}(\varphi(x),\varphi(y))\}}{\min\{\mathrm{dist}(\varphi(x),\varphi(y))\}} \,\middle|\, \mathrm{dist}(x,y)=t \right\} \leq K.$$

The smallest such constant $K \geq 1$ is called the *dilatation* of φ, and is denoted by $K[\varphi]$.

A natural distance function can be defined on Teichmüller space using quasiconformal maps.

Definition 1.16 *Given two points $X_1 = [(X_1,\varphi_1)]$ and $X_2 = [(X_2,\varphi_2)]$ in $\mathcal{T}_g$, their* Teichmüller distance *is defined by*

$$d_{\mathcal{T}}\big(X_1,X_2\big) = \frac{1}{2}\inf_{\varphi} \log K[\varphi],$$

where the infimum is taken over all quasiconformal maps $\varphi : X_1 \to X_2$ isotopic to $\varphi_2 \circ \varphi_1^{-1}$, and $K[\varphi]$ denotes the dilatation of φ.

An important property of quasiconformal maps is that they control the deformation of lengths of closed geodesics. The following result, which will be used later, provides a precise inequality:

Theorem 1.17 ([8, Theorem 6.4.3]) *Let S and S' be compact hyperbolic surfaces of genus $g \geq 2$, and let $\varphi : S \to S'$ be a K-quasiconformal homeomorphism. Then for each closed geodesic γ on S, the unique closed geodesic $\varphi[\gamma]$ in the homotopy class of $\varphi \circ \gamma$ satisfies*

$$\frac{1}{K}\ell(\gamma) < \ell(\varphi[\gamma]) < K\,\ell(\gamma),$$

where $\ell(\cdot)$ denotes the hyperbolic length.

The *moduli space* $\mathcal{M}_g$ is the set of isometry classes of closed oriented hyperbolic surfaces of genus g. In other words, two surfaces belong to the same class in $\mathcal{M}_g$ if and only if they are isometric.

Let F be a fixed closed oriented surface of genus g. We denote by $\mathrm{Mod}(F)$ the group of isotopy classes of orientation-preserving homeomorphisms of F, called the *mapping class group* of genus g. This group acts by isometries on the Teichmüller space $\mathcal{T}_g$ endowed with the Teichmüller metric, and this action is properly discontinuous [15, Theorem 12.2]. Consequently, the moduli space can be realized as the quotient

$$\mathcal{M}_g = \mathcal{T}_g/\mathrm{Mod}(F),$$

which carries a natural orbifold structure of real dimension $6g-6$.

The Teichmüller space and the moduli space describe how hyperbolic structures on a given surface can vary. They serve as a natural setting for understanding geometric deformations and for organizing families of hyperbolic surfaces, including those with arithmetic properties.

1.3.2 Surfaces of signature $(1,1)$

A *surface of signature* (g,n) is a compact, connected, oriented surface of genus g with n boundary components. When the surface is hyperbolic we assume that each boundary component is a simple closed geodesic. In this subsection, we consider the case of hyperbolic surfaces with signature $(1,1)$, a surface of genus one with one geodesic boundary component, also referred to as a *Q-piece*.

Such a surface can be obtained by gluing two boundary components of a pair of pants, that is, a hyperbolic surface of signature $(0,3)$.

Let S be a hyperbolic surface of signature $(1,1)$. Its fundamental group is a free group on two generators, say γ and β, with a single relation:

$$\pi_1(S) = \langle \gamma, \beta \mid [\gamma,\beta] = \eta \rangle,$$

where η represents the boundary component of S. This relation implies that γ and β generate a Fuchsian group Γ such that the commutator η is a hyperbolic element of $\mathrm{PSL}(2,\mathbb{R})$.

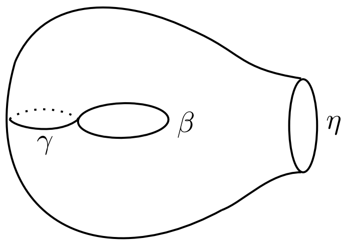

Figure 1.7: *A Q–piece with closed geodesics γ, β, and η.*

By abuse of notation, we use the same symbols γ and β to refer both to the closed geodesics on the surface and to the hyperbolic elements of Γ representing them.

By the Gauss–Bonnet formula, the area of a hyperbolic surface of signature (g, n) with geodesic boundary is given by

$$\mathrm{Area}(S) = 2\pi(2g - 2 + n).$$

In the case $(1, 1)$, this yields $\mathrm{Area}(S) = 2\pi$.

It follows from [20, Theorem 4] that, given real numbers $x, y, z > 2$ satisfying

$$x^2 + y^2 + z^2 - xyz - 2 < 0, \tag{11}$$

there exists a unique hyperbolic surface of signature $(1, 1)$, represented by a Fuchsian group $\Gamma = \langle \gamma, \beta \rangle$, such that

$$\mathrm{tr}(\gamma) = x, \qquad \mathrm{tr}(\beta) = y, \qquad \mathrm{tr}(\gamma\beta) = z,$$

and whose boundary geodesic corresponds to the commutator $\eta = [\gamma, \beta]$, with

$$\mathrm{tr}(\eta) = x^2 + y^2 + z^2 - xyz - 2$$

Moreover, if θ denotes the angle between the axes of γ and β, then

$$\cos\theta = \frac{2z - xy}{\sqrt{x^2 - 4}\,\sqrt{y^2 - 4}}. \tag{12}$$

In particular, geometric quantities such as the twist parameter of the corresponding Q–piece depend continuously on the triple (x, y, z). We can say that the Teichmüller space of hyperbolic surfaces of signature $(1, 1)$ can be parametrized by the set

$$\mathcal{Q} = \big\{(x, y, z) \in \mathbb{R}^3 \;\big|\; x, y, z > 2, \text{ and } x^2 + y^2 + z^2 - xyz - 2 < 0\big\}. \tag{13}$$

1.4 Decomposition of Hyperbolic Surfaces

In the study of moduli space, an important tool is the decomposition of hyperbolic surfaces into simpler geometric pieces, both from a geometric and an analytic point of view. Two approaches will be relevant for us. The first is the classical decomposition into pairs of pants, which provides a natural framework for parametrizing Teichmüller space (see Buser [8]). The second, due to Parlier [28], is based on curve and chain systems, which refine pants decompositions by incorporating additional geodesic information. We now describe both constructions.

1.4.1 Decomposition into pairs of pants

A *pair of pants*, or Y*-piece* is a hyperbolic surface of signature $(0,3)$, that is, a compact surface with three geodesic boundary components. Such a piece can be obtained by gluing two right-angled hexagons along their alternating sides. Given three positive real numbers ℓ_1, ℓ_2, ℓ_3, there exists a unique Y-piece whose boundary geodesics have lengths ℓ_1, ℓ_2, ℓ_3 [8, Theorem 3.1.7].

Every closed hyperbolic surface of genus $g \geq 2$ can be decomposed into Y-pieces. More precisely, there exist $3g-3$ disjoint simple closed geodesics that decompose the surface into $2g-2$ Y-pieces. This is called a *pants decomposition*.

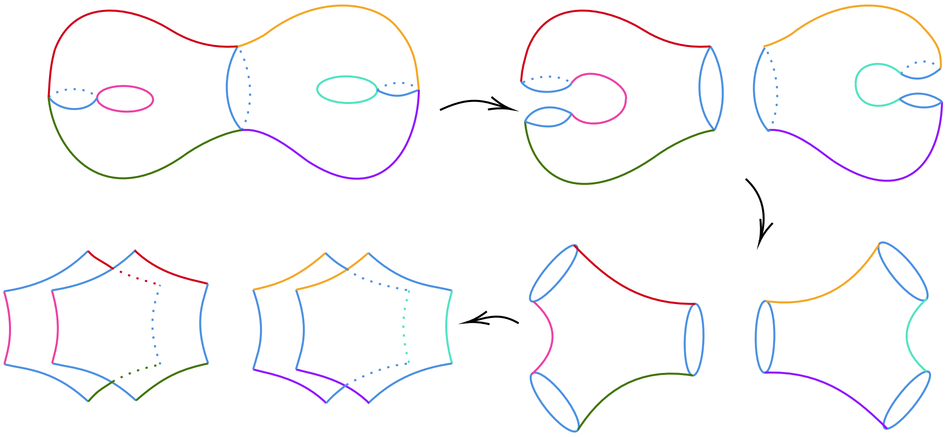

Figure 1.8: *A pants decomposition of a genus 2 surface.*

Given a simple closed geodesic γ on a hyperbolic surface with length $\ell(\gamma)$, its *collar* is the embedded neighborhood

$$\mathfrak{C}(\gamma) = \{\, p \in S \mid d(p,\gamma) < w(\gamma)/2 \,\},$$

where the *width* $w(\gamma)$ is determined by the relation

$$\sinh\left(\tfrac{\ell(\gamma)}{2}\right)\sinh\left(\tfrac{w(\gamma)}{2}\right) = 1.$$

The next result ensures that the neighborhoods around the curves of a pants decomposition do not overlap, so that each curve can be studied independently (see [8, Theorem 4.1.1]).

Theorem 1.18 (Collar Lemma) *Let S be a closed hyperbolic surface of genus $g \geq 2$, and let $\gamma_1, \ldots, \gamma_{3g-3}$ be the collection of disjoint simple closed geodesics that* *define a pants decomposition of S. Then the collars $\mathfrak{C}(\gamma_1), \ldots, \mathfrak{C}(\gamma_{3g-3})$ are pairwise disjoint.*

In particular, the Collar Lemma shows that if all boundary geodesics of a pants decomposition are very short, then any curve intersecting them transversally must necessarily be long. This phenomenon will appear in Chapter 3, where the Collar Lemma is used to justify the hypotheses of the main theorem on the parametrization of Teichmüller space by short curves.

1.4.2 Decomposition into curve and chain

We begin by defining an intermediate structure known as a *curve and arc decomposition.*

Definition 1.19 *A* curve and chain decomposition *of a surface $S \in \mathcal{M}_g$ consists of a collection $\mathcal{C}$ of disjoint simple closed geodesics and a collection $\mathcal{A}$ of simple geodesic arcs on $S \setminus \mathcal{C}$, such that the complement $S \setminus (\mathcal{C} \cup \mathcal{A})$ is a union of geodesic right-angled hexagons.*

Each arc in $\mathcal{A}$ is called an *orthogeodesic* joining two boundary components of $S \setminus \mathcal{C}$ perpendicularly.

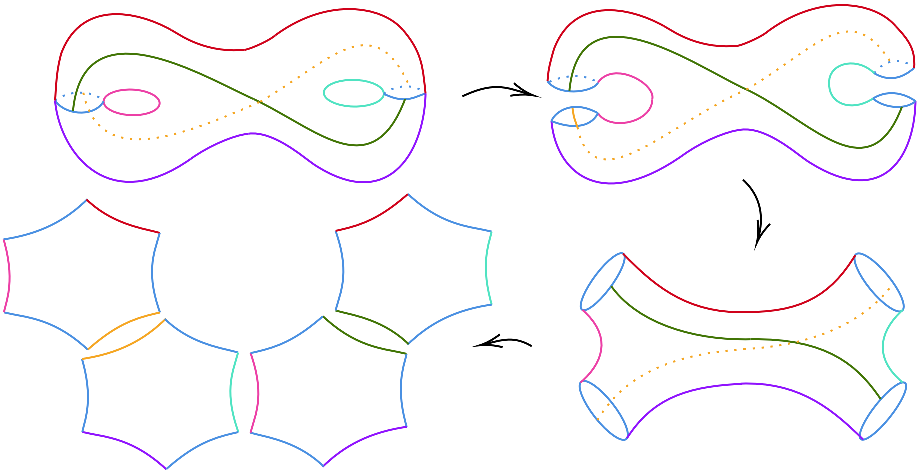

Figure 1.9: *A curve and chain decomposition of a genus* 2 *surface.*

While curve and chain decompositions provide a useful geometric structure, they involve arcs, which are not closed curves. To remedy this, we associate to each arc a closed curve called a *chain*.

Definition 1.20 *Given an arc $a \in \mathcal{A}$ with endpoints on boundary geodesics γ_1 and γ_2 (not necessarily distinct), we define the associated* chain *curve γ_a as the closed curve in the free homotopy class of the concatenation $\gamma_1 * a * \gamma_2 * a^{-1}$.*

Depending on whether γ_1 and γ_2 are distinct or the same, γ_a is either a simple closed curve or a closed curve with a single self-intersection.

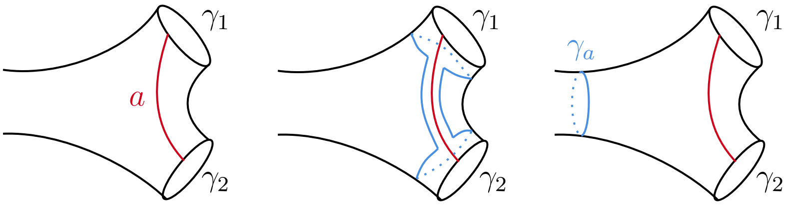

Figure 1.10: *Construction of a chain curve associated to an orthogeodesic arc between boundary geodesics.*

The set $\mathcal{C}_{\mathcal{A}} = \{\gamma_a\}_{a\in\mathcal{A}}$, constructed from a choice of arcs $\mathcal{A}$, together with $\mathcal{C}$ forms what we call a *curve and chain system.*

We observe that a pants decomposition is a special case of a curve and chain decomposition: by cutting each pair of pants along the geodesic arcs orthogonal to its boundary components, one obtains a family of right-angled geodesic hexagons. This observation shows that pants decompositions can be naturally understood as a particular case of curve and chain decompositions. More importantly, this generalization is precisely what allows us to go beyond the result of Dória (see [Proposition 1.24] or [13, Proposition 4.1.7]), providing a parametrization of Teichmüller space in terms of lengths of closed curves that are uniformly bounded by a logarithmic function of the genus.

1.5 Twist Parameters

When a hyperbolic surface is decomposed along simple closed geodesics, the resulting pieces must be reassembled in a way that preserves the global geometry of the surface. Beyond the lengths of the boundary geodesics, this reconstruction requires additional data that records how adjacent pieces are glued together. The twist parameter of each gluing captures this information by measuring the relative displacement along a geodesic during the gluing process, encoding a subtle but essential aspect of the hyperbolic structure that cannot be recovered from length data by itself.

Let $(\mathcal{C}, \mathcal{C}_{\mathcal{A}})$ be a curve and chain system on a hyperbolic surface $S \in \mathcal{M}_g$. Let $\gamma \in \mathcal{C}$ be a closed geodesic. Since γ is simple, a sufficiently small regular neighborhood of γ is an annulus. The two boundary components of this annulus determine the two sides of γ. Fix an orientation on the curves of S. We may choose on each side of γ an arc $a_i \in \mathcal{A}$, $i = 1, 2$, oriented so that it leaves γ. Let $\beta_i \in \mathcal{C}$, $i = 1, 2$, denote the closed geodesic reached at the other endpoint of a_i, and assume that $\beta_i \neq \gamma$ for $i = 1, 2$. The chain γ_{a_i} associated with the arc a_i, together with β_i

and γ, form the boundary components of an immersed pair of pants in S, which we denote by Y_i. We note that the Y–pieces Y_1 and Y_2 are not required to be disjoint, since some of their boundary geodesics may coincide. The boundary component of Y_i isometric to γ will be denoted by γ_i.

In what follows, we adopt the standard abuse of notation of denoting the length of a closed geodesic by the same symbol as the geodesic itself. Thus, for a closed geodesic γ, the symbol γ may also denote its length $\ell(\gamma)$, whenever this causes no ambiguity.

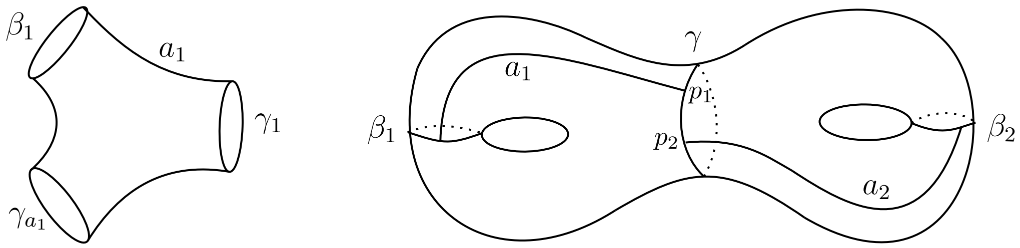

Figure 1.11: *The Y-piece associated with the arc a_1 and how it appears on surface S*

By gluing Y_1 and Y_2 along γ_1 and γ_2, one obtains a four-holed sphere, usually called an X-piece which, in this specific case, we will denote by X_γ. To determine the geometry of the resulting X-piece, it is necessary to know the lengths of its boundary geodesics, including the length of γ, together with an additional twist parameter that specifies how the two copies of γ are identified.

To define this parameter, which describes the gluing of the Y-pieces along γ, let $p_i = a_i \cap \gamma$ be the intersection points of the chosen arcs a_i with γ, $i = 1, 2$, one on each side of γ. The construction we adopt follows Parlier [28], where the twist parameter is defined in the context of moduli space.

Definition 1.21 *Let $\gamma \in \Gamma$ be a closed geodesic equipped with an orientation, and let $p_1, p_2 \in \gamma$ be the reference points determined by the chosen curve and chain system. The* twist parameter *associated with γ, denoted by $\alpha = \alpha(\gamma)$, is the value $\alpha \in [0, 1)$ such that the oriented distance along γ from p_1 to p_2 is equal to $\alpha\,\ell(\gamma)$.*

Assume that the boundary components γ_i, $i = 1, 2$, are parametrized in such a way that $\gamma_i(0) = p_i$, with $t \in \mathbb{S}^1$.

Fix $\gamma \in \mathcal{C}$ and consider the corresponding X–piece X_γ. Following Buser (see §3.3, [8]), one associates to X_γ two simple closed curves, denoted by δ_γ and η_γ, whose geodesic representatives depend on the twist parameter.

More precisely, let b be the oriented arc on γ whose initial point is p_1 and whose terminal point is p_2. By definition, its length satisfies

$$\ell(b) = \alpha\, \ell(\gamma).$$

Let d denote the geodesic arc perpendicular to β_1 and β_2 which is homotopic, relative to endpoints, to the path $a_2\, b\, a_1^{-1}$.

Lifting the configuration to the universal covering of the X–piece X_γ, we obtain a crossed right–angled hexagon as illustrated in Figure 1.12. Applying the hyperbolic cosine rule for right–angled hexagons (see (8)), we obtain

$$\cosh d = \sinh a_1 \sinh a_2 \cosh b - \cosh a_1 \cosh a_2. \tag{14}$$

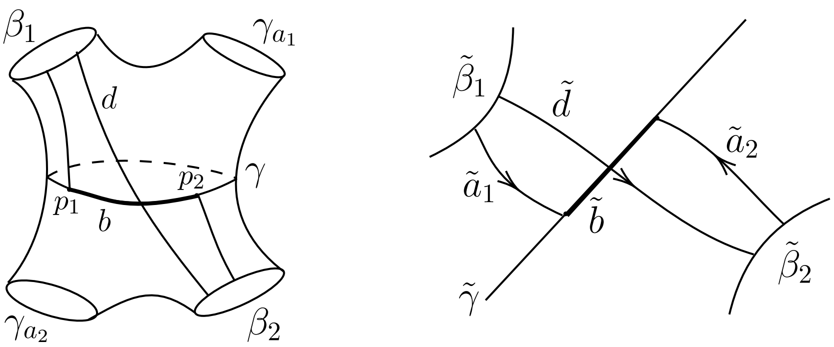

Figure 1.12: *X-piece and the lifting of the associated curves*

Let δ be the closed geodesic in the homotopy class of the curve

$$d\, \beta_2\, d^{-1}\, \beta_1.$$

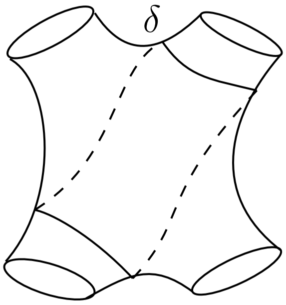

Figure 1.13: *The curve δ.*

Cutting the X–piece X_γ open along δ yields two Y–pieces. One of these has boundary components β_1, β_2, and δ, and contains the geodesic arc d perpendicular to β_1 and β_2. Decomposing this Y–piece into right–angled hexagons and again applying (8), we obtain

$$\cosh\frac{\delta}{2} = \sinh\frac{\beta_1}{2}\,\sinh\frac{\beta_2}{2}\,\cosh d - \cosh\frac{\beta_1}{2}\,\cosh\frac{\beta_2}{2}. \tag{15}$$

Combining (15) with (14), we see that the length δ depends on the quantities

$$\alpha,\ \gamma,\ \beta_1,\ \beta_2,\ a_1,\ a_2.$$

Since each arc a_i can be expressed in terms of the boundary geodesics γ, β_i, and γ_{a_i} by applying (8) to the right–angled hexagon obtained from the pair of pants Y_i it follows that the lengths of the closed geodesic δ is determined by

$$\alpha,\ \gamma,\ \beta_1,\ \beta_2,\ \gamma_{a_1},\ \gamma_{a_2}.$$

In other words, once the curve and chain system is fixed, δ depend only on the twist parameter and on the lengths of closed geodesics.

This allows one to recover the twist parameter α from length data. More precisely, α is characterized by the relation

$$\cosh\frac{\delta}{2} = u + v\,\cosh(\alpha\,\gamma), \tag{16}$$

where the coefficients u and v are real analytic functions of the lengths $\gamma,\ \beta_1,\ \beta_2,\ \gamma_{a_1},\ \gamma_{a_2}$, independent of α, and satisfy $v > 0$ (see [8, Lemma 3.3.12]).

Proposition 1.22 *Let $(\mathcal{C}, \mathcal{C}_{\mathcal{A}})$ be a curve and chain decomposition of a hyperbolic surface $S \in \mathcal{M}_g$. For any closed geodesic $\gamma \in \mathcal{C}$, let $\alpha(\gamma) \in [0,1)$ be the twist parameter defined by the reference points determined by $(\mathcal{C}, \mathcal{C}_{\mathcal{A}})$. Then the twist parameter $\alpha(\gamma)$ is completely determined by the lengths of the closed geodesics*

$$\gamma,\ \beta_1,\ \beta_2,\ \gamma_{a_1},\ \gamma_{a_2},$$

where β_i and γ_{a_i} are the boundary geodesics and associated chains arising from the curve and chain decomposition at γ.

1.6 Parametrization Using Pants Decomposition

Hyperbolic surfaces can be parametrized by numerical data associated to a decomposition into simpler geometric pieces. By recording the lengths of the curves in a pants decomposition together with the corresponding twist parameters, one obtains a concrete coordinate system on Teichmüller space, known as Fenchel–Nielsen coordinates. Variants of this construction, such as parametrizations based on curve and chain systems, allow one to encode the geometry of a surface using only finitely many parameters with effective control on their number and size.

1.6.1 Parametrization via pants decompositions

Given a pants decomposition $\gamma_1, \dots, \gamma_{3g-3}$ of a closed surface S, the geometry of S can be encoded by the collection of the lengths of the curves γ_i together with their corresponding twist parameters. The resulting system of parameters is called the *Fenchel–Nielsen coordinates* associated with this decomposition:

$$\big(\,\ell(\gamma_1), \dots, \ell(\gamma_{3g-3}), \alpha(\gamma_1), \dots, \alpha(\gamma_{3g-3})\,\big).$$

These coordinates provide a global parametrization of the Teichmüller space $\mathcal{T}_g$, showing that $\mathcal{T}_g$ is homeomorphic to $\mathbb{R}^{6g-6}$. Different choices of pants decomposition yield different coordinate systems, related to each other by transformations induced by the mapping class group.

Given a hyperbolic surface S of genus g, there exists a pants decomposition of S whose boundary geodesics all have their lengths bounded above by a constant depending only on g. The existence of such decompositions was established by Bers [5], and a detailed exposition can be found in Buser [8, Theorem 5.1.2].

Theorem 1.23 (Bers) *For every closed hyperbolic surface S of genus $g \geq 2$ there exists a collection of $3g-3$ disjoint simple closed geodesics $\gamma_1, \dots, \gamma_{3g-3}$ such that $S \setminus \{\gamma_1, \dots, \gamma_{3g-3}\}$ is a union of $2g-2$ Y-pieces and each geodesic satisfies*

$$\ell(\gamma_i) \leq 26(g-1), \quad i = 1, \dots, 3g-3.$$

This result ensures that every surface of genus g admits a pants decomposition where the lengths of the cutting curves are linearly controlled in g.

The *systole* of a hyperbolic surface S, denoted by $\mathrm{sys}(S)$, is the length of the shortest closed geodesic on S that is not homotopic to a point. In his thesis, Dória [13, Proposition 4.1.7] established a generalization of Bers theorem, assuming a lower bound on the systole, one can construct a system of closed curves, obtained from a pants decomposition, whose lengths parametrize the Teichmüller space and remain bounded by a linear function of g, with an additional dependence on the systole bound.

Proposition 1.24 *Let S be a closed hyperbolic surface of genus $g \geq 2$ with systole* $\mathrm{sys}(S) \geq s$*. Then there exists a collection of $9g-9$ simple closed geodesics on S whose lengths satisfy*

$$\ell(\gamma_i) \leq ag + b(s), \quad i = 1, \ldots, 9g-9,$$

for some universal constant $a > 0$ and a constant $b(s)$ depending only on s. Moreover, the lengths of these curves provide a parametrization of the Teichmüller space $\mathcal{T}_g$.

In Chapter 3 we generalize this result by exhibiting a new parametrization of the Teichmüller space, in which the lengths of the curves are further controlled, improving the upper bound given in the proposition above.

1.6.2 Parametrization via curve and chain systems

The relevance of curve and chain systems lies in the fact that they can be used to parametrize surfaces, as shown by Parlier in [28]:

Proposition 1.25 *The lengths of the curves in $\mathcal{C}$ and $\mathcal{C}_{\mathcal{A}}$, together with the twist parameters along $\mathcal{C}$, uniquely determine the surface S in moduli space.*

Unlike classical Fenchel–Nielsen coordinates, which rely on lengths and twist parameters associated to pants decompositions, curve and chain systems allow a parametrization of moduli space using only the lengths of finitely many closed curves and a controlled number of twist parameters. Moreover, the lengths involved can be bounded above by a logarithmic function of the genus, providing a significant advantage for quantitative applications.

Thus, the twist parameter describes the way the two sides of γ are reattached after cutting along the curve, specifying the relative displacement between neighboring regions. Together with the lengths of the curves in $\mathcal{C}$ and $\mathcal{C}_{\mathcal{A}}$, the twist parameters provide a complete set of geometric data needed to recover the hyperbolic structure of the surface.

Remark: In the case where no specific global orientation of γ is fixed, the twist parameter is determined up to a choice of sign, but this ambiguity does not affect the uniqueness of the reconstructed surface.

Thus, a curve and chain system provides an effective set of coordinates: the lengths of $\mathcal{C}$ and $\mathcal{C}_{\mathcal{A}}$, and the twists along $\mathcal{C}$.

The number of distinct topological types of curve and chain systems grows with genus, but can be effectively bounded:

Lemma 1.26 *The number $N_{cc}(g)$ of topological types of marked curve and chain systems is bounded above by*

$$N_{cc}(g) \leq \frac{1}{e^6}\left(\frac{126}{e^5}\right)^{g-1}(g-1)^{6g-6}.$$

This result follows from the count of topological types of decompositions into curves and arcs established by Parlier in [28, Lemma 3.2]. Since each curve and chain system is uniquely determined by such a decomposition, the total number of topological types of curve and chain systems coincides with that of the corresponding curve and arc decompositions.

This counting result will be important later when we estimate the number of possible configurations in moduli space.

More importantly, the framework of curve and chain systems introduced by Parlier provides the key tool that allows us to go beyond the result of Dória [13, Proposition 4.1.7]. While Parlier's construction ensures the existence of such systems with logarithmic length bounds, in this work we use this framework to obtain a quantitative parametrization of Teichmüller space in terms of lengths of closed curves that are uniformly bounded by a logarithmic function of the genus.

CHAPTER 2

Arithmetic and Semi-Arithmetic Fuchsian Groups

2.1 Number-Theoretic Preliminaries

This section collects the number–theoretic notions and results that will be used throughout Chapter 2 and in the subsequent chapters. We fix notation and recall basic facts on totally real number fields, their embeddings, and the associated invariants, with particular emphasis on the structure of the group of units of the ring of integers. The topological denseness properties of these units under the real embeddings are recorded here, as they will later provide the arithmetic flexibility needed in the construction and variation of geometric parameters under fixed arithmetic constraints.

2.1.1 Number fields and basic invariants

A *number field* is a finite extension $K/\mathbb{Q}$ of degree $[K:\mathbb{Q}] = n$, and many of its arithmetic properties can be understood by examining its embeddings into $\mathbb{C}$. By the Primitive Element Theorem, there exists an element $\alpha \in K$ such that $K = \mathbb{Q}(\alpha)$. In particular, if f denotes the minimal polynomial of α over $\mathbb{Q}$, then $\deg(f) = n$, and each embedding

$$\sigma : K \hookrightarrow \mathbb{C}$$

is uniquely determined by sending α to one of the n distinct complex roots of f. It follows that there are exactly n such embeddings. These embeddings provide a concrete way to view the elements of K as acting simultaneously on several real or complex copies of $\mathbb{R}$ or $\mathbb{C}$. When all embeddings have image contained in $\mathbb{R}$, one says that K is *totally real*.

The *ring of integers* of K, denoted $\mathcal{O}_K$, consists of all *algebraic integers* in K; that is, all elements of K that satisfy a monic polynomial with coefficients in $\mathbb{Z}$. From an algebraic standpoint, $\mathcal{O}_K$ behaves in many ways like $\mathbb{Z}$, but it accommodates the arithmetic peculiarities of K. It is a free $\mathbb{Z}$-module of rank n, and its structure often reflects the geometry of the embeddings of K. For instance, when d is a squarefree integer, $K = \mathbb{Q}(\sqrt{d})$ is a quadratic extension and the structure of $\mathcal{O}_K$ is determined explicitly by the value of d modulo 4. In this case,

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{d}], & \text{if } d \equiv 2, 3 \pmod 4, \\ \mathbb{Z}\left[\dfrac{1+\sqrt{d}}{2}\right], & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

This classification follows from the fact that, for an element $x = a + b\sqrt{d}$, the integrality of its trace and norm imposes conditions on the coefficients a and b. These conditions depend on the congruence class of d modulo 4, giving rise to the two possible descriptions of $\mathcal{O}_K$.

Given the embeddings $\sigma_1, \ldots, \sigma_n : K \hookrightarrow \mathbb{R}$ of a totally real field K, each element $x \in K$ determines the real numbers

$$\sigma_1(x), \ldots, \sigma_n(x),$$

which are not necessarily distinct. The *trace* and *norm* of x are defined by

$$\mathrm{Tr}_{K/\mathbb{Q}}(x) = \sigma_1(x) + \cdots + \sigma_n(x), \qquad \mathrm{N}_{K/\mathbb{Q}}(x) = \sigma_1(x) \cdots \sigma_n(x).$$

These quantities are algebraic invariants intrinsic to the extension $K/\mathbb{Q}$.

If $x \in K$, its *minimal polynomial* over $\mathbb{Q}$ is the unique monic polynomial of least degree satisfied by x, denoted by $m_x(T)$. The roots of $m_x(T)$ are precisely the distinct conjugates of x, then

$$\prod_{i=1}^{n} \bigl(T - \sigma_i(x)\bigr) = m_x(T)^{[K:\mathbb{Q}(x)]}.$$

From this expression one sees that $\mathrm{Tr}_{K/\mathbb{Q}}(x)$ and $\mathrm{N}_{K/\mathbb{Q}}(x)$ appear as the first and last coefficients of the *characteristic polynomial*

$$\prod_{i=1}^{n} \bigl(T - \sigma_i(x)\bigr).$$

The embeddings also give an injective map

$$\iota : K \longrightarrow \mathbb{R}^n, \qquad x \longmapsto \bigl(\sigma_1(x), \ldots, \sigma_n(x)\bigr),$$

which identifies K with a subfield of $\mathbb{R}^n$ when K is totally real. Under this identification, the trace and norm correspond respectively to the sum and the product of the coordinates of $\iota(x)$.

2.1.2 Units in number fields

An element $u \in \mathcal{O}_K$ is called a *unit* if it is invertible in $\mathcal{O}_K$; that is, if there exists $v \in \mathcal{O}_K$ such that $uv = 1$. The set of all units of $\mathcal{O}_K$ forms a multiplicative group, denoted by $\mathcal{O}_K^*$. We note that

$$\mathcal{O}_K^* = \{\, u \in \mathcal{O}_K \mid \mathrm{N}_{K/\mathbb{Q}}(u) = \pm 1 \,\}.$$

Indeed, since the norm is multiplicative and takes integer values on $\mathcal{O}_K$, an algebraic integer is a unit if and only if its norm is ± 1.

When K is totally real, every embedding $\sigma_i : K \hookrightarrow \mathbb{R}$ sends units to nonzero real numbers. Composing the embeddings with the absolute value and the logarithm yields maps

$$u \longmapsto \log|\sigma_i(u)|, \qquad i = 1, \dots, n,$$

which provide a link between the arithmetic of units and the additive structure of real vector spaces.

Let K be a number field of degree $n = [K : \mathbb{Q}]$. Denote by r_1 the number of real embeddings of K and by r_2 the number of pairs of complex conjugate embeddings, so that $r_1 + 2r_2 = n$.

The following fundamental result describes the algebraic structure of the unit group of $\mathcal{O}_K$.

Theorem 2.1 (Dirichlet's Unit Theorem [27]) *The group of units $\mathcal{O}_K^*$ is a finitely generated abelian group of rank $r_1 + r_2 - 1$. Moreover, its torsion subgroup is the finite cyclic group consisting of the roots of unity contained in K.*

Dirichlet's Unit Theorem shows that the unit group $\mathcal{O}_K^*$ contains free abelian subgroups of controlled rank. Through a fixed real embedding, these give rise to subgroups of one–dimensional real Lie groups.

2.1.3 Denseness of totally positive units

The following lemma explains how the presence of rank at least 2 forces such subgroups to be dense.

Lemma 2.2 *Let G be a one-dimensional topological group with finitely many connected components. If $H < G$ is a subgroup of rank at least 2, then H is dense in G.*

Proof. We may assume without loss of generality that G is connected. We distinguish two cases.

If G is non–compact, then G is topologically isomorphic to $(\mathbb{R}, +)$. Under such an isomorphism, the image of H is an additive subgroup of $\mathbb{R}$. Additive subgroups of $\mathbb{R}$ are either cyclic or dense. Since H has rank at least 2, its image cannot be cyclic and must therefore be dense in $\mathbb{R}$, which implies that H is dense in G.

If G is compact, then G is topologically isomorphic to the circle group $\mathbb{S}^1$. In this case, subgroups of rank at least 2 are necessarily dense, since any non-dense subgroup of $\mathbb{S}^1$ is cyclic. $\square$

Lemma 2.2, together with Dirichlet's Unit Theorem (Theorem 2.1), yields the following consequence.

Corollary 2.3 *Let K be a totally real number field with $[K : \mathbb{Q}] \geq 3$. Then the group of units $\mathcal{O}_K^*$ is dense in $\mathbb{R}$.*

We say that a unit $u \in \mathcal{O}_K^*$ is *totally positive* if $\sigma_i(u) > 0$ for every real embedding $\sigma_i : K \hookrightarrow \mathbb{R}$. The subgroup of totally positive units is denoted by $\mathcal{O}_{K,+}^*$.

Since the subgroup of totally positive units has finite index in the full unit group, the denseness result for $\mathcal{O}_K^*$ immediately extends to $\mathcal{O}_{K,+}^*$, yielding the following corollary.

Corollary 2.4 *Let K be a totally real number field with $[K : \mathbb{Q}] \geq 3$. Then the subgroup of totally positive units*

$$\mathcal{O}_{K,+}^* := \{u \in \mathcal{O}_K^* : \sigma_i(u) > 0 \textit{ for all real embeddings } \sigma_i\}$$

is dense in $\mathbb{R}_{>0}$.

Proof. Let $\sigma_1, \ldots, \sigma_n : K \hookrightarrow \mathbb{R}$ be the (real) embeddings. Consider the sign homomorphism

$$\mathrm{sgn} : \mathcal{O}_K^* \to \{\pm 1\}^n, \qquad u \mapsto \big(\mathrm{sgn}(\sigma_1(u)), \ldots, \mathrm{sgn}(\sigma_n(u))\big).$$

Its kernel is $\mathcal{O}_{K,+}^*$, hence $\mathcal{O}_{K,+}^*$ has finite index in $\mathcal{O}_K^*$. Therefore, $\mathcal{O}_K^* \cap \mathbb{R}_{>0}$ is a finite union of multiplicative translates of $\mathcal{O}_{K,+}^*$ in $\mathbb{R}_{>0}$.

By Corollary 2.3, $\mathcal{O}_K^*$ is dense in $\mathbb{R}^*$, hence $\mathcal{O}_K^* \cap \mathbb{R}_{>0}$ is dense in $\mathbb{R}_{>0}$. Since multiplication by a positive constant is a homeomorphism of $\mathbb{R}_{>0}$, a finite union of translates of a non-dense subset cannot be dense. Thus $\mathcal{O}_{K,+}^*$ is dense in $\mathbb{R}_{>0}$. $\square$

2.2 Quaternion Algebras

Quaternion algebras provide the algebraic framework underlying the arithmetic Fuchsian groups considered in this chapter. In this section we recall the basic structure and classification of quaternion algebras over number fields, describe their behavior under real embeddings, and introduce the notions of splitting and ramification at real places. These concepts will be used to determine when a quaternion algebra gives rise to discrete subgroups of $\mathrm{PSL}(2,\mathbb{R})$. We also introduce orders in quaternion algebras, which supply the integral structure needed to construct arithmetic groups and will be essential in the discussion of arithmetic and semi-arithmetic Fuchsian groups in the next section.

Throughout this section, K denotes a field of characteristic different from 2.

2.2.1 Quaternion algebras over number fields

Definition 2.5 *A* K-algebra *is a ring* A *with unit, equipped with the structure of a vector space over* K *such that*

$$\lambda(xy) = (\lambda x)y = x(\lambda y) \qquad \text{for all } \lambda \in K,\ x, y \in A.$$

This condition ensures that multiplication in A is bilinear with respect to the scalar multiplication arising from the K-vector space structure. Thus A combines the linear structure of a K-vector space with the multiplicative structure of a ring in a compatible way.

Basic examples include the field K itself, finite products K^n, matrix algebras $M_m(K)$, and field extensions L/K, where L is naturally a K-algebra via scalar multiplication by elements of K. In each case, the ring operations act linearly with respect to the K-vector space structure.

We say that a ring A is *simple* if its only two-sided ideals are $\{0\}$ and A.

Definition 2.6 *Given* $a, b \in K^*$*, the* quaternion algebra over K with parameters (a, b) *is the* K*-algebra*

$$A = \left(\frac{a, b}{K}\right) = K \oplus Ki \oplus Kj \oplus Kij,$$

with generators i *and* j *satisfying*

$$i^2 = a, \qquad j^2 = b, \qquad ij = -ji.$$

Every element of A can be written uniquely as

$$x = x_0 + x_1 i + x_2 j + x_3 ij, \qquad x_k \in K,$$

so that A is a K-vector space of dimension 4. We identify naturally K as a subfield of A.

For any arbitrary ring R, its *center* is the set

$$Z(R) = \{\, z \in R : za = az \text{ for all } a \in R \,\},$$

which is a commutative subring of R. A K-algebra is called *central* over K if its center is exactly K. A *central simple K-algebra* is a K-algebra that is both simple and central over K.

A ring R is called a *division algebra* if every nonzero element of A is invertible. Equivalently, R has no zero divisors and each nonzero element admits a multiplicative inverse. When R is finite-dimensional over K, being a division algebra means that the multiplicative structure of R behaves like that of a field, except that multiplication in R need not be commutative.

Quaternion algebras over K are precisely the central simple K-algebras of dimension 4; see, for example, [33, Proposition 7.6.1] or [32, Chapter I]. In particular, the construction

$$A = \left(\frac{a, b}{K}\right)$$

described above gives a central simple K-algebra, and every central simple K-algebra of dimension 4 is K-isomorphic to one of this form.

Example 2.7 *The classical Hamilton quaternion algebra is obtained by taking $K = \mathbb{R}$ and parameters $a = b = -1$. One writes*

$$\mathcal{H} = \left(\frac{-1, -1}{\mathbb{R}}\right).$$

Its generators i and j satisfy

$$i^2 = j^2 = -1, \qquad ij = -ji,$$

and every element has a unique expression

$$x = x_0 + x_1 i + x_2 j + x_3 ij, \qquad x_k \in \mathbb{R}.$$

This is a division algebra. It is the unique quaternion division algebra over $\mathbb{R}$, and *it is the model that motivates the terminology "quaternion algebra."*

Example 2.8 *Let* $K = \mathbb{R}$ *and consider the algebra*

$$A = \left(\frac{1,1}{\mathbb{R}}\right).$$

The generators now satisfy

$$i^2 = j^2 = 1, \qquad ij = -ji.$$

A direct computation shows that the map

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad i \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad j \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

extends to an isomorphism

$$A \;\cong\; M_2(\mathbb{R}).$$

This example illustrates that changing the signs of the parameters can drastically alter the structure of the algebra: although its defining relations are presented similarly to Hamilton's quaternions, the resulting algebra is not a division algebra.

Definition 2.9 *Let* $A = \left(\frac{a,b}{K}\right)$ *be a quaternion algebra over* K*. For an element*

$$x = x_0 + x_1 i + x_2 j + x_3 ij \in A,$$

its conjugate *is defined by*

$$\overline{x} = x_0 - x_1 i - x_2 j - x_3 ij.$$

The reduced trace *and* reduced norm *of* x *are given by*

$$\mathrm{tr}(x) = x + \overline{x}, \qquad \mathrm{n}(x) = x\,\overline{x}.$$

Both quantities, $\mathrm{tr}(x)$ and $\mathrm{n}(x)$, belong to K, and the reduced norm is multiplicative. The caracteristic polynomial over K of any element $x \in A$ is

$$p_x(T) = T^2 - \mathrm{tr}(x)\,T + \mathrm{n}(x),$$

showing that every element of a quaternion algebra satisfies a quadratic polynomial over its base field.

Theorem 2.10 *Let A be a quaternion algebra over $\mathbb{R}$. Then A is a central simple K-algebra of dimension 4, and there are exactly two possibilities:*

$$A \simeq M_2(K) \qquad \text{or} \qquad A \simeq \mathcal{H}.$$

This result follows from the Wedderburn theory of central simple algebras and is stated explicitly in [25, Theorem 2.1.7].

2.2.2 Real embeddings and ramification

Let A be a K-algebra and let F/K be a field extension. The *scalar extension* of A from K to F is the F-algebra

$$A_F \;=\; A \otimes_K F,$$

obtained by replacing the coefficients of elements of A by their images in F. The natural map $A \to A_F$ sends an element $x \in A$ to $x \otimes 1$, and A_F is generated as an F-vector space by elements of the form $x \otimes f$ with $x \in A$ and $f \in F$. The algebraic relations in A_F are the same as those in A, except that scalar multiplication now takes place in F.

From now on, let K be a totally real field with real embeddings $\sigma_1, \ldots, \sigma_n : K \hookrightarrow \mathbb{R}$. Given such an embedding σ_i, one may extend scalars from K to $\mathbb{R}$ by forming the $\mathbb{R}$-algebra

$$A_{\sigma_i} \;=\; A \otimes_{\sigma_i} \mathbb{R}.$$

This construction replaces the coefficients of elements of A by their images under σ_i, and produces an $\mathbb{R}$-algebra of dimension 4 whose multiplicative structure reflects that of A.

Under the scalar extension determined by σ_i, one obtains the real algebra $A_{\sigma_i} = A \otimes_{\sigma_i} \mathbb{R}$. If an element of $A = \left(\frac{a,b}{K}\right)$ is written as

$$x = x_0 + x_1 i + x_2 j + x_3 ij, \qquad x_k \in K,$$

then its image in A_{σ_i} is obtained by applying σ_i to the coefficients and interpreting the generators i and j as the generators of the extended algebra. More precisely, A_{σ_i} is generated over $\mathbb{R}$ by elements $i_{\sigma_i}, j_{\sigma_i}$ satisfying

$$i_{\sigma_i}^2 = \sigma_i(a), \qquad j_{\sigma_i}^2 = \sigma_i(b), \qquad i_{\sigma_i} j_{\sigma_i} = -\, j_{\sigma_i} i_{\sigma_i},$$

and the image of x is

$$\sigma_i(x_0) + \sigma_i(x_1)i_{\sigma_i} + \sigma_i(x_2)j_{\sigma_i} + \sigma_i(x_3)i_{\sigma_i}j_{\sigma_i}.$$

In this way one obtains a natural identification

$$A_{\sigma_i} \;\cong\; \left(\frac{\sigma_i(a),\, \sigma_i(b)}{\mathbb{R}}\right),$$

where the coefficients are transported through the embedding σ_i and the defining relations of the generators are modified accordingly.

Example 2.11 *Let $K = \mathbb{Q}(\sqrt{2})$ and consider the quaternion algebra*

$$A = \left(\frac{-1,\, \sqrt{2}}{K}\right).$$

The field K has two real embeddings,

$$\sigma_1(\sqrt{2}) = \sqrt{2}, \qquad \sigma_2(\sqrt{2}) = -\sqrt{2}.$$

Under each embedding, the algebra A becomes a quaternion algebra over $\mathbb{R}$ with parameters

$$A_{\sigma_1} \;\cong\; \left(\frac{-1,\, \sqrt{2}}{\mathbb{R}}\right) \;\cong\; M_2(\mathbb{R}), \qquad A_{\sigma_2} \;\cong\; \left(\frac{-1,\, -\sqrt{2}}{\mathbb{R}}\right) \;\cong\; \mathcal{H}.$$

Thus the behaviour of A over $\mathbb{R}$ depends on which embedding of K is used: the parameter $\sqrt{2}$ becomes positive under σ_1 and negative under σ_2. This example illustrates how the signs of the parameters, when viewed through different real embeddings of a totally real field, can affect the structure of the corresponding real quaternion algebras.

By Theorem 2.10, for each real embedding σ_i, the algebra A_{σ_i} falls into one of two possibilities:

$$A_{\sigma_i} \;\simeq\; M_2(\mathbb{R}) \qquad \text{or} \qquad A_{\sigma_i} \;\simeq\; \mathcal{H}.$$

In the first case one says that A *splits* at σ_i; in the second, that A is *ramified* at σ_i. This dichotomy at the real embeddings reflects how the multiplicative structure of A interacts with the different real forms of its base field and will be used in the construction of arithmetic groups associated with A.

2.2.3 Orders in quaternion algebras

Let K be a totally real number field with ring of integers $\mathcal{O}_K$, and let A be a quaternion algebra over K. In this section we introduce the notion of an order, which provides an integral structure inside A, which is compatible with the arithmetic of K.

Since A has dimension 4 as a K-vector space, an $\mathcal{O}_K$-submodule $\mathcal{L} \subset A$ is said to have *full rank* if it contains a K-basis of A. Equivalently, every element of A can be written as a K-linear combination of elements of $\mathcal{L}$.

Definition 2.12 *An* order *in A is a subring $\mathcal{O} \subset A$ containing* 1 *such that:*

1. *$\mathcal{O}$ is a finitely generated $\mathcal{O}_K$-module; and*
2. *$\mathcal{O}$ has full rank in A.*

The notion of an order isolates precisely those integral structures in A that are compatible with both its multiplicative structure and the arithmetic of $\mathcal{O}_K$, and it will serve as the starting point for constructing arithmetic groups in the next section.

An order $\mathcal{O}$ is called *maximal* if it is not properly contained in any other order of A.

Example 2.13 *Let $A = \left(\frac{a,b}{K}\right)$ with generators i and j. The $\mathcal{O}_K$-submodule*

$$\mathcal{O}_0 = \mathcal{O}_K \oplus \mathcal{O}_K i \oplus \mathcal{O}_K j \oplus \mathcal{O}_K ij$$

is an order in A, often called the standard order. *In general, however, $\mathcal{O}_0$ is not maximal. To see this, note that the element*

$$\frac{1+i+j+ij}{2}$$

typically does not lie in $\mathcal{O}_0$ (unless 2 is a unit in $\mathcal{O}_K$), but the $\mathcal{O}_K$-module generated by $\mathcal{O}_0$ together with this element is still closed under multiplication. Hence one obtains a strictly larger order

$$\mathcal{O}_0' \;=\; \mathcal{O}_K\big[\,1,\, i,\, j,\, ij,\, \tfrac{1+i+j+ij}{2}\big]\,,$$

showing that $\mathcal{O}_0$ cannot be maximal. Thus the standard order provides a basic example of an order that admits a proper enlargement within the quaternion algebra.

We record the following standard facts about orders; see [25, Lemma 2.2.7].

Lemma 2.14

1. *An* $\mathcal{O}_K$*-subring* $\mathcal{O} \subset A$ *containing* 1 *is an order if and only if it is a finitely generated* $\mathcal{O}_K$*-module whose* K*-span equals* A*.*
2. *Every order in* A *is contained in an* $\mathcal{O}_K$*-maximal order.*

Maximal orders serve as the natural integral objects from which arithmetic groups will be constructed in the next section.

2.3 Arithmetic and Semi-Arithmetic Fuchsian Groups

Arithmetic and semi-arithmetic Fuchsian groups arise naturally from quaternion algebras over totally real number fields. Starting from norm–one elements of suitable orders, one obtains subgroups of $\mathrm{PSL}(2,\mathbb{R})$, whose commensurability invariants include the invariant trace field and the associated invariant quaternion algebra. These algebraic data provide the framework for distinguishing arithmetic, semi-arithmetic, and subarithmetic hyperbolic surfaces, and they will be used in the subsequent sections.

2.3.1 Commensurability and invariant trace fields

Two subgroups $\Gamma_1, \Gamma_2 \subset \mathrm{PSL}(2,\mathbb{R})$ are said to be *commensurable* if their intersection has finite index in each of them. This relation identifies Fuchsian groups that, although possibly distinct as subgroups of $\mathrm{PSL}(2,\mathbb{R})$, determine hyperbolic surfaces with closely related geometric and arithmetic properties.

For a Fuchsian group $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$, its *trace set* is defined by

$$\mathrm{Tr}(\Gamma) = \{\, |\operatorname{tr}(\gamma)| : \gamma \in \Gamma \,\}.$$

The *trace field* of Γ is the field generated over $\mathbb{Q}$ by its trace set, namely

$$\mathbb{Q}(\mathrm{Tr}(\Gamma)).$$

One then considers the subgroup

$$\Gamma^{(2)} = \langle \gamma^2 : \gamma \in \Gamma \rangle,$$

which is a normal subgroup of finite index in Γ. Among all finite index subgroups of Γ, the group $\Gamma^{(2)}$ has the distinguished property of producing the smallest possible trace field. Accordingly, the *invariant trace field* of Γ is defined as

$$k\Gamma = \mathbb{Q}(\mathrm{Tr}(\Gamma^{(2)})).$$

2.3.2 Invariant quaternion algebras

Associated with Γ, one also considers the *invariant quaternion algebra*

$$A(\Gamma^{(2)}) = \left\{ \sum_i a_i \gamma_i \ : \ a_i \in k\Gamma,\ \gamma_i \in \Gamma^{(2)} \right\},$$

that is, the $k\Gamma$-subalgebra of $M_2(\mathbb{R})$ generated by the matrices representing the elements of $\Gamma^{(2)}$.

The use of $\Gamma^{(2)}$ in both constructions is motivated by the fact that the invariant trace field and the associated quaternion algebra remain unchanged within the commensurability class of Γ. This also explains the terminology "invariant" in the definition of the trace field and the algebra.

We begin by recalling some basic notions regarding subgroups of $\mathrm{PSL}(2,\mathbb{R})$.

Definition 2.15 *Let* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ *be a subgroup.*

- *We say that* Γ *is* reducible *if all its elements share a common fixed point on the boundary* $\partial\mathbb{H}$*. Otherwise,* Γ *is* irreducible*.*
- *We say that* Γ *is* elementary *if it has a finite orbit in its action on* $\mathbb{H} \cup \partial\mathbb{H}$*. Otherwise,* Γ *is* non-elementary*.*

Proposition 2.16 *Let* Γ *be a finitely generated, non-elementary subgroup of* $\mathrm{PSL}(2,\mathbb{R})$*. Then both* $k\Gamma$ *and* $A(\Gamma^{(2)})$ *are invariant under commensurability.*

Proof. See [25, Theorem 3.3.4 and Corollary 3.3.5]. ☐

The next proposition provides a useful way to construct explicit bases for the invariant algebra associated to a Fuchsian group. We thank Gregory Cosac for communicating the proof of this result.

Proposition 2.17 *Let* $\Gamma = \langle \gamma, \beta \rangle$ *be an irreducible group, with* $tr(\beta\gamma) \neq 0$*. Then,* $\{1, \gamma, \beta, \eta = [\gamma, \beta]\}$ *is a basis for* $M_2(\mathbb{C})$*.*

Proof. We make use of two identities for matrices $g, h \in \mathrm{SL}_2(\mathbb{C})$ that we shall develop as follows:

$$hg^2 = (\mathrm{tr}(g))hg - h \tag{1}$$

Since $gh \in \mathrm{SL}(2,\mathbb{C})$, the Cayley–Hamilton theorem applied to the characteristic polynomial

$$x^2 - \mathrm{tr}(gh)x + 1$$

yields

$$(gh)^2 = \mathrm{tr}(gh)\, gh - I.$$

Multiplying on the right by h^{-1} and using that $h^{-1} = (\mathrm{tr}(h)) \cdot 1 - h$ we obtain:

$$ghg = (\mathrm{tr}(gh))g + h - (\mathrm{tr}(h)) \cdot 1.$$

Multiplying on the right by g^{-1} and using that $g^{-1} = (\mathrm{tr}(g)) \cdot 1 - g$ then gives us:

$$gh + hg = [\mathrm{tr}(gh) - \mathrm{tr}(h)\mathrm{tr}(g)] \cdot 1 + (\mathrm{tr}(h))g + (\mathrm{tr}(g))h \tag{2}$$

Note that any $T \in GL(2,\mathbb{C})$ induces an isomorphism of $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$ defined by $X \mapsto TX$, and another defined by $X \mapsto XT$.

We know that $\{1, \beta, \gamma^{-1}, \beta\gamma^{-1}\}$ is a basis for $M_2(\mathbb{C})$, since $\langle \beta, \gamma^{-1} \rangle$ is irreducible (see [25, Lemma 1.2.4]). From (2), by making $g = \gamma^{-1}$ and $h = \beta$ we obtain

$$\begin{aligned} \gamma^{-1}\beta &= (\mathrm{tr}(\gamma^{-1}\beta) - \mathrm{tr}(\gamma^{-1})\mathrm{tr}(\beta)) \cdot 1 + (\mathrm{tr}(\gamma^{-1}))\beta + (\mathrm{tr}(\beta))\gamma^{-1} - \beta\gamma^{-1} \\ &= \mathrm{tr}(\beta\gamma) \cdot 1 + (\mathrm{tr}(\gamma^{-1}))\beta + (\mathrm{tr}(\beta))\gamma^{-1} - \beta\gamma^{-1} \end{aligned}$$

thus $\{\gamma^{-1}\beta, \beta, \gamma^{-1}, \beta\gamma^{-1}\}$ is also a basis. By (1), making $h = \gamma^{-1}$ and $g = \beta$, we get:

$$\gamma^{-1}\beta^2 = (\mathrm{tr}(\beta))\gamma^{-1}\beta - \gamma^{-1}$$

and so $\{\gamma^{-1}\beta, \beta, \gamma^{-1}\beta^2, \beta\gamma^{-1}\}$ is basis fot $M_2(\mathbb{C})$. Aplying the linear isomorphism $\begin{array}{ccc} M_2(\mathbb{C}) & \longrightarrow & M_2(\mathbb{C}) \\ X & \longmapsto & \gamma X \beta^{-1} \end{array}$ takes this basis to $\{1, \gamma, \beta, \gamma\beta\gamma^{-1}\beta^{-1}\} = \{1, \gamma, \beta, \eta\}$.

□

Proposition 2.18 *Let* $\Gamma = \langle \gamma_1, \dots, \gamma_n \rangle$, *and suppose that* $g = \gamma_1$ *and* $h = \gamma_2$ *generate an irreducible group, with* $tr(\gamma_2\gamma_1) \neq 0$. *Define* $\eta = [g,h]$, *and let*

$$K = \mathbb{Q}(\{tr(\gamma_i), tr(g\gamma_i), tr(h\gamma_i), tr(\eta\gamma_i) \mid i = 1, \dots, n\}).$$

Then, we have $K = \mathbb{Q}(tr(\Gamma))$.

Proof. By Proposition 2.17, we know that the set $\{1, g, h, \eta\}$ forms a basis for $M_2(\mathbb{C})$. Therefore, for each $i = 1, \dots, n$, there exist coefficients $a_{ij} \in \mathbb{C}$, with $j = 1, \dots, 4$, such that

$$\gamma_i = a_{i1} + a_{i2}g + a_{i3}h + a_{i4}\eta.$$

Multiplying the relation above by each element of the basis and applying the trace, we obtain:

$$\underbrace{\begin{pmatrix} \operatorname{tr}(\gamma_i) \\ \operatorname{tr}(\gamma_i g) \\ \operatorname{tr}(\gamma_i h) \\ \operatorname{tr}(\gamma_i \eta) \end{pmatrix}}_{\in M_{4\times 1}(K)} = \underbrace{\begin{pmatrix} 2 & \operatorname{tr}(g) & \operatorname{tr}(h) & \operatorname{tr}(\eta) \\ 2\operatorname{tr}(g) & \operatorname{tr}(g^2) & \operatorname{tr}(hg) & \operatorname{tr}(\eta g) \\ 2\operatorname{tr}(h) & \operatorname{tr}(gh) & \operatorname{tr}(h^2) & \operatorname{tr}(\eta h) \\ 2\operatorname{tr}(\eta) & \operatorname{tr}(g\eta) & \operatorname{tr}(h\eta) & \operatorname{tr}(\eta^2) \end{pmatrix}}_{\in M_4(K)} \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \\ a_{i4} \end{pmatrix}$$

We therefore conclude that $a_{ij} \in K$ for all $i = 1, \dots, n$ and all $j = 1, \dots, 4$. Hence, the group Γ lies in a K-vector space A generated by the basis $B = \{1, g, h, \eta\}$. Since A is closed under multiplication, it is a K-algebra. This yields the inclusion $\mathbb{Q}(\operatorname{tr}(\Gamma)) \subset K$. Consequently, we have $K = \mathbb{Q}(\operatorname{tr}(\Gamma))$.

□

Lemma 2.19 *If* $\Gamma = \langle \gamma_1, \dots, \gamma_n \rangle$ *and the set* $\{\operatorname{tr}(\gamma_i), \operatorname{tr}(\gamma_i\gamma_j) \mid 1 \le i, j \le n\}$ *consists of algebraic integers, then every trace* $\operatorname{tr}(\gamma)$ *with* $\gamma \in \Gamma$ *is an algebraic integer.*

Proof. Every element of Γ is a word in the generators $\gamma_1, \dots, \gamma_n$. We will then prove that the property of having an algebraic integer trace is preserved under multiplication, inversion, and taking commutators.

First, for matrices $X, Y \in \mathrm{SL}_2(\mathbb{C})$, the trace identity

$$\operatorname{tr}(XY) = \operatorname{tr}(X)\operatorname{tr}(Y) - \operatorname{tr}(XY^{-1})$$

holds. Thus, if $\operatorname{tr}(X)$, $\operatorname{tr}(Y)$, and $\operatorname{tr}(XY)$ are algebraic integers, then $\operatorname{tr}(XY^{-1})$ also is.

Next, the Cayley–Hamilton identity for $X \in \mathrm{SL}_2(\mathbb{C})$,

$$X^2 - \operatorname{tr}(X)\,X + I = 0,$$

shows that $\operatorname{tr}(X^k)$ is an algebraic integer whenever $\operatorname{tr}(X)$ is, by induction on k.

Now let γ be any word in the generators. Using the identities above, one can systematically rewrite $\operatorname{tr}(\gamma)$ as a polynomial with integer coefficients in the quantities $\operatorname{tr}(\gamma_i)$ and $\operatorname{tr}(\gamma_i\gamma_j)$ for $1 \le i, j \le n$. Since these are assumed to be algebraic integers, and algebraic integers are closed under addition and multiplication, it follows that $\operatorname{tr}(\gamma)$ is an algebraic integer. □

2.3.3 Arithmetic Fuchsian groups from quaternion algebras

Let K be a totally real field of degree n, and let

$$\sigma_1, \dots, \sigma_n : K \hookrightarrow \mathbb{R}$$

be its real embeddings. Let A be a quaternion algebra over K. For each $i = 1, \dots, n$, set

$$A_{\sigma_i} = A \otimes_{\sigma_i} \mathbb{R}.$$

Then each A_{σ_i} is isomorphic either to $M_2(\mathbb{R})$ (splits) or to Hamilton's quaternion algebra $\mathcal{H}$ (ramifies).

Assume that A splits at exactly $r \geq 1$ real places. Reordering the embeddings if necessary, we may suppose that

$$A_{\sigma_i} \cong M_2(\mathbb{R}), \qquad i = 1, \dots, r,$$

and

$$A_{\sigma_i} \cong \mathcal{H}, \qquad i = r+1, \dots, n.$$

For each $i = 1, \dots, r$, fix an isomorphism

$$\varphi_i : A_{\sigma_i} \to M_2(\mathbb{R}).$$

The reduced norm on A is a multiplicative map

$$\mathrm{n} : A \longrightarrow K.$$

We denote by

$$A^1 = \{x \in A : \mathrm{n}(x) = 1\}$$

the subgroup of norm-one elements of A. If $\mathcal{O} \subset A$ is an order, we also define

$$\mathcal{O}^1 = \mathcal{O} \cap A^1,$$

the group of norm-one units of $\mathcal{O}$.

The embeddings $\sigma_1, \dots, \sigma_r$, together with the fixed isomorphisms $\varphi_1, \dots, \varphi_r$, determine a homomorphism

$$\rho : A^1 \longrightarrow \mathrm{SL}_2(\mathbb{R})^r, \qquad x \longmapsto \big(\varphi_1(\sigma_1(x)), \dots, \varphi_r(\sigma_r(x))\big).$$

Restricting ρ to $\mathcal{O}^1$, we obtain a subgroup

$$\rho(\mathcal{O}^1) \subset \mathrm{SL}_2(\mathbb{R})^r.$$

Its image in $\mathrm{PSL}_2(\mathbb{R})^r$ is an arithmetic subgroup in the sense of Borel, and therefore it is a lattice; see [6]. Let

$$\pi_1 : \mathrm{SL}_2(\mathbb{R})^r \longrightarrow \mathrm{PSL}_2(\mathbb{R})$$

denote the natural projection of the first coordenate map. We set

$$\Gamma(A, \mathcal{O}) = \pi_1\big(\rho(\mathcal{O}^1)\big) \subset \mathrm{PSL}_2(\mathbb{R}).$$

We say that $\Gamma(A, \mathcal{O})$ is an *arithmetic group derived from a quaternion algebra acting on* $\mathbb{H}^r$.

In the special case $r = 1$, this construction yields an arithmetic Fuchsian group. In that case, we write

$$\Gamma(A, \mathcal{O}) \subset \mathrm{PSL}_2(\mathbb{R}).$$

2.3.4 Subarithmetic and semi-arithmetic hyperbolic surfaces

In this subsection we introduce subarithmetic hyperbolic surfaces, adapting from the notion adopted by Kontorovich and Zhang in [22].

Definition 2.20 *Let* $\Gamma \subseteq \mathrm{PSL}(2, \mathbb{R})$ *a finitely generated, non-elementary Fuchsian group. We say that* Γ *is* subarithmetic *if* Γ *is commensurable with some subgroup of an arithmetic group derived from a quaternion algebra acting on* $\mathbb{H}^r$*, for some* $r \geq 1$.

By Proposition 2.16 if Γ is subarithmetic commensurable with a subgroup of $\Gamma(A, \mathcal{O})$ then A is isomorphic to the invariant algebra of Γ. Hence, the number of real places of invariant number field of Γ such that the invariant quaternion algebra splits is invariant by commensurability. We will call this number the *arithmetic dimension* of Γ, denoted by $\mathrm{a.dim}(\Gamma)$.

Although subarithmetic groups fail to satisfy the strong rigidity properties characteristic of arithmetic groups, they retain several fundamental arithmetic features inherited from the ambient arithmetic structure:

- **Invariant trace field.** The invariant trace field

$$k\Gamma = \mathbb{Q}(\mathrm{Tr}(\Gamma^{(2)}))$$

is a number field. In particular, $k\Gamma$ is totally real for subarithmetic Fuchsian groups.

- **Integrality of traces.** The traces of elements of Γ lie in the ring of integers $\mathcal{O}_{k\Gamma}$. This mirrors the arithmetic situation and provides strong number-theoretic constraints on the algebraic data associated to Γ.
- **Associated quaternion algebra.** The invariant quaternion algebra $A(\Gamma^{(2)})$ is defined over $k\Gamma$ and remains constant within the commensurability class of Γ. Thus, even though Γ is not arithmetic, it still determines a well-defined arithmetic object.
- **Algebraic description of generators.** Irreducible subarithmetic groups admit generating sets whose traces and commutators generate both the invariant trace field and the associated quaternion algebra, as described in Propositions 2.16 and 2.17. This provides an explicit algebraic framework similar to the arithmetic case.

The principal distinction between arithmetic and subarithmetic surfaces lies in the loss of rigidity. Arithmetic surfaces satisfy strong classification results up to commensurability, while subarithmetic surfaces admit large deformation spaces inside moduli space.

The invariants $k\Gamma$ and $A(\Gamma^{(2)})$ allow one to describe arithmeticity intrinsically, without reference to the ambient quaternion algebra. This intrinsic characterization is due to Takeuchi [31] and serves as the starting point for the theory of semi-arithmetic Fuchsian groups developed in [29].

Theorem 2.21 (Takeuchi) *Let Γ be a cofinite Fuchsian group. Then Γ is arithmetic if and only if the following conditions hold:*

1. *The invariant trace field $k\Gamma = \mathbb{Q}(\mathrm{Tr}(\Gamma^{(2)}))$ is a totally real number field, and every element of $\mathrm{Tr}(\Gamma^{(2)})$ is an algebraic integer of $k\Gamma$.*
2. *For every embedding $\varphi : k\Gamma \hookrightarrow \mathbb{C}$ different from the identity embedding, one has*
$$\varphi(\mathrm{Tr}(\Gamma^{(2)})) \subset [-2, 2].$$

Borel [6] proved that, up to conjugacy, there exist only finitely many arithmetic Fuchsian groups of finite covolume.

Motivated by the scarcity of arithmetic groups, Schmutz Schaller and Wolfart introduced in [29] a broader class of Fuchsian groups that generalizes arithmeticity while retaining key number-theoretic features. These are the so-called semi-arithmetic groups.

Definition 2.22 *A cofinite Fuchsian group* Γ *is called* semi-arithmetic *if its invariant trace field* $k\Gamma$ *is a totally real number field and every element of* $\mathrm{Tr}(\Gamma^{(2)})$ *is an algebraic integer of* $k\Gamma$.

It follows from [29, Proposition 1] that a Fuchsian group is semi-aritmetic if, and only if, it is a subarithmetic group cofinite. If $n = [k\Gamma : \mathbb{Q}]$, then $1 \leq \mathrm{a.dim}(\Gamma) \leq n$. We note that a semi-arithmetic group Γ is arithmetic only when $\mathrm{a.dim}(\Gamma) = 1$. Thus, in the semi-arithmetic setting, $\mathrm{a.dim}(\Gamma)$ measures the deviation of Γ to be arithmetic.

We now describe some classes of examples illustrating the notion of subarithmeticity.

Example 2.23 ***(Surfaces of signature** $(1,1)$**)*** *Let* S *be a hyperbolic surface of signature* $(1,1)$, *that is, a compact surface of genus one with one geodesic boundary component. As discussed in Section 1.3.2, such a surface can be written as*

$$S = \Gamma\backslash\mathbb{H},$$

where $\Gamma = \langle\gamma, \beta\rangle$ *is a free group of rank two, and the commutator* $[\gamma, \beta]$ *corresponds to the boundary geodesic of* S.

Since Γ *is a free group, it cannot be a cofinite Fuchsian group. In particular,* S *is never arithmetic or semi-arithmetic in the sense of Definition 2.22, which is formulated for cofinite groups. Nevertheless, it is convenient to single out those Q-pieces for which the* algebraic *semi-arithmetic conditions hold, namely:*

$$k\Gamma \textit{ is a totally real number field} \qquad \textit{and} \qquad \mathrm{Tr}(\Gamma^{(2)}) \subset \mathcal{O}_{k\Gamma}.$$

In this thesis we will refer to such a surface as a subarithmetic Q-piece.

For a subarithmetic Q-piece, the group Γ *is subarithmetic. Indeed, let* $k = k\Gamma$ *and let* $A = A(\Gamma^{(2)})$ *be the invariant quaternion algebra over* k *generated by* $\Gamma^{(2)}$. *Let* $\mathcal{O} \subset A$ *be the* $\mathcal{O}_k$*-order generated by* $\Gamma^{(2)}$. *Then* $\Gamma^{(2)} \subset \mathcal{O}^1$, *and via the distinguished real embedding* $k \hookrightarrow \mathbb{R}$ *coming from the given realization* $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$, *the image of* $\mathcal{O}^1$ *in* $\mathrm{PSL}(2,\mathbb{R})$ *is an arithmetic group* $\Gamma(A,\mathcal{O})$ *acting on* $\mathbb{H}^r$ *for some* $r \geq 1$, *derived from a quaternion algebra and containing* $\Gamma^{(2)}$. *Therefore,* Γ *is commensurable with a subgroup of the arithmetic group* $\Gamma(A,\mathcal{O})$ *acting on* $\mathbb{H}^r$, *and hence* Γ *is subarithmetic.*

Surfaces of signature $(1,1)$ *satisfying the semi-arithmetic trace conditions thus provide explicit and flexible examples of subarithmetic hyperbolic surfaces.*

Example 2.24 ***(Hecke triangle groups and their subgroups)*** *For an integer* $q \geq 3$, *the* Hecke triangle group *(or triangle group of type* $(2, q, \infty)$*) is the Fuchsian* *group*

$$H_q = \Delta(2, q, \infty) \subset \mathrm{PSL}(2, \mathbb{R})$$

generated by the two Möbius transformations

$$S(z) = -\frac{1}{z}, \qquad T(z) = z + \lambda_q, \qquad \lambda_q = 2\cos\left(\frac{\pi}{q}\right).$$

It admits the presentation

$$H_q = \langle S, T \mid S^2 = (ST)^q = \mathrm{id} \rangle,$$

and geometrically it uniformizes the hyperbolic orbifold obtained from a triangle with angles $(\pi/2, \pi/q, 0)$*.*

The arithmeticity of the Hecke triangle group $H_q = \Delta(2, q, \infty)$ *is completely classified. By Takeuchi's criterion [31],* H_q *is arithmetic if and only if* $q \in \{3, 4, 6\}$*. For* $q \notin \{3, 4, 6\}$ *the group is non-arithmetic, despite the fact that its invariant trace field*

$$kH_q = \mathbb{Q}\left(2\cos\left(\frac{\pi}{q}\right)\right)$$

is totally real and that the traces of $H_q^{(2)}$ *satisfy the usual integrality conditions. In particular, for* $q \geq 5$*,* $q \neq 6$*, the group* H_q *is a cofinite Fuchsian group which is semi-arithmetic but not arithmetic.*

The obstruction to arithmeticity lies in the behavior under nontrivial real embeddings of the trace field: Takeuchi's criterion further requires that, for every embedding $\sigma : k_{H_q} \hookrightarrow \mathbb{R}$ *distinct from the identity, the corresponding conjugate group have bounded traces, or equivalently be relatively compact in* $\mathrm{PSL}(2, \mathbb{R})$*. This compactness condition fails for* $q \notin \{3, 4, 6\}$*.*

Nevertheless, these groups still exhibit an ambient arithmetic structure. In the sense of Kontorovich, the Hecke triangle group H_q *is subarithmetic: it embeds as an infinite-index subgroup of an arithmetic Fuchsian group associated o a quaternion algebra over* kH_q *(see, for instance, [22]). As a consequence, any torsion-free subgroup* $\Gamma \leq H_q$ *of finite index yields a finite-area hyperbolic surface* $S = \Gamma\backslash\mathbb{H}$ *which is non-arithmetic but semi-arithmetic.*

The existence of infinitely many semi-arithmetic Fuchsian groups was already observed by Takeuchi, who exhibited explicit families of non-arithmetic triangle groups satisfying the algebraic conditions defining semi-arithmeticity. These examples show that semi-arithmetic surfaces form a strictly larger class than arithmetic ones. However, the abundance of such examples does not, by itself, describe how extensively semi-arithmetic surfaces populate the moduli space of hyperbolic surfaces. A much stronger manifestation of this richness is provided by the following result, due to Cosac and Dória (see [11]), which shows that semi-arithmetic surfaces occupy a genuinely large region of moduli space from the geometric point of view.

Theorem 2.25 *The set*

$$\{\, \mathrm{sys}(S) \mid S \textit{ is a closed semi-arithmetic hyperbolic surface}\,\}$$

is dense in the positive real numbers.

Definition 2.26 *A closed hyperbolic surface is called* arithmetic *(resp.,* semi-arithmetic*) if it is of the form $S = \Gamma\backslash\mathbb{H}$, where Γ is an arithmetic (resp., semi-arithmetic) Fuchsian group.*

2.4 The Stretch Invariant

Semi-arithmetic Fuchsian groups form a class which is significantly larger than the arithmetic ones, and their geometric behaviour cannot be controlled solely by the coarea or by the arithmetic dimension. In particular, bounded coarea does not force finiteness of commensurability classes: for each genus $g \geq 2$, there exist infinitely many semi-arithmetic surfaces of genus g, hence of the same coarea; see Theorem 2.25. Motivated by this phenomenon, Belolipetsky, Cosac, Dória and Teixeira Paula introduced in [3] a new geometric invariant, called *stretch*, which quantifies how far a semi-arithmetic group is from admitting an isometric or holomorphic embedding into an arithmetic ambient space. This invariant restores the type of finiteness results known for the arithmetic case.

Let Γ be a semi-arithmetic Fuchsian group of arithmetic dimension r. Then $\Gamma^{(2)}$ embeds into an arithmetic lattice acting on $\mathbb{H}^r = \mathbb{H} \times \cdots \times \mathbb{H}$ via a discrete representation $\rho_0 : \Gamma^{(2)} \to \mathrm{PSL}(2,\mathbb{R})^r$. We consider maps $u : \mathbb{H} \to \mathbb{H}^r$ which are ρ_0–equivariant, i.e. $u(\gamma z) = \rho_0(\gamma)u(z)$ for all $\gamma \in \Gamma^{(2)}$ and $z \in \mathbb{H}$. If Γ is torsion-free, standard arguments yield a smooth ρ_0–equivariant map. In the presence of torsion, one passes to a torsion-free finite-index subgroup and applies a construction due to Karcher based on the *Riemannian center of mass*, producing a smooth equivariant map with controlled Lipschitz constant.

Definition 2.27 *Let Γ be a semi-arithmetic Fuchsian group of arithmetic dimension r. We say that Γ has* stretch *at most L if there exists a ρ_0–equivariant L–Lipschitz map $u : \mathbb{H} \to \mathbb{H}^r$. The* stretch *of Γ, denoted $\delta(\Gamma)$, is the infimum of all such constants. For the hyperbolic orbifold $S = \Gamma\backslash\mathbb{H}$ we write $\delta(S) = \delta(\Gamma)$.*

Intuitively, the stretch measures the minimal geometric distortion required for the surface $\Gamma\backslash\mathbb{H}$ to embed into $\mathbb{H}^r$ along the representation determined by the invariant trace field of Γ.

The stretch satisfies several natural properties that clarify its geometric behaviour.

Proposition 2.28 *If Γ_1 and Γ_2 are commensurable semi-arithmetic Fuchsian groups, then $\delta(\Gamma_1) = \delta(\Gamma_2)$.*

The proof, given in [3], relies on the center-of-mass construction, which allows a Lipschitz equivariant map defined on a finite-index subgroup to be extended to the entire group without increasing its Lipschitz constant.

When the arithmetic dimension is $r = 1$, the embedding into the hyperbolic plane is isometric. In this case one may take $u = \mathrm{id}$, showing that arithmetic Fuchsian groups always satisfy $\delta(\Gamma) = 1$.

Example 2.29 (The triangle group $\Delta(2,5,\infty)$) *A Fuchsian group Γ is said to admit a* modular embedding *if there exists a holomorphic ρ_0–equivariant map*

$$F : \mathbb{H} \longrightarrow \mathbb{H}^r,$$

that is,

$$F(\gamma z) = \rho_0(\gamma)\, F(z)$$

for all $\gamma \in \Gamma^{(2)}$ and $z \in \mathbb{H}$, where $\rho_0 : \Gamma^{(2)} \to \mathrm{PSL}(2,\mathbb{R})^r$ is the representation arising from the invariant quaternion algebra via the real embeddings at which it splits.

The Hecke triangle group $\Gamma = \Delta(2,5,\infty)$ provides an example that illustrate this phenomenon. Although Γ is a non-arithmetic lattice in $\mathrm{PSL}(2,\mathbb{R})$, it is semi-arithmetic and admits a modular embedding. Its invariant trace field is the real quadratic field $k = \mathbb{Q}(\sqrt{5})$, and there exists a holomorphic Γ–equivariant map

$$F : \mathbb{H} \longrightarrow \mathbb{H} \times \mathbb{H}$$

whose image projects to a complex geodesic on a Hilbert modular surface, as described by McMullen [26].

By the Schwarz-Pick lemma, any holomorphic map between hyperbolic spaces is 1–Lipschitz. Consequently, the stretch of Γ satisfies

$$\delta(\Gamma) \leq 1.$$

Since $\delta(\Gamma) \geq 1$ *for any lattice, it follows that*

$$\delta(\Delta(2,5,\infty)) = 1.$$

This example shows that stretch equal to one is a consequence of the existence of a modular embedding and so it does not characterize arithmeticity.

The introduction of the stretch invariant provides a missing geometric constraint that restores finiteness phenomena analogous to those which were already known for arithmetic lattices. The principal result of [3] is the following.

Theorem 2.30 *For fixed constants* $L \geq 1$, $\mu > 0$ *and integer* $r \geq 1$, *there exist only finitely many conjugacy classes of semi-arithmetic Fuchsian groups* Γ *satisfying* $\mathrm{a.dim}(\Gamma) \leq r$, $\delta(\Gamma) \leq L$ *and* $\mathrm{coarea}(\Gamma) \leq \mu$.

2.5 Number Theoretic Facts About Semi-Arithmetic Fuchsian Groups

Several arithmetic properties of semi-arithmetic Fuchsian groups can be reduced to elementary facts about trace fields, quaternion algebras, and their behavior under field embeddings. The results collected here record number–theoretic consequences that will be used in later arguments. This section serves as a technical reference for the proofs in the subsequent chapters.

Let α be an algebraic integer, and let $P \in \mathbb{Z}[X]$ be its minimal polynomial. A complex number β is said to be a Galois conjugate of α if $P(\beta) = 0$. The *house* of α, denoted by $\overline{|\alpha|}$, is defined as

$$\overline{|\alpha|} := \max\{|\beta| : \beta \text{ is a Galois conjugate of } \alpha\}.$$

We say that an algebraic integer α is *reciprocal* if it is Galois conjugate to its multiplicative inverse α^{-1}. In this case, the set of roots of the minimal polynomial P is invariant under the inversion map $x \mapsto x^{-1}$. It follows that P must have even degree and satisfies the functional equation

$$P(X) = X^n P(1/X),$$

where n is the degree of P.

In what follows, we extend a proposition originally proved in [3, Proposition 4.2]. Our aim is to explore the arithmetic properties of the eigenvalues of hyperbolic elements in semi-arithmetic Fuchsian groups. As an application, we derive a lower bound for the systole of semi-arithmetic hyperbolic surfaces.

We record the following estimate from Belolipetsky, Cosac, Dória and Teixeira Paula, which relates stretch to the spectral data of a semi-arithmetic group (see [3, Proposition 3.3]).

Proposition 2.31 *Let* Γ *be a semi-arithmetic Fuchsian group with representation* ρ *as above, and suppose that* Γ *has stretch at most* L. *Let* $\mathrm{Spec}(\Gamma^{(2)})$ *denote the set of largest eigenvalues of the hyperbolic elements of* $\Gamma^{(2)}$. *Then*

$$\sup\left\{\frac{\log \overline{|\lambda|}}{\log\lambda} : \lambda \in \mathrm{Spec}(\Gamma^{(2)})\right\} \leq L.$$

We will apply this proposition to the subgroup under consideration in order to bound the stretch from below in terms of the spectral behaviour of a suitable hyperbolic element. This will provide the key input in the proof of proposition below.

Proposition 2.32 *Let* S *be a semi-arithmetic closed hyperbolic surface with invariant trace field of degree at most* d, *and stretch bounded above by* L. *Then:*

1. *For any closed geodesic* $\beta \subset S$, *the quantity* $e^{\ell(\beta)}$ *is a reciprocal algebraic integer of degree at most* $2d$, *and house bounded above by* $e^{\ell(\beta)L}$.
2. *There exists a constant* $s > 0$, *depending only on* d *and* L, *such that the injectivity radius of* S *is at least* s.

Proof. We can write $S = \Gamma\backslash\mathbb{H}$, where $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ is a semi-arithmetic Fuchsian group with invariant trace field F of degree at most d.

As previously noted, for any closed geodesic $\beta \subset S$, there exists a hyperbolic element $T_\beta \in \Gamma$ whose axis projects to β. The length of β is related to the largest eigenvalue of T_β^2 via the formula:

$$\ell(\beta) = \log\lambda(T_\beta),$$

where $\lambda(T)$ denotes the spectral radius (largest eigenvalue) of T^2, for any hyperbolic element $T \in \Gamma$.

Now observe that $e^{\pm\ell(\beta)}$ are the two real roots of the quadratic polynomial

$$X^2 - \mathrm{tr}(T_\beta^2)X + 1 \in \mathcal{O}_F[X],$$

so $e^{\ell(\beta)}$ is a real algebraic integer of degree at most $2d$. Moreover, this polynomial is reciprocal, since its roots satisfy $\alpha \cdot \alpha^{-1} = 1$, and its coefficients lie in the ring of integers of F. Therefore, $e^{\ell(\beta)}$ is a reciprocal algebraic integer of degree at most $2d$.

To estimate the house of $e^{\ell(\beta)}$, we use the fact that the stretch bound L provides a uniform control on the geometric distortion of lengths under the equivariant map associated to ρ_0. In particular, this implies that $e^{\ell(\beta)}$ is bounded in terms of L and the algebraic data of the representation.

Using this together with the algebraic nature of the length functions, we may apply Proposition 2.31 to conclude that the house of $e^{\ell(\beta)}$ is at most $e^{\ell(\beta)L}$, which proves part (1).

For part (2), it suffices to give a uniform lower bound on the length of any closed geodesic $\beta \subset S$, which in turn gives a lower bound for the injectivity radius of S. By a result of Dimitrov [12, Theorem 1], which proves the Schinzel–Zassenhaus conjecture, the house of any algebraic integer of degree $2d$ that is not a root of unity is bounded below by $2^{1/(4d)}$. Since the spectral radius $\lambda(T_\beta)$ is given by the maximum modulus of its eigenvalues, this estimate yields

$$\lambda(T_\beta) > 2^{1/(4d)}.$$

Therefore, for any closed geodesic $\beta \subset S$, we obtain:

$$\ell(\beta) = \frac{\log \lambda(T_\beta)^L}{L} \geq \frac{\log \lambda(T_\beta)}{L} \geq \frac{\log 2}{4dL}.$$

By setting $s = \frac{\log 2}{4dL}$, which depends only on d and L, we conclude that the injectivity radius of S is bounded below by s, proving part (2). $\square$

We now consider a quantitative estimate on the number of reciprocal algebraic integers under degree and size constraints. For each $m \in \mathbb{N}$ and $X \geq 1$, let $\mathcal{U}_m(X)$ denote the set of real reciprocal algebraic integers of degree at most $2m$ and house at most X.

Lemma 2.33 *Given a positive integer m and a real number $X > 1$, the set $\mathcal{U}_m(X)$ is finite and contains at most*

$$2m \cdot (4mX)^{m^2}$$

elements.

Proof. To each $\lambda \in \mathcal{U}_m(X)$, we associate its minimal polynomial $P_\lambda \in \mathbb{Z}[X]$. Since any reciprocal algebraic integer of degree at most $2m$ has at most $2m$ distinct conjugates, it follows that each such polynomial corresponds to at most $2m$ elements in $\mathcal{U}_m(X)$. Thus, it suffices to bound the number of possible reciprocal minimal polynomials of degree at most $2m$.

Let $P \in \mathbb{Z}[X]$ be the minimal polynomial of a real reciprocal algebraic integer of degree $2d \leq 2m$. By definition of reciprocity, P has at most d coefficients which are different from 1. More precisely, we may write

$$P(X) = X^{2d} + a_1 X^{2d-1} + \cdots + a_d X^d + a_d X^{d-1} + \cdots + a_1 X + 1,$$

so that P is fully determined by the coefficients $a_1, \ldots, a_d \in \mathbb{Z}$.

Let $\alpha_1, \ldots, \alpha_{2d}$ be the roots of P. Since $\lambda \in \mathcal{U}_m(X)$, all roots satisfy $|\alpha_i| \leq X$. Each coefficient a_i is (up to sign) equal to the i-th elementary symmetric function in $\alpha_1, \ldots, \alpha_{2d}$, and hence satisfies the bound:

$$|a_i| \leq \binom{2d}{i} X^i \leq (2dX)^i.$$

Therefore, the total number of possible choices for the vector $(a_1, \ldots, a_d)$ is at most:

$$2^d (2dX)^{\frac{d(d+1)}{2}}.$$

Hence, there exist at most

$$\sum_{d=1}^{m} 2^d \cdot (2dX)^{\frac{d(d+1)}{2}} \leq m 2^m \cdot (2mX)^{\frac{m(m+1)}{2}} \leq (4mX)^{m^2}.$$

possibilities for a minimal polynomial of an element of $\mathcal{U}_m(X)$ (the last inequality can be verified applying log in both sides).

Since each such polynomial corresponds to at most $2m$ elements of $\mathcal{U}_m(X)$, we conclude that:

$$\#\mathcal{U}_m(X) \leq 2m \cdot (4mX)^{m^2}.$$

□

In [10], Calegari and Huang establish more general counting results for algebraic integers, which yield sharper upper bounds than the one obtained in the lemma above. However, for the purposes of our application, their estimates do not lead to a significant improvement, and the simpler bound stated here suffices for our analysis.

CHAPTER 3

Determining Surfaces by Short Geodesics

3.1 Length Coordinates with Logarithmic Bound

In Chapter 1, we explored how hyperbolic surfaces can be decomposed using systems of curves and arcs, forming what is known as a *curve and chain system*. We now use this combinatorial structure to construct a new global coordinate system for the Moduli space $\mathcal{M}_g$ based entirely on the lengths of finitely many closed geodesics. The central result of this section asserts that these lengths can be chosen uniformly short in a precise, logarithmic sense and still determine the point in moduli space uniquely.

The existence of curve and chain systems with controlled lengths, as established by Parlier [28], is a key ingredient in the constructions we develop here. His result ensures the existence of systems where the lengths of the main curves and their associated chains are bounded logarithmically in the genus.

In what follows, we present a slight refinement of his construction. In addition to bounding the lengths of the curves in $\mathcal{C}$ and $\mathcal{C}_{\mathcal{A}}$, we also include explicit bounds on the lengths of the arcs themselves. This additional control will be important later, particularly when we parametrize surfaces using auxiliary geodesics associated with the arcs.

3.1.1 A controlled curve and chain system

For $\varepsilon > 0$, we denote by

$$\mathcal{M}_g^{\geq \varepsilon} = \{S \in \mathcal{M}_g : \operatorname{sys}(S) \geq \varepsilon\}$$

the ε-thick part of the moduli space. Equivalently, $\mathcal{M}_g^{\geq \varepsilon}$ consists of those closed hyperbolic surfaces whose systole has length at least ε. By the Collar Lemma, surfaces in $\mathcal{M}_g^{\geq \varepsilon}$ admit uniform geometric bounds depending only on ε.

Proposition 3.1 *Let $\varepsilon > 0$. Then any surface $S \in \mathcal{M}_g^{\geq \varepsilon}$ admits a curve and chain system $\mathcal{C}, \mathcal{C}_{\mathcal{A}}$ satisfying:*

$$\begin{aligned} &i)\ \ \ell(\gamma) < 2\log(4g) \quad \textit{for every } \gamma \in \mathcal{C},\\ &ii)\ \ \ell(\gamma_a) < 8\log(4g) + 4\operatorname{arcsinh}\left(\frac{1}{\sinh\left(\frac{\varepsilon}{2}\right)}\right) \quad \textit{for every } a \in \mathcal{A},\\ &iii)\ \ \ell(a) < 2\log(4g) + 2\operatorname{arcsinh}\left(\frac{1}{\sinh\left(\frac{\varepsilon}{2}\right)}\right) \quad \textit{for every } a \in \mathcal{A}. \end{aligned}$$

Moreover, the system can be chosen so that for each curve $\gamma \in \mathcal{C}$ and for each side of γ, there exists at least one arc $a \in \mathcal{A}$ joining γ to another curve $\delta \in \mathcal{C}$ with $\delta \neq \gamma$.

Before proceeding with the proof of Proposition 3.1, we recall two auxiliary results that will be instrumental in controlling the geometry of the curves and arcs involved in the decomposition. The first one follows from estimates obtained by Bavard [1] on the radii of embedded disks in compact hyperbolic surfaces. In particular, it implies the existence of a based geodesic loop. Whose length is asymptotically bounded by $2\log(4g)$, which is the limit we will need in the proof of Proposition 3.1.

Lemma 3.2 *Let $S \in \mathcal{M}_g$ be a closed hyperbolic surface, and let $x \in S$. Then there exists a geodesic loop δ_x based at x such that*

$$\ell(\delta_x) < 2\cosh^{-1}\left(\frac{1}{2\sin\left(\frac{\pi}{12g-6}\right)}\right).$$

The next result provides an estimate on the distance between a simple geodesic loop and the collar of its associated simple closed geodesic. This type of estimate, which is adapted from Parlier [28], will be used to control the geometry of arcs in the curve and chain decomposition.

Lemma 3.3 *Let $S \in \mathcal{M}_g$ and let c be a simple geodesic loop on S. Let γ be the unique simple closed geodesic freely homotopic to c, and denote by $\mathfrak{C}(\gamma)$ its collar. Then*

$$\sup_{p \in c} d(p, \mathfrak{C}(\gamma)) < \log\left(\sinh\left(\frac{\ell(c)}{2}\right)\right).$$

Proof of Proposition 3.1 - The proof follows the same line of reasoning as in [28, Theorem 1.3]. For every point $x \in S$, let δ_x denote the shortest nontrivial loop based at x. As the minimal such loop, δ_x satisfies Lemma 3.2, which yields

$$\ell(\delta_x) < 2\ \operatorname{arccosh}\left(\frac{1}{2\sin\frac{\pi}{12g-6}}\right) < 2\log(4g).$$

Now consider the collection $\tilde{\mathcal{C}}$ of all simple closed geodesics in S that are freely homotopic to some δ_x. Let $\mathcal{C}$ be a maximal disjoint subset of $\tilde{\mathcal{C}}$.

To construct the desired set of arcs, we begin by decomposing S into Voronoi cells around the boundary curves of $S \setminus \mathcal{C}$. More precisely, for each boundary component $\mathcal{C}$ of $S \setminus \mathcal{C}$, define the cell

$$V_\gamma = \{x \in S \mid d(x,\gamma) \leq d(x,\delta),\ \delta \text{ a boundary of } S \setminus \mathcal{C}\}\,.$$

Then, by Lemma 3.3 and standard estimates, for any $x \in V_\gamma$, we have:

$$\begin{aligned} d(x,\gamma) &< \log\left(\sinh\left(\frac{\ell(\delta_x)}{2}\right)\right) + \operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{\ell(\gamma)}{2})}\right) && (1)\\ &< \log(4g) + \operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{\varepsilon}{2})}\right). && (2) \end{aligned}$$

The set of points that lie in more than one cell forms the *cut locus* of the decomposition. Points in the cut locus that belong to exactly two cells are edges, while those belonging to three or more cells are vertices. A vertex lying in exactly three cells is called *trivalent*. For each edge of the cut locus emanating from a trivalent vertex, we construct a geodesic arc connecting the corresponding boundary components of $S \setminus \mathcal{C}$, crossing the edge. These arcs are added to the set $\mathcal{A}$ under construction. For further details, see [28].

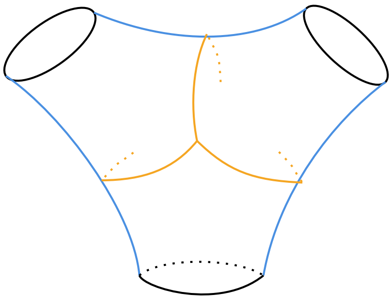

Figure 3.1: *Trivalent vertex*

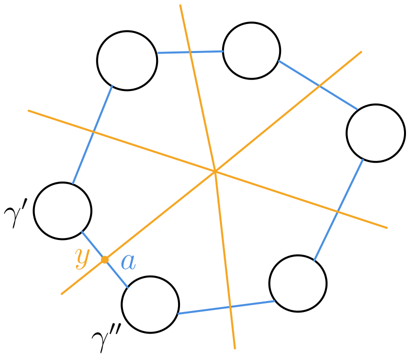

Figure 3.2: *Vertex with high degree*

If a vertex v has degree $q \geq 4$, we construct q arcs as in the trivalent case.

Given one such arc $a \in \mathcal{A}$, let γ' and γ'' be the boundary curves of $S \setminus \mathcal{C}$ joined by a. Let y be the point where a intersects the edge separating the Voronoi cells associated to γ' and γ''. Then by using (2) we obtain (iii) :

$$\ell(a) = 2d(y,\gamma') < 2\log(4g) + 2\operatorname{arcsinh}\left(\frac{1}{\sinh(\frac{\varepsilon}{2})}\right).$$

Now consider the q simple arcs $\{c_i\}_{i=1}^{q}$ joining v to the respective boundary curves of $S \setminus \mathcal{C}$. Orient c_1 toward v, and the remaining c_i (for $i \geq 2$) away from v. For each $i = 3, \ldots, q-1$, we concatenate c_1 with c_i and include an arc in its homotopy class into $\mathcal{A}$, thereby adding $q-3$ new arcs.

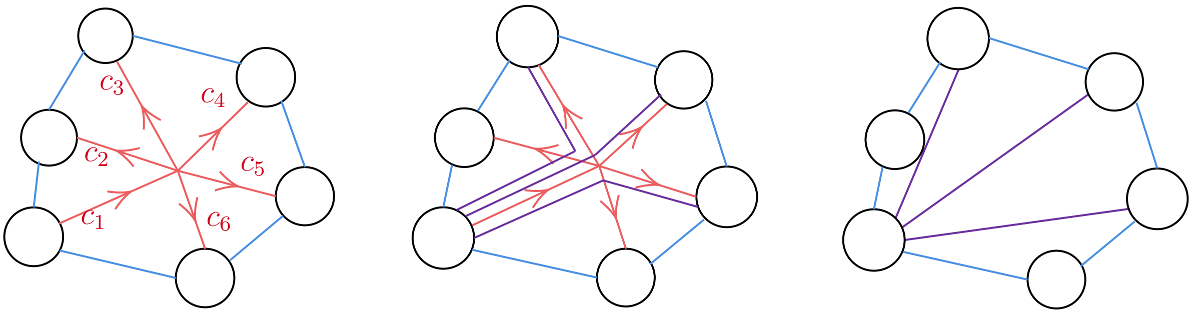

Figure 3.3: *Construction of new arcs at a vertex of degree greater than* 4.

Hence, for each such arc a joining γ' and γ'', we have $\ell(a) \leq 2d(v, \gamma')$. Since $\ell(\gamma_a) \leq 2\ell(a) + \ell(\gamma') + \ell(\gamma'')$, the above estimates provide the required upper bounds on $\ell(\gamma)$, $\ell(a)$, and $\ell(\gamma_a)$.

Finally, the construction guarantees that in every connected component of $S \setminus \mathcal{C}$, each boundary curve is connected to at least two arcs. Consequently, after gluing the surface back together, each geodesic in $\mathcal{C}$ has, on both sides, at least one arc linking it to a different curve in $\mathcal{C}$. This ensures that the arcs that where selected to define the twist parameters meet the necessary adjacency conditions.

□

3.1.2 Length bounds and injectivity

The proof of Theorem A follows directly from the next result, which provides an explicit description of the data determining a hyperbolic surface in terms of bounded-length curves and arcs.

Theorem 3.4 *Let $\varepsilon > 0$, and let $S \in \mathcal{M}_g^{\geq \varepsilon}$ be a closed hyperbolic surface. Then there exists a curve and chain system $(\mathcal{C}, \mathcal{C}_{\mathcal{A}})$ on S, together with a collection of closed geodesics $\Lambda = \{\delta(\gamma)\}_{\gamma \in \mathcal{C}}$, such that the surface S is uniquely determined by the lengths $\ell(\mathcal{C})$, $\ell(\mathcal{C}_{\mathcal{A}})$, and $\ell(\Lambda)$. Moreover, the lengths of all curves in $\mathcal{C}$, $\mathcal{C}_{\mathcal{A}}$, and Λ, as well as the lengths of the arcs in $\mathcal{A}$, are bounded above by a constant depending only on g and ε, namely*

$$16 \log(4g) + 8 \operatorname{arcsinh}\left(\frac{1}{\sinh\left(\frac{\varepsilon}{2}\right)}\right).$$

Finally, the total number of closed geodesics involved in this parametrization is at most $12g - 12$*.*

Proof. Let $S \in \mathcal{M}_g^{\geq\varepsilon}$ be a hyperbolic surface. By Proposition 3.1, there exists a curve and chain system $(\mathcal{C}, \mathcal{C}_\mathcal{A})$ on S whose curves have uniformly bounded length, depending only on g and ε.

Fix $\gamma \in \mathcal{C}$. By construction, there exist two arcs $a_1, a_2 \in \mathcal{C}_\mathcal{A}$ issuing from γ on opposite sides, with endpoints on distinct curves $\beta_1, \beta_2 \in \mathcal{C}$, where $\beta_i \neq \gamma$ for $i = 1, 2$. For each i, the curves γ, β_i, and $\gamma_{a_i} \sim \gamma a_i \beta_i a_i^{-1}$ bound an immersed pair of pants $Y_i \subset S$.

Let $p_1, p_2 \in \gamma$ be the reference points determined by the chosen curve and chain system, as in Section 1.5, and let $\alpha = \alpha(\gamma) \in [0, 1)$ denote the twist parameter of γ, defined by the oriented distance along γ from p_1 to p_2.

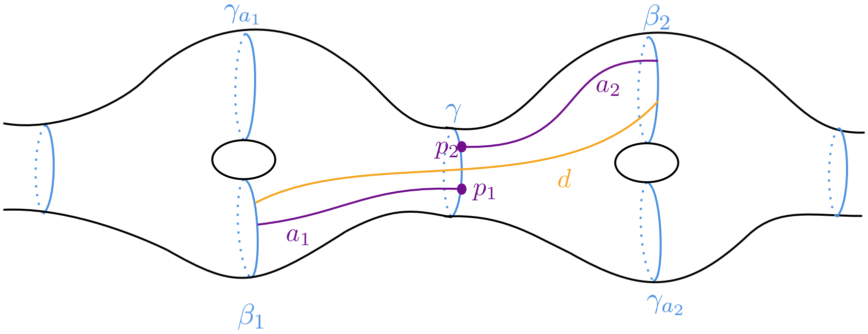

Figure 3.4: *Configuration of the geodesics and auxiliary arcs used to compute the traces of the cross-products.*

Consider the arc $b \subset \gamma$ joining p_1 to p_2, whose length is $\ell(b) = \alpha\, \ell(\gamma)$. Let d be the unique geodesic arc perpendicular to β_1 and β_2 in the homotopy class of the concatenation $a_2\, b\, a_1^{-1}$. Gluing the two Y–pieces Y_1 and Y_2 along γ according to the parameter α yields an immersed X–piece $X_\gamma \subset S$.

We now define an auxiliary closed geodesic. Let δ be the closed geodesic freely homotopic to the curve

$$d\, \beta_2\, d^{-1}\, \beta_1.$$

By construction, the curve δ is contained in the X–piece X_γ.

As shown in Proposition 1.22, the length of δ depend real analytically on

$$\alpha,\ \gamma,\ \beta_1,\ \beta_2,\ \gamma_{a_1},\ \gamma_{a_2},$$

and, conversely, the twist parameter α can be recovered from the length data of these closed geodesics. In particular, there exists a real analytic function F such that

$$\alpha = F\big(\ell(\gamma), \ell(\delta), \ell(\beta_1), \ell(\beta_2), \ell(\gamma_{a_1}), \ell(\gamma_{a_2})\big).$$

We now obtain uniform bounds on the lengths of the auxiliary curves. Using the triangle inequality and the length bounds from Proposition 3.1, we obtain

$$\ell(\delta) \leq \ell(\beta_1) + \ell(\beta_2) + 2\ell(\gamma) + 2\ell(a_1) + 2\ell(a_2) < 16\log(4g) + 8 \operatorname{arcsinh}\left(\frac{1}{\sinh(\varepsilon/2)}\right).$$

Thus, the geometry of S is completely determined by the lengths of the curves in $\mathcal{C}$, the chains in $\mathcal{C}_{\mathcal{A}}$, and the auxiliary family

$$\Lambda = \{\delta\}_{\gamma\in\mathcal{C}}.$$

Finally, we estimate the number of curves involved. Since S has area $4\pi(g-1)$, it admits a decomposition into $4g-4$ right–angled hexagons. Each hexagon contributes with three arcs, yielding $6g-6$ distinct arcs after gluing. Moreover, the system $\mathcal{C}$ contains at most $3g-3$ curves, and each $\gamma \in \mathcal{C}$ gives rise to one auxiliary curve δ. Therefore,

$$\#\mathcal{C} + \#\mathcal{C}_{\mathcal{A}} + \#\Lambda \leq (3g-3) + (6g-6) + (3g-3) = 12g - 12,$$

which completes the proof. □

We conclude this section with a combinatorial lemma that provides a uniform bound on the number of topological configurations arising from curve and chain systems. A family Ω of closed curves on a surface R is said to admit a *parametrization by a marked curve and chain system* if it is obtained through the construction described in the proof of the preceding theorem. By construction, such a family Ω is determined by a unique curve and chain system, which we call the *underlying curve and chain system* of Ω.

Let $\mathcal{F}_g$ denote the set of all families Ω arising in this way on a closed surface of genus g, considered up to the action of the group of orientation-preserving homeomorphisms of the surface. Since this action is trivial on the moduli space $\mathcal{M}_g$, replacing a curve and chain system by its image under a homeomorphism does not change its topological type. But, topologically, the set of all systems can be decomposed into finitely many orbits. This allows us to obtain a quantitative upper bound on the cardinality of $\mathcal{F}_g$.

Lemma 3.5 *The set $\mathcal{F}_g$ is finite and satisfies the estimate*

$$\#\mathcal{F}_g \leq B^g g^{6g},$$

for some universal constant $B > 0$.

Proof. To each $\Omega \in \mathcal{F}_g$ we associate its underlying curve and chain system. Suppose that this system contains k closed curves in Γ. Once the underlying system is fixed, there are at most $k \cdot (\#\mathcal{A})^2$ possible choices of chains that recover the family Ω.

Let $N_{cc}(g)$ denote the number of orbits of curve and chain systems on a surface of genus g under the action of the orientation-preserving homeomorphism group. It follows that

$$\#\mathcal{F}_g \leq (3g-3)(4g-4)^2 \cdot N_{cc}(g).$$

By [28, Lemma 3.2], the number $N_{cc}(g)$ satisfies the estimate

$$N_{cc}(g) \leq \frac{1}{e^6}\left(\frac{12^6}{e^5}\right)^{g-1}(g-1)^{6g-6}.$$

Combining these inequalities yields

$$\#\mathcal{F}_g \leq \frac{48(g-1)^2}{e^6}\left(\frac{12^6}{e^5}\right)^{g-1}(g-1)^{6g-6} \leq B^g g^{6g},$$

for some universal constant $B > 0$, as claimed. □

3.2 Minimal Distance Between Coverings of Arithmetic Hyperbolic Surfaces

The results of this section provide a geometric separation principle for arithmetic and semi-arithmetic hyperbolic surfaces. By establishing a uniform lower bound on the distance between distinct coverings, one obtains effective control on how such surfaces can accumulate in moduli space. These separation estimates form the geometric input needed for the counting arguments developed at the end of the section.

3.2.1 Geometric separation of coverings

Given a hyperbolic surface S, for each integer $n > 1$, the number of n-sheeted coverings of S is equal to the number of conjugacy classes of subgroups of index n in the Fuchsian group $\mathcal{C}$ that uniformizes S. On the other hand, it is a classical result that any finitely generated group admits only finitely many subgroups of a given index. Therefore, for fixed n, the number of n-sheeted coverings of S is finite.

Moreover, by the Gauss–Bonnet theorem, there is a well-defined correspondence between the genus of a covering surface and its degree over S. In particular, for each genus g, the number of points in the moduli space $\mathcal{M}_g$ corresponding to coverings of

S coincides with the number of m-sheeted coverings of S, for some explicit integer m depending only on g and on the genus of S.

This allows us to define the function

$$\delta_S(g) = \begin{cases} 0, \text{ if there does not exist a genus } g \text{ surface covering } S, \\ \min\{d_{\mathcal{T}}(\widetilde{S}, \bar{S}) : \widetilde{S}, \bar{S} \in \mathcal{M}_g \text{ are distinct coverings of } S\}, \text{ otherwise,} \end{cases}$$

where $d_{\mathcal{T}}$ denotes the Teichmüller metric on $\mathcal{M}_g$.

From the celebrated work of Kahn and Markovic [16], who proved the Ehrenpreis conjecture, it follows that for any $\varepsilon > 0$, there exist distinct coverings of a fixed surface S whose Teichmüller distance is at most ε. In particular, this implies that

$$\lim_{g\to\infty} \delta_S(g) = 0 \quad \text{for any hyperbolic surface } S.$$

This naturally leads to the question of whether it is possible to establish a uniform qualitative upper bound for the function $\delta_S(g)$ across all genera. More precisely, does there exist a polynomial Q, depending only on the base surface S, such that

$$\delta_S(g) \leq Q\left(\frac{1}{g}\right) \qquad \text{for all } g \gg 1\ ?$$

If the answer is affirmative, this would represent the strongest possible asymptotic control for $\delta_S(g)$, as shown by the next theorem. Before stating the theorem, we recall the following consequence of the minimal spacing property established in [24, Lemma 2.1].

Lemma 3.6 *Let G be an arithmetic Fuchsian group with invariant trace field F. Then there exists a constant $c > 0$, depending only on F, such that for any elements $T, T' \in G$ with $|tr(T)| \neq |tr(T')|$, we have*

$$\left|tr(T)^2 - tr(T')^2\right| > c.$$

Proof. Let F be the invariant trace field of the arithmetic Fuchsian group G. By definition, the traces of elements in G (or more precisely, of squares of elements) lie in F, and for every nontrivial embedding $\varphi : F \hookrightarrow \mathbb{C}$, the image of any such trace under φ is bounded in absolute value by 2:

$$|\varphi(\mathrm{tr}(T^2))| \leq 2, \quad \text{for all } T \in G.$$

Recall the identity

$$\operatorname{tr}(T^2) = \operatorname{tr}(T)^2 - 2, \quad \text{for all } T \in \mathrm{SL}(2, \mathbb{R}).$$

Thus, if $|\operatorname{tr}(T)| \neq |\operatorname{tr}(T')|$, then $\operatorname{tr}(T)^2 \neq \operatorname{tr}(T')^2$, which implies

$$\operatorname{tr}(T^2) \neq \operatorname{tr}(T'^2).$$

Now consider the difference $\operatorname{tr}(T^2) - \operatorname{tr}(T'^2) \in F$. Since F is a number field of degree $d = [F : \mathbb{Q}]$, let $\varphi_1 = \mathrm{id}, \varphi_2, \dots, \varphi_d$ be the real embeddings of F into $\mathbb{R}$. The *norm* of an element $x \in F$ is given by

$$\mathrm{N}_{F/\mathbb{Q}}(x) = \prod_{i=1}^{d} \varphi_i(x).$$

If $x = \operatorname{tr}(T^2) - \operatorname{tr}(T'^2) \in \mathcal{O}_F \setminus \{0\}$, then $\mathrm{N}_{F/\mathbb{Q}}(x) \in \mathbb{Z} \setminus \{0\}$, and therefore

$$|\mathrm{N}_{F/\mathbb{Q}}(x)| \geq 1.$$

On the other hand, since the traces $\operatorname{tr}(T^2)$ and $\operatorname{tr}(T'^2)$ have absolute value at most 2 under any embedding φ_i, we have

$$|\varphi_i(\operatorname{tr}(T^2) - \operatorname{tr}(T'^2))| \leq 4, \quad \text{for all } i = 2, \dots, d.$$

Hence,

$$\prod_{i=2}^{d} |\varphi_i(\operatorname{tr}(T^2) - \operatorname{tr}(T'^2))| \leq 4^{d-1}.$$

Consequently, using the inequality

$$|\operatorname{tr}(T^2) - \operatorname{tr}(T'^2)| \cdot \prod_{i=2}^{d} |\varphi_i(\operatorname{tr}(T^2) - \operatorname{tr}(T'^2))| \geq 1,$$

we deduce that

$$|\operatorname{tr}(T^2) - \operatorname{tr}(T'^2)| \geq \frac{1}{4^{d-1}}.$$

Finally, since

$$|\operatorname{tr}(T^2) - \operatorname{tr}(T'^2)| = |\operatorname{tr}(T)^2 - \operatorname{tr}(T')^2|,$$

it follows that

$$|\operatorname{tr}(T)^2 - \operatorname{tr}(T')^2| \geq \frac{1}{4^{d-1}}.$$

Therefore, we may take $c = 4^{-(d-1)} > 0$, which depends only on the degree

$d = [F : \mathbb{Q}]$, concluding the proof. □

Theorem 3.7 *Let S be a closed arithmetic surface of genus $g \geq 2$ with invariant trace field F. Then there exists a constant $A = A(d) > 0$, depending only on the degree $d = [F : \mathbb{Q}]$, such that for any other closed arithmetic surface $S' \in \mathcal{M}_g$ with the same invariant trace field F, we have*

$$d_{\mathcal{T}}(S, S') \geq A g^{-192}.$$

Proof. Let $S = G\backslash\mathbb{H} \in \mathcal{M}_g$ be an arithmetic closed surface with invariant trace field F, where $\deg(F/\mathbb{Q}) = d$. Since any arithmetic surface has stretch equal to 1, it follows from Proposition 2.32 that there exists a constant $s > 0$, depending only on d, such that $S \in \mathcal{M}_g^{\geq s}$.

By applying Theorem 3.4, we obtain a system of curves and chains $\mathcal{C} = \{\gamma_1, \dots, \gamma_k\}$, $\mathcal{C}_{\mathcal{A}} = \{\gamma_a \mid a \in \mathcal{A}\}$, such that the surface S is completely determined by the set

$$\Omega = \{\gamma_i, \delta_i, \gamma_a \mid 1 \leq i \leq k,\ a \in A\}$$

and satisfies

$$\ell_S(\omega) \leq 48 \log g + c(d), \quad \text{for every } \omega \in \Omega,$$

where $c(d) > 0$ is an explicit constant depending only on d.

Let $S' \in \mathcal{M}_g$ be another arithmetic surface with the same invariant trace field F, and assume $S' \neq S$. If

$$d_{\mathcal{T}}(S, S') \geq \frac{\log 2}{2},$$

then the theorem holds with $A = 1$, and we are done. So we may assume that

$$d_{\mathcal{T}}(S, S') < \frac{\log 2}{2}.$$

Let $f : S \to S'$ be an $e^{2\varepsilon}$-quasiconformal homeomorphism, with $\varepsilon < \frac{\log 2}{2}$ and isotopic to the identity. Since $S \neq S'$, there exists some $\omega \in \Omega$ such that $\ell_S(\omega) \neq \ell_{S'}(\omega)$.

Let $T = T_\omega(S)$ and $T' = T_\omega(S')$ be the hyperbolic elements in the respective uniformizing groups associated to the curve ω. Denote $t = \mathrm{tr}(T)$, $t' = \mathrm{tr}(T')$. Then $|t| \neq |t'|$, and by Lemma 3.6 we have

$$|t^2 - (t')^2| > c,$$

for some constant $c > 0$ depending only on d.

On the other hand, Theorem 1.17 yields

$$e^{-2\varepsilon} \leq \frac{\cosh^{-1}\left(\frac{(t')^2-2}{2}\right)}{\cosh^{-1}\left(\frac{t^2-2}{2}\right)} \leq e^{2\varepsilon}.$$

Taking logarithms and reorganizing gives

$$\left|\log\cosh^{-1}\left(\frac{(t')^2-2}{2}\right) - \log\cosh^{-1}\left(\frac{t^2-2}{2}\right)\right| < 2\varepsilon.$$

Let us define the function $L(x) = \log\cosh^{-1}\left(\frac{x-2}{2}\right)$. By the mean value theorem, there exists a point θ between t^2 and $(t')^2$ such that

$$|(t')^2 - t^2| \cdot |L'(\theta)| = |L((t')^2) - L(t^2)| < 2\varepsilon.$$

Therefore,

$$c \cdot |L'(\theta)| < 2\varepsilon.$$

We now estimate $L'(\theta)$ from below. Since $\log(x) \leq \cosh^{-1}(x)$ for any $x > 1$, and from the bound on the length of $\omega \in \Omega$, we have:

$$2\log\left(\frac{|t|}{2}\right) \leq 2\cosh^{-1}\left(\frac{|t|}{2}\right) = \ell_\omega(S) \leq 48\log g + c(d) = \log(C_d \cdot g^{48}),$$

for some constant $C_d = e^{c(d)}$. Moreovere,

$$2\log\left(\frac{|t'|}{2}\right) \leq 2\cosh^{-1}\left(\frac{|t'|}{2}\right) = \ell_{S'}(\omega) \leq e^{2\varepsilon}\ell_S(\omega).$$

We recall that we are assuming $e^{2\varepsilon} < 2$. Thus,

$$t^2, (t')^2 \leq 4C_d^2 g^{96}.$$

Hence,

$$\theta \leq \max\{t^2, (t')^2\} \leq 4C_d^2 g^{96}.$$

For all $x > 4$, a direct computation shows:

$$L'(x) = \frac{1}{\sqrt{x(x-4)} \cdot \cosh^{-1}\left(\frac{x-2}{2}\right)} > \frac{1}{x^2}.$$

Therefore,

$$L'(\theta) > \frac{1}{\theta^2} \geq \frac{1}{16\ C_d^4\ g^{192}}.$$

Substituting into the inequality above yields:

$$\varepsilon > \frac{c}{16\ C_d^4 g^{192}}.$$

It follows that

$$d_{\mathcal{T}}(S, S') > A \cdot g^{-192},$$

where $A = \frac{c}{16\ C_d^4} > 0$ depends only on d, as desired. ☐

The lower bound obtained in Theorem 3.7 is not sharp, and several aspects of the proof indicate that the exponent 192 can be significantly improved. This exponent arises from the application of Theorem 3.4, where the lengths of the curves used to determine the surface are bounded by $48 \log g + c(d)$. Since the trace of a hyperbolic element grows exponentially with the length of the associated geodesic, we obtain a polynomial bound of degree 96 for t, and hence of degree 192 for t^2, which explains the origin of the exponent.

Additionally, the constant A in the conclusion depends inversely on powers of C_d, which itself depends exponentially on the constant $c(d)$ from Theorem 3.4. This reflects the compounded effect of using non-optimal constants at various steps in the argument.

A more refined analysis could improve both the exponent and the constant. For instance, if one assumes that $d_{\mathcal{T}}(S, S') < \varepsilon$ for arbitrarily small $\varepsilon > 0$, the upper bound on the lengths of curves on S' can be made uniformly close to those on S, allowing better control on the trace difference $|t^2 - (t')^2|$ and consequently improving the final lower bound on $d_{\mathcal{T}}(S, S')$. Such improvements, however, would require sharper length bounds and a deeper understanding of the geometry of arithmetic surfaces.

This suggests that while the current lower bound is effective and explicit, it should be seen as a proof of concept that arithmetic surfaces with the same invariant trace field cannot accumulate arbitrarily close in Teichmüller space without eventually becoming isometric, rather than as a definitive quantitative result.

Theorem 3.3 shows that any effective approximation by finite coverings necessarily requires at least polynomial growth in the genus, thereby ruling out any subpolynomial version of the effective Ehrenpreis theorem in the arithmetic setting. On the other hand, Luo proved in [23, Theorem 1] that such approximations can always be achieved with coverings of polynomial degree. Taken together, these results show that polynomial growth is optimal in order.

3.3 Counting Semi-Arithmetic Hyperbolic Surfaces

We now establish a lower bound for the cardinality of $\mathcal{SA}[g,d,L]$. Let $\mathcal{AS}_{g,d}$ denote the set of arithmetic hyperbolic surfaces of genus g whose invariant trace field has degree at most d. Since every arithmetic surface is semi-arithmetic with stretch equal to 1, we have

$$\mathcal{AS}_{g,d} \subseteq \mathcal{SA}[g,d,L]$$

for every $L \geq 1$. In particular, fixing the Bolza surface [18], which is an arithmetic hyperbolic surface of genus 2 with quadratic invariant trace field, the number of genus g coverings of the Bolza surface provides a lower bound for $\#\mathcal{SA}[g,d,L]$ whenever $d \geq 2$. Moreover, it follows from the proof of [4, Corollary 1.4] that there exists a universal constant $b > 0$ such that

$$\#\mathcal{SA}[g,d,L] \geq (bg)^{2g}. \tag{3}$$

The following theorem shows that this growth rate is essentially optimal.

Theorem 3.8 *For any integers $g \geq 2$, $d \geq 2$, and any real number $L \geq 1$, there exists a constant $C > 0$, depending only on d and L, such that*

$$2 \leq \limsup_{g\to\infty} \frac{\log\left(\#\mathcal{SA}[g,d,L]\right)}{g \log g} \leq C.$$

Moreover, the constant C satisfies the upper bound $C \leq ULd^2$, for some universal constant $U \geq \frac{1}{2}$.

Proof. The lower bound in the statement follows directly from inequality (3), which provides an explicit construction of sufficiently many pairwise distinct semi-arithmetic surfaces with the required invariants.

We now focus on establishing the upper bound. Fix integers $g \geq 2$, $d \geq 2$, and a real number $L \geq 1$. For each element $S^* \in \mathcal{SA}[g,d,L]$, Proposition 2.32 ensures the existence of a constant $s = s(d,L) > 0$ such that $S^* \in \mathcal{M}_g^{\geq s}$, i.e., all systoles are uniformly bounded from below.

Let $S \in \mathcal{T}_g$ be a marked representative of S^*. By Theorem 3.4, there exists a family of length functions

$$L_1, \ldots, L_n : \mathcal{T}_g \to \mathbb{R},$$

with $n \leq 12(g-1)$, such that the functions $\{L_i(\cdot)\}$ provide a coordinate system for $\mathcal{T}_g$, and each $L_i(S) \leq 16\log(\sigma g)$, where $\sigma = \sigma(d,L) > 1$ is a constant depending only on d and L.

Moreover, Proposition 2.32 guarantees that for each i, the number $\exp(L_i(S))$ is an algebraic integer of degree at most $2d$, and satisfies

$$\overline{|\exp(L_i(S))|} \leq (\sigma g)^{16L}.$$

Let $P(S)$ denote the list of exponentials $\big(\exp(L_1(S)), \ldots, \exp(L_n(S))\big) \in \mathbb{R}^n$. Each coordinate of $P(S)$ lies in the ring of algebraic integers of degree at most $2d$, and is bounded above by $(\sigma g)^{16L}$.

By Lemma 2.33, the number of such algebraic integers (of degree at most $2d$, and height bounded by $(\sigma g)^{16L}$) is at most:

$$\left[2d \cdot (4d)^{d^2} \cdot (\sigma g)^{16Ld^2}\right].$$

Since the dimension n satisfies $n \leq 12(g-1)$, the total number of possible vectors $P(S)$ is bounded by:

$$\left[2d \cdot (4d)^{d^2} \cdot (\sigma g)^{16Ld^2}\right]^{12(g-1)}.$$

Now, suppose that two different points $S, S' \in \mathcal{T}_g$, representing distinct semi-arithmetic surfaces $S^*, S'^* \in \mathrm{SA}[g, d, L]$, satisfy $P(S) = P(S')$. Then these must correspond to different topological types of marked surfaces with respect to the curve and chain decomposition used in the parametrization.

According to Lemma 3.5, the number of topological types of such decompositions for a genus g surface is at most $Bg \cdot g^{6g}$, where $B > 0$ is a universal constant.

Combining both contributions, we obtain:

$$\#\mathrm{SA}[g, d, L] \leq B^g \cdot g^{6g} \cdot \left[2d \cdot (4d)^{d^2} \cdot (\sigma g)^{16Ld^2}\right]^{12(g-1)}.$$

So there exist constants $B_1 = B_1(d, L)$ and $B_2 = B_2(d, L)$ such that:

$$\#\mathrm{SA}[g, d, L] \leq B_1^g \cdot g^{B_2 g}.$$

Taking logarithms and dividing by $g \log g$, we obtain:

$$\frac{\log\left(\#\mathrm{SA}[g, d, L]\right)}{g \log g} \leq \frac{\log(B_1)}{\log g} + B_2 \leq C,$$

for all $g \geq 2$, where

$$C = C(d, L) = \frac{\log(B_1)}{\log 2} + B_2.$$

Finally, all constants B_1, B_2, and hence C, depend only on d and L. More precisely, the construction shows that there exists a universal constant $U > 0$ such that

$$C(d, L) \leq ULd^2.$$

This completes the proof. $\square$

CHAPTER 4

Denseness of Subarithmetic Surfaces of Signature $(1,1)$

This chapter is devoted to denseness phenomena for families of hyperbolic surfaces with arithmetic constraints. Building on the parametrization results and on the arithmetic framework developed in previous chapters, we study how subarithmetic surfaces are distributed inside moduli space. The main results show that, despite their rigid arithmetic origin, these surfaces occur densely when length and twist parameters that are not constrained by the arithmetic invariants are allowed to vary.

4.1 Geometry of a Q-Piece

We begin by recalling the geometric construction of a Q-piece, that is, a hyperbolic surface of signature $(1,1)$, as described in Subsection 1.3.2. Throughout this section, we adopt the common abuse of notation and write $\ell(\gamma)=\gamma$ whenever γ denotes a geodesic. The basic building blocks are two isometric right-angled hexagons whose three pairwise non-consecutive sides have lengths $\gamma_1/2$, $\gamma_2/2$, and $\eta/2$. The side joining the edges $\gamma_1/2$ and $\gamma_2/2$ will be denoted by a parametrized with speed 1.

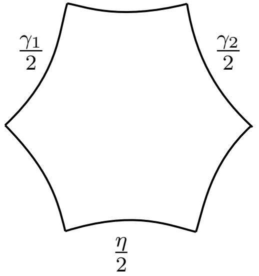

Figure 4.1: *A right-angled hexagon used in the construction of a Q-piece.*

Gluing the two hexagons along the prescribed non-consecutive sides produces a Y-piece whose geodesic boundary components are γ_1, γ_2, and η. Assume that $\gamma_1 = \gamma_2$. In order to obtain a surface of signature $(1,1)$, we must further identify the two boundary components of equal length, γ_1 and γ_2. This identification, however, is not unique and is governed by a twist parameter.

Following [8, Section 3.4], we define a twist parameter using the endpoints of the geodesic arc a that conect γ_1 with γ_2. Parametrize each boundary component γ_i, $i = 1, 2$, so that $\gamma_i(0)$ coincides with the intersection point of a with γ_i. The *twist parameter* associated with this Q-piece is the real number α defined by the identification

$$\gamma(t) = \gamma_1(t) = \gamma_2(\alpha - t), \qquad t \in \mathbb{S}^1.$$

The resulting surface S is completely determined, up to isometry, by the lengths of the closed geodesics γ and η, together with the twist parameter α. Let b denote the geodesic arc on γ realizing the distance between the feet of the arc a on γ, as illustrated below, so $\ell(b) = \alpha\gamma$.

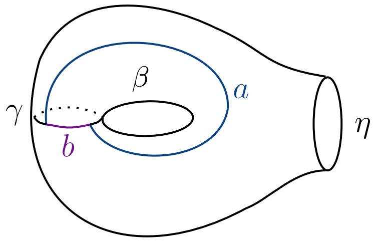

Figure 4.2: *Q–piece with twist parameter α.*

We define β to be the unique closed geodesic in the free homotopy class of the concatenation $a * b$. With this choice, the third boundary component η of the original Y-piece is represented by the commutator

$$\eta = [\gamma, \beta].$$

Consequently, using Keen's description of Q-pieces in [20], we may realize S as

$$S \cong \Gamma \backslash D,$$

where $D \subseteq \mathbb{H}$ is a convex domain and Γ is a free Fuchsian group of rank 2, generated by hyperbolic elements γ and β whose axes intersect in D. The projections of these axes to S are the closed geodesics represented by γ and β.

We denote by θ the angle between the geodesics γ and β. Observe that we obtain from this construction geometric relations among the parameters of the Q-piece. One of these relations arises from the right-angled triangle with side lengths $a/2$, $\beta/2$, and $b/2$.

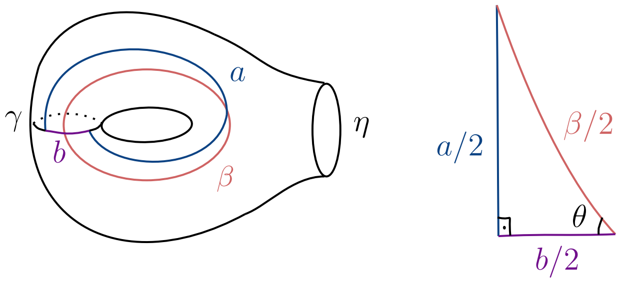

Figure 4.3: *A right-angled triangle associated with a Q-piece.*

The geometric identities arising from the surface yield explicit algebraic expressions for the traces associated with the generators of Γ, which will be used in the lemma below.

We recall that, given a number field K, $\mathcal{O}^*_{K,+}$ denotes the group of totally positive units of K.

Let

$$x = \mathrm{tr}(\gamma), \quad y = \mathrm{tr}(\beta), \quad z = \mathrm{tr}(\gamma\beta) \quad \text{and} \quad r = \mathrm{tr}(\eta).$$

Lemma 4.1 *Let K be a totally real number field, and let S be a Q-piece as described above. Suppose that there exist $\zeta \in \mathcal{O}^*_{K,+}$ and $\delta, \xi \in \mathcal{O}_{K,+}$ such that*

$$x = \sqrt{2(\zeta + 2)}, \tag{1}$$

$$r = -2\xi \tag{2}$$

$$\lambda = \frac{\xi + 1}{\zeta} + 1, \tag{3}$$

$$\cosh\left(\frac{\alpha\gamma}{2}\right) = \sqrt{4\delta + 1}. \tag{4}$$

Then the following quantities are totally real algebraic integers:

i) $\cosh\left(\frac{a}{2}\right) = \sqrt{\lambda}$,
ii) $y = 2\sqrt{\lambda(4\delta + 1)}$,
iii) $z = \sqrt{\lambda(4\delta + 1)(\zeta + 4)} + \sqrt{2\delta\zeta\lambda}$.

Proof. For the first item we use the equation (8) of Chapter 1 for the hexagon in

Figure 4.1 and then

$$\begin{aligned}\cosh(a) &= \frac{\cosh\left(\frac{\eta}{2}\right)+\cosh^2\left(\frac{\gamma}{2}\right)}{\sinh^2\left(\frac{\gamma}{2}\right)}\\ &= \frac{\cosh\left(\frac{\eta}{2}\right)+\cosh^2\left(\frac{\gamma}{2}\right)}{\cosh^2\left(\frac{\gamma}{2}\right)-1}\\ &= \frac{2\,|\mathrm{tr}(\eta)|+\mathrm{tr}^2(\gamma)}{\mathrm{tr}^2(\gamma)-4} \qquad (5)\\ &= \frac{4\,\xi+x^2}{x^2-4}.\end{aligned}$$

But $\cosh(a) = 2\cosh^2\left(\frac{a}{2}\right)-1$. Therefore

$$\begin{aligned}\cosh^2\left(\frac{a}{2}\right) &= \frac{1}{2}\left[\cosh(a)+1\right] \qquad (6)\\ &= \frac{1}{2}\left[\frac{4\,\xi+x^2}{x^2-4}+1\right]\\ &= \lambda. \qquad (7)\end{aligned}$$

Since $\zeta = \frac{\mathrm{tr}^2(\gamma)-4}{2}$ defined in (1) was chosen as a unit we have that λ is totally positive algebraic integer, and consequently $\cosh\left(\frac{a}{2}\right)$ is a totally real algebraic integer.

Item $ii)$ follows from the hyperbolic Pythagorean Theorem (3) in Chapter 1, applied to the right–angled triangle depicted in Figure 4.3. Substituting the expressions for $\cosh\left(\frac{\alpha\gamma}{2}\right)$ given in (4) and for $\cosh\left(\frac{a}{2}\right)$ given in (7), we obtain

$$\begin{aligned}y &= 2\cosh\left(\frac{\beta}{2}\right)\\ &= 2\cosh\left(\frac{a}{2}\right)\cosh\left(\frac{\alpha\gamma}{2}\right) \qquad (8)\\ &= 2\sqrt{\lambda(4\delta+1)}.\end{aligned}$$

Finally, the third item is obtained from [2, Theorem 7.38.6]:

$$\frac{1}{2}\,\mathrm{tr}(\gamma\beta) = \cosh\left(\frac{\gamma}{2}\right)\cosh\left(\frac{\beta}{2}\right)+\sinh\left(\frac{\gamma}{2}\right)\sinh\left(\frac{\beta}{2}\right)\cos(\theta),$$

where θ is the angle between γ and β, which we rewrite as

$$z = \sqrt{\lambda(4\delta+1)(\zeta+4)}+2\cos(\theta)\sqrt{\left(\frac{\zeta}{2}\right)(\lambda(4\delta+1)-1)}. \qquad (9)$$

We then apply the trigonometric formula for a right–angled triangle (6) which gives

$$\cos(\theta) = \tanh\left(\frac{b}{2}\right) \coth\left(\frac{\beta}{2}\right).$$

Thus,

$$\begin{aligned}\cos^2(\theta) &= \frac{\cosh^2\left(\frac{\alpha}{2}\gamma\right) - 1}{\cosh^2\left(\frac{\alpha}{2}\gamma\right)} \times \frac{\cosh^2\left(\frac{\beta}{2}\right)}{\cosh^2\left(\frac{\beta}{2}\right) - 1} \\ &= \frac{4\delta\lambda}{\lambda(4\delta+1) - 1}.\end{aligned}$$

Now substituting $\cos(\theta)$ and the items $i)$ and $ii)$ in (9) we obtain the result. □

4.2 Constructing a Dense Family of Semi-Arithmetic Q-Pieces

Let $\mathcal{M}_{1,1}$ be the moduli space of marked hyperbolic surfaces of signature $(1,1)$. The elements of $\mathcal{M}_{1,1}$ will be called *Q-pieces* or *one-holed tori*. We recall from (13) that $\mathcal{M}_{1,1}$ has an analytic structure given by the open set

$$\mathcal{Q} = \{(x,y,z) \mid x,y,z > 2 \text{ and } x^2+y^2+z^2-2xyz < 0,\} \tag{10}$$

where x, y and z denote the traces of the hyperbolic elements induced by the closed geodesics γ, β and η defined in section 4.1. For any list $(x,y,z) \in \mathcal{Q}$, we denote by $S(x,y,z)$ the respective Q-piece.

In this section we will show that the set of parameters which produce subarithmetic surfaces is dense in $\mathcal{M}_{1,1}$.

We begin with a totally real number field K of degree $n \geq 3$. For example, consider $K = \mathbb{Q}(\Theta)$, where Θ is a root of

$$f(x) = x^3 - 3x - 1.$$

This polynomial is irreducible over $\mathbb{Q}$ and has three distinct real roots, as it can be seen from its discriminant:

$$\Delta = -4(-3)^3 - 27(-1)^2 = 108 - 27 = 81 > 0.$$

Therefore, all the roots are real, and K is a totally real cubic field.

Theorem 4.2 *The set of subarithmetic Q-pieces is dense in the moduli space $\mathcal{M}_{1,1}$.*

Proof. Let $S_0 = S(x_0, y_0, z_0)$ be a Q-piece, with $(x_0, y_0, z_0) \in \mathcal{Q}$, and let

$$\Gamma_0 = \langle \gamma_0, \beta_0 \rangle < \mathrm{PSL}(2, \mathbb{R})$$

be its uniformizing Fuchsian group, where

$$x_0 = \mathrm{tr}(\gamma_0), \qquad y_0 = \mathrm{tr}(\beta_0), \qquad z_0 = \mathrm{tr}(\gamma_0\beta_0), \qquad r_0 = \mathrm{tr}([\gamma_0, \beta_0]) = \mathrm{tr}(\eta_0).$$

By equation (11) in Chapter 1, we have

$$x_0^2 + y_0^2 + z_0^2 - x_0 y_0 z_0 - 2 = r_0 < -2.$$

Let $0 \leq \alpha_0 < 1$ be the twist parameter of S_0. Since K is a totally real number field with $[K : \mathbb{Q}] \geq 3$, Corollary 2.4 implies that both $\mathcal{O}_{K,+}$ and $\mathcal{O}^*_{K,+}$ are dense in $\mathbb{R}_{>0}$. We can choose parameters

$$\delta, \xi \in \mathcal{O}_{K,+}, \qquad \zeta \in \mathcal{O}^*_{K,+},$$

such that $r = -2\xi < -2$, $\sqrt{2(\zeta + 2)}$ and $\sqrt{4\delta + 1}$ are arbitrarily close to r_0, x_0 and $\cosh\left(\frac{\alpha_0\gamma_0}{2}\right)$ respectively.

Now we construct a Q-piece S as in Section 4.1 with geodesic γ, twist parameter α, and boundary η such that

$$\cosh\left(\frac{\ell(\gamma)}{2}\right) = \frac{\sqrt{2(\zeta + 2)}}{2}, \qquad \cosh\left(\frac{\ell(\eta)}{2}\right) = \xi, \qquad \cosh\left(\frac{\alpha\ell(\gamma)}{2}\right) = \sqrt{4\delta + 1}.$$

Then $S = S(x, y, z)$, where $x = \sqrt{2(\zeta + 2)}$, and

$$x^2 + y^2 + z^2 - xyz - 2 = r. \tag{11}$$

It remains to show that y and z can be made arbitrarily close to y_0 and z_0, respectively, and that S is a subarithmetic Q-piece.

From equations (5), (6) and (8), we obtain

$$y = \frac{2|r| + x^2}{x^2 - 4} \cosh\left(\frac{\alpha\gamma}{2}\right)$$

which shows that y can be chosen arbitrarily close to y_0. The proximity of z to z_0 follows from the relation (11).

The traces of all elements of $\Gamma = \langle \gamma, \beta \rangle$, the uniformization group of S, are algebraically determined by the triple (x, y, z). Indeed, by Lemma 2.19, for every $\mu \in \Gamma$, the trace $\mathrm{tr}(\mu)$ is an algebraic integer whenever x, y, and z are algebraic integers. Moreover, the invariant trace field of Γ is

$$k\Gamma = \mathbb{Q}\big(x^2, y^2, xyz\big).$$

Therefore, for any choice of parameters ζ, ξ and δ satisfying the above conditions, Lemma 4.1 ensures that the corresponding triple (x, y, z) consists of totally real algebraic integers. It follows from Example 2.23 that the associated Q-piece $S(x, y, z)$ is subaritmetic. ☐

Finally, we briefly indicate a natural continuation of the present work. The denseness result established in this chapter concerns subarithmetic Q–pieces of signature $(1, 1)$. A natural problem is to extend this phenomenon to more general classes of hyperbolic surfaces. In particular, one is led to investigate whether analogous denseness results hold for semi-arithmetic surfaces in higher genus.

This question is part of an ongoing research project, developed in collaboration with Cayo Dória and Gregory Cosac, whose goal is to establish denseness results in a more general setting. While the full picture remains to be understood, the results obtained here provide a first step in this direction and suggest that similar phenomena may persist beyond the case of Q–pieces.

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