Universidade Federal de Goiás (UFG)
Instituto de Matemática e Estatística (IME)
Programa de Pós Graduação em Matemática

Luan Lima da Silva

Piecewise Slow-Fast Systems and Applications

Goiânia
2024

UNIVERSIDADE FEDERAL DE GOIÁS
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA

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LUAN LIMA DA SILVA

Piecewise Slow-Fast Systems and Applications

Sistemas Lento-Rápido por Partes e Aplicações

Dissertação apresentada ao Programa de Pós-Graduação do Instituto de Matemática e Estatística (IME) da Universidade Federal de Goiás (UFG), como requisito parcial para obtenção do título de Mestre em Matemática.

Área de concentração: Sistemas Dinâmicos

Orientador: Prof. Dr. Durval José Tonon

Coorientador: Prof. Dr. Rodrigo Donizete Euzébio

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2024

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Silva, Luan Lima da
Piecewise Slow-Fast Systems and Applications [manuscrito] /
Luan Lima da Silva. - 2024.
111 f.: il.

Orientador: Prof. Dr. Durval José Tonon; co-orientador Dr.
Rodrigo Donizete Euzébio.
Dissertação (Mestrado) - Universidade Federal de Goiás, Instituto
de Matemática e Estatística (IME), Programa de Pós-Graduação em
Matemática, Goiânia, 2024.
Bibliografia.

1. Fast-slow system. 2. Fenichel's Theorem. 3. Discontinuous
systems. 4. Filippov's convention. 5. Sotomayor-Teixeira's
regularization. I. Tonon, Durval José, orient. II. Título.

CDU 51

UNIVERSIDADE FEDERAL DE GOIÁS

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA

ATA DE DEFESA DE DISSERTAÇÃO

Ata nº **5** da sessão de Defesa de Dissertação de **Luan Lima da Silva**, que confere o título de Mestre em **Matemática**, na área de concentração em **Sistemas Dinâmicos**.

Aos **[18/03/2024] décimo oitavo dia do mês de março do ano de dois mil e vinte de quatro**, a partir das **14:00h**, via Web Videoconferência realizou-se a sessão pública de Defesa de Dissertação intitulada **"Piecewise Slow-Fast Systems and Applications"**. Os trabalhos foram instalados pelo Orientador, Professor Doutor **Durval José Tonon - IME/UFG** com a participação dos demais membros da Banca Examinadora: Professor Doutor **Rodrigo Donizete Euzébio - IME/UFG**, membro titular interno; Professor Doutor **Luiz Fernando Goncalves - IME/UFG**, membro titular interno; Professor Doutor **Tiago de Carvalho - DCM/USP,** membro titular externo**.** Durante a arguição os membros da banca **não fizeram** sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Dissertação, tendo sido o candidato **aprovado** pelos seus membros. Proclamados os resultados pelo Professor Doutor **Durval José Tonon - IME/UFG**, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos Membros da Banca Examinadora, aos **[18/03/2024] décimo oitavo dia do mês de março do ano de dois mil e vinte de quatro**.

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Às minhas mães.

Agradecimentos

Agradeço a Deus por me abençoar e permitir que eu alcançasse mais este objetivo em minha vida.

Agradeço às minhas mães: Ádelia, que contribuiu para que eu me tornasse a pessoa que sou; e Genilda, que contribuiu para que eu pudesse realizar meus estudos.

Agradeço ao meu irmão Lucas pelo apoio, por me ouvir nos momentos difíceis e estar sempre disposto a me ajudar.

Agradeço ao corpo docente do Instituto de Matemática e Estatística da Universidade Federal de Goiás, àqueles que contribuíram para minha formação. Em especial, aos professores Romildo Pina e Kaye Silva.

Agradeço ao meu coorientador, o professor Rodrigo Donizete Euzébio, pelas valiosas contribuições para minha formação, pelas orientações, por ser muito atencioso e sempre estar disposto a ajudar. Obrigado por tudo.

Destino um agradecimento especial ao meu orientador, o professor Durval José Tonon. Agradeço por estar ao meu lado desde o começo da minha carreira acadêmica; por contribuir muito para minha formação, sempre dando valiosas orientações e, sobretudo, conselhos que me ajudaram muito a conseguir avançar na academia. É um excelente pesquisador, professor, orientador e, como pessoa, alguém admirável, que sempre terá meu respeito e consideração. Serei eternamente grato por ser meu orientador, pela confiança depositada em mim e pelas inúmeras oportunidades. Muito obrigado por tudo.

À CAPES e ao CNPq, pelo suporte financeiro.

“A matemática é uma filosofia, uma arte, um sonho sem começo nem fim... Cuja essência reside em si mesma, como criação ou como descoberta. É o que nos deixa mais próximos de decifrar os enigmas de Deus.”

Luan Lima

Resumo

Silva, L. L. **Piecewise Slow-Fast Systems and Applications**. Goiânia, 2024. 116p. Dissertação de Mestrado. Programa de Pós Graduação em Matemática, Instituto de Matemática e Estatística (IME), Universidade Federal de Goiás (UFG).

Nesta dissertação será apresentado um estudo voltado aos campos vetoriais suaves por partes (ou descontínuos), com ênfase na convenção de Filippov e no processo de regularização de Sotomayor-Teixeira. Além disso damos uma atenção especial aos sistemas lento-rápido, que são uma classe de campos vetoriais suave a 1-parâmetro definidos em mais de uma escala de tempo e que estabelecem relações com os campos vetoriais descontínuos. Inicialmente são apresentados resultados e definições da teoria clássica de campos vetoriais suave. O objetivo é estabelecer uma conexão natural entre o conjunto dos campos vetoriais suave e o conjunto dos campos vetoriais descontínuos. No desenvolvimento, apresentamos os principais conceitos e resultados associados aos sistemas lento-rápido e aos sistemas suaves por partes. Por fim, é apresentado e discutido um modelo de campo vetorial suave por partes aplicado ao estudo da dinâmica climática.

Palavras–chave

Sistemas Lento-Rápido, Teorema de Fenichel, Sistemas Descontínuos, Conveção de Filippov, Regularização de Sotomayor-Teixeira.

Abstract

Silva, L. L. **Piecewise Slow-Fast Systems and Applications**. Goiânia, 2024. 116p. MSc. Dissertation. Programa de Pós Graduação em Matemática, Instituto de Matemática e Estatística (IME), Universidade Federal de Goiás (UFG).

This dissertation presents a study of piecewise-smooth (or discontinuous) vector fields, with an emphasis on the Filippov's convention and the Sotomayor-Teixeira's regularization process. We also pay special attention to slow-fast systems, which are a class of 1-parameter smooth vector fields defined on more than one time scale and which establish relations with discontinuous vector fields. Initially, results and definitions are presented from the classical theory of smooth vector fields. The aim is to establish a natural connection between the set of smooth vector fields and the set of discontinuous vector fields. Then we present the main concepts and results associated with slow-fast systems and piecewise-smooth systems. Finally, a piecewise-smooth vector field model applied to the study of climate dynamics is presented and discussed.

Keywords

Slow-Fast System, Fenichel's Theorem, Discontinuous Systems, Filippov's Convention, Sotomayor-Teixeira's Regularization.

Contents

Introduction

The subject of this dissertation has its roots in the work carried out by scientists, especially physicists and mathematicians, throughout history. Among these scientists, we can highlight Aristotle (4th century BC), Galileo Galilei (1564 - 1642) and Kepler (1571 - 1630), who dedicated themselves to understanding the dynamics of celestial bodies - how they interact and position themselves in relation to each other as time passes. This problem was responsible for giving rise to various methods of research and study in mathematics, in which we highlight the area of dynamic systems.

The basic object of study in this dissertation is ordinary differential equations. The history of differential equations began with the study of calculus by mathematicians Isaac Newton (1642 - 1727) and Gottfried Wilhelm Leibniz (1646 - 1716). Newton was responsible for introducing the numerous applications of differential equations in the 18th century, with the development of calculus and the description of the basic principles of mechanics. Unlike Newton, who considered variables changing over time, Leibniz studied the variables x and y varying over sequences of infinitely close values, introducing the notation dx and dy, as well as the integral sign (see [12]).

At the end of the 19th century, Henri Poincaré published the work "*Les méthodes nouvelles de la Mécanique Céleste*", where he made valuable contributions by presenting innovative methods for studying the solutions of differential equations or systems of differential equations. Most differential equations cannot be solved analytically. In view of this, Poincaré understood that the most important thing would be to study the global behavior of the solutions; if it were possible to solve the equation explicitly, the solution would have little purpose in describing the global behavior of the orbits. With this, Poincaré argued that it was more important to dedicate oneself to describing the qualitative behavior of the solutions of a differential equation than to search for an explicit expression for them.

The area of dynamical systems, which has historical roots long before those founded by Poincaré, was only properly recognized and gained a place of autonomy in mathematics thanks to the contributions made by him, who presented a completely new, original and authentic way of approaching problems involving differential equations.

This dissertation will present general and specific elements of the theory of piecewise-smooth systems. The main aim of the dissertation is to show how we can treat non-smooth vector fields, precisely in the region of space where the vector field is multivalued (takes more than one value). We will refer to the vector field as the discontinuous vector field and the region where the vector field is not smooth as the discontinuity region. It is important to note that the theory that proposes formalizing the study of piecewise-smooth systems is still under development. Thus, in this scenario there are various perspectives that propose methods for studying the dynamics of a discontinuous vector field.

When we study the phase portrait of a differential equation, we are mainly interested in the behavior of the solutions. It is important to note that the concept of a solution to a differential equation with a discontinuous second member is not universal. It usually obeys a certain convention stipulated a priori or is defined according to the problem. Since we do not have uniqueness of solutions, we need to pay more attention to the analysis of the problem. For convenience, we have chosen Filippov's convention as the basis for our theoretical study, due to its essentially geometric nature. Today, the book of Filippov [5] is unanimously accepted as an important contribution to the theory of dynamical systems.

One of these discontinuous systems, which we find in nature, can be observed if we consider the trajectory of light on surfaces with different refractive indices. For example, if we consider a lake and the air (outside the lake), the transition region from one to the other, i.e. the surface of the lake, can be understood as the Σ discontinuity region. When a beam of light hits the surface of the lake, from the air to the lake or vice versa, there is a change in its trajectory due to the refractive index of the lake or the air. In this phenomenon we observe a dynamic system with two regions: the inside of the lake and its outside. This is a simple example of discontinuous dynamics. Few phenomena in nature have discontinuous dynamics, but it is a very common phenomenon in technology, for example when it comes to studying the interaction of electrical and mechanical circuits (see [1]). There are various practical interests in studying the dynamics of discontinuous vector fields. Some of them are presented throughout this dissertation.

Before dealing with the main subject of the dissertation, which is piecewise-smooth vector fields, Chapter 1 presents classical results from the theory of smooth vector fields (see [14] and [16]). The results presented are essential for a good understanding of the theory and concepts presented in subsequent chapters.

Chapter 2 will present some elements and results of the geometric theory of singular perturbations, with emphasis on Fenichel's Theorem (see [9]). This chapter pays special attention to fast-slow systems, where we present a detailed analysis of how to interpret fast flow and slow flow, followed by several examples.

Chapter 3 presents the main concepts associated with the study of the dynamics of discontinuous vector fields and some results (see [3], [6] and [8]). This chapter presents two methodologies for analyzing discontinuous vector fields: the Filippov's convention and the Utkin's convention. In the theory of discontinuous vector fields presented in this dissertation, we consider the Filippov's convention. The Utkin's convention is presented only to illustrate the existence of other methods for studying discontinuous vector fields. We also present a comparison between the two conventions - their similarities and differences. In this chapter we also introduce the concept of equivalence and topological conjugation for discontinuous vector fields.

Chapter 4 presents discontinuous vector fields as singular perturbation problems obtained from the Sotomayor-Teixeira's regularization process (see [2] and [15]). The Sotomayor-Teixeira regularization process is a methodology for studying discontinuous vector fields, which consists of determining a continuous vector field that approximates the discontinuous vector field. This technique has a limitation because it depends on the choice of a function, called a transition function, to regularize the discontinuous vector field. In this context, an analysis is also presented considering an appropriate change of coordinates applied to the regularized vector field; with the aim of obtaining information on the dynamics of the discontinuous vector field regardless of the choice of transition function. This technique is called polar blow-up. Unlike what we observed in [2], in this dissertation we present an alternative analytical representation for the change of polar coordinates, which allows for a better understanding and visualization of the object of study.

Chapter 5 presents and discusses a piecewise-smooth vector field model applied to the study of climate dynamics (see [21]). In this chapter we present a little of the modeling process of the climate dynamics problem that led us to the mathematical model in question, which is a discontinuous system where each of the smooth systems involved is a fast-slow vector field. The essential point of the mathematical model presented is that there is no consistent theory to be applied to the study of the problem. Even though the model can be partially associated with systems that have been previously studied, it has a configuration that does not allow any results from discontinuous vector field theory or geometric singular perturbation theory to be used. To this end, we present a study of the model using analytical tools, by which we prove that the discontinuous system admits a single stable periodic orbit. We also explore other elements of the dynamics of mathematical model.

CHAPTER 1

Classical Results from Smooth Vector Fields Theory

This chapter presents some of the main results of the qualitative theory of smooth vector fields, together with some analytical results that are essential for the development of this theory and the theory of piecewise-smooth vector fields. The results of this chapter are very important for the study of piecewise-smooth vector fields.

1.1 Preliminary - Some Results of Analysis

In this section we present some results of analysis that are for understanding the theory of dynamical systems. The reader can find the proofs of the results presented in this section in [11].

Definition 1.1.1 *We denote by*

$$Lin(\mathbb{R}^n,\mathbb{R}^m)$$

the linear space of all linear map from $\mathbb{R}^n$ *to* $\mathbb{R}^m$*, thus* $A \in Lin(\mathbb{R}^n,\mathbb{R}^m)$ *if and only if*

$$A:\mathbb{R}^n \to \mathbb{R}^m, \qquad A(\lambda x+y)=\lambda Ax+Ay$$

with $\lambda \in \mathbb{R}$ *and* $x,y \in \mathbb{R}^n$.

Lemma 1.1.1 *For all* $A=(a_{ij}) \in Lin(\mathbb{R}^n,\mathbb{R}^m)$ *and* $h \in \mathbb{R}^n$,

$$|Ah| \leq |A||h|,$$

where $|A|$ *and* $|h|$ *corresponds, respectively, to the usual Euclidean norm in* $\mathbb{R}^{nm}$ *and* $\mathbb{R}^n$.

Let $f:U \to \mathbb{R}^n$ be a map of class C^k $(k \geq 1)$ on the open $U \subset \mathbb{R}^m$ and $p \in U$. In order to establish a clear notation, we assume that $J(f)(p)=J(f)\Big|_p$, which corresponds to the derivative or jacobian of f in the point p.

Theorem 1.1.1 ***(Mean Value Inequality).***
Given $U \subset \mathbb{R}^m$ open, let $f : U \to \mathbb{R}^n$ be differentiable in each point of the open line segment (a,b) and such that its restriction to the closed segment $[a,b] \subset U$ is continuous. If $|J(f)(x)| \leq M$ for all $x \in (a,b)$ then $|f(b) - f(a)| \leq M \cdot |b-a|$.

Corollary 1.1.1 *Let $U \subset \mathbb{R}^m$ open and convex. If $f : U \to \mathbb{R}^n$ is differentiable, with $|J(f)(x)| \leq M$ for all $x \in U$ then f is Lipschitz, with $|f(y) - f(x)| \leq M \cdot |y-x|$ for all $x,y \in U$.*

Theorem 1.1.2 ***(Inverse Function Theorem).***
Let $f : U \to \mathbb{R}^m$, defined on the open $U \subset \mathbb{R}^m$, $f \in C^k(U)$ $(k \geq 1)$, $p \in U$ and $f'(p) : \mathbb{R}^m \to \mathbb{R}^m$ an isomorphism (i.e. $det[J(f)(p)] \neq 0$). Then f is a C^k-diffeomorphism of an open V containing p onto an open W containing $f(p)$. The inverse diffeomorphism $f^{-1} : W \to V$ is of class C^k and its derivative in $f(p)$ is $[f'(p)]^{-1}$, in others words $J(f^{-1})(f(p)) = [J(f)(p)]^{-1}$.

Definition 1.1.2 *(a) An* ***immersion*** *of the open $U \subset \mathbb{R}^m$ into $\mathbb{R}^n$ is a differentiable map $f : U \to \mathbb{R}^n$ such that, for each $x \in U$, the derivative $f'(x) : \mathbb{R}^m \to \mathbb{R}^n$ is an injective linear transformation. Of course $m \leq n$.*

(b) A ***submersion*** *of the open $U \subset \mathbb{R}^m$ onto $\mathbb{R}^n$ is a differentiable map $f : U \to \mathbb{R}^n$ such that, for each $x \in U$, the derivative $f'(x) : \mathbb{R}^m \to \mathbb{R}^n$ is a surjective linear transformation. Of course $m \geq n$.*

Theorem 1.1.3 ***(Local Form of Immersion).***
Let $f : U \to \mathbb{R}^{m+n}$ defined on the open $U \subset \mathbb{R}^m$, $f \in C^k(U)$ $(k \geq 1)$ and $p \in U$. If the derivative $f'(p) : \mathbb{R}^m \to \mathbb{R}^{m+n}$ is injective, exists a C^k-diffeomorphism $h : Z \to V \times W$, of an open $Z \ni f(p)$ in $\mathbb{R}^{m+n}$ onto an open $V \times W \ni (p,0)$ in $\mathbb{R}^m \times \mathbb{R}^n$, such that $(h \circ f)(x) = (x,0)$ for all $x \in V$.

Corollary 1.1.2 *Let $f : U \to \mathbb{R}^{m+n}$ defined on the open $U \subset \mathbb{R}^m$, $f \in C^k(U)$ $(k \geq 1)$ and $p \in U$, with $f'(p) : \mathbb{R}^m \to \mathbb{R}^{m+n}$ injective. There exists an open V, with $p \in V \subset U$, such that f restricted to V onto $f(V)$ is a C^k-diffeomorphism. The inverse C^k-diffeomorphism $f^{-1} : f(V) \to V$ is the restriction of a C^k-map $\xi : Z \to V$, defined in an open $Z \ni f(p)$ in $\mathbb{R}^{n+m}$.*

Theorem 1.1.4 ***(Local Form of Submersion).***
Let $f : U \to \mathbb{R}^n$ defined on the open $U \subset \mathbb{R}^{m+n}$, $f \in C^k(U)$ $(k \geq 1)$ and $p \in U$. If the derivative $f'(p) : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is surjective or, more precisely, if is given a decomposition in direct sum of the type $\mathbb{R}^{m+n} = \mathbb{R}^m \bigoplus \mathbb{R}^n$ such that $p = (p_1, p_2)$ and the partial derivative $\partial_2 f(p) = f'(p)|\mathbb{R}^n : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then there are opens V,W,Z, with $p \in Z \subset \mathbb{R}^{n+m}$, $p_1 \in V \subset \mathbb{R}^m$, $f(p) \in W \subset \mathbb{R}^n$, and a C^k-diffeomorphism $h : V \times W \to Z$ such that $(f \circ h)(x,w) = w$ for all $(x,w) \in V \times W$.

Theorem 1.1.5 ***(Implicit Function Theorem).***
Let $f : U \to \mathbb{R}^n$ defined on the open $U \subset \mathbb{R}^{m+n}$, $f \in C^k(U)$ $(k \geq 1)$ and $p \in U$. If $f'(p) : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is surjective or, more precisely, if is given a decomposition in direct sum of the type $\mathbb{R}^{m+n} = \mathbb{R}^m \bigoplus \mathbb{R}^n$ such that $p = (p_1, p_2)$ and the partial derivative $\partial_2 f(p) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then there are opens V, Z, in which $p_1 \in V \subset \mathbb{R}^m$ and $p \in Z \subset U$ satisfying the following property: for each $x \in V$ there is only one $\xi(x) \in \mathbb{R}^n$ such that $(x, \xi(x)) \in Z$ and $f(x, \xi(x)) = c$ (it is constant). The map $\xi : V \to \mathbb{R}^n$ thus defined is of class C^k and its derivative in the arbitrary point $x \in V$ is given by

$$\xi'(x) = -[\partial_2 f(x, \xi(x))]^{-1} \cdot [\partial_1 f(x, \xi(x))].$$

Note that $f^{-1}(c) \cap Z$ is the graphic of the map $\xi : V \to \mathbb{R}^n$.

The following is a definition that provides a characterization of manifolds in $\mathbb{R}^n$. The definition of manifold, in a more general context, is also presented in Section 1.3.

Definition 1.1.3 *(a) A **parametrization** (or chart) of class C^k $(k \geq 0)$ of a set $V \subset \mathbb{R}^n$ is a homeomorphism $\phi : V_0 \to V$, which is also an immersion of class C^k when $k \geq 1$, defined on the open $V_0 \subset \mathbb{R}^m$. One necessarily has $m \leq n$.*

*(b) A **surface** of dimension m and class C^k in $\mathbb{R}^n$ is a set $M \subset \mathbb{R}^n$ that can be covered by a collection of open $U \subset \mathbb{R}^n$, such that each $V = U \cap M$ admit a parametrization $\phi : V_0 \to V$, of class C^k, defined on an open $V_0 \subset \mathbb{R}^m$. Each of these sets V is an open in M. For each $p \in M$, V is said to be a **parametrized neighborhood** of p.*

The reader should realize that Corollary 1.1.2 is a generalization of the Theorem 1.1.2. There is also a more general result when the function $f : S_1 \to S_2$ is defined between two manifolds (see [18]), which in this scenario we call surfaces. We just point out that if f is of class C^1, and S_1, S_2 are surfaces of class C^k, $k \geq 1$. Given $p \in S_1$, it follows that $f'(p) : T_p S_1 \to T_{f(p)} S_2$ (a linear transformation that goes from the tangent space of S_1 at p into the tangent space of S_2 at $f(p)$). To display the derivative of f in coordinates, we need of charts $\phi : U \to S_1$ in p on S_1 and $\varphi : V \to S_2$ in $f(P)$ on S_2, such that $\phi(u) = p$ and $\varphi(v) = f(p)$. We define $\xi = \varphi^{-1} \circ f \circ \phi : U_p \subset U \to V_p \subset V$, then

$$\xi'(u) = [\varphi^{-1}]'(f \circ \phi(u)) \circ f'(\phi(u)) \circ \phi'(u) = [\varphi^{-1}]'(f(p)) \circ f'(p) \circ \phi'(u)$$

thus,

$$f'(p) = \varphi'(f(p)) \circ \xi'(u) \circ [\phi^{-1}]'(u).$$

Some results presented in this section assume by hypothesis that the function $f : U \to \mathbb{R}^q$ is of class C^k $(k \geq 1)$ on an open $U \subset \mathbb{R}^w$. But, in a more general context, the results are valid when we assume that f is ***strongly differentiable*** at a point $p \in U$. In this case, by not assuming the continuity of the differential of f on some open from U containing p, the differentiability properties of f only hold at the point where it is strongly differentiable. So, in the cases where we guarantee the existence of local diffeomorphisms, we would only have a guarantee of the existence of local homeomorphisms strongly differentiable at point p. Also there are maps which are strongly differentiable at a certain point but which are not differentiable in any neighborhood of that point. For details on the concept of strongly differentiable maps the reader can consult [11].

Theorem 1.1.6 *Let $f : U \to \mathbb{R}^{n-m}$ of class C^k on the $U \subset \mathbb{R}^n$. If $c \in \mathbb{R}^{n-m}$ is a regular value of f (i.e. $f'(x)$ is surjective for all $x \in f^{-1}(c)$) then $f^{-1}(c)$ is a surface of class C^k and dimension m in $\mathbb{R}^n$. In each point $p \in f^{-1}(c)$, the tangent space $T_p[f^{-1}(c)]$ is the kernel of the derivative $f'(p) : \mathbb{R}^n \to \mathbb{R}^{n-m}$.*

Theorem 1.1.7 *All surface of class C^k is locally the graphic of a map of class C^k.*

Corollary 1.1.3 *All surface $M \subset \mathbb{R}^n$, of class C^k, is locally the inverse image of a regular value by a map of class C^k.*

1.2 Flow and Phase Portrait of Vector Fields

Let E be an open subset of $\mathbb{R}^n$. Consider the vector field $F : E \to \mathbb{R}^n$ with $F \in C^k(E)$, $k \geq 1$. To the vector field we associate the system of autonomous differential equations

$$\dot{x} = F(x), \qquad (1\text{-}1)$$

with $x = (x_1, x_2, \ldots, x_n)$ and $x_i = x_i(t)$ for all $i = 1, 2, \ldots, n$. The solutions of (1-1) are differentiable functions $\varphi : I \subset \mathbb{R} \to E$ such that

$$\frac{d\varphi}{dt}(t) = F(\varphi(t)),$$

for all $t \in I$. They are called *integral curves* of F.

A point $p \in E$ is called a *singular point, equilibrium point* or *critical point* of F if $F(p) = \vec{0}$ and it is called a *regular point* of F if $F(p) \neq \vec{0}$. Note that $p \in E$ is a singular point of F if, and only if, $\varphi(t) = p$ is a solution of (1-1).

A integral curve $\varphi : I \to E$ of F is called *maximum solution* if for all integral curve $\psi : J \to E$ such that $I \subset J$ and $\varphi = \psi|I$ (i.e. φ is equals ψ restricted to I) then $I = J$ and, consequentially, $\varphi = \psi$. In this case I is called *maximum interval* of φ.

Theorem 1.2.1 *(a) (Existence and uniqueness of maximum solutions). For each $x \in E$ exist an open interval I_x where is defined the unique maximum solution φ_x of (1-1) such that $\varphi_x(0) = x$.*

(b) (Group property). If $y = \varphi_x(s)$ e $s \in I_x$, then $I_y = I_x - s = \{r - s;\, r \in I_x\}$, $\varphi_y(0) = y$ e $\varphi_y(t) = \varphi_x(t+s)$ for all $t \in I_y$.

(c) (Differentiability in relation to the initial conditions). The set $D = \{(t,x);\, x \in E,\, t \in I_x\}$ is open in $\mathbb{R}^{n+1}$ and $\varphi : D \to \mathbb{R}^n$ given by $\varphi(t,x) = \varphi_x(t)$ is a C^k map. Even more, φ satisfies the equation

$$\frac{\partial^2 \varphi}{\partial t \partial x}(t,x) = J(F)(\varphi(t,x)) \cdot \frac{\partial \varphi}{\partial x}(t,x), \quad \frac{\partial \varphi}{\partial x}(0,x) = id$$

for all $(t,x) \in D$. Here id denotes the identity of $\mathbb{R}^n$ and J the jacobian of F (derivative of F).

The proof of the Theorem 1.2.1 can be found in [16].

Definition 1.2.1 *The map $\varphi : D \to \mathbb{R}^n$ defined on Theorem 1.2.1 is called flow generated to F. In some cases we write $\varphi(t,p) = \varphi_p(t)$.*

Definition 1.2.2 *Let $\varphi_p(t)$ be an integral curve of the vector field F from point p. The set $\gamma_p = \{\varphi_p(t);\, t \in I_p\}$ is called orbit or trajectory of F by the point p.*

Corollary 1.2.1 *If $E = \mathbb{R}^n$ and $|F(x)| < c$ for some $c \in \mathbb{R}$ and for all $x \in \mathbb{R}^n$, so $I_x = \mathbb{R}$ for all $x \in \mathbb{R}^n$.*

The proof of the Corollary 1.2.1 can be found in [16].

Often when we do qualitative analysis of a vector field, we consider the maximum interval of the solution to be $\mathbb{R}$. But analytically, in many cases, the maximum interval of the solution does not coincide with $\mathbb{R}$. From perspective of the qualitative analysis, we can make a rescale in time of the vector field $F \in C^1(\mathbb{R})$, given by

$$G(x) = \frac{F(x)}{1 + |F(x)|}$$

such that orbits of G have the same geometric location of the orbits of F and by the Corollary 1.2.1 the maximum interval of any solution of G coincides with $\mathbb{R}$. Hence, to studying the behavior of orbits from F is the same as to studying the behavior of orbits from G and vice versa.

Theorem 1.2.2 *If $\varphi_x(t)$ is a maximum solution of (1-1) in I_x, then only one of the following alternatives occurs* *(a)* φ_x *is injective;*

(b) $I_x = \mathbb{R}$ *and* $\varphi_x(t)$ *is constant;*

(c) $I_x = \mathbb{R}$ *and* φ_x *is periodic, that is, exist* $\tau > 0$ *such that* $\varphi_x(t+\tau) = \varphi_x(t)$ *for all* $t \in \mathbb{R}$ *and* $\varphi_x(t_1) \neq \varphi_x(t_2)$ *if* $|t_1 - t_2| < \tau$.

The proof of the Theorem 1.2.2 can be found in [16].

Definition 1.2.3 *The open set E, provided by the decomposition onto orbits of F is called phase portrait of F. The orbits are oriented towards the integral curves of the vector field F; the singular points receive the trivial orientation.*

Example 1.2.1 *Consider the vector field* $F(x,y) = (x, -y)$.

Note that the origin is the only singularity of F and is of saddle type. The phase portrait of F is given at the Figure 1.1 and the arrows indicate the orientation of the trajectories. ℵ

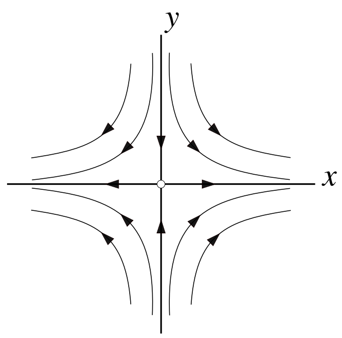

Figure 1.1: *Phase portrait of F.*

Example 1.2.2 *Consider the system*

$$\begin{aligned} \dot{x} &= -y + x(\mu - x^2 - y^2), \\ \dot{y} &= \ \ x + y(\mu - x^2 - y^2), \end{aligned} \tag{1-2}$$

with $\mu \in \mathbb{R}$*. Denote by F the vector field associated with* (1-2).

Note that the only singularity of F is the origin. Analyzing F in polar coordinates we obtain its phase portrait, given at the Figure 1.2. ℵ

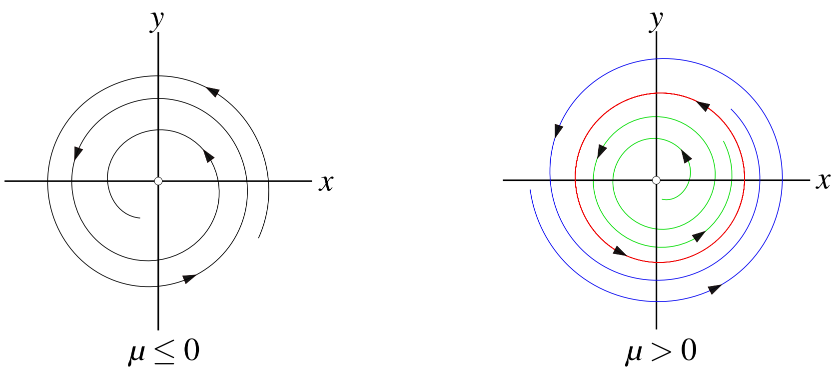

Figure 1.2: *Phase portrait of F.*

1.3 Smooth Vector Fields on Manifolds

We will present some definitions and results about manifolds and smooth vector fields on manifolds.

Definition 1.3.1 *A topological space M is locally Euclidean of dimension m if every point p in M has a neighborhood U such that there is homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a chart, U a coordinate neighborhood, and ϕ a coordinate map on U.*

Definition 1.3.2 *A topological manifold is a topological, Hausdorff, second countable, locally Euclidean space. It is said to be of dimension m if it is locally Euclidean of dimension m.*

For the reader to better understand the topological concepts involved in Definition 1.3.2, see [13].

Definition 1.3.3 *Two charts $(U, \phi : U \to \mathbb{R}^n)$, $(V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are C^r-compatible if the two maps*

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V),$$
$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V),$$

are C^r.

Definition 1.3.4 *A C^r atlas on a locally Euclidean space M is a collection $\mathfrak{I} = \{(U_\alpha, \phi_\alpha) : \alpha \in I,$ where I is a set of indices$\}$ of C^r-compatibles charts that cover M, that is, $M = \bigcup_{\alpha \in I} U_\alpha$.*

Definition 1.3.5 *An atlas $\Im$ on a locally Euclidean space is said to be maximal if it is not contained in a larger atlas; in other words, if $\wp$ is any other atlas containing $\Im$, then $\wp = \Im$. A C^r manifold is a topological manifold M together with a C^r maximal atlas. We said that the maximal atlas is the differential structure on M.*

In particular any surface $M \subset \mathbb{R}^n$ of dimension m and class C^k, according to the Definition 1.1.3 (item b), is a C^rmanifold of dimension m. The next example shows some types of manifolds.

Example 1.3.1 *Some manifolds.*

(a) $\mathbb{R}^n$ together with the usual topology (coming from the Euclidean metric) is a C^∞ $n-$manifold. Any open subset of $\mathbb{R}^n$ is also a C^∞ $n-$manifold.

(b) Let $A \subset \mathbb{R}^n$ be an open and $f : A \to \mathbb{R}^m$ a function, with $f \in C^r(A)$. The set $M = \{(x, f(x)) \in A \times \mathbb{R}^m\}$ is a C^r $n-$manifold. In particular, if $f(x) = |x|$, $A = \mathbb{R}$, so M is a C^0 curve; if $f(x) = |x|^2$, $A = \mathbb{R}^2$, then M is a C^∞ surface of dimension 2.

(c) The sphere and the torus are examples of C^∞ orientable compact surfaces without boundary. The 2-dimensional real projective space and the Klein bottle are examples of C^∞ compact surfaces no orientable without boundary. $\aleph$

Definition 1.3.6 *Given $p \in M$, M a C^r manifold (or smooth manifold), $r \geq 1$, the tangent space to M in p is defined by*

$$T_pM = \{\alpha'(0) |\ \alpha : (-\varepsilon, \varepsilon) \to M \textit{ is a differentiable curve and } \alpha(0) = p\}.$$

The tangent bundle of M is defined by

$$TM = \bigcup_{p \in M} T_pM.$$

Definition 1.3.7 *Let M be a smooth manifold. A C^r vector field on M is a C^r map $F : M \to TM$, such that $F(p) \in T_pM$ for all $p \in M$.*

For more information about manifolds the reader can consult [18].

The definitions presented at the Section 1.2 naturally extend to vector fields in any manifold, as do many results. If $f : M \to N$ is a diffeomorphism of class C^{r+1}, where M and N are smooth manifolds, and F a vector field of class C^r on M, so $G = f_*F$ defined by $G(q) = df_p \cdot F(p)$ with $q = f(p)$ is a vector field of class C^r on N, as $f_*F = df \circ F \circ f^{-1}$. If $\alpha : I \to M$ is an integral curve of F, then $f \circ \alpha : I \to N$ is an integral curve of G, and vice versa (this discussion is resumed in the Example 1.4.1). In particular, f takes trajectories from F in trajectories from G. Hence, if $\phi : U \to U_0 \subset \mathbb{R}^m$

is a local chart of M, $G = \phi_* F$ is a C^r vector field on U_0; we said that G is an expression from F at the local chart (U, ϕ). With these considerations, the local theorems about existence, uniqueness and differentiability of solutions extend to vector fields on manifolds. Whenever a compact manifold is said, it is implied that it does not have a boundary, unless otherwise stated. Below are presented some results for vector fields on compact manifolds.

Teorema 1.3.1 *Let M be a compact manifold and F be a C^r vector field on M, $r \geq 1$. There exists in M a global flow of class C^r for F. In others words, there exist a map $\varphi : \mathbb{R} \times M \to M$ such that $\varphi(0, p) = p$ and $\frac{d\varphi}{dt}(t, p) = F(\varphi(t, p))$. Furthermore, for each $t \in \mathbb{R}$ the map $f_t : M \to M$, $f_t(p) = \varphi(t, p)$, is a diffeomorphism of class C^r such that $f_0(p) = p$ and $f_{t+s}(p) = (f_t \circ f_s)(p)$ for all $p \in M$.*

The proof of the Theorem 1.3.1 can be found in [14].

1.4 Equivalence and Conjugation of Vector Fields and Local Structure of Hyperbolic Singular Points

Next, we introduce several notions of equivalence between two vector fields, which allow us to compare their phase portraits.

Definition 1.4.1 *Let F_1, F_2 be vector fields defined in the open sets of $\mathbb{R}^n$, E_1, E_2 respectively. F_1 is said to be topologically equivalent (respectively C^r-equivalent) to F_2 when there exists a homeomorphism (respectively a C^r-diffeomorphism) $h : E_1 \to E_2$ which takes an orbit of F_1 into an orbit of F_2 preserving the orientation. More precisely, let $p \in E_1$ and $\gamma_1(p)$ be the oriented orbit of F_1 passing through p; then $h(\gamma_1(p))$ is the oriented orbit $\gamma_2(h(p))$ of F_2 passing through $h(p)$.*

Definition 1.4.2 *Let $\varphi_1 : D_1 \to \mathbb{R}^n$ and $\varphi_2 : D_2 \to \mathbb{R}^n$ be the flows generated by the vector fields $F_1 : E_1 \to \mathbb{R}^n$ and $F_2 : E_2 \to \mathbb{R}^n$ respectively. F_1 is said to be topologically conjugated (respectively C^r-conjugated) to F_2 when there exists a homeomorphis (respectively a C^r-diffeomorphism) $h : E_1 \to E_2$ such that $h(\varphi_1(t, x)) = \varphi_2(t, h(x))$ for all $(t, x) \in D_1$.*

In this case, we necessarily have $I_1(x) = I_2(h(x))$, where $I_1(x)$ and $I_2(h(x))$ denote the maximum intervals of the respective maximum solutions. From the Definition 1.4.2 the homeomorphism h is called topological conjugation (respectively C^r-conjugation) between F_1 and F_2. The conjugation relation is also an equivalence relation between vector fields defined in opens of $\mathbb{R}^n$. It is clear that every conjugation is an equivalence. An equivalence h between F_1 and F_2 takes singular point in singular point and periodic orbit in periodic orbit. If h is a conjugation, the period of the periodic orbits is also preserved.

Lemma 1.4.1 *Let $F_1 : E_1 \to \mathbb{R}^n$ and $F_2 : E_2 \to \mathbb{R}^n$ be vector fields C^k and $h : E_1 \to E_2$ a C^k-diffeomorphism. Then h is a conjugation between F_1 and F_2 if, and only if,*

$$J(h)(p)\cdot F_1(p) = F_2(h(p)), \quad \forall p \in E_1. \tag{1-3}$$

Proof: Let $\varphi_1 : D_1 \to E_1$ and $\varphi_2 : D_2 \to E_2$ the flows of F_1 and F_2, respectively. We suppose that h satisfies (1-3). Given $p \in E_1$, let $\psi = h(\varphi_1(t,p)), t \in I_1(p)$. We have that

$$\psi'(t) = J(h)\Big|_{\varphi_1(t,p)} \cdot \frac{d\varphi_1}{dt}(t,p) = J(h)\Big|_{\varphi_1(t,p)} \cdot F_1(\varphi_1(t,p)) = F_2(h(\varphi_1(t,p))) = F_2(\psi(t)).$$

Hence $h(\varphi_1(t,p)) = \varphi_2(t,h(p))$, that is, h is a topological conjugation between F_1 and F_2.

Reciprocally, we suppose that h is a C^r-conjugation. Given $p \in E_1$, we have

$$h(\varphi_1(t,p)) = \varphi_2(t,h(p)), \quad t \in I_1(p).$$

Thus,

$$\frac{d(h\circ\varphi_1)}{dt}(t,p) = \frac{d\varphi_2}{dt}(t,h(p)) \;\Rightarrow\; J(h)\Big|_{\varphi_1(t,p)} \cdot \frac{d\varphi_1}{dt}(t,p) = F_2(\varphi_2(t,h(p))).$$

When $t = 0$, we have

$$J(h)\Big|_{\varphi_1(0,p)} \cdot \frac{d\varphi_1}{dt}(0,p) = F_2(\varphi_2(0,h(p))) \;\Rightarrow\; J(h)(p)\cdot F_1(p) = F_2(h(p)). \qquad \blacksquare$$

Example 1.4.1 *(Changing variables). Constructing a topological conjugation for a given vector field.*

The conjugation between vector fields is a very useful property for studying the behavior of orbits of this vector fields. In some cases it is simpler to study the topological conjugate of a given vector field than the vector field itself. In particular, if F is a vector field, $F \in C^1(U)$, where U is an open of $\mathbb{R}^2$ and $h : U \to V$ is a C^2-diffeomorphism, we can define a vector field $G \in C^1(V)$, from F and h, in such a way that G and F are topologically conjugated and h is the topological conjunction between them. Let is define the vector field G.

From the function h, we have that

$$(u,v) = h(x,y) \;\Leftrightarrow\; h^{-1}(u,v) = (x,y).$$

Derivating with respect to the variable t,

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = J(h)\Big|_{(x,y)} \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J(h)\Big|_{(x,y)} \cdot F(x,y) = J(h)\Big|_{h^{-1}(u,v)} \cdot (F \circ h^{-1})(u,v).$$

By the Inverse map Theorem 1.1.2, we can write

$$J(h)\Big|_{h^{-1}(u,v)} = \left[J(h^{-1})\Big|_{(u,v)}\right]^{-1}.$$

Thus,

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \left[J(h^{-1})\Big|_{(u,v)}\right]^{-1} \cdot (F \circ h^{-1})(u,v).$$

Define

$$G(u,v) = \left[J(h^{-1})\Big|_{(u,v)}\right]^{-1} \cdot (F \circ h^{-1})(u,v).$$

Having defined the vector field G, see that if α is an integral curve of F then $\beta(t) = (h \circ \alpha)(t)$ is an integral curve of G. In fact,

$$\begin{aligned} \beta'(t) &= J(h)\Big|_{\alpha(t)} \cdot \alpha'(t) = J(h)\Big|_{(h^{-1}\circ\beta)(t)} \cdot [(F \circ h^{-1}) \circ \beta](t) \\ &= J(h)\Big|_{h^{-1}(\beta(t))} \cdot (F \circ h^{-1})(\beta(t)) = (G \circ \beta)(t) = G(\beta(t)). \end{aligned}$$

Therefore β is an integral curve of G. Conversely, if β is an integral curve of G then $\alpha(t) = (h^{-1} \circ \beta)(t)$ is an integral curve of F. In fact,

$$\begin{aligned} \alpha'(t) &= J(h^{-1})\Big|_{\beta(t)} \cdot \beta'(t) = J(h^{-1})\Big|_{\beta(t)} \cdot (G \circ \beta)(t) \\ &= \left[J(h^{-1})\Big|_{\beta(t)}\right] \cdot \left[J(h^{-1})\Big|_{\beta(t)}\right]^{-1} \cdot (F \circ h^{-1})(\beta(t)) \\ &= id \cdot [F \circ (h^{-1} \circ \beta)](t) = [F \circ \alpha](t) = F(\alpha(t)). \end{aligned}$$

Therefore α is an integral curve of F. ℵ

The Example 1.4.1 can be naturally generalized to $\mathbb{R}^n$ and we say that h is a ***change of variables*** that induces the vector field G from the vector field F. This example provides us with a way to construct a topological conjugation of a vector field using diffeomorphism. Furthermore, it is a diffeomorphism with very good properties and which allows us, in many situations, to study the behavior orbits of the original vector field without knowing its integral curves.

Let p be a regular point of a vector field F of class C^k, $k \geq 1$. From the Tubular Flow Theorem (see [16]), we know that there is a diffeomorphism of class C^k that conjugates F, in a neighborhood of p with the constant vector field $G = (1,0,\ldots,0)$.

Consequently, two vector fields F and H are locally C^kconjugate around regular points. Because of this observation, we can consider local qualitative knowledge of the orbits of a vector field around regular points to be satisfactory, given that there is only one local differentiable conjugation class.

The reader should note that, by the Lemma 1.4.1, the reciprocal of Example 1.4.1 is also true. In fact, if we have two smooth vector fields $F \in C^1(U)$ and $G \in C^1(V)$ with their respective flows $\varphi_1(t,p)$ and $\varphi_2(t,q)$, they are C^r-conjugated if there exists a C^r homeomorphism $h : U \to V$ such that $h(\varphi_1(t,p)) = \varphi_2(t,h(p))$. In this case, it can be seen that

$$G(q) = J(h)\Big|_{h^{-1}(q)} \cdot (F \circ h^{-1})(q),$$

for all $q \in V \subset \mathbb{R}^n$. So, h is just a change of variables.

On the other hand, if p is a singular point of F we can have several different situations. Even in linear systems there are already several different classes of differentiable conjugation. In $\mathbb{R}^2$ we have the saddle, the center, the node, etc.

Definition 1.4.3 *A singular point p of a vector field F of class C^k, $k \geq 1$, is called hyperbolic if all eigenvalues of $J(F)(p)$ have real part non-zero.*

Theorem 1.4.1 ***(The Hartman-Grobman Theorem).***
Let $F : E \to \mathbb{R}^n$ be a vector field of class C^1 and p a hyperbolic singular point. There are neighborhoods U of p in E and V of 0 in $\mathbb{R}^n$ such that F restricted in U is topologically conjugate with $J(F)(p)$ in V.

For more details about the Theorem 1.4.1 see [16].

Example 1.4.2 *Let F be a vector field given by*

$$\begin{aligned} \dot{x} &= ax + by + P(x,y), \\ \dot{y} &= cx + dy + Q(x,y), \end{aligned} \tag{1-4}$$

where $a,b,c,d \in \mathbb{R}$ and P,Q are polynomials of deg n,k respectively, with $n,k > 1$, such that

$$\begin{aligned} P(x,y) &= a_{20}x^2 + a_{11}xy + a_{02}y^2 + \ldots + a_{0n}y^n, \\ Q(x,y) &= b_{20}x^2 + b_{11}xy + b_{02}y^2 + \ldots + b_{0k}y^k. \end{aligned}$$

Note that the origin is a singular point of F. We have that

$$J(F)(x,y) = J(F)\Big|_{(x,y)} = \begin{pmatrix} a + P_x(x,y) & b + P_y(x,y) \\ c + Q_x(x,y) & d + Q_y(x,y) \end{pmatrix}.$$

Thus,

$$J(F)\Big|_{(0,0)} = J(F)\Big|_{\vec{0}} = \begin{pmatrix} a+P_x(0,0) & b+P_y(0,0) \\ c+Q_x(0,0) & d+Q_y(0,0) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If the eigenvalues of $J(F)(0,0)$ have a non-zero real part, then by Theorem 1.4.1 there is a neighborhood of the origin in which F and $G(x,y) = (ax+by, cx+dy)$ are topologically conjugate . In other words, in the neighborhood of the origin, when we observe the phase portrait of F, it is as if we were observing the phase portrait of G. So if we know the local topology of the phase portrait of G near the origin, we also know the local topology of the phase portrait of F near the origin.

For example, if $b=c=0$ and $d=-a$, with $a \neq 0$, then locally the origin of (1-4) is a saddle point, since the origin is a saddle point of G. ℵ

The Theorem 1.4.2 presented below is a complement to the Hartman-Grobman Theorem.

Theorem 1.4.2 ***(The Stable and Unstable Manifold Theorem).***
Consider the vector field $F: E \to \mathbb{R}^n$, where E is an open subset of $\mathbb{R}^n$ and $F \in C^k(E)$, $k \geq 1$. Suppose that $x_0 \in E$ is a hyperbolic point for F, i.e, the associated linearized system

$$\dot{x} = \left[J(F)\Big|_{x_0} \right] x$$

has a matrix $J(F)(x_0)$ with r negative-real-part and $n-r$ positive-real-part eigenvalues with corresponding eigenspaces E^s and E^u. Then there exists a neighborhood U of x_0 containing locally stable and unstable manifolds $W^s(x_0)$ and $W^u(x_0)$ given by

$$\begin{aligned} W^s(x_0) &= \{x \in U : \varphi_t(x) \to x_0 \text{ as } t \to \infty \text{ and } \varphi_t(x) \in U \text{ for all } t \geq 0\}, \\ W^u(x_0) &= \{x \in U : \varphi_t(x) \to x_0 \text{ as } t \to -\infty \text{ and } \varphi_t(x) \in U \text{ for all } t \leq 0\}. \end{aligned}$$

Furthermore, $W^s(X_0)$ and $W^u(x_0)$ are tangent to E^s and E^u at x_0, that is, we have $T_{x_0}(W^s(x_0)) = E^s$ and $T_{x_0}(W^u(x_0)) = E^u$. The manifolds $W^s(x_0)$ and $W^u(x_0)$ are at least as smooth as F.

For more details about the Theorem 1.4.2 see [9].

Example 1.4.3 *Let F be a vector field on $\mathbb{R}^3$ given by*

$$\begin{aligned} \dot{x} &= x + x^2 + y^2 + z^2, \\ \dot{y} &= y + x^2 + y^2 + z^2, \\ \dot{z} &= -z + x^2 + y^2 + z^2. \end{aligned}$$

He have to

$$J(F)\Big|_{\vec{0}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

By the Theorem 1.4.1, the vector field F is conjugate topologically to linear vector field $G(x,y,z) = (x,y,-z)$ in an neighborhood of the origin. From the Theorem 1.4.2, the Figure 1.3 show a sketch of $W^s(\vec{0}), W^u(\vec{0}), E^s$ and E^u associated to F at $\vec{0}$. ℵ

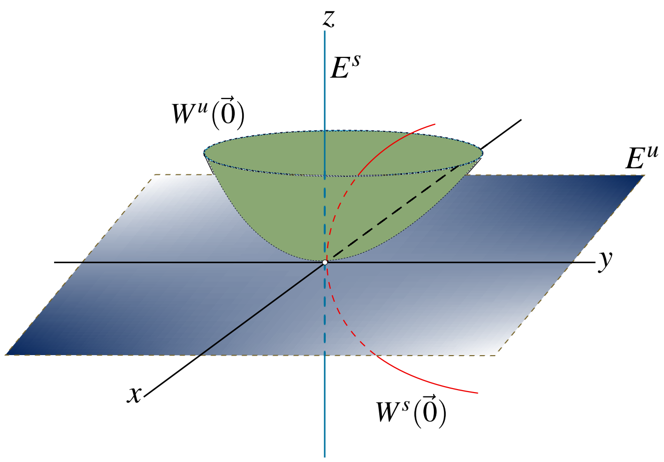

Figure 1.3: *Unstable and stable manifold of F at $\vec{0}$.*

1.5 The Poincaré Map and the Limit Cycles in the Plane

The Poincaré map or map associated with a closed orbit γ of a vector field is a diffeomorphism π, which we will define below. This transformation describes the behavior of the vector field in a neighborhood of γ.

Let $\gamma = \{\varphi(t,p);\ 0 \leq t \leq \tau_0\}$ be a periodic orbit of period τ_0 of a vector field F of class C^k, $k \geq 1$, defined on $E \subset \mathbb{R}^n$. Let Σ be a transversal section of F at p. Because of the continuity of the flow φ of F, for every point $q \in \Sigma$ near p the trajectory $\varphi(t,q)$ remains close to γ, with t in a pre-fixed compact interval, for example, $[0, 2\tau_0]$. We define $\pi(q)$ as the first point where this orbit, starting from q, intersects the section again. Let Σ_0 be the domain of π. Naturally $p \in \Sigma_0$ and $\pi(p) = p$.

Many properties of the phase portrait of F near γ are reflected in π and vice versa. For example, the periodic orbits of F near γ correspond to the periodic points of π, which are points $q \in \Sigma_0$ for which $\pi^n(q) = q$ for some integer $n \geq 1$. The asymptotic behavior of the orbits of F near γ is also described by π. Thus, $\lim\limits_{n\to\infty} \pi^n(q) = p$ implies $\lim\limits_{t\to\infty} d(\varphi(t,q),\gamma) = 0$, where $d(\varphi(t,q),\gamma) = \inf\{|\varphi(t,q) - r|;\ r \in \gamma\}$.

As a consequence of the Tubular Flow Theorem, we can show that π is a diffeomorphism with the same differentiability class as F. Displaying the map π is not an easy job, but we can understand many properties of this map without having to display it.

The section Σ taken above is a $(n-1)$-dimensional differentiable submanifold of the open $E \subset \mathbb{R}^n$. We can assume that the manifold that appears here is a disk or affine subspace of $\mathbb{R}^n$.

Definition 1.5.1 *Let E be an open in $\mathbb{R}^2$ and $F : E \to \mathbb{R}^2$ a vector field of class C^1. A periodic orbit γ of F is called limit cycle if exists a neighborhood V of γ such that γ is the unique closed orbit of F that intersects V.*

Proposition 1.5.1 *With the notations from the previous definition, there are only the following types of limit cycles (reducing V if necessary).*

(a) Stable, when $\lim_{t\to\infty} d(\varphi(t,q),\gamma) = 0$ for all $q \in V$;

(b) Unstable, when $\lim_{t\to-\infty} d(\varphi(t,q),\gamma) = 0$ for all $q \in V$;

(c) Semi-stable, when $\lim_{t\to\infty} d(\varphi(t,q),\gamma) = 0$ for all $q \in V \cap Ext(\gamma)$; and $\lim_{t\to-\infty} d(\varphi(t,q),\gamma) = 0$ for all $q \in V \cap Int(\gamma)$, or the opposite.

The proof of the Proposition 1.5.1 can be found in [16].

Remark 1.5.1 *Using the notations from the Proposition 1.5.1, we have that γ is a limit cycle if and only if p is an isolated fixed point of π. Also,*

(a) γ is stable if, and only if, $|\pi(x) - p| < |x - p|$ for all $x \neq p$ near of p;

(b) γ is unstable if, and only if, $|\pi(x) - p| > |x - p|$ for all $x \neq p$ near of p;

(c) γ is semi-stable if, and only if, $|\pi(x) - p| < |x - p|$ for all $x \in \Sigma \cap Ext(\gamma)$ near of p and $|\pi(x) - p| > |x - p|$ for all $x \in \Sigma \cap Int(\gamma)$ near of p, or the opposite.

Consider the map $d : \Sigma_0 \to \mathbb{R}^n$ by

$$d(x) = \pi(x) - p, \tag{1-5}$$

where $\varphi(t,p)$ is a periodic orbit of F. In particular, if $|J(\pi)(p)| < 1$, we can apply Mean Value Theorem and conclude that γ is stable, on the other hand, γ is unstable if $|J(\pi)(p)| > 1$. Let us show the first case. From (1-5) follows that

$$J(d)\Big|_q = J(\pi)\Big|_q - \vec{0} = J(\pi)\Big|_q.$$

We assume that $|\pi'(p)| < 1$ and know that d is a diffeomorphism of class C^k onto its image, since F of class C^k implies at π be a diffeomorphism of class C^k. Since F is of class C^k, $k \geq 1$, it follows that $J(d)$ is continuous in Σ_0. Defines $\eta : \Sigma_0 \to \mathbb{R}$ by

$$\eta(x) = |J(d)(x)| = |J(\pi)(x)|.$$

Since η is continuous and $\eta(p) = |J(\pi)(p)| < 1$, it follows that exists a neighborhood V of p in Σ_0 such that $\eta(x) < 1$ for all $x \in V$, in fact, there is $0 < \alpha < 1$ such that the inverse image of η on $(\eta(p) - \alpha, \eta(p) + \alpha) \subset (-1, 1)$ is an open of Σ_0 that contain p. This ensures the existence of V, furthermore we can assume also that V is a convex set. Hence, $|J(d)(x)| < 1$ for all $x \in V$. By the Mean Value Inequality 1.1.1 follows that

$$\frac{|d(x) - d(y)|}{|x - y|} < 1,$$

for all $x, y \in V$. If $y = p$ we have

$$\frac{|d(x) - d(p)|}{|x - p|} = \frac{|\pi(x) - p|}{|x - p|} < 1.$$

Then,

$$|\pi(x) - p| < |x - p|,$$

for all $x \in V$. Therefore from the Remark 1.5.1 follows that γ is stable. The other case is analogous.

Theorem 1.5.1 *(Derivative of the Poicaré map) Let $E \subset \mathbb{R}^2$ be an open and $F = (P, Q) : E \to \mathbb{R}^2$ a vector field of class C^1. Let γ be a periodic orbit of F with period T and $\pi : \Sigma_0 \to \Sigma$ the Poicaré map in a transversal section Σ at $p \in \gamma$. Then*

$$\pi'(p) := |J(\pi)(p)| = \exp\left[\int_0^T \nabla \cdot F(\varphi(t, p))dt\right], \tag{1-6}$$

where $\nabla \cdot F(x, y) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$. In particular, if $\int_0^T \nabla \cdot F(\varphi(t, p))dt < 0$ then γ is stable and if $\int_0^T \nabla \cdot F(\varphi(t, p))dt > 0$, γ is unstable.

The proof of the Theorem 1.5.1 can be found in [16]. Note that the previous theorem is only true for planar vector fields.

Example 1.5.1 *Let F be a vector field given by*

$$\begin{aligned} \dot{x} &= \ \ y + x(1 - x^2 - y^2), \\ \dot{y} &= -x + y(1 - x^2 - y^2). \end{aligned} \tag{1-7}$$

Let us prove that F has an unique periodic orbit γ, find a Poicaré map π associated to γ and show that $\pi'(p) \neq 0$, whatever it is $p \in \mathbb{R}^2$ such that $|p| = 1$.

Let us express F at polar coordinates. We have that

$$\begin{aligned} \dot{r} &= \dot{x}\cos\theta + \dot{y}\sin\theta, \\ r\dot{\theta} &= \dot{y}\cos\theta - \dot{x}\sin\theta. \end{aligned}$$

Thus, we can write (1-7) as

$$\begin{aligned} \dot{r} &= r(1-r^2), \\ \dot{\theta} &= \quad -1. \end{aligned} \tag{1-8}$$

From the system (1-8), it follows that $\dot{r} > 0$ if $r < 1$, $\dot{r} < 0$ if $r > 1$ and $\dot{r} = 0$ if $r = 1$. Thus, F has an unique periodic orbit, the circle of radius 1 centered at the origin. Given the sign of $\dot{r}$ if we suppose that exists other periodic orbit let is get to a contradiction. The sign of $\dot{\theta}$ indicate that the orbits of F are oriented clockwise.

We have that

$$\frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}} = r(r^2-1).$$

Thus,

$$\int \frac{1}{r(r^2-1)}dr = \int d\theta \;\Rightarrow\; \frac{1}{2}\ln\left(\frac{|1-r^2|}{r^2}\right) = \theta + k, \tag{1-9}$$

where k is a real constant. Let is consider $\Sigma_0 = \{(r\cos\alpha, r\sin\alpha) \in \mathbb{R}^2;\ 1-\varepsilon < r < 1+\varepsilon,$ where α is a real constant between 0 and $2\pi,$ and $\varepsilon > 0$ small$\}$. Note that Σ_0 is a line segment orthogonal to $\mathbb{S}^1$ at $p = (\cos\alpha, \sin\alpha)$. Let is take $a \in \Sigma_0$ and define for (1-9), $r(\alpha) = a$ (initial condition). We can conclude from (1-9) that

$$r(\theta) = \frac{ae^{-(\theta-\alpha)}}{\sqrt{e^{-2(\theta-\alpha)}a^2 - a^2 + 1}}.$$

The Poicaré map $\pi : \Sigma_0 \to \Sigma$ of F with respect to $\mathbb{S}^1$ at p is given by

$$\pi(a) = r(-2\pi+\alpha) = \frac{ae^{2\pi}}{\sqrt{e^{4\pi}a^2 - a^2 + 1}}.$$

Note that Poincaré map is obtained applying r in $\theta = -2\pi + \alpha$, because from the initial condition the first return is obtained by advancing -2π in polar coordinates, negative because the orbits are oriented clockwise.

We have that

$$\pi'(a) = \frac{(1-e^{4\pi})[\pi(a)]^3 + \pi(a)e^{4\pi}}{ae^{4\pi}}.$$

If $a = 1$ then $\pi(a) = 1$, thus

$$\pi'(1) = e^{-4\pi} < 1.$$

Therefore $\mathbb{S}^1$ is a stable orbit of F.

Using the Theorem 1.5.1, we have that

$$\nabla \cdot F(x,y) = 2 - 4(x^2 + y^2).$$

$\mathbb{S}^1$ can be parameterized by $\gamma(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$. Thus,

$$\pi'(p) = \exp\left[\int_0^{2\pi} \nabla \cdot F(\gamma(t))dt\right] = \exp\left[\int_0^{2\pi} -2dt\right] = e^{-4\pi}. \qquad \aleph$$

1.6 The Poincaré-Bendixson's Theorem

The proof of the results presented in this section can be found in [16].

Let E be an open subset of $\mathbb{R}^n$ and $F : E \to \mathbb{R}^n$ a vector field of class C^k, $k \ge 1$. Let $\varphi(t) = \varphi(t,p)$ be the integral curve of F passing through p, defined in its maximum interval $I = (a,b)$. If $b = \infty$, we define the set

$$\omega(p) = \{x \in E;\ \exists\ (t_n) \text{ with } t_n \to \infty \text{ and } \varphi(t_n) \to x, \text{ when } n \to \infty\}.$$

Analogously, if $a = -\infty$, we define the set

$$\alpha(p) = \{x \in E;\ \exists\ (t_n) \text{ with } t_n \to -\infty \text{ and } \varphi(t_n) \to x, \text{ when } n \to \infty\}.$$

The sets $\omega(p)$ and $\alpha(p)$ are called, respectively, of $\omega-$limit set and $\alpha-$limit set of p.

Remark 1.6.1 *(a) If p is a singular point of F, then whatever the point p, $\alpha(p) = \omega(p) = \{p\}$, since in this case $\varphi(t) = p$, for all $t \in \mathbb{R}$.*

(b) If γ_p is the orbit of F by the point p and $q \in \gamma_p$, then $\omega(p) = \omega(p)$. In fact, if $q \in \gamma_p$, exists $c \in \mathbb{R}$ such that $\varphi(t,p) = \varphi(t+c,q)$. Analogously, $\alpha(p) = \alpha(q)$.

(c) Let $\varphi(t) = \varphi(t,p)$ be the integral curve of F by the point p and $\psi(t) = \psi(t,p)$ the integral curve of $-F$ by the point p, then $\varphi(t) = \psi(-t)$. Consequently the $\omega-$limit of φ is equal to $\alpha-$limit of ψ, and vice versa.

Theorem 1.6.1 ***(Topological Structure of the Limit Sets).***
Let $F : E \to \mathbb{R}^n$ a vector field of class C^k, $k \ge 1$, defined in an open $E \subset \mathbb{R}^n$ and be $\gamma^+ = \{\varphi(t,p);\ t \ge 0\}$ (respectively, $\gamma^- = \{\varphi(t,p);\ t \le 0\}$) the positive semi-orbit (respectively, the negative semi-orbit) of the vector field F by the point p. If γ^+ (respectively γ^-) is contained in a compact subset $K \subset E$, then

(a) $\omega(p) \neq \emptyset$ *(respectively,* $\alpha(p)$*);*
(b) $\omega(p)$ *is compact (respectively,* $\alpha(p)$*);*

(c) $\omega(p)$ *is invariant by* F *(respectively,* $\alpha(p)$*), that is, if* $q \in \omega(p)$ *then the orbit of* F *by* q *is contained in* $\omega(p)$*;*

(d) $\omega(p)$ *is connected (respectively,* $\alpha(p)$*).*

The Theorem 1.6.1 is also true when E is a compact manifold and F is a vector field of class C^k on E, $k \geq 1$.

Corollary 1.6.1 *Under the conditions of the previous theorem, if* $q \in \omega(p)$*, then the integral curve of* F *by the point* q *is defined for all* $t \in \mathbb{R}$*.*

Theorem 1.6.2 ***(The Poincaré-Bendixson's Theorem).***
Let $F : E \to \mathbb{R}^2$ *a vector field of class* C^k*,* $k \geq 1$*, defined in an open* $E \subset \mathbb{R}^2$ *and be* $\varphi(t) = \varphi(t, p)$ *an integral curve of* F*, defined for all* $t \geq 0$*, such that* γ^+ *is contained in a compact* $K \subset E$*. Assumes that the vector field* F *has a finite number of equilibrium points in* $\omega(p)$*. We have the following alternatives:*

(a) *If* $\omega(p)$ *contains only regular points, then* $\omega(p)$ *is a periodic orbit;*

(b) *If* $\omega(p)$ *contains regular and singular points, then* $\omega(p)$ *consists of a set of orbits, each of which tends to one of these singular points when* $t \to \infty$*;*

(c) *If* $\omega(p)$ *does not contain regular points, then* $\omega(p)$ *is a singular point.*

Theorem 1.6.3 ***(The Poincaré-Bendixson's Theorem at Sphere*** $\mathbb{S}^2$***).***
Let $F : E \subset \mathbb{R}^3 \to \mathbb{R}^3$ *be a vector field of class* C^1*, where* E *is an open of* $\mathbb{R}^3$ *that contain* $\mathbb{S}^2$*. Suppose that if* $p \in \mathbb{S}^2$ *then* $\varphi(t, p) \in \mathbb{S}^2$ *for all* $t \in \mathbb{R}$*. In other words,* F *restricted to* $\mathbb{S}^2$ *is a well-defined vector field on* $\mathbb{S}^2$ *(as defined in the section 1.3).*

If F *has a finite number of singular points in* $\mathbb{S}^2$*, then the* $\omega-$*limit set of an orbit by* $p \in \mathbb{S}^2$ *present the same possibility (a), (b) and (c) as at Theorem 1.6.2.*

Example 1.6.1 *Let* F *be a vector field given by*

$$\begin{aligned} \dot{x} &= -y + x(1 - x^2 - y^2), \\ \dot{y} &= \ \ x + y(1 - x^2 - y^2), \end{aligned} \tag{1-10}$$

and $p \in \mathbb{R}^2$*. Let us determine the limit sets* $\omega(p)$ *and* $\alpha(p)$*.*

Note that the origin is the only singular point of F. In polar coordinates, we can write (1-10) as

$$\begin{aligned} \dot{r} &= r(1 - r^2) \\ \dot{\theta} &= \quad 1. \end{aligned} \tag{1-11}$$

Analyzing the sign of $\dot{r}$ and observing that $\dot{\theta}$ is constant, we can conclude that (1-10) has only one periodic orbit, $\mathbb{S}^1$. Furthermore, γ^+ of $\varphi(t, p)$ is contained in a compact $K \ni \vec{0}$

for all $p \in \mathbb{R}^2$ such that $|p| > 1$. Since $|p| = 1$ corresponds to a periodic orbit, then by the existence and uniqueness theorem, it follows that $\varphi(t, p)$ belongs to interior of $\mathbb{S}^1$ for all $t \in \mathbb{R}$ and $p \in \mathbb{R}^2$ such that $|p| < 1$.

Therefore, from the Theorem 1.6.2, we conclude that
If $|p| > 1$, then $\omega(p) = \mathbb{S}^1$ and $\alpha(p) = \emptyset$;
If $|p| = 1$, then $\omega(p) = \alpha(p) = \mathbb{S}^1$;
If $0 < |p| < 1$, then $\omega(p) = \mathbb{S}^1$ and $\alpha(p) = \{\vec{0}\}$;
If $p = \vec{0}$, then $\omega(p) = \alpha(p) = \{\vec{0}\}$.

For the purposes of qualitative analysis, we can assume that the orbits of the vector field are parametrized in $\mathbb{R}$. This is a consequence of the Corollary 1.2.1. ℵ

Example 1.6.2 *Let F be a vector field of class C^1 on $\mathbb{R}^2$ that has no singular points in $B_{r,R} = \{(x,y);\ r^2 \leq x^2 + y^2 \leq R^2\}$, with $0 < r < R$. If F points to the interior of $B_{r,R}$, at every point on its boundary, then F has a periodic orbit in $B_{r,R}$.*

This follows from Theorem 1.6.2 applied to any positive semi-orbit about a point on the boundary of $B_{r,R}$. ℵ

Theorem 1.6.4 *Let F be a vector field of class C^1 in an open set $E \subset \mathbb{R}^2$. If γ is a closed orbit of F such that Int $(\gamma) \subset E$ then there is a singular point of F contained in Int (γ).*

Theorem 1.6.4 is an immediate consequence of Theorem 1.6.3 and one of the classic results of the qualitative theory of smooth vector fields.

CHAPTER 2

Geometric Singular Perturbation Theory

This chapter presents a study of a special class of smooth vector fields the 1-parameter F_ε ($\varepsilon > 0$), defined on an open set from $\mathbb{R}^p$ ($p > 1$). The main aim of the theory presented in this chapter is to determine elements of the dynamics of F_ε when ε is sufficiently small. To do this, we turn to the study of F_ε when $\varepsilon = 0$. The main reference used in this chapter is [9].

2.1 Fast-Slow Vector Fields and Fenichel's Theorem

We shall begin by studying ordinary differential equations (ODEs) in which some variables have derivatives of much larger magnitude than those of other variables. This scenario yields a system with different time scales.

Definition 2.1.1 *A fast-slow vector field or (m,n)-fast-slow system is a system of ordinary differential equations taking the form*

$$\begin{array}{rcccl} \varepsilon\frac{dx}{d\tau} & = & \varepsilon\dot{x} & = & f(x,y,\varepsilon), \\ \frac{dy}{d\tau} & = & \dot{y} & = & g(x,y,\varepsilon), \end{array} \tag{2-1}$$

where $f : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, $g : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, and $0 < \varepsilon << 1$. Furthermore, the x variable is called fast variable, and the y variable is called slow variable. Setting $\tau = \varepsilon t$ gives the equivalent form

$$\begin{array}{rcccl} \frac{dx}{dt} & = & \dot{x} & = & f(x,y,\varepsilon), \\ \frac{dy}{dt} & = & \dot{y} & = & \varepsilon g(x,y,\varepsilon). \end{array} \tag{2-2}$$

We refer to t as the fast time scale or fast time and to τ as the slow time scale or slow time.

The vector field F_ε in the slow time scale, τ, can be write as

$$F_\varepsilon(x,y) = \left(\frac{f(x,y,\varepsilon)}{\varepsilon}, g(x,y,\varepsilon) \right),$$

for each small $\varepsilon > 0$.

We define the vector field G_ε by

$$G_\varepsilon(x,y) = \varepsilon \cdot F_\varepsilon(x,y) = (f(x,y,\varepsilon), \varepsilon g(x,y,\varepsilon)).$$

Since ε is positive, it follows that the geometric local and orientation of the orbits of G_ε and F_ε are equal, because the spatial orientation of the vectors is the same at each point on space. System(2-1) or (2-2) is also called a ***singular perturbation problem***.

Definition 2.1.2 *The differential-algebraic equation obtained by setting $\varepsilon = 0$ in the formulation of the slow time scale (2-1) is called the* ***slow subsystem***, ***slow vector field*** *or* ***reduced problem***:

$$\begin{aligned} 0 &= f(x,y,0), \\ \dot{y} &= g(x,y,0). \end{aligned} \tag{2-3}$$

The parameterized system of ODEs obtained by setting $\varepsilon = 0$ on the fast time scale formulation (2-2) is called a ***fast subsystem***, ***fast vector field*** *or* ***layer problem***:

$$\begin{aligned} \dot{x} &= f(x,y,0), \\ \dot{y} &= \quad 0. \end{aligned} \tag{2-4}$$

Note that (2-3) is not an ODE, but an ODE with an algebraic constraint $f(x,y,0) = 0$. Therefore, we have a differential-algebraic equation (DAE). Initial conditions $x(t) = x_0$ and $y(t) = y_0$ must satisfy the constraint for solutions to exist. A slow-fast system is called *singular limit* where $\varepsilon = 0$.

Definition 2.1.3 *We call the set*

$$C_0 = \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n;\ f(x,y,0) = 0\}$$

the ***critical set***. *If C_0 is a submanifold of $\mathbb{R}^m \times \mathbb{R}^n$, we refer to C_0 as the* ***critical manifold***.

Proposition 2.1.1 *Equilibrium points of the fast subsystem are in one-to-one correspondence with points in C_0.*

Proof: The fast subsystem is given by

$$\begin{aligned} \dot{x} &= f(x,y,0), \\ \dot{y} &= \quad 0. \end{aligned} \tag{2-5}$$

The set of critical points of (2-5) is given by

$$H = \{(x,y) \in \mathbb{R}^m \times \mathbb{R}^n;\ f(x,y,0) = 0\}.$$

By Definition 2.1.3, follows that $C_0 = H$. ■

Example 2.1.1 *Consider the (2,1)-fast-slow system given by*

$$\begin{aligned} \varepsilon\dot{x} &= cx - by + az, \\ \varepsilon\dot{y} &= x^2 + y^2 + z^2 - R^2, \\ \dot{z} &= bx + cy. \end{aligned} \tag{2-6}$$

where $a, b, c \in \mathbb{R}$ *not all zeros and* $R > 0$.

The reduced problem and layer problem from (2-6) are given respectively by

$$\begin{aligned} 0 &= cx - by + az, \\ R^2 &= x^2 + y^2 + z^2, \\ \dot{z} &= bx + cy, \end{aligned} \tag{2-7}$$

and

$$\begin{aligned} \dot{x} &= cx - by + az, \\ \dot{y} &= x^2 + y^2 + z^2 - R^2, \\ \dot{z} &= 0. \end{aligned} \tag{2-8}$$

The critical manifold is given by

$$C_0 = \{(x, y, z) \in \mathbb{R}^2 \times \mathbb{R};\ cx - by + az = 0 \text{ and } x^2 + y^2 + z^2 = R^2\}.$$

From (2-7) follows that

$$\begin{aligned} c\dot{x} - b\dot{y} + a\dot{z} &= 0, \\ x\dot{x} + y\dot{y} + z\dot{z} &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} \dot{x} &= -ay - bz, \\ \dot{y} &= ax - cz. \end{aligned}$$

Therefore the solutions of (2-7) are solutions of the system (no reciprocally)

$$\begin{aligned} \dot{x} &= -ay - bz, \\ \dot{y} &= ax - cz, \\ \dot{z} &= bx + cy. \end{aligned} \tag{2-9}$$

The singular points of (2-9) are given by $A = \{t(c, -b, a) \in \mathbb{R}^3;\ t \in \mathbb{R}\}$. Note that C_0 is a circle and $C_0 \cap A = \emptyset$, thus C_0 is a periodic orbit of (2-9) whose orientation is given by the upward orientation of the vector $v = (c, -b, a)$. Therefore, we know the dynamics of (2-7).

The dynamics of (2-8) is characterized by the algebraic sign of $\dot{x}$ and $\dot{y}$ inside and outside the sphere $x^2 + y^2 + z^2 = R^2$, and above and below the plane $cx - by + az = 0$.

Note also that the plane $z = k$, k a real constant, is invariant by the flow of (2-8) for all $k \in \mathbb{R}$. The Figure 2.1 shows the geometric location of the orbits of (2-9). ℵ

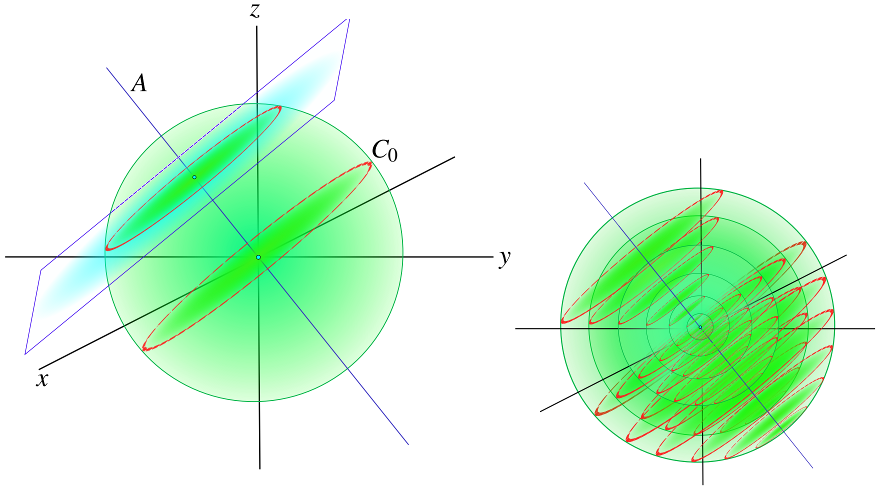

Figure 2.1: *Orbits of* (2-9).

Consider the (m,n)-fast-slow-system given by

$$\begin{aligned} \varepsilon\dot{x} &= f(x,y,\varepsilon), \\ \dot{y} &= g(x,y,\varepsilon). \end{aligned} \tag{2-10}$$

Defines $F_\varepsilon = (f(x,y,\varepsilon), \varepsilon g(x,y,\varepsilon))$, which corresponds to system (2-10) on the fast time scale.

From (2-10) on the slow time scale, taking $\varepsilon = 0$, we obtain the reduced problem. On the fast time scale, taking $\varepsilon = 0$, we obtain the layer problem. From these two different singular limits ($\varepsilon = 0$) we hope to recover information about the case $0 < \varepsilon \ll 1$. Let F_R denote the reduced flow on C_0.

The next step to connect the slow flow F_R to F_ε is to linearize F_ε on C_0 in the phase space variables. Fix some $p = (x_0, y_0) \in C_0$. Then

$$J(F_\varepsilon)(p)|_{\varepsilon=0} = \begin{pmatrix} f_x(p,0) & f_y(p,0) \\ 0 & 0 \end{pmatrix}. \tag{2-11}$$

Let us discuss some general properties of (m,n)-fast-slow-system. Fist, for the reduced problem (2-3) note that it is a differential equation with algebraic constraint, in other words a DAE. Solving (2-3) means finding a curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^n$, $\alpha(t) = (x(t), y(t))$, that satisfies both equations, i.e. this curve must satisfy the differential equation $\dot{y} = g(x,y,0)$ and must be points that belong to the surface or curve (object of dimension k, $k \geq n$) of equation $0 = f(x,y,0)$.

On the other hand, we have the layer problem (2-4). This problem is a system of differential equations as we know it, in other words an ODE. But this system has some special characteristics because the associated vector field has a zero component in the direction of the variable y. Define the set

$$V_y = \{(x,y) \in \mathbb{R}^{m+n};\ x \in \mathbb{R}^m\} \tag{2-12}$$

for each fixed $y \in \mathbb{R}^n$. Note that

$$v = (x,y) \in V_y \quad \Leftrightarrow \quad v = (x,0) + (0,y) \in V_y,$$

thus V_y is an affine vector space of dimension m. We can see that V_y is invariant by the flow of (2-4). In fact, let $p = (a,b) \in V_b$ and $\varphi(t,p)$ be the flow of (2-4) by the point p. Then necessarily $\varphi(t,p) = (x(t),y(t)) = (x(t),b)$ for all $t \in I_p$, thus $\varphi(t,p) \in V_b$ for all $t \in I_p$. Therefore V_y is invariant by the flow of (2-4) for all $y \in \mathbb{R}^n$.

Let F_l be the vector field of the layer problem (2-4). Note that

$$\mathbb{R}^m \times \mathbb{R}^n = \bigcup_{y \in \mathbb{R}^n} V_y$$

is a disjoined union. We can restrict F_l to V_y and we also note that V_y is a regular submanifold of $\mathbb{R}^{m+n}$. Denote by Δ the phase portrait of F_l on $\mathbb{R}^{m+n}$ and Δ_y the phase portrait of F_l on V_y. We have that

$$\Delta = \bigcup_{y \in \mathbb{R}^n} \Delta_y$$

is a disjoined union. With this split of the phase portrait, we can see that understanding the global dynamics of F_l is equivalent to understanding its dynamics in each V_y. Let G_y denote the vector field F_l restrict to Δ_y, thus

$$G_y(x,y) = G_y(x),$$

and we put $G_y(x) := f(x,y,0)$. It is a vector field defined in $\mathbb{R}^m$, for each $y \in \mathbb{R}^n$ fix.

Note that the set of singular points of G_y is given by

$$S_y = C_0 \cap V_y.$$

We have that

$$J(G_{y_0})(x_0) = (f_x(p,0)). \tag{2-13}$$

So instead of using (2-11) to study the nature of p with respect to F_l we just study the nature of x_0 with respect to G_{y_0}. That is, the eigenvalues of (2-13). We do this because in the direction of the variable x we do not have trivial dynamics, since the vector field cancels out in the direction of the variable y.

Motivated by the previous discussions, below we give a definition that allows us to evaluate the nature of the equilibrium points of a layer problem, in the sense that we do this for the vector field G_y on Δ_y.

Definition 2.1.4 *A subset $M \subset C_0$ is called **normally hyperbolic** if the $m \times m$ matrix $(f_x(p,0))$ of first partial derivatives with respect to the fast variable has no eigenvalues with zero real part for all $p \in M$.*

Definition 2.1.5 *A normally hyperbolic subset $M \subset C_0$ is called **attracting** if all eigenvalues of $(f_x(p,0))$ have negative real part for all $p \in M$; similarly, M is called **repelling** if all eigenvalues have positive real part. If M is normally hyperbolic and neither attracting nor repelling, it is of **saddle** type.*

Let is now talk more specifically about some of the characteristics of the reduced problem:

$$\begin{aligned} 0 &= f(x,y,0), \\ \dot{y} &= g(x,y,0). \end{aligned} \tag{2-14}$$

Definition 2.1.6 *Let C_0 be the critical manifold. We call*

$$C_{0,s} = \{p \in C_0;\ (f_x)(p,0) \text{ is not invertible}\}$$

(fast-slow) singular points** and $C_{0,r} := C_0 - C_{0,s}$ **(fast-slow) regular points.

Proposition 2.1.2 *Let $p = (x_0, y_0) \in C_0 \subset \mathbb{R}^m \times \mathbb{R}^n$, and consider that $(f_x)(p,0)$ has maximal rank, so that $p \in C_{0,r}$. Then there exists a neighborhood $V \subset C_0$ of p such that the slow subsystem for $\varepsilon = 0$ on V is given by*

$$\begin{aligned} \dot{x} &= -[(f_x)(q,0)]^{-1} \cdot (f_y)(q,0) \cdot g(q,0), \\ \dot{y} &= \ \ g(q,0), \end{aligned}$$

for all $q \in V$.

Proof: Since p is a regular point of C_0, then Implicit Function Theorem 1.1.5 yields the existence of a map

$$x : U \subset \mathbb{R}^n \to \mathbb{R}^m$$

such that the graphic of x on U is equal to some open $V \subset C_0$ that contain p. Thus,

$$f(x(y),y,0) = f(x,y) = 0$$

for all $y \in U$; equivalently,

$$f(q,0) = 0$$

for all $q \in V$. We have (at $\varepsilon = 0$)

$$
\begin{aligned}
& f(x(y),y) = 0 \\
\Rightarrow \quad & J(f)(x(y),y) \cdot \begin{pmatrix} x'(y) \\ id \end{pmatrix} = 0 \\
\Rightarrow \quad & \begin{pmatrix} f_x(x,y) & f_y(x,y) \end{pmatrix} \cdot \begin{pmatrix} x'(y) \\ id \end{pmatrix} = 0 \\
\Rightarrow \quad & \left[\begin{pmatrix} f_x(x,y) & 0 \end{pmatrix} + \begin{pmatrix} 0 & f_y(x,y) \end{pmatrix} \right] \cdot \left[\begin{pmatrix} x'(y) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ id \end{pmatrix} \right] = 0 \\
\Rightarrow \quad & \begin{pmatrix} f_x(x,y) & 0 \end{pmatrix} \cdot \begin{pmatrix} x'(y) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & f_y(x,y) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ id \end{pmatrix} = 0 \\
\Rightarrow \quad & \begin{pmatrix} f_x(x,y) \end{pmatrix} \cdot \begin{pmatrix} x'(y) \end{pmatrix} + \begin{pmatrix} f_y(x,y) \end{pmatrix} \cdot \begin{pmatrix} id \end{pmatrix} = 0 \\
\Rightarrow \quad & (f_x)(x,y) \cdot J(x)(y) + (f_y)(x,y) \cdot id = 0 \\
\Rightarrow \quad & J(x)(y) = -[(f_x)(x,y)]^{-1} \cdot (f_y)(x,y).
\end{aligned}
$$

If $x(t) = x(y(t))$ is the solution from (2-14) in $V \subset C_0$, follows that

$$\dot{x} = J(x)(y) \cdot \dot{y} = -[(f_x)(x,y,0)]^{-1} \cdot (f_y)(x,y,0) \cdot g(x,y,0). \qquad \blacksquare$$

Example 2.1.2 *Consider the fast-slow system given on the slow time scale by*

$$
\begin{aligned}
\varepsilon \dot{x} &= x + y^2 + \varepsilon, \\
\dot{y} &= y^2 - x + \varepsilon xy.
\end{aligned}
\tag{2-15}
$$

The reduced problem and layer problem of (2-15) are given respectively by

$$
\begin{aligned}
0 &= x + y^2, \\
\dot{y} &= y^2 - x
\end{aligned}
\tag{2-16}
$$

and

$$
\begin{aligned}
\dot{x} &= x + y^2, \\
\dot{y} &= 0.
\end{aligned}
\tag{2-17}
$$

The critical manifold of (2-15) is given by $C_0 = \{(x,y) \in \mathbb{R}^2; x = -y^2\}$.

Note that slow dynamics is characterized by (2-16). Therefore, note that (2-16) is a particular case of the problem

$$
\begin{aligned}
k &= h(x,y), \\
\dot{y} &= w(x,y),
\end{aligned}
\tag{2-18}
$$

where h is of class C^2, w of class C^1 and k is a real constant. (2-18) is a differential-algebraic equation whose solution is a curve $\alpha : I \subset \mathbb{R} \to \mathbb{R}^2$.
We have that

$$k = h(x(t), y(t)) \Leftrightarrow \dot{x}h_x(x,y) + \dot{y}h_y(x,y) = 0.$$

From (2-16), it follows that

$$\dot{x} = 2yx - 2y^3.$$

Note that the solution of (2-16) is the solution of the system

$$\begin{aligned} \dot{x} &= 2yx - 2y^3, \\ \dot{y} &= \quad y^2 - x. \end{aligned} \tag{2-19}$$

when $k = 0$ in (2-18). In this particular case, using the first equation in (2-16), yet we can restrict the vector field from (2-19) along its orbit given by the algebraic restriction and we get

$$\begin{aligned} \dot{x} &= -4y^3, \\ \dot{y} &= \; 2y^2. \end{aligned} \tag{2-20}$$

Note that (2-20) can be obtained directly from Proposition 2.1.2. The calculations that led to (2-20) start from the same principle as the proof of Proposition 2.1.2.

In addition, if $y > 0$ then $\dot{x} < 0$ and $\dot{y} > 0$; if $y < 0$ then $\dot{x}, \dot{y} > 0$; if $y = 0$ then $\dot{x} = \dot{y} = 0$. Thus we obtain a characterization of the dynamics of (2-16), illustrated at Figure 2.2. We have that

$$f_x(p, 0) = 1 > 0,$$

for all $p \in C_0$. Then C_0 is normally hyperbolic and repelling. The dynamic of (2-17) is easy to understand and is also illustrated in the Figure 2.2. ℵ

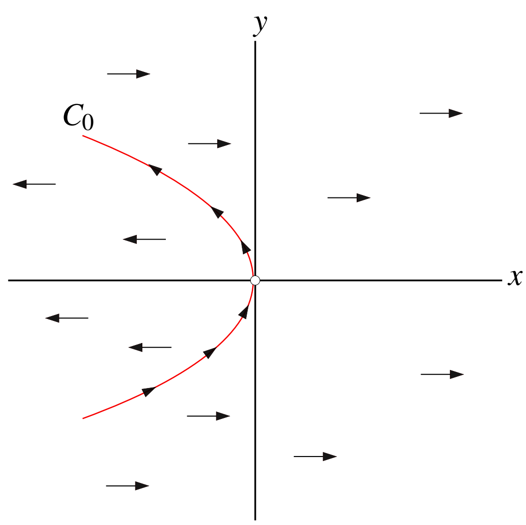

Figure 2.2: *Critical manifold and fast-slow dynamics of* (2-15).

Example 2.1.3 *Consider the (unforced) van der Pol equation given by*

$$z'' + \mu(x^2 - 1)z' + x = 0, \tag{2-21}$$

with $\mu > 0$.

Take the time scale $t = \mu\tau$ in (2-21), we have that

$$x(\tau) = z(\mu\tau) \quad \Rightarrow \quad \dot{x}(\tau) = \mu\dot{z}(\mu\tau),$$

thus

$$\frac{\ddot{x}}{\mu^2} + (x^2 - 1)\dot{x} + x = 0.$$

Considering the change of variable

$$y = \frac{\dot{x}}{\mu^2} + \int (x^2 - 1)dx = \frac{\dot{x}}{\mu^2} + \frac{x^3}{3} - x,$$

we have

$$\dot{y} = \frac{\ddot{x}}{\mu^2} + (x^2 - 1)\dot{x} = -x$$

and

$$\frac{\dot{x}}{\mu^2} = y - \frac{x^3}{3} + x.$$

Setting $\varepsilon = \frac{1}{\mu^2}$ we obtain a singular perturbation problem equivalent to (2-21) given on the slow time scale by

$$\begin{aligned} \varepsilon\dot{x} &= y - \frac{x^3}{3} + x, \\ \dot{y} &= -x. \end{aligned} \tag{2-22}$$

In the fast scale we write (2-22) as

$$\begin{aligned} \dot{x} &= y - \frac{x^3}{3} + x, \\ \dot{y} &= -\varepsilon x. \end{aligned} \tag{2-23}$$

We obtain then the reduced problem and layer problem given respectively by

$$\begin{aligned} 0 &= y - \frac{x^3}{3} + x, \\ \dot{y} &= -x. \end{aligned} \tag{2-24}$$

and

$$\begin{aligned} \dot{x} &= y - \frac{x^3}{3} + x, \\ \dot{y} &= 0. \end{aligned} \tag{2-25}$$

The critical manifold is given by

$$C_0 = \left\{(x,y) \in \mathbb{R}^2;\, y = \frac{x^3}{3} - x\right\}.$$

Analogous to what was done in Example 2.1.2, we have that

$$y - \frac{x^3}{3} + x = 0 \;\Rightarrow\; \dot{y} + (-x^2+1)\dot{x} = 0 \;\Leftrightarrow\; \dot{x} = \frac{\dot{y}}{x^2-1}.$$

Considering the second equation in (2-24), it follows that

$$\dot{x} = \frac{\dot{y}}{x^2-1} = \frac{-x}{x^2-1}.$$

Therefore the solutions of (2-24) are solutions of the system (no reciprocally)

$$\begin{aligned} \dot{x} &= \frac{x}{1-x^2}, \\ \dot{y} &= -x. \end{aligned} \tag{2-26}$$

Note also that C_0 is invariant by the flow of (2-26). Thus,
if $x < -1$ then $\dot{x} > 0$ and $\dot{y} > 0$;
if $-1 < x < 0$ then $\dot{x} < 0$ and $\dot{y} > 0$;
if $0 < x < 1$ then $\dot{x} > 0$ and $\dot{y} < 0$;
if $x > 1$ then $\dot{x} < 0$ and $\dot{y} < 0$;
if $x = 0$ then $\dot{x} = \dot{y} = 0$ (singular point);
if $x = -1$ ou $x = 1$ then the dynamics are not well defined.

The dynamic of (2-25) is easy to understand. In fact, $\dot{y} = 0$ and, if $y > \frac{x^3}{3} - x$ then $\dot{x} > 0$; if $y < \frac{x^3}{3} - x$ then $\dot{x} < 0$. The Figure 2.3 below illustrate the critical manifold and dynamics fast-slow of (2-22).

Taking a closer look at the fast dynamics, from the previous discussion (as at (2-12)), we have

$$V_y = \{(x,y) \in \mathbb{R}^2;\, x \in \mathbb{R}\}.$$

Note that V_y is a line parallel to the x-axis for each $y \in \mathbb{R}$. We have that $G_y : V_y \to \mathbb{R}$ is given by

$$G_y(x) = y - \frac{x^3}{3} + x,$$

and note that $V_y - (0,y)$ is isomorphic to $\mathbb{R}$; in other words, V_y is the vector space $\mathbb{R}^m$ (in this case $m = 1$) minus a translation (this is also true in the general case). The vector field G_y is one-dimensional. We have

$$S_y = C_0 \cap V_y = \{x \in \mathbb{R};\, \frac{x^3}{3} - x = y\}.$$

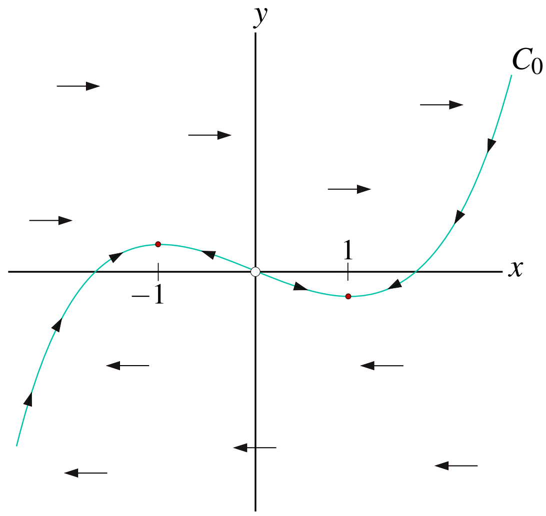

Figure 2.3: *Critical manifold and fast-slow dynamics of* (2-22).

Let $p = (x_0, y_0) \in S_{y_0}$; we have

$$J(G_{y_0})(x_0) = (f_x(p,0)) = -x_0^2 + 1.$$

Note that

- if $x_0 < -1$ or $x_0 > 1$ then $f_x(p,0) < 0$ (negative eigenvalue) and thus x_0 is a stable equilibrium point for G_{y_0} on V_{y_0};
- if $|x_0| < 1$ then $f_x(p,0) > 0$ (positive eigenvalue) and thus x_0 is an unstable equilibrium point for G_{y_0} on V_{y_0};
- if $|x_0| = 1$ then $f_x(p,0) = 0$ (eigenvalue with zero real part) and thus x_0 is a degenerate equilibrium point.

The sets

$$M_1 = \{(x,y) \in C_0;\, x < -1\},$$

$$M_2 = \{(x,y) \in C_0;\, x > 1\},$$

and

$$M_3 = \{(x,y) \in C_0;\, |x| < 1\}$$

are normally hyperbolic. Furthermore they are smooth regular submanifolds of $\mathbb{R}^2$. Any subset of C_0 that contain the point $\left(-1, \frac{2}{3}\right)$ or $\left(1, -\frac{2}{3}\right)$ is non-normally hyperbolic.

The Figure 2.4 shows the dynamics of (2-25) along the affine vector spaces. It is easy to see that the degenerate equilibrium points of (2-25) are $\left(-1, \frac{2}{3}\right)$ and $\left(1, -\frac{2}{3}\right)$. The red line in Figure 2.4 corresponds to one of the affine spaces that has a degenerate equilibrium point. ℵ

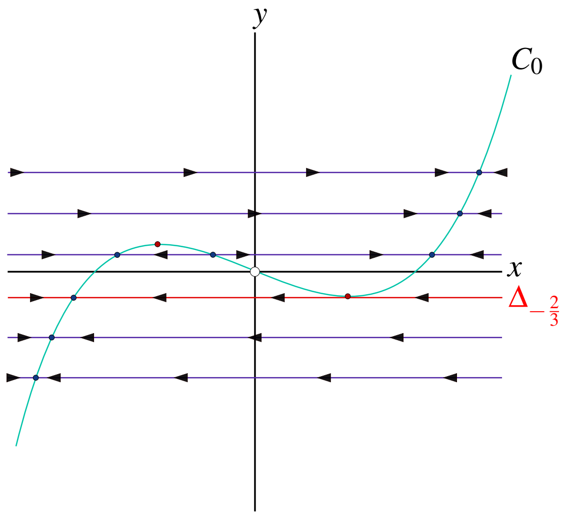

Figure 2.4: *Critical manifold and dynamics of G_y.*

Definition 2.1.7 *Let F be a vector field of class C^1 in an open $E \subset \mathbb{R}^n$ and $M \subset E$ a compact connected C^1-manifold with boundary embedded in $\mathbb{R}^n$. Let $\varphi_p(t)$ denote the flow defined by the vector field F in $p \in E$.*

*(a) M is called an **inflowing invariant manifold** if for every $p \in \partial M$ (boundary of M), the vector field is pointing strictly inward and for all $p \in M$, $\varphi_p(t) \in M$ for all $t \geq 0$;*

*(b) M is called an **overflowing invariant manifold** if for every $p \in \partial M$ (boundary of M), the vector field is pointing strictly outward and for all $p \in M$, $\varphi_p(t) \in M$ for all $t \leq 0$;*

*(c) M is called an **invariant manifold** if for every $p \in M$, we have $\varphi_p(t) \in M$ for all $t \in \mathbb{R}$;*

*(d) M is called a **locally invariant manifold** if for each $p \in M$, there exists a time interval $I_p = (t_1, t_2)$ such that $0 \in I_p$ and $\varphi_p(t) \in M$ for all $t \in I_p$.*

Definition 2.1.8 *The Hausdorff distance between two nonempty sets $V, W \subset \mathbb{R}^n$ is defined by*

$$d_H(V,W) = \left\{ \sup_{v \in V} \inf_{w \in W} |v - w|, \sup_{w \in W} \inf_{v \in V} |v - w| \right\}.$$

Theorem 2.1.1 ***(Fenichel's Theorem).***
Suppose M_0 is a compact normally hyperbolic submanifold (possibly with boundary) of the critical manifold C_0 of (2-10) and that $f, g \in C^r$ $(r \geq 2)$. Then for $\varepsilon > 0$ sufficiently small, the following hold:

(a) There exists a locally invariant manifold M_ε (called slow manifold) diffeomorphic to M_0. Local invariance means that trajectories can enter or leave M_ε only through its boundaries (see Figure 2.5). M_ε is not unique.

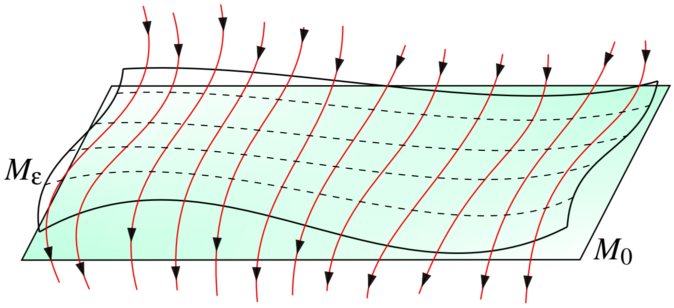

Figure 2.5: *Slow manifold M_ε and compact normally hyperbolic manifold $M_0 \subset C_0$.*

(b) M_ε has Hausdorff distance $O(\varepsilon)$ (as $\varepsilon \to 0$) from M_0.

(c) The flow on M_ε converges to the slow flow as $\varepsilon \to 0$.

(d) M_ε is C^r-smooth.

(e) M_ε is normally hyperbolic and has the same stability properties with respect to the fast variable as M_0 (attracting, repelling or of saddle type).

The proof of the Theorem 1.5.1 can be found in [9].

Example 2.1.4 *Consider the fast-slow system given by*

$$\begin{aligned} \dot{x} &= y^2 - x, \\ \dot{y} &= \ -\varepsilon y. \end{aligned} \tag{2-27}$$

The critical manifold from (2-27) is given by

$$C_0 = \{(x,y) \in \mathbb{R}^2;\, x = y^2\}.$$

Furthermore

$$(f_x)(q,0) = (-1) < 0,$$

for all $q \in C_0$. Thus C_0 is a normally hyperbolic manifold and attracting everywhere.

We can solve the system (2-27) explicitly. Using the method of separable variables, we have

$$\dot{y} = \frac{dy}{dt} = -\varepsilon y \ \Leftrightarrow\ \frac{dy}{y} = -\varepsilon dt \ \Leftrightarrow\ \ln|y| = -\varepsilon t + c \ \Leftrightarrow\ y(t) = \pm e^{-\varepsilon t + c}, \quad c \in \mathbb{R}.$$

Thus,

$$y(t) = e^{-\varepsilon t}k, \quad k \in \mathbb{R}.$$

On the other hand taking $\varepsilon \neq \frac{1}{2}$, by the integral factor method, we have

$$\dot{x} + x = y^2 \quad \Leftrightarrow \quad e^t\dot{x} + e^t x = e^t y^2 \quad \Leftrightarrow \quad \frac{d[e^t x]}{dt} = e^t y^2(t) = e^{(1-2\varepsilon)t}k^2$$

$$\Leftrightarrow \quad e^t x = \frac{k^2}{1-2\varepsilon}e^{(1-2\varepsilon)t} + u \quad \Leftrightarrow \quad x(t) = \frac{k^2}{1-2\varepsilon}e^{-2\varepsilon t} + ue^{-t}, \quad u \in \mathbb{R}.$$

The integral curve of (2-27) by the point $p = (x_0, y_0)$ is given by

$$\varphi_\varepsilon(t,p) = \left(\frac{k^2}{1-2\varepsilon}e^{-2\varepsilon t} + ue^{-t}, e^{-\varepsilon t}k\right),$$

where $k = y_0$ and $u = x_0 - \dfrac{y_0^2}{1-2\varepsilon}$. Tanking $k = y_0$ and $u = 0$, follows that

$$x_0 = \frac{y_0^2}{1-2\varepsilon}. \tag{2-28}$$

Therefore, whatever $p = (x_0, y_0)$ satisfying (2-28), follows that

$$\varphi_\varepsilon(t,p) = \left(\frac{y_0^2}{1-2\varepsilon}e^{-2\varepsilon t}, y_0 e^{-\varepsilon t}\right).$$

Note that $\varphi_\varepsilon(t,p) \in C_\varepsilon = \left\{(x,y) \in \mathbb{R}^2; x = \dfrac{y^2}{1-2\varepsilon}\right\}$ for $t \in \mathbb{R}$. Furthermore in this case we can take M_0 as any compact submanifold in C_0 and then, for $\varepsilon > 0$ sufficiently small, we have $M_\varepsilon \subset C_\varepsilon$. ℵ

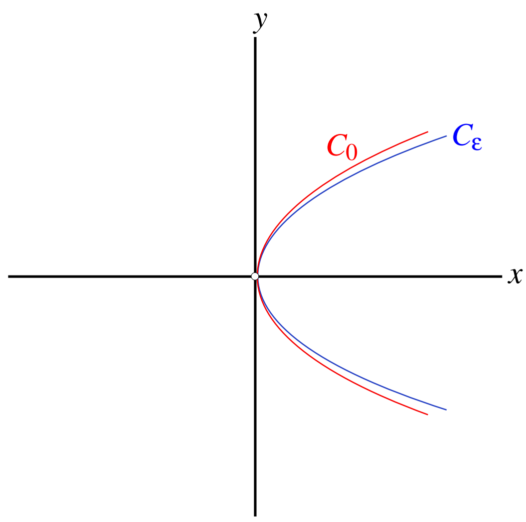

Figure 2.6: *Critical manifold C_0 and perturbed manifold C_ε.*

So far we have dealt with the case where the critical set C_0 is a manifold consisting of regular or hyperbolic points. In many cases we come across the scenario where C_0 has no structure of manifold or, with the loss of regularity and normal hyperbolicity in C_0. We will present some situations in which this can occur. Suppose that $p \in C_0$ is a singular point, i.e., $p \in C_{0,s}$, so that

$$f_x(p,0) : \mathbb{R}^m \to \mathbb{R}^m$$

is not of maximal rank. The simplest possible rank deficiency arises when $f_x(p,0)$ has rank $m-1$ with zero as an eigenvalue of multiplicity one.

Example 2.1.5 *Consider the* $(1,1)$*-fast-slow system given by*

$$\begin{aligned} \dot{x} &= \ \ y - x^2, \\ \dot{y} &= \varepsilon g(x,y,\varepsilon). \end{aligned} \tag{2-29}$$

The critical manifold is a parabola

$$C_0 = \{(x,y) \in \mathbb{R}^2;\, y = x^2\}.$$

We consider the origin $\vec{0} \in C_0$ and obtain

$$f_x(\vec{0},0) = -2x\big|_{x=0} = 0.$$

Therefore, $\vec{0} \in C_{0,s}$. Observe that

$$f_{xx}(\vec{0},0) = -2 \neq 0,$$

which is a nondegeneracy condition. The fast subsystem of (2-29) is

$$\begin{aligned} \dot{x} &= y - x^2, \\ \dot{y} &= \ \ 0. \end{aligned}$$

In this case, $y \in \mathbb{R}$ is a parameter and $\dot{x} = y - x^2$ is the normal form for a **fold bifurcation** at $y = 0$; alternative terms for a fold bifurcation are **saddle-node bifurcation**, **turning point**, and **limit point**. ℵ

The last example can be extended to a general (m,n)-fast-slow system.

Definition 2.1.9 *Suppose* $p \in C_0$*. Then* p *is a* ***fold point*** *if*

$$f_x(p,0) \quad \textit{is of rank } m-1.$$

A fold point is called **nondegenerate** *if for vectors w and v, which are in the left and right nullspaces of $f_x(p,0)$ respectively, one has*

$$w \cdot [f_{xx}(p,0)(v,v)] \neq 0 \quad \text{and} \quad w \cdot [f_y(p,0)] \neq 0.$$

In addition to the important fold point, there are many other singularities/bifurcations (see [9]). Here are some examples that do not (completely) satisfy the condition of normal hyperbolicity.

Example 2.1.6 *Consider the following $(1,1)$-fast-slow system:*

$$\begin{aligned} \dot{x} &= x(y-x), \\ \dot{y} &= \varepsilon g(x,y,\varepsilon). \end{aligned}$$

The critical set $C_0 = \{(x,y) \in \mathbb{R}^2;\, x = 0 \text{ or } y = x\}$ is a manifold except at the origin $\vec{0}$. This yields a **transcritical point**, or **transcritical singularity**; see [9]. ℵ

Example 2.1.7 *Consider the $(1,2)$-fast-slow system*

$$\begin{aligned} \dot{x} &= x(y-x), \\ \dot{y}_1 &= \varepsilon g_1(x,y,\varepsilon), \\ \dot{y}_2 &= \varepsilon g_2(x,y,\varepsilon), \end{aligned}$$

where $y = (y_1, y_2)$.

The critical set $C_0 = \{(x,y) \in \mathbb{R}^2;\, y_1 = -y_2 x + x^3\}$ can be seen as the graph of a function $y_1 = y_1(x, y_2)$ and therefore is a manifold, but it contains a curve of fold points given by

$$L = \{(x,y) \in C_0;\, f_x(x,y,0) = y_2 - 3x^2 = 0\}.$$

As $f_{xx}(x,y,0) = -6x$ is null only at the origin, follows that the fold points L are nondegenerate except at $\vec{0}$, which is a **cusp point** or **cusp singularity**; see [9]. ℵ

Example 2.1.8 *Consider the $(2,1)$-fast-slow system*

$$\begin{aligned} \dot{x}_1 &= yx_1 - x_2 - x_1(x_1^2 + x_2^2) = f_1(x,y), \\ \dot{x}_2 &= x_1 + yx_2 - x_2(x_1^2 + x_2^2) = f_2(x,y), \\ \dot{y} &= \varepsilon g(x,y,\varepsilon), \end{aligned}$$

where $x = (x_1, x_2)$.

The critical manifold $C_0 = \{(x,y) \in \mathbb{R}^2;\, x_1 = x_2 = 0\}$ is simply the y-axis. The linearization with respect to the fast variables at $p \in C_0$ is

$$f_x(p,0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} y & -1 \\ 1 & y \end{pmatrix}.$$

Therefore, C_0 consists only of regular points, but $f_x(\vec{0},0)$ has a pair of complex eigenvalues $\pm i$, so that C_0 is not normally hyperbolic at the origin, which is a **Hopf bifurcation** of the fast subsystem. ℵ

Degenerate cases can present a great variety in the dynamics of fast-slow systems, more specifically in the dynamics of the fast subsystem. As described in the previous examples, to study these cases, we need to resort to the study of bifurcations. The reader can find more information on this subject in [9].

CHAPTER 3

Piecewise-Smooth Vector Fields

In this chapter, in the sense of Filippov and in the sense of Utkin, the definition of piecewise-smooth vector fields and some properties will be presented. It will be possible for the reader to associate the definitions presented in this chapter with the definitions given for smooth vector fields in Chapter 1. The main references used in this chapter are [3], [6] and [8].

3.1 Why Study Piecewise-Smooth Vector Fields and How to Manage These Systems?

Let us start this section by presenting a practical example that illustrates the necessity to know the behavior of piecewise-smooth vector field in order to describe certain physical phenomena. Readers interested in more examples of the application of vector field-smooth vector field can consult [1].

DC-DC (direct current) converters are circuits used to change one DC voltage into another. In the past, this procedure was carried out by converting the DC voltage into an AC (alternating current) voltage by means of a transformer and then converting the resulting AC voltage back into a DC voltage. This procedure is not efficient, as there is a large loss of energy and we need large devices. To change voltages with household electronic devices, such as laptops, we need something smaller and that loses less energy. The most commonly used DC-DC converters need electronic switches to change one DC voltage to another, with small energy losses. The use of switches means that DC-DC converters represent non-smooth dynamic systems, which, when pushed beyond their intended operating limits, can give rise to complex dynamics. We are interested in the peculiar dynamics that arise due to the non-smooth nature of the switching process.

The simple DC-DC converter circuit illustrated in Figure 3.1 converts a constant input voltage E to a constant higher or lower voltage γ, by switching the part of the circuit containing the input on and off. Disregarding time-delays and circuit latency, the movement would slide along the surface $V = \gamma$. In reality, this would lead to fast switching, which would lead to undesirable effects such as inefficiency, overheating and the excitation of overtones. Instead, a standard method is applied, in which the output $V(t)$ is compared to a reference voltage $V_r(t)$ that incorporates the form of a centered low-amplitude periodic ramp function around γ.

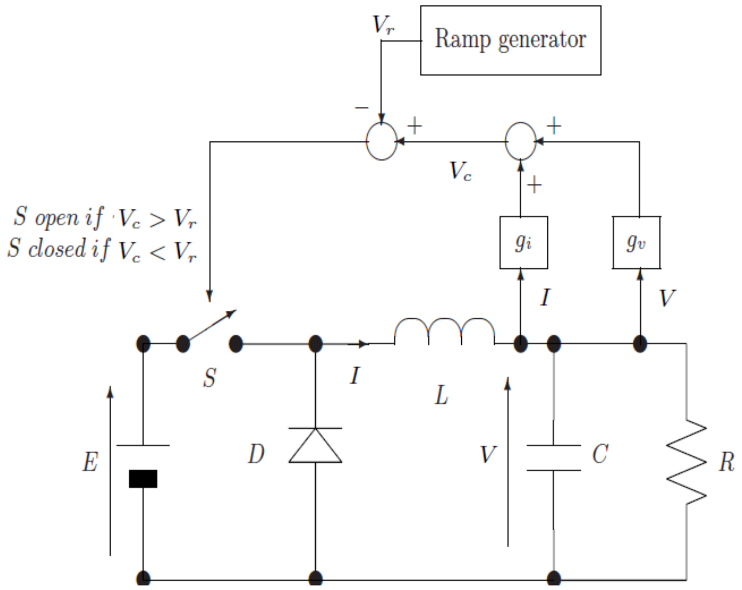

Figure 3.1: *Schematic diagram of a simple DC-DC buck converter circuit. (Figure from [1]).*

The current $I(t)$ and voltage $V(t)$ inside the circuit evolve in a smooth manner between switching events, which arise when $V(t) = V_r(t)$. Applying Kirchhoff's Laws, we obtain the equations that describe the dynamics of this circuit:

$$\begin{aligned} \dot{V} &= -\frac{V}{RC} + \frac{I}{C}, \\ \dot{I} &= -\frac{V}{L} + \alpha, \end{aligned} \tag{3-1}$$

with $\alpha = 0$ if $V \geq V_r$; or $\alpha = \dfrac{E}{L}$ if $V < V_r$. For the output voltage $V(t)$ and corresponding current $I(t)$. Here C, E, L and R are positive constants representing the capacitance, battery voltage, inductance and resistance, respectively, of the components depicted in Figure 3.1.

The reader should realize that (3-1) is a piecewise-smooth vector field, and the region of discontinuity in the phase space $V \times I$ is given by the line $V = V_r$. Leaving the scenario of smooth vector fields, how could we study the behavior of (3-1) in the region of discontinuity: are there solutions and are unique? How can we understand the dynamics of the vector field in this region? In the next section, we will begin by presenting some of the methods and definitions we have adopted to mathematically study the behavior of piecewise-smooth vector field in the region of discontinuity.

There are several methods we can use to study piecewise-smooth vector fields. In this dissertation we will talk about Filippov's convention, Utkin's convention and the regularization process developed by Sotomayor and Teixeira. Depending on the vector field and the dynamics of the phenomenon it describes, it may be more appropriate to adopt a certain method (or convention). An of the motivations for adopting the Filippov's convention is to model of various practical problems, such as stick-slip processes, anti-lock braking systems, relay systems and, in general, a substantial part of control systems (see [1]). The Utkin's convention also has several uses in the context of applications, especially in the field of physics. For the development of a consistent mathematical theory that describes the dynamics of piecewise-smooth vector fields, we have predominantly adopted Filippov's convention.

3.2 General Concepts of Piecewise-smooth Vector Fields, Filippov's Convention and Utkin's Convention

In the following we have a general definition for piecewise-smooth vector fields on manifolds. For more information associated with it is available at [8].

Definition 3.2.1 *A piecewise-smooth vector field is a triple* (M,Σ,Z) *where*

(a) M *is a smooth manifold of dimension* m*;*

(b) Σ *is called* ***switching manifold*** *and is formed by a finite disjoint union of simple curves* $\Sigma=\cup_{i=1}^{n}\Sigma_i$*, which separate* M *into* $n+1$ *components (or zones)* R_i*, where* $\Sigma_i=\gamma_i^{-1}(0)$ *and* $\gamma_i : M\to\mathbb{R}$*,* $i=1,\dots,n$*, are smooth functions with* 0 *as a regular value (i.e. for any* i*,* $\nabla\gamma_i(p)\neq 0$ *for all* $p\in\gamma_i^{-1}(0)$*);*

(c) Z *is a set of* $n+1$ *vector fields of class* C^r*,* $r\geq 1$*, in* M*, say* $Z=(X_1,\dots,X_{n+1})$ *where each* X_i *is defined in the closure of* R_i*.* Z *is multivalued in* Σ_i*,* $i=1,\cdots,n$*.*

In item (b) of the previous definition, $\nabla\gamma_i(p)$ is in the direction of the vector orthogonal to the space $T_p\gamma_i^{-1}(0)$ in T_pM. The Theorem 1.1.6 also holds for functions defined between manifolds in the sense that they were defined in Section 1.3. With this, the reader should realize that $\gamma_i^{-1}(0)$ is a regular submanifold of M of codimension 1, so $T_p\gamma_i^{-1}(0)\subset T_pM$ has dimension $m-1$. The following example illustrates a case of a piecewise-smooth vector field in the general sense presented by the Definition 3.2.1.

Example 3.2.1 *Let* (S^2,Σ,Z) *be a piecewise-smooth vector field, where* $\Sigma=\Sigma_1\cup\Sigma_2$*, with* $\gamma_1,\gamma_2 : S^2\to\mathbb{R}$ *given by* $\gamma_1(x,y,z)=z-\frac{1}{2}$ *and* $\gamma_2(x,y,z)=z+\frac{1}{2}$*; and* $Z=(X_1,X_2,X_3)$ *is such that*

$X_1 : R_1 \to T_pS^2$ is given by

$$\begin{aligned} \dot{x} &= z, \\ \dot{y} &= 0, \\ \dot{z} &= -x; \end{aligned}$$

$X_2 : R_2 \to T_pS^2$ is given by

$$\begin{aligned} \dot{x} &= -2y+z, \\ \dot{y} &= 2x, \\ \dot{z} &= -x; \end{aligned}$$

$X_3 : R_3 \to T_pS^2$ is given by

$$\begin{aligned} \dot{x} &= z, \\ \dot{y} &= 0, \\ \dot{z} &= -x, \end{aligned}$$

with $R_1 = \{(x,y,z) \in S^2;\, z \geq \frac{1}{2}\}$, $R_2 = \{(x,y,z) \in S^2;\, -\frac{1}{2} \leq z \leq \frac{1}{2}\}$ and $R_3 = \{(x,y,z) \in S^2;\, z \leq -\frac{1}{2}\}$.

Note that $\nabla\gamma_i(p) : T_pS^2 \to \mathbb{R}$, $i = 1,2$, is surjective for all $p \in \gamma_i^{-1}(0)$. In fact, $\nabla\gamma_i(p) = (0,0,1)$ for all $p \in \gamma_i^{-1}(0)$ and since T_pS^2 is not parallel to plane xy follows that $\nabla\gamma_i(p)(T_pS^2) \neq \{0\}$. We have that $\Sigma_1 = \{(x,y,z) \in S^2;\, z = \frac{1}{2}\}$ and $\Sigma_2 = \{(x,y,z) \in S^2;\, z = -\frac{1}{2}\}$. The vector fields X_1, X_2 and X_3 are particular cases of the vector field given in the Example 2.1.1. Knowing this, we provide a sketch of the phase portrait of Z in the Figure 3.2. ℵ

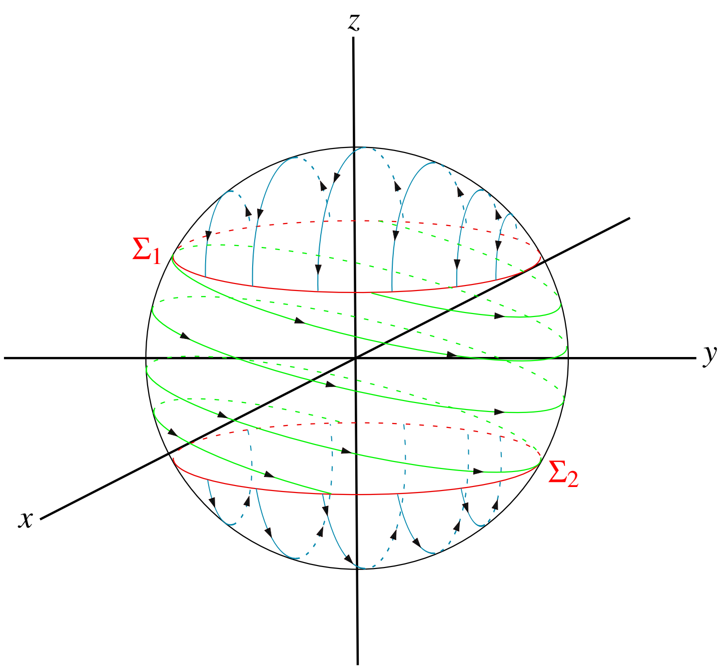

Figure 3.2: *Dynamics of Z in S^2.*

The following concepts associated with piecewise-smooth vector fields and the Filippov's convention were obtained from the references [5], [6] and [3].

In this dissertation we will focus our attention on the scenario in which the manifold $M = U$ is an open in $\mathbb{R}^n$ and $\Sigma = f^{-1}(0)$, where $f : U \to \mathbb{R}$ is of class C^k ($k \geq 1$) and 0 a regular value of f. Thus, $R_1 = \Sigma^+ = \{p \in U;\ f(p) \geq 0\}$ and $R_2 = \Sigma^- = \{p \in U;\ f(p) \leq 0\}$. Then $Z = (X,Y)$, where $X : \Sigma^+ \to \mathbb{R}^n$ and $Y : \Sigma^- \to \mathbb{R}^n$ are of class C^r on U, $r \geq 1$. Let $\Lambda(U)$ be the space of C^r-vector fields on an open $U \subset \mathbb{R}^n$, $r \geq 1$. Call $\Omega(U)$ the space of vector fields $Z : U \to \mathbb{R}^n$ such that

$$Z(p) = \begin{cases} X(p), & \text{for } p \in \Sigma^+, \\ Y(p), & \text{for } p \in \Sigma^-, \end{cases} \tag{3-2}$$

with $X,Y \in \Lambda(U)$. Note that Z is multivalued in Σ. To understand the dynamics of Z in Σ we can adopt certain conventions, inspired by the geometric structure or physics of our object of study. Before telling how we will visualize the interaction of X and Y in Σ, let is classify the points p of Σ based on the orientation of $X(p)$ and $Y(p)$ with respect to $\nabla f(p)$.

We will highlight the three most important properties of the gradient of a differentiable function f. In this discussion, we will fix a point p and assume that $\nabla f(p) \neq 0$. Then

(i) The gradient points in a direction in which the function f is increasing;
(ii) Of all the directions along which the function f grows, the direction of the gradient is the fastest growing;
(iii) The gradient of f at point p is orthogonal to the level surface of f that passes through this point.

From the above, we can see that the gradient of f always points to the region Σ^+. This is important for the items (ii) and (iii) of the Definition 3.2.3.

Definition 3.2.2 *The **Lie derivatives** of f at p in the direction of X at $q \in U$ is defined by*

$$X(q)f(p) = \langle X(q), \nabla f(p) \rangle,$$

and

$$X^i(q)f(p) = X(q)(X^{i-1}(q)f(p)) = \langle X(q), \nabla(X^{i-1}(q) \cdot f(p)) \rangle, \quad i \geq 2,$$

where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^n$. When $q = p$, we write $X^i(q)f(p) = X^i f(p)$, for all $i \geq 1$.

Definition 3.2.3 *Considering the definition of Z given in (3-2). We distinguish the following regions on the discontinuity set Σ:*

(i) ***Sewing region****:* $\Sigma^c = \{p \in \Sigma;\, Xf(p) \cdot Yf(p) > 0\}$ *(see Figure 3.3).*

(ii) ***Escaping region****:* $\Sigma^e = \{p \in \Sigma;\, Xf(p) > 0$ *and* $Yf(p) < 0\}$ *(see Figure 3.4).*

(iii) ***Sliding region****:* $\Sigma^s = \{p \in \Sigma;\, Xf(p) < 0$ *and* $Yf(p) > 0\}$ *(see Figure 3.4).*

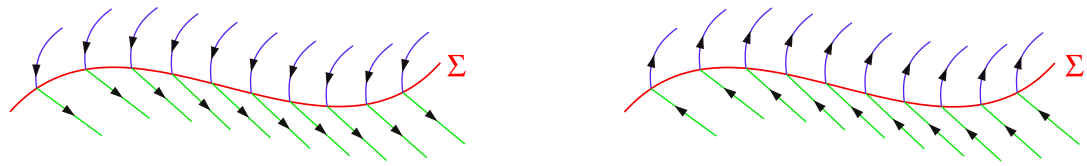

Figure 3.3: *Sewing region.*

Figure 3.4: *Sliding and escape region, respectively.*

Let $p \in \Sigma^s \cup \Sigma^e$ and $r_p : [0,1] \to \mathbb{R}^n$ given by

$$r_p(\alpha) = (1-\alpha)Y(p) + \alpha X(p).$$

Note that r_p corresponds to the line connecting the points $X(p)$ and $Y(p)$, as illustrate at Figure 3.5

The ***sliding vector field*** associated to $Z \in \Omega$ is the vector field Z^Σ tangent to Σ and defined at $p \in \Sigma^s \cup \Sigma^e$ by

$$Z^\Sigma(p) = r_p(\alpha_p), \tag{3-3}$$

where $\alpha_p \in (0,1)$ is such that $r_p(\alpha_p) \in T_p\Sigma$, i.e.

$$\langle r_p(\alpha_p), \nabla f(p) \rangle = 0.$$

The Figure 3.6 illustrate the vector $Z^\Sigma(p)$.

Let is show that α_p exists and is unique in $(0,1)$ for each $p \in \Sigma^s \cup \Sigma^e$. Let is determine for which α,

$$\langle r_p(\alpha), \nabla f(p) \rangle = 0. \tag{3-4}$$

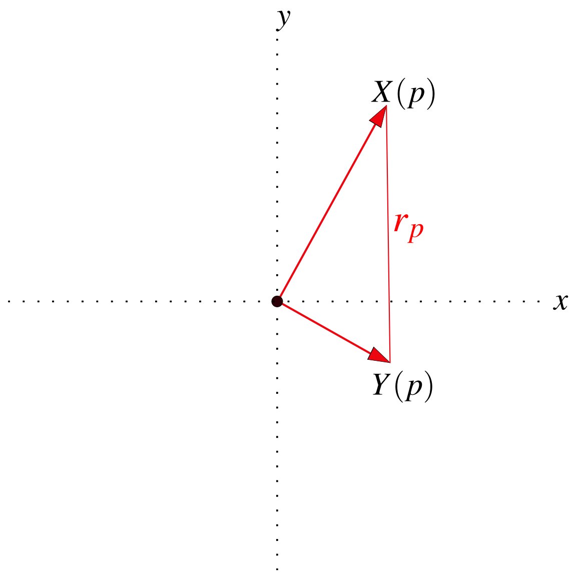

Figure 3.5: *Line r_p connecting the points $X(p)$ and $Y(p)$.*

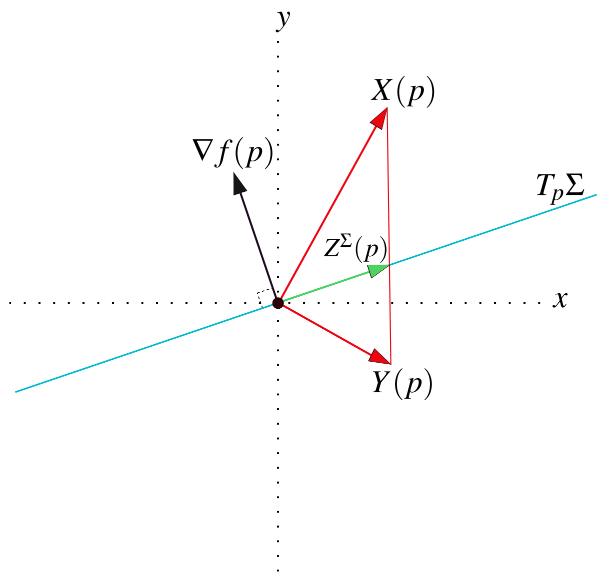

Figure 3.6: *Definition of Z^{Σ}.*

We have that

$$\begin{aligned}
\langle r_p(\alpha), \nabla f(p)\rangle = 0 \quad &\Leftrightarrow \quad \langle (1-\alpha)Y(p) + \alpha X(p), \nabla f(p)\rangle = 0 \\
&\Leftrightarrow \quad (1-\alpha)\langle Y(p), \nabla f(p)\rangle + \alpha\langle X(p), \nabla f(p)\rangle = 0 \\
&\Leftrightarrow \quad \alpha(\langle X(p), \nabla f(p)\rangle - \langle Y(p), \nabla f(p)\rangle) + \langle Y(p), \nabla f(p)\rangle = 0 \\
&\Leftrightarrow \quad \alpha = \frac{-\langle Y(p), \nabla f(p)\rangle}{\langle X(p), \nabla f(p)\rangle - \langle Y(p), \nabla f(p)\rangle}.
\end{aligned}$$

It follows from the last equality that there is only one α that satisfies (3-4). We then define

$$\alpha_p = \frac{-\langle Y(p), \nabla f(p)\rangle}{\langle X(p), \nabla f(p)\rangle - \langle Y(p), \nabla f(p)\rangle} = \frac{\langle Y(p), \nabla f(p)\rangle}{\langle Y(p) - X(p), \nabla f(p)\rangle}. \tag{3-5}$$

To simplify the notation, denote:

$$a = a(p) = \langle Y(p), \nabla f(p)\rangle \text{ and } b = b(p) = \langle X(p), \nabla f(p)\rangle. \tag{3-6}$$

Thus,

$$\alpha_p = \frac{a}{a-b}.$$

Note that $ab < 0$ (since $p \in \Sigma^s \cup \Sigma^e$). If $a > 0$ so $b < 0$, thus $(a-b) > 0$. Thus,

$$a > 0 \;\Leftrightarrow\; \frac{a}{a-b} > 0 \;\Leftrightarrow\; \alpha_p > 0.$$

On the other hand,

$$a < a-b \;\Leftrightarrow\; \frac{a}{a-b} < 1 \;\Leftrightarrow\; \alpha_p < 1.$$

Hence, $0 < \alpha_p < 1$. The other case is analogous. Therefore for each $p \in \Sigma^s \cup \Sigma^e$ exists an unique $\alpha_p \in (0,1)$ given at (3-5) such that $r_p(\alpha_p) \in T_p\Sigma$. Then Z^Σ is well defined at (3-3). Thus,

$$Z^\Sigma(p) = r_p(\alpha_p) = (1-\alpha_p)Y(p) + \alpha_p X(p).$$

Therefore

$$Z^\Sigma(p) = \frac{Yf(p)\cdot X(p) - Xf(p)\cdot Y(p)}{Yf(p) - Xf(p)}. \tag{3-7}$$

From the Figure 3.6, the Figure 3.7 illustrates the vector field Z^Σ applied to p in the phase space of Z.

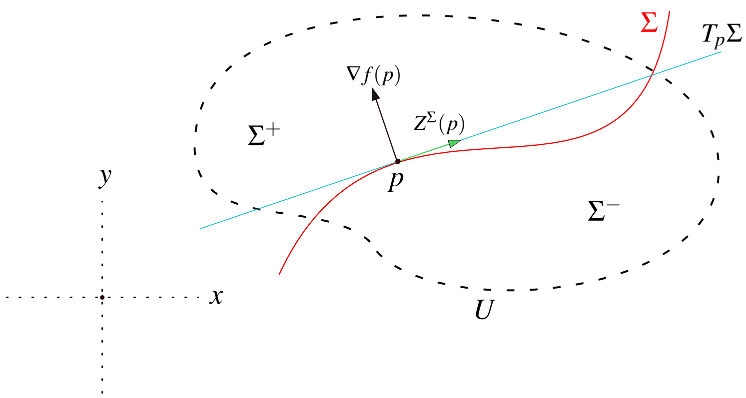

Figure 3.7: *Z^Σ applied at p in the phase space of Z.*

The definition of Z^Σ given in (3-3) is the Filippov's convention. Next we will introduce the Utkin's convention (or Equivalent Control Method), see [20]. Consider the vector field

$$\dot{x} = F(x,\mu),$$

with $x \in \mathbb{R}^n$ and $\mu = \mu(x) \in \mathbb{R}$ given by

$$\mu(x) = \begin{cases} \mu^+(x), & \text{if } f(x) > 0, \\ \mu^-(x), & \text{if } f(x) < 0. \end{cases}$$

Let be $p \in \Sigma$. According to Utkin, we define the sliding vector field as

$$W^\Sigma(p) = F(p,\widetilde{\mu}) \tag{3-8}$$

where $\langle F(p,\widetilde{\mu}), \nabla f(p)\rangle = 0$ for some $\widetilde{\mu} = \widetilde{\mu}(p) \in I \subset \mathbb{R}$. Note that depending on the law of formation of F and $p \in \Sigma$, $\widetilde{\mu}(p)$ may not exist, it may exist and be unique or it may exist and not be unique. The set I can vary depending on the problem.

If control power μ is linear in F, we can write

$$F(x,\mu) = H(x) + G(x)\mu.$$

If $p \in \Sigma^s \cup \Sigma^e$, from (3-7), follows that

$$Z^\Sigma(p) = H(p) - G(p)\frac{Hf(p)}{Gf(p)}.$$

From (3-8), we have that

$$\widetilde{\mu}(p) = -\frac{Hf(p)}{Gf(p)}.$$

Then if μ is linear in F we conclude that

$$Z^\Sigma(p) = W^\Sigma(p), \quad \forall\, p \in \Sigma^s \cup \Sigma^e.$$

Note that if $p \notin \Sigma^s \cup \Sigma^e$ and $Gf(p) \neq 0$ then $W^\Sigma(p)$ is well defined, which is not the case in Filippov's convention, since Z^Σ is only defined for the escaping and sliding points. Furthermore, when μ is not linear in F we can get different results between W^Σ and Z^Σ.

Filippov's and Utkin's conventions are algebraically equivalent when μ is linear in F. Filippov's convention is that just the right convex combination of the vector fields needs to be taken for the resulting vector field $Z^\Sigma(p)$ to lie in $T_p\Sigma$; Utkin's convention has the interpretation that $\widetilde{\mu}(p)$ is precisely the control power that is needed to pull the flow back to being in a direction that is tangent to Σ at p.

The Utkin's convention depends on a control power introduced into the formation law of F, and this control power is responsible for determining the sliding region and the sliding vector field associated (see [19]). The control power allows us to tell when the discontinuous vector field, in the region of discontinuity, can take on a vector tangent to the discontinuity. In other words, at a point $q \in \Sigma$, the power control $\widetilde{\mu}$ may or may not take on values that allow $F(q,\widetilde{\mu})$ to be tangent to Σ at q. Utkin's convention presents a characterization of the sliding vector field W^{Σ} that is strongly linked to what the model proposes to describe, and in this scenario we are faced with many practical problems. The following example (see [7]) presents a model in which we use the concept of power control in the sense to Utkin's convention for determine the vector field W^{Σ}.

Example 3.2.2 *Consider the piecewise-smooth vector field $F(x,y,\mu)$ given by*

$$\begin{aligned} \dot{x} &= \quad 2\mu^2 - 1, \\ \dot{y} &= 2\mu^2 - \mu - x, \end{aligned} \tag{3-9}$$

where $\mu(x,y) = sign(y)$, $y \neq 0$, and power control $\widetilde{\mu} \in I = [-1,1]$.

The region of discontinuity of F is given by $f^{-1}(0) = \Sigma = \{(x,y) \in \mathbb{R}^2;\ y = 0\}$, where $f(x,y) = y$. Then $\nabla f(p) = (0,1)$ for all $p \in \Sigma$. Let be $p = (x,y) \in \Sigma$. From (3-8), follows that vector field W^{Σ} at p is defined by

$$W^{\Sigma}(p) = (2\widetilde{\mu}^2 - 1, 2\widetilde{\mu}^2 - \widetilde{\mu} - x),$$

where $\langle F(x,y,\widetilde{\mu}), \nabla f(p)\rangle = 0$ for some $\widetilde{\mu} = \widetilde{\mu}(p) \in [-1,1]$. We have

$$\langle F(x,y,\widetilde{\mu}), \nabla f(p)\rangle = 0 \;\Leftrightarrow\; 2\widetilde{\mu}^2 - \widetilde{\mu} - x = 0 \;\Leftrightarrow\; \widetilde{\mu}(x,y) = \widetilde{\mu}(x) = \frac{1 \pm \sqrt{1+8x}}{4}.$$

We must have $x \geq -\frac{1}{8}$. Note that there is a solution $\widetilde{\mu}(x) \in I$ only if $x \in J = [-\frac{1}{8}, 3]$. If $-\frac{1}{8} \leq x \leq 1$ then

$$\widetilde{\mu}(x) = \frac{1 \pm \sqrt{1+8x}}{4};$$

if $1 < x \leq 3$ then

$$\widetilde{\mu}(x) = \frac{1 - \sqrt{1+8x}}{4}.$$

Notice that there are values of x in J for which $\widetilde{\mu}(x)$ has two values in I. This is different from the Filippov's convention that assumes only one value for the sliding vector field in each point $q \in \Sigma^s \cup \Sigma^e$. We can also see that the sliding vector field according to Filippov applies only to the points of Σ such that $1 < x < 3$, unlike what happens in this example, in which the "sliding vector field" is defined at all points of Σ such that $-\frac{1}{8} \leq x \leq 3$. If we consider that

$$\widetilde{\mu}(x) = \frac{1 - \sqrt{1+8x}}{4},$$

for all $x \in J$, then

$$W^{\Sigma}(x,0) = \left(x + \frac{\sqrt{1+8x}-3}{4}, 0 \right),$$

for all $x \in J$.

The vector field F from (3-9) can be write as $Z = (X,Y)$, where $X(x,y) = (1, 1-x)$ and $Y(x,y) = (1, 3-x)$. The vector field Z^{Σ} from (3-7) is given by

$$Z^{\Sigma}(x,0) = (1,0),$$

for all $x \in (1,3)$. ℵ

As mentioned at the beginning of this dissertation, we will adopt the Filippov's convention and the regularization process to treat piecewise-smooth systems. In the following definitions, unless otherwise stated, the sliding vector field considered is the from the Filippov's convention.

Definition 3.2.4 *We say that $p \in \Sigma$ is a Σ-**regular point** if*

(i) $p \in \Sigma^c$ or

(ii) $p \in \Sigma^s \cup \Sigma^e$ and $Z^{\Sigma}(p) \neq 0$.

*The points of Σ which are not Σ-regular are called Σ-**singular**. We distinguish two subsets in the set of Σ-singular points: Σ^t and Σ^d. They are given by*

*(i) $\Sigma^t = \{p \in \Sigma;\ Xf(p) \cdot Yf(p)) = 0\}$. A point $p \in \Sigma^t$ is called a **tangential singularity** or **tangency point** of Z;*

*(ii) $\Sigma^d = \{p \in \Sigma^s \cup \Sigma^e;\ Z^{\Sigma}(p) = 0\}$. A point $p \in \Sigma^d$ is called a **pseudo equilibrium** of Z.*

Theorem 3.2.1 *Let $g : I \subset \mathbb{R} \to \mathbb{R}$ be a function n times derivable at a point p, inside to I. Suppose that $g'(p) = g''(p) = \cdots = g^{(n-1)}(p) = 0$, but $g^n(p) \neq 0$. We claim that*

(i) If n is even, then p will be a point of local maximum if $g^n(p) < 0$, or a point of local minimum if $g^n(p) > 0$.

(ii) If n is odd, the point p is neither a local maximum nor a local minimum.

The proof of the Theorem 3.2.1 can be found in [10].

Definition 3.2.5 *Let $p \in \Sigma$. For a given $W \in \Lambda(U)$ we say that r is the **contact order** of the trajectory Γ_W^p of W with Σ at p if $W^k f(p) = 0$, for all $k = 1, \ldots, r-1$ and $W^r f(p) \neq 0$.*

Considering the vector field Z given at (3-2). Let Γ_X^p be the trajectory of X at $p \in \Sigma^t$, with $X(p) \neq 0$ and $\alpha_p : A_p \to \mathbb{R}^n$ the integral curve associated, i.e, $\alpha_p(0) = p$ and $\alpha'_p(t) = X(\alpha_p(t))$. Suppose that r is the contact order of the trajectory Γ_X^p with Σ at p. Let is analyze the behavior of Γ_X^p in relation to $T_p\Sigma$ at the point p. By Corollary 1.1.2, since α_p is an immersion on I_p, it is easy to see that Γ_X^p is a manifold of dimension one according to the Definition 1.1.3. So by Theorem 1.1.7 it follows that, for each $q \in \Gamma_X^p$, Γ_X^p is locally the graphical of a map $g_q : I_q \subset \mathbb{R} \to \mathbb{R}^{n-1}$, i.e., for each $q \in \Gamma_X^p$ there is an open $V_q \subset \mathbb{R}^n$ such that $V_q \cap \Gamma_X^p = G(g_q) = \{(x, g_q(x)) \in \mathbb{R}^n; x \in I_q\}$.

Since $p \in \Sigma^t$ follows that $X(p) \in T_p\Sigma$, then Γ_X^p is tangent to $T_p\Sigma$ at p. To understand the nature of the tangency of Γ_X^p in $T_p\Sigma$ at the point p, let is rotate the canonical system of coordinates of $\mathbb{R}^n$ so that one of the directions coincides with the direction of $\nabla f(p)$, and the other (orthogonal) directions are given in $T_p\Sigma$. The Figure 3.8 illustrate the construction of the system of coordinates.

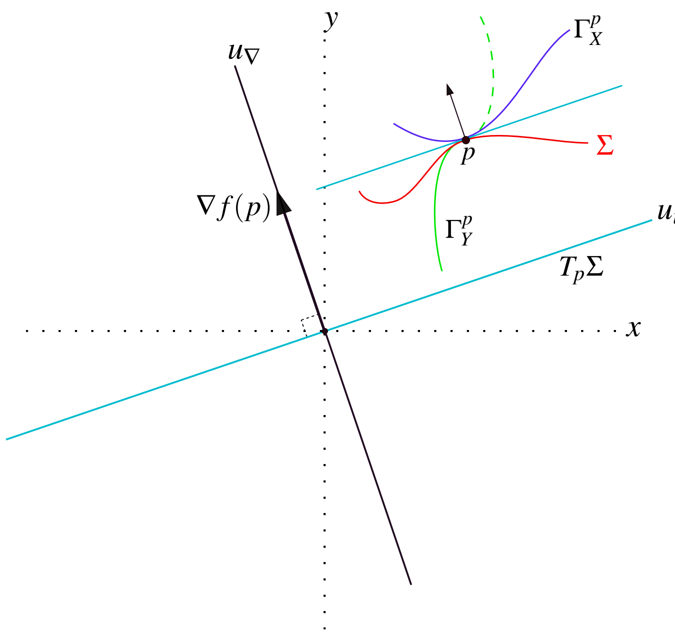

Figure 3.8: *Abstract representation of the system of coordinate obtained from $\nabla f(p)$ and $T_p\Sigma$.*

We can write α_p in the new system of coordinates as $\beta_{p_u}(t) = (u_1(t), u_2(t), \ldots, u_{n-1}(t), u_\nabla(t))$, where $p_u = (p_1, p_2, \ldots, p_{n-1}, p_\nabla)$ is the point p in the new coordinates and $u_\nabla = u_\nabla(t)$ is the component of α_p in the direction of $\nabla f(p)$, given by

$$u_\nabla = \frac{\langle \nabla f(p), \alpha_p \rangle}{|\langle \nabla f(p), \alpha_p \rangle|} |\vec{u}_\nabla(t)| = \frac{\langle \nabla f(p), \alpha_p \rangle}{|\langle \nabla f(p), \alpha_p \rangle|} \left| \frac{\langle \nabla f(p), \alpha_p \rangle}{\langle \nabla f(p), \nabla f(p) \rangle} \cdot \nabla f(p) \right| = \left\langle \frac{\nabla f(p)}{|\nabla f(p)|}, \alpha_p \right\rangle.$$

Denote by $v(p) = \frac{\nabla f(p)}{|\nabla f(p)|}$. Then $u_\nabla(t) = \langle v(p), \alpha_p(t) \rangle$.

The reader should realize that in order to understand the nature of tangency, it is sufficient to understand its nature in the direction of $\nabla f(p)$. Note that $\beta'_{p_u}(t) \neq \vec{0}$ for all $t \in A_p$, because $\alpha'_p(t) \neq \vec{0}$ for all $t \in A_p$. In the new system of coordinates we will write Γ_X^p locally as the graph of function at the point p_u. Consider that $\Gamma_X^{p_u}$ is the curve Γ_X^p in the new system of coordinates and $g : I_{p_u} \subset \mathbb{R} \to \mathbb{R}^{n-1}$ is such that $G(g_{p_u})$ is an open of $\Gamma_X^{p_u}$ at p_u and we write $g(u_i) = (g_1(u_i), \ldots, g_{n-1}(u_i))$. So we can parameterize an open of $\Gamma_X^{p_u}$ at p_u as $\gamma : I_{p_u} \to \mathbb{R}^n$ given by $\gamma(u_i) = (g_1(u_i), g_2(u_i), \ldots, g_{i-1}(u_i), u_i, g_{i+1}(u_i), \ldots, g_{n-1}(u_i), g_\nabla(u_i))$, with $i \neq \nabla$. Note that $u'_\nabla(0) = 0$ (i.e $\beta'_{p_u}(0)$ is orthogonal to $\nabla f(p)$); if $i = \nabla$ then the component of $\gamma'(p_\nabla)$ in the direction of $\nabla f(p)$ is not zero, which contradicts the fact that $u'_\nabla(0) = 0$, hence $i \neq \nabla$.

Of curse $p_i \in I_{p_u}$ and from the continuity of α_p it follows that exists $\varepsilon > 0$ such that $\gamma(u_i(t)) = \beta_{p_u}(t)$ for all $t \in (-\varepsilon, \varepsilon) \subset A_p$. Thus,

$$g_\nabla(u_i(t)) = u_\nabla(t), \quad \forall t \in (-\varepsilon, \varepsilon).$$

We have that

$$g'_\nabla(u_i(t))u'_i(t) = u'_\nabla(t) = \langle v(p), \alpha'_p(t) \rangle = \langle v(p), X(\alpha_p(t)) \rangle. \tag{3-10}$$

Note that

$$\beta'_{p_u}(t) = \gamma'(u(t))u'_i(t),$$

thus

$$\beta'_{p_u}(0) = \gamma'(u_i(0))u'_i(0).$$

Since $\alpha'_p(0) = X(p) \neq \vec{0}$, follows that $\beta'_{p_u}(0) \neq \vec{0}$, so $u'_i(0) \neq 0$. Then from (3-10), we have

$$g'_\nabla(u_i(t)) = \frac{\langle v(p), \alpha'_p(t) \rangle}{u'_i(t)} \quad \Rightarrow \quad g'_\nabla(u_i(0)) = g'_\nabla(p_i) = \frac{\langle v(p), \alpha'_p(0) \rangle}{u'_i(0)} = \frac{\langle v(p), X(p) \rangle}{u'_i(0)} = 0.$$

Note that

$$g'_\nabla(p_i) = 0 \;\Leftrightarrow\; \langle \nabla f(p), X(p) \rangle = 0 \;\Leftrightarrow\; Xf(p) = 0.$$

To understand the nature of the tangency of Γ_X^p in $T_p\Sigma$ at the point p, we just need to understand the nature of the critical point p_i of the function g_∇. We have that

$$g'_\nabla(u_i(t))u'_i(t) = \langle v(p), \alpha'_p(t) \rangle = u'_\nabla(t).$$

On the other hand,

$$g''_\nabla(u_i(t)) \cdot [u'_i(t)]^2 + g'_\nabla(u_i(t)) \cdot u''_i(t) = u''_\nabla(t); \tag{3-11}$$

thus

$$g''_{\nabla}(p_i)\cdot[u'_i(0)]^2+g'_{\nabla}(p_i)\cdot u''_i(0)=u''_{\nabla}(0) \;\Rightarrow\; g''_{\nabla}(p_i)=\frac{u''_{\nabla}(0)}{[u'_i(0)]^2};$$

then

$$g''_{\nabla}(p_i)=0 \;\Leftrightarrow\; u''_{\nabla}(0)=0 \;\Leftrightarrow\; \langle v(p),\alpha''_p(0)\rangle=0 \;\Leftrightarrow\; \langle\nabla f(p),\alpha''_p(0)\rangle=0.$$

Note that

$$\alpha''_p(t)=J(X)(\alpha_p(t))\cdot X(\alpha_p(t))=[J(X)\cdot X](\alpha_p(t)), \tag{3-12}$$

thus

$$g''_{\nabla}(p_i)=0 \;\Leftrightarrow\; \langle\nabla f(p),J(X)(\alpha_p(0))\cdot X(\alpha_p(0))\rangle=\langle[J(X)]^T(p)\nabla f(p),X(p)\rangle=0.$$

In the last equality 'T' denotes the transposed matrix. Denote by $\xi(q)=\langle X(q),\nabla f(p)\rangle$, $q\in U$. The differential of ξ at q in the direction of h is given by

$$D(\xi)(q)\cdot h=\langle J(X)(q)\cdot h,\nabla f(p)\rangle=\langle[J(X)]^T(q)\cdot\nabla f(p),h\rangle.$$

Hence,

$$\nabla\xi(q)=[J(X)]^T(q)\cdot\nabla f(p),\quad \forall q\in U.$$

Particularly,

$$\nabla\xi(p)=\nabla(\langle X(p),\nabla f(p)\rangle)=[J(X)]^T(p)\nabla f(p).$$

Thus,

$$g''_{\nabla}(p_i)=0 \;\Leftrightarrow\; \langle X(p),\nabla(\langle X(p),\nabla f(p)\rangle)\rangle=\langle X(p),\nabla(Xf(p))\rangle=0 \;\Leftrightarrow\; X^2f(p)=0.$$

Now deriving (3-11) again with respect to t and evaluating at $t=0$, we get

$$g^{(3)}_{\nabla}(p_i)=\frac{u^{(3)}_{\nabla}(0)}{[u'_i(0)]^3},$$

given that $g'_{\nabla}(p_i)=g''_{\nabla}(p_i)=0$. Furthermore, similarly to what was done,

$$g^{(3)}_{\nabla}(p_i)=0 \;\Leftrightarrow\; \langle\nabla f(p),\alpha^{(3)}_p(0)\rangle=0.$$

From (3-12) denote by $X_1(q)=[J(X)\cdot X](q)$, so $X_1(\alpha_p(t))=\alpha^{(2)}_p(t)$. We have

$$\alpha^{(3)}(t)=J[X_1](\alpha_p(t))\cdot X(\alpha_p(t)).$$

Note that

$$\langle \nabla f(p), \alpha_p^{(3)}(t)\rangle = \langle \nabla f(p), J[X_1](\alpha_p(t))\cdot X(\alpha_p(t))\rangle = \langle J[X_1]^T(\alpha_p(t))\cdot \nabla f(p), X(\alpha_p(t))\rangle$$

and

$$\nabla \xi_1(q) = J[X_1]^T(q)\cdot \nabla f(p),$$

where $\xi_1(q) = \langle X_1(q), \nabla f(p)\rangle$, $q \in U$. Thus,

$$\langle \nabla f(p), \alpha_p^{(3)}(0)\rangle = \langle \nabla \xi_1(p), X(p)\rangle = \langle X(p), \nabla \xi_1(p)\rangle.$$

We can also see that

$$\nabla \xi_1(p) = \nabla(X^2 f(p)).$$

Hence,

$$\langle \nabla f(p), \alpha_p^{(3)}(0)\rangle = \langle X(p), \nabla(X^2 f(p))\rangle = X^3 f(p).$$

Therefore,

$$g_\nabla^{(3)}(p_i) = 0 \;\Leftrightarrow\; X^3 f(p) = 0.$$

Using an argument by induction, assuming that $g'_\nabla(p_i) = g''_\nabla(p_i) = g_\nabla^{(3)}(p_i) = \cdots = g_\nabla^{(n-1)}(p_i) = 0$, it follows that the equivalence

$$g_\nabla^{(n)}(p_i) = 0 \;\Leftrightarrow\; X^n f(p) = 0, \tag{3-13}$$

is true for all $n \in \mathbb{N}$. In addition, in the condition presented, it is worth noting that

$$g_\nabla^{(n)}(p_i) = \frac{u_\nabla^{(n)}(0)}{[u'_i(0)]^n}.$$

Since the contact order of the trajectory Γ_X^p with Σ at p is r, so $X^k f(p) = 0$ for all $k = 1, \cdots, r-1$, then from (3-13) follows $g_\nabla^k(p_i) = 0$ for all $k = 0, \cdots, r-1$ and $g_\nabla^r(p_i) \neq 0$. We have that

$$g_\nabla^{(r)}(p_i) = \frac{u_\nabla^{(r)}(0)}{[u'_i(0)]^r}. \tag{3-14}$$

Then if r is odd, from the Theorem 3.2.1, follows that the point p_i is neither a local maximum nor a local minimum for g_∇. In this case p_i will be a turning point. The nature of the tangency of the trajectory Γ_X^p is characterized by something analogous to the green curve illustrated in Figure 3.8. In this case we say that p is a ***visible tangency*** of Γ_X^p passing through p, because the trajectory not only tangents but also crosses $T_p\Sigma$ at p. If $p \in \Sigma^t$ is such that $Y(p) \neq 0$ and the contact order m of the trajectory Γ_Y^p is odd, then we have the same conclusions for Γ_Y^p, i.e, p is a visible tangency of Γ_Y^p passing through p. On the other hand if r is even, from the Theorem 3.2.1, p_i will be a point of local maximum if $g^r_\nabla(p_i) < 0$, or a point of local minimum if $g^r_\nabla(p_i) > 0$. Since r is even, from (3-14), we see that the sign of $g^{(r)}_\nabla(p_i)$ is the same of $u^{(r)}_\nabla(0)$; next, we note that the sign of $u^{(r)}_\nabla(0)$ is the same as that of $X^r f(p)$. Therefore, when r is even, we have

$$g^{(r)}_\nabla(p_i) > 0 \;\Leftrightarrow\; X^r f(p) > 0$$

and

$$g^{(r)}_\nabla(p_i) < 0 \;\Leftrightarrow\; X^r f(p) < 0.$$

If $X^r f(p) > 0$ then p_i is a point of local minimum for g_∇, thus the trajectory Γ^p_X in a neighborhood of p, does not cross $T_p\Sigma$ at p and remains in the half-space of $\mathbb{R}^n$ defined by $T_p\Sigma$ at p, in which the gradient of f at p points. In this case also we say that p is a visible tangency of Γ^p_X passing through p (the blue curve in Figure 3.8 illustrates this situation). If $X^r f(p) < 0$ then p_i is a point of local maximum for g_∇, thus the trajectory Γ^p_X in a neighborhood of p, does not cross $T_p\Sigma$ at p and remains in the half-space of $\mathbb{R}^n$ defined by $T_p\Sigma$ at p, in which the gradient of f at p does not points. In this case we say that p is an ***invisible tangency*** of Γ^p_X passing through p. If $p \in \Sigma^t$ is such that $Y(p) \neq 0$ and the contact order m of the trajectory Γ^p_Y is even, we have similar conclusions for Γ^p_Y. If $Y^m f(p) > 0$ we say that p is an invisible tangency of Γ^p_Y passing through p. If $Y^m f(p) < 0$ we say that p is a visible tangency of Γ^p_Y passing through p. This discussion leads us to the following definition:

Definition 3.2.6 *Let $p \in \Sigma^t$ and r, m the contact order of Γ^p_X and Γ^p_Y, respectively.*

*(i) We say that p is **singular tangential point** if p is an invisible tangency for both X and Y, in other words, r and m are even, $X^r f(p) < 0$ and $Y^m f(p) > 0$.*

*(ii) We say that p is **regular tangential point** if p is a visible tangency for X or Y, in other words, r or m is odd; or r and m are even and, $X^r f(p) > 0$ or $Y^m f(p) < 0$.*

It is important for the reader to be aware of the fact that if $p \in \Sigma^t$ and $X(p) = 0$ then $X^i f(p) = 0$, for all $i \in \mathbb{N}$. In this case we say that the contact order r of Γ^p_X is $r = \infty$. The same goes for the Γ^p_Y if $Y(p) = 0$. When the contact order of Γ^p_X (or Γ^p_Y) is $r = \infty$ we cannot apply the Definition 3.2.6 at p. In this situation we must analyze the behavior of the vector field X (or Y) in a neighborhood of p in Σ^+ (or Σ^-) to tell how the orbits behave near it in relation to Σ.

Figures 3.9 and 3.10 illustrate all possible cases for regular and singular tangencies, respectively.

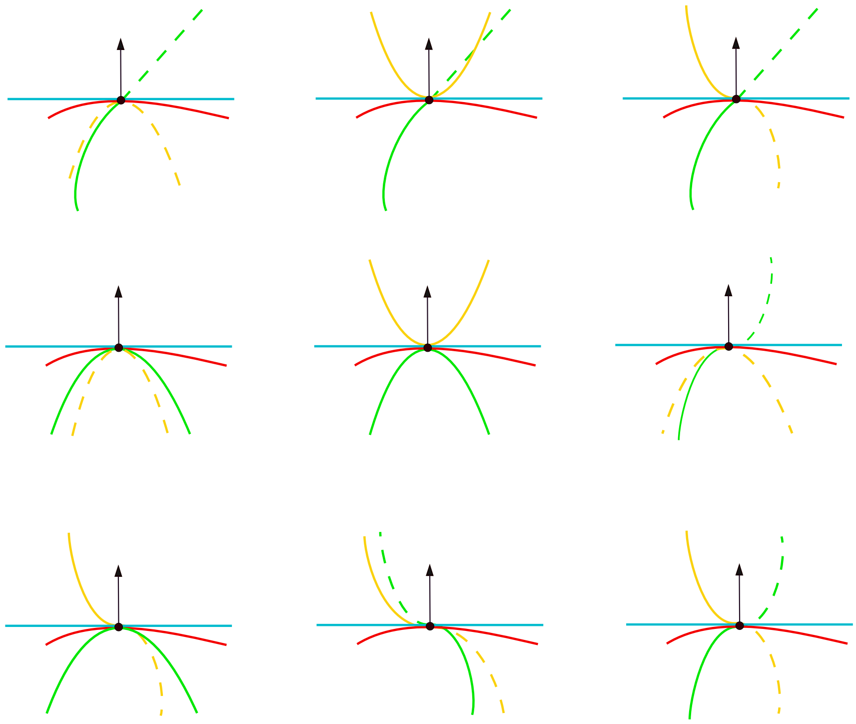

Figure 3.9: *Regular tangencies - the orange curve represents Γ_X^p, the green curve is Γ_Y^p, the red curve is Σ, the blue line is $T_p\Sigma$ and the black vector is $\nabla f(p)$.*

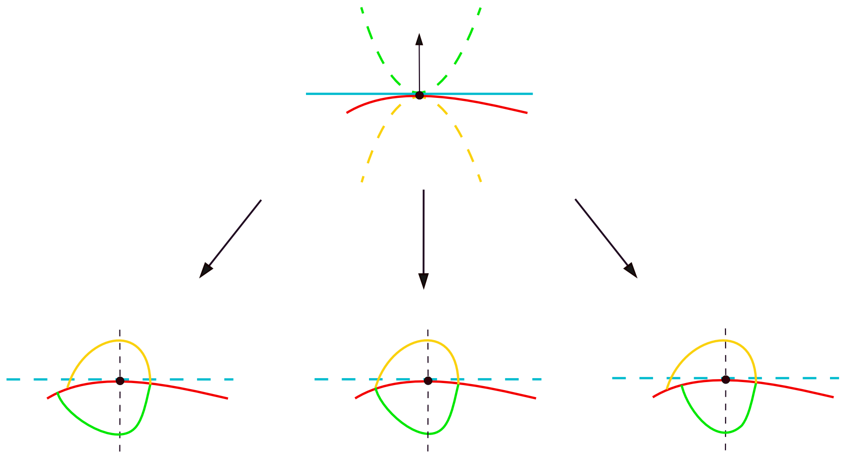

Figure 3.10: *Singular tangency - the color of the curves has the same meaning as in the Figure 3.9.*

Let be $W \in \Lambda(U)$ and $p \in U$. Then we denote its flow by $\varphi_W(t,p)$. Thus,

$$\begin{cases} \dfrac{d}{dt}\varphi_W(t,p) = W(\varphi_W(t,p)), \\ \varphi_W(0,p) = p, \end{cases}$$

where $t \in I = I_W^p \subset \mathbb{R}$. The interval I depends on the point p and the vector field W.

Definition 3.2.7 *The local integral curve (or simply* ***local trajectory****) $\varphi_Z(t,p)$ from (3-2) is defined as follows:*

- *For $p \in \Sigma^+ \backslash \Sigma$ or $p \in \Sigma^- \backslash \Sigma$ the trajectory is given by $\varphi_Z(t,p) = \varphi_X(t,p)$ and $\varphi_Z(t,p) = \varphi_Y(t,p)$ respectively, where $t \in I$. It is implied that I can be I_X^p or I_Y^p, whichever makes sense. This also applies to the other items.*
- *For $p \in \Sigma^c$ such that $Xf(p) > 0$, $Yf(p) > 0$ and taking the origin of the time at p, the trajectory is defined as $\varphi_Z(t,p) = \varphi_Y(t,p)$ for all $t \in I_Y^p \cap \{t \leq 0\}$ and $\varphi_Z(t,p) = \varphi_X(t,p)$ for all $t \in I_X^p \cap \{t \geq 0\}$. For the case $Xf(p) < 0$ and $Yf(p) < 0$ the definition is the same reversing the time.*
- *For $p \in \Sigma^e$ and taking the origin of the time at p, the trajectory is defined as $\varphi_Z(t,p) = \varphi_{Z^\Sigma}(t,p)$ for $t \in I_{Z^\Sigma}^p \cap \{t \leq 0\}$ and $\varphi_Z(t,p)$ is either $\varphi_X(t,p)$ or $\varphi_Y(t,p)$ or $\varphi_{Z^\Sigma}(t,p)$ for $t \in I \cap \{t \geq 0\}$. For the case $p \in \Sigma^s$ the definition is the same but reversing time.*
- *For p a regular tangency point and taking the origin of the time at p, the trajectory is defined as $\varphi_Z(t,p) = \varphi_1(t,p)$ for $t \in I \cap \{t \leq 0\}$ and $\varphi_Z(t,p) = \varphi_2(t,p)$ for $t \in I \cap \{t \geq 0\}$ for $t \in I \cap \{t \geq 0\}$ where each φ_1, φ_2 is either φ_X or φ_Y or φ_{Z^Σ}.*
- *For p a singular tangency point we have $\varphi_Z(t,p) = p$ for all $t \in I$.*

Definition 3.2.8 *A* ***global trajectory*** *(or orbit) $\phi_Z(t,p_0)$ of $Z \in \Omega(U)$ passing through p_0 is an union*

$$\phi_Z(t,p_0) = \bigcup_{i \in \mathbb{Z}} \{\varphi_i(t,p_i); t_i \leq t \leq t_{i+1}\}$$

of preserving-orientation local trajectories $\varphi_i(t,p_i)$ satisfying $\varphi_i(t_{i+1},p_i) = \varphi_{i+1}(t_{i+1},p_{i+1}) = p_{i+1}$ and $t_i \to \pm\infty$ as $i \to \pm\infty$. A global trajectory is a positive (respectively, negative) global trajectory if $i \in \mathbb{N}$ (respectively, $-i \in \mathbb{N}$) and $t_0 = 0$.

The reader can find some examples of the description of integral curves of piecewise-smooth vector fields according to Definition 3.2.7 at [6].

Next we present the concept of a periodic orbit for Z, and then the concept of a closed trajectory. It's important to note that in the context of piecewise-smooth vector fields Z of the form (3-2), every periodic orbit is a closed trajectory, but not every closed trajectory is a periodic orbit.

Definition 3.2.9 *Let $\phi_Z(t,p)$ a global trajectory of Z (given in (3-2)). We say that ϕ_Z is periodic if ϕ_Z is periodic in the variable t, i.e., if there exist $T > 0$ such that $\phi_Z(t+T,p) = \phi_Z(t,p)$, for all $t \in \mathbb{R}$.*

Definition 3.2.10 *Consider the piecewise-smooth vector field Z given in (3-2). A closed global trajectory Γ of Z is a:*

(i) ***pseudo cycle*** *if* $\Gamma \cap \Sigma \neq \emptyset$ *and it does not contain neither equilibrium nor pseudo equilibrium point (see Figure 3.11).*

(ii) ***pseudo graph*** *if* $\Gamma \cap \Sigma \neq \emptyset$ *and it is an union of equilibrium, pseudo equilibrium and orbit-arcs of Z joining these points (see Figure 3.12).*

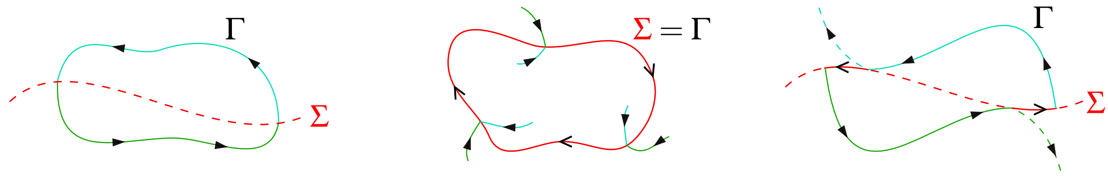

Figure 3.11: *Possible kinds of pseudo cycles.*

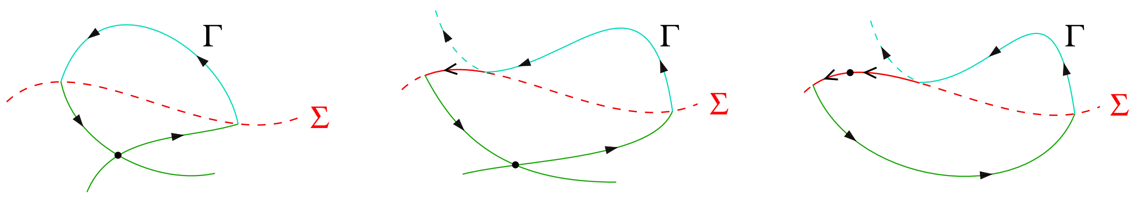

Figure 3.12: *Examples of pseudo graphs.*

3.3 Topological Equivalence Between Distinct Filippov Vector Fields

The definitions and results presented in this section can be found at [6]. In order for the reader to also have a clear understanding of the subject matter presented, it is recommended to refer to Section 1.4. Considering the definition of a discontinuous vector field $Z=(X,Y)$ given in (3-2). In this section, in order to simplify the notation in the proof of the results, we will consider that $\Sigma^+=\{p\in U;\, f(p)>0\}$ and $\Sigma^-=\{p\in U;\, f(p)<0\}$; with X defined on $\Sigma^+\cup\Sigma$ and Y defined on $\Sigma^-\cup\Sigma$.

Definition 3.3.1 *Two Filippov vector fields Z and $\widetilde{Z}$ of Ω defined in open sets U and V and with discontinuity curves $\Sigma\subset U$ and $\widetilde{\Sigma}\subset V$ respectively are Σ-equivalent if there exists an orientation preserving homeomorphism $h:U\to V$ which sends Σ to $\widetilde{\Sigma}$ and sends orbits of Z to orbits of $\widetilde{Z}$.*

Definition 3.3.2 *Two Filippov vector fields Z and $\widetilde{Z}$ of Ω defined in open sets U and V and with discontinuity curves $\Sigma\subset U$ and $\widetilde{\Sigma}\subset V$ respectively are topologically equivalent if there exists an orientation preserving homeomorphism $h:U\to V$ which sends orbits of Z to orbits of $\widetilde{Z}$.*

Proposition 3.3.1 *Let us consider any diffeomorphism $h:U\to V$ which conjugates on one hand, X in $\Sigma^+\subset U$ and $\widetilde{X}$ in $\widetilde{\Sigma}^+\subset V$ and, in the other hand, Y in $\Sigma^-\subset U$ and $\widetilde{Y}$ in $\widetilde{\Sigma}^-\subset V$. Then, it also conjugates the sliding vector fields Z^Σ and $\widetilde{Z}^{\widetilde{\Sigma}}$, and therefore h gives a topological equivalence between $Z=(X,Y)$ and $\widetilde{Z}=(\widetilde{X},\widetilde{Y})$.*

Proof: We have that $\Sigma=\{q\in U;\, f(q)=0\}$, so $\widetilde{\Sigma}=\{p\in V;\, \widetilde{f}(p)=f(h^{-1}(p))=0\}$. $\widetilde{X}$ is given by

$$\widetilde{X}(q)=J(h)\Big|_{h^{-1}(q)}\cdot(X\circ h^{-1})(q)$$

for all $q\in\widetilde{\Sigma}^+$; and $\widetilde{Y}$ by

$$\widetilde{Y}(q)=J(h)\Big|_{h^{-1}(q)}\cdot(Y\circ h^{-1})(q)$$

for all $q \in \widetilde{\Sigma}^-$. Given $p \in \widetilde{\Sigma}$, we have

$$\begin{aligned}
\widetilde{X}\widetilde{f}(p) &= \langle \widetilde{X}(p), \nabla \widetilde{f}(p) \rangle \\
&= \langle J(h)(h^{-1}(p)) \cdot X(h^{-1}(p)), \nabla f(h^{-1}(p)) \cdot [J(h)(h^{-1}(p))]^{-1} \rangle \\
&= \langle J(h)(h^{-1}(p)) \cdot X(h^{-1}(p)), [\nabla f(h^{-1}(p)) \cdot [J(h)(h^{-1}(p))]^{-1}]^T \rangle \\
&= \langle X(h^{-1}(p)), [J(h)(h^{-1}(p))]^T \cdot [\nabla f(h^{-1}(p)) \cdot [J(h)(h^{-1}(p))]^{-1}]^T \rangle \\
&= \langle X(h^{-1}(p)), \nabla f(h^{-1}(p)) \rangle \\
&= Xf(h^{-1}(p)).
\end{aligned}$$

Similarly, we can conclude that $\widetilde{Y}\widetilde{f}(p) = Yf(h^{-1}(p))$ whatever $p \in \widetilde{\Sigma}$. Thus, the sewing, escaping, sliding and tangency regions from Σ are preserved by h in $\widetilde{\Sigma}$. The sliding vector field $\widetilde{Z}^{\widetilde{\Sigma}}$ is given by

$$\begin{aligned}
\widetilde{Z}^{\widetilde{\Sigma}}(p) &= \frac{\widetilde{Y}\widetilde{f}(p) \cdot \widetilde{X}(p) - \widetilde{X}\widetilde{f}(p) \cdot \widetilde{Y}(p)}{\widetilde{Y}\widetilde{f}(p) - \widetilde{X}\widetilde{f}(p)} \\
&= [J(h)(h^{-1}(p))] \cdot \frac{Yf(h^{-1}(p)) \cdot X(h^{-1}(p)) - Xf(h^{-1}(p)) \cdot Y(h^{-1}(p))}{Yf(h^{-1}(p)) - Xf(h^{-1}(p))} \\
&= [J(h)(h^{-1}(p))] \cdot Z^{\Sigma}(h^{-1}(p)).
\end{aligned}$$

Then h sends orbits to orbits. ■

All the topological equivalences defined using the Proposition 3.3.1 preserve Σ and therefore are also Σ-equivalences. The following two propositions are important to exemplify the existence of smooth vector fields with topologically equivalent parts, but which are not Σ-equivalent.

Consider the Filippov vector field given by

$$Z_{\varepsilon,\mu}(x,y) = \begin{cases} X_{\varepsilon,\mu}(x,y) = \begin{pmatrix} -1 & \varepsilon \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y-\mu \end{pmatrix}, & \text{if } y > 0, \\ Y(x,y) = (1,1), & \text{if } y < 0, \end{cases}$$

where $\varepsilon, \mu \in \mathbb{R}$.

Proposition 3.3.2 *The Filippov vector fields $Z_{\varepsilon,\mu}(x,y)$ and $Z_{\widetilde{\varepsilon},\widetilde{\mu}}(x,y)$ with $\mu\widetilde{\mu} > 0$, $\varepsilon < 0$ and $\widetilde{\varepsilon} > 0$ are not Σ-equivalent.*

Proposition 3.3.3 *For μ and ε small enough, the Filippov vector fields $Z_{\varepsilon,\mu}(x,y)$ and $Z_{\widetilde{\varepsilon},\widetilde{\mu}}(x,y)$ with $\mu\widetilde{\mu} > 0$ are topologically equivalent.*

The proofs of Propositions 3.3.2 and 3.3.3 can be found in [6].

If we remove the hypothesis of differentiability in Proposition 3.3.1, that is, if we consider that h is only a homeomorphism in Proposition 3.3.1, it is no more true. We consider the following two vector fields defined in a neighborhood U of the origin and taking as discontinuity curve $\Sigma = \{(x,y);\, y=0\}$:

$$Z(x,y) = \begin{cases} X(x,y) = (0,-1), & \text{if } y > 0,\\ Y(x,y) = (0,1), & \text{if } y < 0, \end{cases}$$

and

$$\widetilde{Z}(x,y) = \begin{cases} \widetilde{X}(x,y) = (-1,-1), & \text{if } y > 0,\\ \widetilde{Y}(x,y) = (-1,1), & \text{if } y < 0. \end{cases}$$

In this case $\Sigma = \Sigma^s = \widetilde{\Sigma} = \widetilde{\Sigma}^s$ and the homeomorphism

$$h(x,y) = \begin{cases} (x-y,y), & \text{if } y > 0,\\ (x,y), & \text{if } y = 0,\\ (x+y,y), & \text{if } y < 0, \end{cases}$$

which is C^0 but not C^1, conjugates X with $\widetilde{X}$ for $y>0$ and Y and $\widetilde{Y}$ for $y<0$ but is not a topological equivalence of Z and $\widetilde{Z}$, since the corresponding sliding vector fields are $Z^{\Sigma}(x,y) = (0,0)$ and $\widetilde{Z}^{\widetilde{\Sigma}}(x,y) = (-1,0)$ which cannot be topologically equivalent.

Let us consider two smooth vector fields X and Y. Then, we denote by $X(p) \parallel Y(p)$ the fact that X and Y are parallel at p and by $X(p) \nparallel Y(p)$ the fact that X and Y are non-parallel at p. It is easy to see that $p \in \Sigma^s \cup \Sigma^e$ is a pseudo equilibrium of $Z = (X,Y) \in \Omega$ if, and only if, $X(p) \parallel Y(p)$.

Next proposition gives the normal form for regular points belonging to $\Sigma^c \cup \Sigma^s \cup \Sigma^e$. It is clear that around regular points which do not belong to Σ applies the Tubular Flow Theorem so that, in this study we only have to deal with points belonging to the discontinuity curve.

Proposition 3.3.4 *Given a Filippov planar vector field $Z = (X,Y)$ with discontinuity curve Σ and $\vec{0} \in \Sigma$, then:*

(i) If $\vec{0} \in \Sigma^c$, then in a neighborhood U of $\vec{0}$, Z is Σ-equivalent to the normal form

$$\widetilde{Z}(x,y) = \begin{cases} \widetilde{X}(x,y) = (0,1), & \textit{if } y > 0,\\ \widetilde{Y}(x,y) = (0,1), & \textit{if } y < 0, \end{cases}$$

in a neighborhood $\widetilde{U}$ of $\vec{0}$. Note that $\widetilde{Z}$ is a smooth vector field.

(ii) If $\vec{0} \in \Sigma^s$ and satisfies $X(\vec{0}) \nparallel Y(\vec{0})$, then in a neighborhood U of $\vec{0}$, Z is Σ-equivalent to the normal form

$$\widetilde{Z}(x,y) = \begin{cases} \widetilde{X}(x,y) = (1,-1), & \textit{if } y > 0,\\ \widetilde{Y}(x,y) = (1,1), & \textit{if } y < 0, \end{cases}$$

in a neighborhood $\widetilde{U}$ *of* $\vec{0}$.

(iii) If $\vec{0} \in \Sigma^e$ *and satisfies* $X(\vec{0}) \nparallel Y(\vec{0})$, *then in a neighborhood* U *of* $\vec{0}$, Z *is* Σ*-equivalent to the normal form*

$$\widetilde{Z}(x,y) = \begin{cases} \widetilde{X}(x,y) = (1,1), & \text{if } y > 0, \\ \widetilde{Y}(x,y) = (1,-1), & \text{if } y < 0, \end{cases}$$

in a neighborhood $\widetilde{U}$ *of* $\vec{0}$.

Proof: In the first case, we construct the equivalence Σ considering φ_X, φ_Y, $\varphi_{\widetilde{X}}$ and $\varphi_{\widetilde{Y}}$ the fluxes of the smooth components of both vector fields. As $\vec{0} \in \Sigma^c$ these vector fields are transversal to $\Sigma \cap U$ and $\widetilde{\Sigma} \cap \widetilde{U}$ respectively. Then, whatever $p \in \Sigma^+ \cap U$, from the Implicit Function Theorem 1.1.5 follows that there exists a time $t(p) \in \mathbb{R}$ such that $\varphi_X(t(p),p) \in \Sigma$, and the same for $\Sigma^- \cap U$ and φ_Y . Thus, imposing that the Σ-equivalence is the identity restricted to Σ, it can be given by

$$h(x,y) = \begin{cases} \varphi_X(-t(p), \varphi_X(t(p),p)), & \text{if } p \in \Sigma^+, \\ p, & \text{if } p \in \Sigma, \\ \varphi_Y(-t(p), \varphi_Y(t(p),p)), & \text{if } p \in \Sigma^+, \end{cases}$$

which it can be seen that is C^0, and satisfies $\varphi_{\widetilde{Z}}(t, h(p)) = h(\varphi_Z(t,p))$.

In the second case, since $X(\vec{0}) \nparallel Y(\vec{0})$ and $\widetilde{X}(\vec{0}) \nparallel \widetilde{Y}(\vec{0})$, $\vec{0}$ is a regular point of both sliding vector fields Z^Σ and $\widetilde{Z}^\Sigma$. Then, by the Tubular flow Theorem (see 1.4), there exists a homeomorphism $\widetilde{h}$ which locally conjugates them. As for any point $p \in \Sigma^+ \cap U$ there is a time $t(p)$ such that $\varphi_X(t(p),p) \in \Sigma$, the same for $p \in \Sigma^- \cap U$. For points far from Σ, the homeomorphism can be extended as in the previous cases by the flow. Thus, the homeomorphism that gives the equivalence Σ can be given by

$$h(x,y) = \begin{cases} \varphi_{\widetilde{X}}(-t(p), \widetilde{h}(\varphi_X(t(p),p))), & \text{if } p \in \Sigma^+, \\ \widetilde{h}(p), & \text{if } p \in \Sigma, \\ \varphi_{\widetilde{Y}}(-t(p), \widetilde{h}(\varphi_Y(t(p),p))), & \text{if } p \in \Sigma^-. \end{cases}$$

The third case is analogous to the second. ■

Definition 3.3.3 *A singular point* $p \in \Sigma$ *of* Z^Σ *is* ***hyperbolic*** *if* $(Z^\Sigma)'(p) \neq 0$; p *is said to be a* ***saddle*** *provided one of the following conditions is satisfied:*

(i) $p \in \Sigma^e$ *and* p *is an attractor for* Z^Σ;

(ii) $p \in \Sigma^s$ *and* p *is a repeller for* Z^Σ.

A point $p \in \Sigma$ *is a* ***fold point*** *of* X *(resp.* Y*) if* $Xf(p) = 0$ *(resp.* $Yf(p) = 0$*) but* $X^2 f(p) \neq 0$ *(resp.* $Y^2 f(p) \neq 0$*).* $p \in \Sigma$ *is a* ***fold-regular*** *point of* X *(resp.* Y*) if it is a fold point of* X *(resp.* Y*) but* $Yf(p) \neq 0$ *(resp.* $Xf(p) \neq 0$*).*

Proposition 3.3.5 *The following Σ-equivalences hold:*

(i) If $\vec{0} \in \Sigma$ is a fold-regular point of the vector field $Z = (X,Y) \in \Omega$ defined in a neighborhood U of $\vec{0}$, then Z is Σ-equivalent in a neighborhood V of $\vec{0}$ to its normal form

$$Z_{a,b} = \begin{cases} X_a = (1, ax), & \text{if } y > 0, \\ Y_b = (0, b), & \text{if } y < 0, \end{cases}$$

where $a = sign(X^2 f(p))$ and $b = sign(Y f(p))$.

(ii) If $\vec{0} \in \Sigma^e \cup \Sigma^s$ is a hyperbolic critical point of the sliding vector field Z^Σ of $Z = (X,Y) \in \Omega$ defined in a neighborhood U of $\vec{0}$, then Z is Σ-equivalent in a neighborhood V of $\vec{0}$ to its normal form

$$Z_{a,b} = \begin{cases} X_{a,ba} = (ax, b), & \text{if } y > 0, \\ Y_{a,b} = (ax, -b), & \text{if } y < 0, \end{cases}$$

where $a = sign(Z^\Sigma)'(p)$ and $b = sign(X f(p))$.

The proof of Proposition 3.3.5 can be found in [6].

CHAPTER 4

Piecewise-Smooth Vector Fields as Singular Perturbation Problems from Regularization

This chapter presents the Sotomayor-Teixeira's regularization process (see [15] and [2]), which is a methodology for studying piecewise-smooth vector fields. Subsequently, results will be presented that relate regularized piecewise-smooth vector fields to the singular perturbation problems presented in Chapter 2.

4.1 Regularization of Piecewise-smooth Vector Fields

Unlike the Filippov's (or Utkin's) convention, the regularization process consists of obtaining a family of smooth vector fields at a parameter $\varepsilon > 0$ which, for ε sufficiently small, the elements of this family will provide an approximation to the discontinuous problem.

As in the previous chapter, let $U \subset \mathbb{R}^n$ be an open set and $\Sigma = f^{-1}(0)$, where $f : U \to \mathbb{R}$ is of class $C^k(k \geq 1)$ and 0 a regular value of f; $\Sigma^+ = \{p \in U;\, f(p) \geq 0\}$ and $\Sigma^- = \{p \in U;\, f(p) \leq 0\}$. Let be $Z \in \Omega(U)$ given by

$$Z(p) = \begin{cases} X(p), & \text{for } p \in \Sigma^+, \\ Y(p), & \text{for } p \in \Sigma^-, \end{cases} \tag{4-1}$$

where $X, Y \in \Lambda(U)$.

Definition 4.1.1 *A C^k function $\varphi : \mathbb{R} \to \mathbb{R}$ ($k \geq 1$) is a transition function if $\varphi(x) = -1$ for $x \leq -1$, $\varphi(x) = 1$ for $x \geq 1$ and $\varphi'(x) > 0$ if $x \in (-1, 1)$. The φ-regularization of $Z = (X, Y)$ given in (4-1) is the 1-parameter family $Z_\varepsilon \in C^r(U)$ given by*

$$Z_\varepsilon(q) = \left(\frac{1}{2} + \frac{\varphi_\varepsilon(f(q))}{2}\right) X(q) + \left(\frac{1}{2} - \frac{\varphi_\varepsilon(f(q))}{2}\right) Y(p), \tag{4-2}$$

with $\varphi_\varepsilon(x) = \varphi(\frac{x}{\varepsilon})$, for $\varepsilon > 0$.

Under the hypotheses of φ and f we see that $Z_\varepsilon \in C^1(U)$ for all $\varepsilon > 0$. Define the set $V_\varepsilon = f^{-1}(I_\varepsilon)$, where $I_\varepsilon = (-\varepsilon, \varepsilon)$. Of course, $f^{-1}(0) \subset V_\varepsilon$. Note that, whatever $\varepsilon > 0$, $Z_\varepsilon(q) = X(q)$ for all $q \in \Sigma^+ \backslash V_\varepsilon$; $Z_\varepsilon(q) = Y(q)$ for all $q \in \Sigma^- \backslash V_\varepsilon$ and $Z_\varepsilon(q)$ is a linear combination of $X(q)$ and $Y(q)$ if $q \in V_\varepsilon$, satisfying

$$|Z_\varepsilon(q)| < |X(q)| + |Y(q)|.$$

The set V_ε is a band around Σ as illustrated in Figure 4.1.

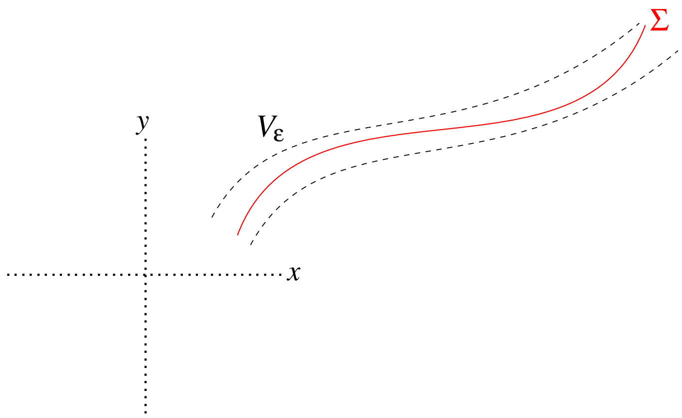

Figure 4.1: *Illustration of the set V_ε.*

Example 4.1.1 *Consider the vector field $Z \in \Omega(\mathbb{R}^2)$ given by*

$$Z(p) = \begin{cases} X(p) = (3, -2), & \text{for } p \in \Sigma^+, \\ Y(p) = (1, 0), & \text{for } p \in \Sigma^-, \end{cases}$$

where $\Sigma = f^{-1}(0)$, with $f(x,y) = y - x$.

Let is determine the family of vector fields Z_ε, associated with a given transition function φ. By the Definition 4.1.1, follows

$$Z_\varepsilon(q) = \frac{X(q) + Y(q)}{2} + \frac{X(q) - Y(q)}{2} \varphi_\varepsilon(f(q)) = (2, -1) + (1, -1)\varphi_\varepsilon(y - x).$$

Therefore,

$$Z_\varepsilon(x,y) = (\varphi_\varepsilon(y-x) + 2, -\varphi_\varepsilon(y-x) - 1).$$

We have that $\nabla f(q) = (-1, 1)$ for all $q \in \mathbb{R}^2$, so $Xf(q) = -5 < 0$ and $Yf(q) = -1 < 0$ whatever $q \in \Sigma$. Thus, $\Sigma^c = \Sigma$. Note that

$$-\varepsilon < f(x,y) < \varepsilon \;\Leftrightarrow\; -\varepsilon + x < y < \varepsilon + x,$$

then $V_\varepsilon = \{(x,y) \in \mathbb{R}^2; -\varepsilon + x < y < \varepsilon + x\}$. We see that the vector field Z_ε depends on the formation law of the function φ. Figure 4.2 illustrates a possible phase portrait for Z_ε (in V_ε). ℵ

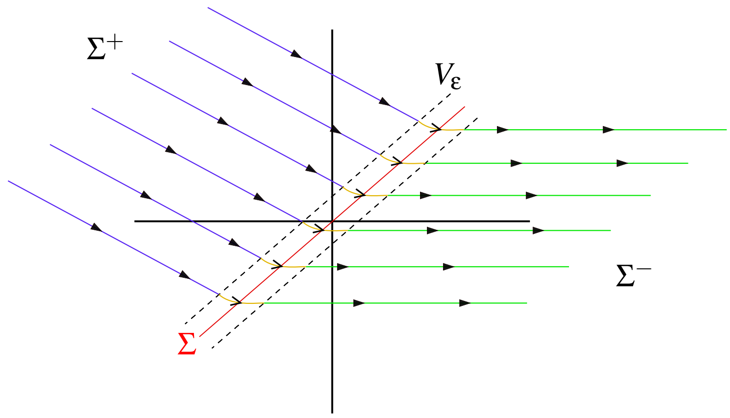

Figure 4.2: *Possible phase portrait setting for Z_ε (in V_ε).*

Example 4.1.2 *Building a transition function of class C^1.*

To construct a transition function φ, let is consider a function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = (x^2-1)(x-a)$, with $a \in \mathbb{R}$. We have that

$$f(x) = \frac{x^4}{4} - \frac{ax^3}{3} - \frac{x^2}{2} + ax + b,$$

where $b \in \mathbb{R}$. Let is determine a and b in such a way that $f'(x) > 0$ for all $x \in (-1,1)$, $f(-1) = -1$ and $f(1) = 1$. We have that

$$\left\{ \begin{array}{rcl} f(-1) & = & -1, \\ f(1) & = & 1, \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} 12b - 8a = -9, \\ 12b + 8a = 15, \end{array} \right.$$

then $a = \dfrac{3}{2}$ and $b = \dfrac{1}{4}$. It's easy to see that, for $a = \dfrac{3}{2}$, $f'(x) > 0$ for all $x \in (-1,1)$. Therefore,

$$f(x) = \frac{x^4}{4} - \frac{x^3}{2} - \frac{x^2}{2} + \frac{3x}{2} + \frac{1}{4}. \tag{4-3}$$

Considering the function f in (4-3), we can define a C^1 transition function $\varphi : \mathbb{R} \to \mathbb{R}$ as

$$\varphi(s) = \left\{ \begin{array}{cl} -1, & \text{if } s \leq -1, \\ f(s), & \text{if } |s| < 1, \\ 1, & \text{if } s \geq 1. \end{array} \right.$$

Figure 4.3 illustrates the graphic of the functions f, $y = -1$ and $y = 1$. Figure 4.4 illustrates the graphic of the transition function φ. ℵ

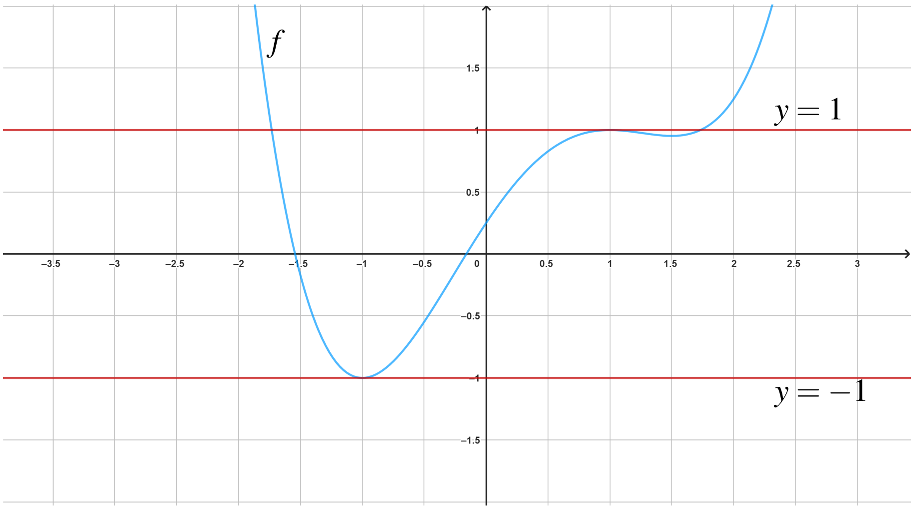

Figure 4.3: *Graphic of the functions f, $y = -1$ and $y = 1$.*

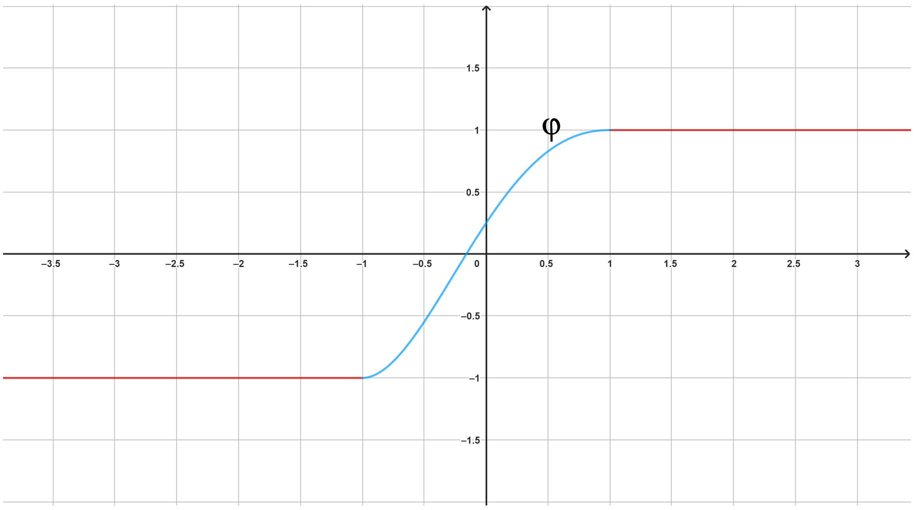

Figure 4.4: *Graphic of the transition function φ.*

4.2 Piecewise-smooth Vector Fields as Singular Perturbation Problems

In this section we will present the relationship between the singular perturbation problems discussed in Section 2 and the vector field Z_ε obtained from the φ-regularization of Z (see [2] and [17]). In other words, we will treat the vector field Z_ε (given in (4-2)) as a singular perturbation problem.

Let $U \subset \mathbb{R}^2$ be an open and $Z \in \Omega(U)$ (as in (4-1)) given by

$$Z(x,y) = \begin{cases} X(x,y), & \text{for } (x,y) \in \Sigma^+, \\ Y(x,y), & \text{for } (x,y) \in \Sigma^-, \end{cases} \tag{4-4}$$

with $X(x,y) = (f_1, g_1)$ and $Y(x,y) = (f_2, g_2)$.

Consider $A(U) = \{\xi : U \to \mathbb{R};\ \xi \in C^r(U),\ L(\xi) = 0\}$ where $L(\xi)$ denotes the linear part of ξ at $(0,0)$; and let $\Omega_d \subset \Omega$ be the set of vector fields $Z=(X,Y)$ in Ω such that there exists $\xi \in A$ that is a solution of

$$\nabla\xi(X-Y)=\Pi_i(X-Y),$$

where Π_i denote the canonical projections, for $i=1$ or $i=2$.

Theorem 4.2.1 *Consider $Z\in\Omega$ given in (4-4), Z_ε its φ-regularization, and $p\in\Sigma$. Suppose that φ is a polynomial of degree k in a small interval $I\subset(-1,1)$ with $0\in I$. Then the trajectories of Z_ε in $U_\varepsilon=\{q\in U;\, f(q)/\varepsilon\in I\}\subset V_\varepsilon$ are in correspondence with the solutions of an ordinary differential equation $\dot{w}=h(w,\varepsilon)$, $w\in\mathbb{R}^2$, satisfying that h is smooth in both variables and $h(w,0)=0$ for any $w\in\Sigma$. Moreover, if $k=1$ and $\langle(X-Y)(p),\nabla f(p)\rangle\neq 0$ then we can take a C^{r-1}-local coordinate system $\{(\partial/\partial x)(p),(\partial/\partial y)(p)\}$ such that this smooth ordinary differential equation is a singular perturbation problem.*

Proof: Consider $Z\in\Omega$ as in (4-4). Suppose that $\varphi(s)=a_1s+a_2s^2+\cdots+a_ks^k$ on $I\subset\mathbb{R}$ with $0\in I$. The trajectories of Z_ε on U_ε are the solutions of the differential system

$$\begin{aligned}\dot{x} &= (f_1+f_2)/2+\varphi(f/\varepsilon)(f_1-f_2)/2,\\ \dot{y} &= (g_1+g_2)/2+\varphi(f/\varepsilon)(g_1-g_2)/2.\end{aligned}$$

The time rescaling $\tau=\frac{t}{\varepsilon^k}$ gives

$$\begin{aligned}\dot{x}=h_1 &= \varepsilon^k(f_1+f_2)/2+(a_1f\varepsilon^{k-1}+a_2f^2\varepsilon^{k-2}+\cdots+a_kf^k)(f_1-f_2)/2\\ \dot{y}=h_2 &= \varepsilon^k(g_1+g_2)/2+(a_1f\varepsilon^{k-1}+a_2f^2\varepsilon^{k-2}+\cdots+a_kf^k)(g_1-g_2)/2.\end{aligned} \tag{4-5}$$

Thus we take $h=(h_1,h_2)$ and have that $h(x,y,0)=\widetilde{h}=(\widetilde{h}_1,\widetilde{h}_2)=0$ for all $(x,y)\in\Sigma$. For $\varepsilon=0$, from (4-5), it follows

$$\begin{aligned}\dot{x}=\widetilde{h}_1 &= (a_kf^k)(f_1-f_2)/2 := a_ku_1f^k\\ \dot{y}=\widetilde{h}_2 &= (a_kf^k)(g_1-g_2)/2 := a_ku_2f^k.\end{aligned}$$

We have that

$$\nabla\widetilde{h}_1=a_kf^k\nabla u_1+a_ku_1\nabla f^k,$$

and

$$\nabla\widetilde{h}_2=a_kf^k\nabla u_2+a_ku_2\nabla f^k.$$

Then

$$J(\widetilde{h})(p)=\begin{pmatrix}\nabla\widetilde{h}_1(p)\\ \nabla\widetilde{h}_2(p)\end{pmatrix}=\begin{pmatrix}a_ku_1(f^k)_x & a_ku_1(f^k)_y\\ a_ku_2(f^k)_x & a_ku_2(f^k)_y\end{pmatrix},$$

whatever $p\in\Sigma$.

The eigenvalues of $J(\widetilde{h})(p)$ for $p \in \Sigma$ are the solutions of the equation

$$\lambda^2 - (a_k/2)\langle (X-Y)(p), \nabla f^k(p)\rangle \lambda = 0.$$

It follows that zero is an eigenvalue of multiplicity at least one. From the other side, if if $k=1$ and $\langle (X-Y)(p), \nabla f(p)\rangle \neq 0$, there exists a non-zero eigenvalue. So we get a normally hyperbolic scenario and we may apply the Fenichel theory (see [4], Lemma 5.3, p. 67]) to get the desired coordinates. ■

The Theorem 4.2.1 says that we can transform a discontinuous vector field in a singular perturbation problem from its φ-regularization. In general this transition can not be done explicitly. Theorem 4.2.2 provides an explicit formula of the singular perturbation problem for a suitable class of vector fields.

Lemma 4.2.1 *Consider $Z \in \Omega_d$ and $\xi \in A$ satisfying $\nabla\xi(X-Y) = \Pi_2(X-Y)$. If $u = x$ and $v = y - \xi(x,y)$ then the differential system*

$$\begin{aligned} \dot{x} &= (f_1+f_2)/2 + \varphi(f/\varepsilon)(f_1-f_2)/2, \\ \dot{y} &= (g_1+g_2)/2 + \varphi(f/\varepsilon)(g_1-g_2)/2. \end{aligned} \tag{4-6}$$

is written as

$$\begin{aligned} \dot{u} &= (\widetilde{f}_1+\widetilde{f}_2)/2 + \varphi(\widetilde{f}/\varepsilon)(\widetilde{f}_1-\widetilde{f}_2)/2, \\ \dot{v} &= (\widetilde{g}_1+\widetilde{g}_2)/2 - (\widetilde{\xi_x})(\widetilde{f}_1+\widetilde{f}_2)/2 - (\widetilde{\xi_y})(\widetilde{g}_1+\widetilde{g}_2)/2, \end{aligned}$$

where the tilde over the function means that the respective function is given in such new coordinates.

Proof: If $u = x$ and $v = y - \xi(x,y)$ we get

$$\dot{u} = (\widetilde{f}_1+\widetilde{f}_2)/2 + \varphi(\widetilde{f}/\varepsilon)(\widetilde{f}_1-\widetilde{f}_2)/2$$

and

$$\begin{aligned} \dot{v} &= \langle \nabla(v), Z_\varepsilon\rangle = \langle(-\xi_x, 1-\xi_y), (\dot{x},\dot{y})\rangle \\ &= (\widetilde{g}_1+\widetilde{g}_2)/2 - (\widetilde{\xi_x})(\widetilde{f}_1+\widetilde{f}_2)/2 - (\widetilde{\xi_y})(\widetilde{g}_1+\widetilde{g}_2)/2 \\ &\quad -\varphi(\widetilde{f}/\varepsilon)/2(\nabla\xi(X-Y) - \Pi_2(X-Y)). \end{aligned}$$

Using the partial differential equation we get the desired formula.

Analogously if $\nabla\xi(X-Y) = \Pi_1(X-Y)$ and $u = x - \xi(x,y)$ and $v = y$ then the differential system (4-6) is written as

$$\begin{aligned}\dot{u} &= (\widetilde{f_1}+\widetilde{f_2})/2-(\widetilde{\xi_x})(\widetilde{f_1}+\widetilde{f_2})/2-(\widetilde{\xi_y})(\widetilde{g_1}+\widetilde{g_2})/2,\\ \dot{v} &= (\widetilde{g_1}+\widetilde{g_2})/2+\varphi(\widetilde{f}/\varepsilon)(\widetilde{g_1}-\widetilde{g_2})/2.\end{aligned}$$ ■

Theorem 4.2.2 *Consider $Z\in\Omega_d$ and Z_ε its φ-regularization. Suppose that φ is a polynomial of degree k in a small interval $I\subset\mathbb{R}$ with $0\in I$. Then the trajectories of Z_ε on $U_\varepsilon=\{q\in U;\, F(q)/\varepsilon\in I\}$ are solutions of a singular perturbation problem.*

Proof: Consider $Z\in\Omega_d$ and suppose that $\varphi(s)=a_1s+a_2s^2+\cdots+a_ks^k$ on $I\subset\mathbb{R}$ with $0\in I$. There exists $\xi\in A$ such that $\nabla\xi(X-Y)=\Pi_i(X-Y)$, for $i=1$ or $i=2$. We suppose without lost of generality that $i=2$. The trajectories of the regularized vector field Z_ε, on U_ε, are the solutions of the differential system (4-6).

We consider the coordinates (u,v) given by $u=x$ and $v=y-\xi(x,y)$, and then we apply Lemma 4.2.1 and we obtain

$$\begin{aligned}\dot{u} &= (\widetilde{f_1}+\widetilde{f_2})/2+\varphi(\widetilde{f}/\varepsilon)(\widetilde{f_1}-\widetilde{f_2})/2,\\ \dot{v} &= (\widetilde{g_1}+\widetilde{g_2})/2-(\widetilde{\xi_x})(\widetilde{f_1}+\widetilde{f_2})/2-(\widetilde{\xi_y})(\widetilde{g_1}+\widetilde{g_2})/2.\end{aligned}$$

Denotes $\omega=\widetilde{f}$. The time rescaling $\tau=\frac{t}{\varepsilon^k}$ gives

$$\begin{aligned}\dot{u} &= \varepsilon^k(\widetilde{f_1}+\widetilde{f_2})/2+(a_1\omega\varepsilon^{k-1}+a_2\omega^2\varepsilon^{k-2}+\cdots+a_k\omega^k)(\widetilde{f_1}-\widetilde{f_2})/2=h_1(u,v,\varepsilon),\\ \dot{v} &= \varepsilon^k[(\widetilde{g_1}+\widetilde{g_2})/2-(\widetilde{\xi_x})(\widetilde{f_1}+\widetilde{f_2})/2-(\widetilde{\xi_y})(\widetilde{g_1}+\widetilde{g_2})/2]=\varepsilon h_2(u,v,\varepsilon).\end{aligned}$$

Then

$$\begin{aligned}\dot{u} &= h_1(u,v,\varepsilon),\\ \dot{v} &= \varepsilon h_2(u,v,\varepsilon),\end{aligned}\tag{4-7}$$

or equivalently

$$\begin{aligned}\varepsilon\dot{u} &= h_1(u,v,\varepsilon),\\ \dot{v} &= h_2(u,v,\varepsilon).\end{aligned}$$ ■

If $k=1$, the reduced problem from (4-7) is given by

$$\begin{aligned}0 &= a_1\omega(\widetilde{f_1}-\widetilde{f_2})/2,\\ \dot{v} &= (\widetilde{g_1}+\widetilde{g_2})/2-(\widetilde{\xi_x})(\widetilde{f_1}+\widetilde{f_2})/2-(\widetilde{\xi_y})(\widetilde{g_1}+\widetilde{g_2})/2;\end{aligned}\tag{4-8}$$

if $k>1$, it is given by

$$\begin{aligned}0 &= a_k\omega^k(\widetilde{f_1}-\widetilde{f_2})/2,\\ \dot{v} &= 0.\end{aligned}$$

The layer problem from (4-7), for $k \geq 1$, is given by

$$\begin{aligned} \dot{u} &= a_k \omega^k (\widetilde{f}_1 - \widetilde{f}_2)/2, \\ \dot{v} &= 0. \end{aligned}$$

Note that the critical set C_0 from (4-8) is given by

$$C_0 = \{q \in \mathbb{R}^2;\ \omega(q) = 0 \text{ or } \widetilde{f}_1(q) = \widetilde{f}_2(q)\}.$$

Furthermore the discontinuity set Σ in the coordinates u and v, $\widetilde{\Sigma} = \{q \in \mathbb{R}^2;\ \omega(q) = 0\}$, from (4-8) is a subset of C_0.

Example 4.2.1 *Consider the regularization function φ given by $\varphi(s) = s$, when $-1/2 < s < 1/2$. Take $X(x,y) = (1,x)$, $Y(x,y) = (-1,-3x)$ and $f(x,y) = y$. The discontinuity set is $\Sigma = \{(x,0);\ x \in \mathbb{R}\}$.*

We have $Xf(p) = x$, $Yf(p) = -3x$ for all $p \in \Sigma$, and then the unique non-regular point is $(0,0)$. We apply Theorem 4.2.2. The vector field Z_ε is given by

$$Z_\varepsilon(x,y) = \left(\frac{y}{\varepsilon}, \frac{2xy}{\varepsilon} - x\right)$$

on U_ε.

The partial differential equation $\nabla\xi(X - Y) = \Pi_i(X - Y)$, with $i = 2$, becomes $2(\xi_x) + 4x(\xi_y) = 4x$. A solution $\xi \in A$ is given by $\xi(x,y) = x^2$. Thus we take the coordinate change $u = x$, $v = y - x^2$. The trajectories of Z_ε in these coordinates are the solutions of the singular system $\widetilde{Z}_\varepsilon$ given by

$$\begin{aligned} \varepsilon\dot{u} &= v + u^2, \\ \dot{v} &= -u. \end{aligned} \tag{4-9}$$

The critical manifold C_0 from (4-9) is given by

$$C_0 = \{(u,v) \in \mathbb{R}^2;\ v = -u^2\}.$$

In this case, we have $\widetilde{\Sigma} = C_0$.

The reduced and layer problem from (4-9) are given, respectively, by

$$\begin{aligned} \dot{u} &= \tfrac{1}{2}, \\ \dot{v} &= -u, \end{aligned} \tag{4-10}$$

and

$$\begin{aligned} \dot{u} &= v + u^2, \\ \dot{v} &= 0. \end{aligned} \tag{4-11}$$

The reader should realize that the change of coordinates considered, $\eta(x,y) = (u,v) = (x, y-x^2)$, is a diffeomorphism. Then Z_ε and $\widetilde{Z}_\varepsilon$ are conjugate topologically on $\mathbb{R}^2$. Note that $\eta(\Sigma) = \widetilde{\Sigma} \subset \widetilde{U}_\varepsilon = \eta(U_\varepsilon)$.

To understand the behavior of Z_ε near to Σ, it is sufficient to understand the behavior of $\widetilde{Z}_\varepsilon$ near to $\widetilde{\Sigma}$. Any compact manifold $\widetilde{M}_0 \subset \widetilde{\Sigma}\backslash\{0\}$ is normally hyperbolic. By Fenichel's Theorem 2.1.1, it follows that there exists a locally invariant manifold $\widetilde{M}_\varepsilon$ (for $\widetilde{Z}_\varepsilon$) diffeomorphic to $\widetilde{M}_0$. Then $M_\varepsilon = \eta^{-1}(\widetilde{M}_\varepsilon)$ is a locally invariant manifold (for Z_ε) diffeomorphic to $M_0 = \eta^{-1}(\widetilde{M}_0) \subset \Sigma$. Figure 4.5 illustrates the change in coordinate systems through η. ℵ

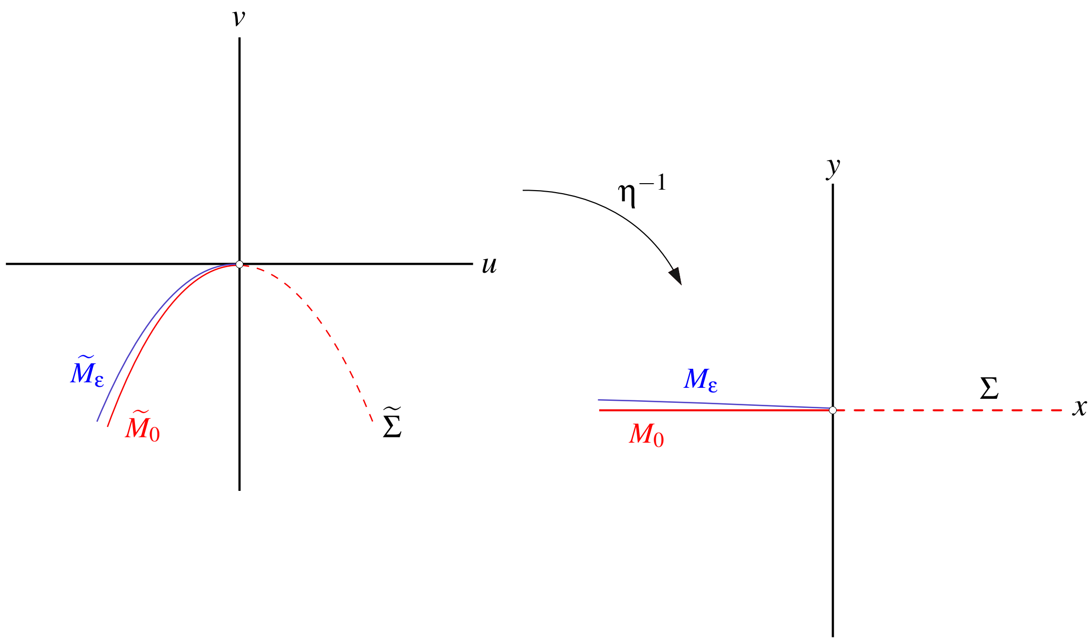

Figure 4.5: *Change of coordinates* η.

Let be $Z = (X,Y) \in \Omega$, with $X(x,y) = (f_1, g_1)$ and $Y(x,y) = (f_2, g_2)$; $\varphi : \mathbb{R} \to \mathbb{R}$ a transition function and Z_ε the $\varphi-$regularization of Z given by

$$\begin{aligned} \dot{x} &= (f_1+f_2)/2 + \varphi(f/\varepsilon)(f_1-f_2)/2, \\ \dot{y} &= (g_1+g_2)/2 + \varphi(f/\varepsilon)(g_1-g_2)/2. \end{aligned} \tag{4-12}$$

Consider $f(x,y) = x$, so $\Sigma = \{(0,y);\ y \in \mathbb{R}\}$.

We will now present another analysis methodology for (4-12), which also consists of obtaining a singular perturbation problem equivalent to Z_ε. With the restriction on f, we write Z_ε as

$$\begin{aligned} \dot{x} &= (f_1+f_2)/2 + \varphi(x/\varepsilon)(f_1-f_2)/2, \\ \dot{y} &= (g_1+g_2)/2 + \varphi(x/\varepsilon)(g_1-g_2)/2, \\ \dot{\varepsilon} &= 0. \end{aligned} \tag{4-13}$$

We observe that system (4-13) is not defined for $\varepsilon = 0$ and it is not an explicit singular perturbation problem according to Definition 2.1.1. However if we consider the ***directional blow-up*** $\beta : \mathbb{R}^3 \to \mathbb{R}^3$ given by $(x, y, \varepsilon) = \beta(u, v, \mu) = (u\mu, v, \mu)$, system (4-13) becomes

$$\begin{aligned} \mu\dot{u} &= (\widetilde{f}_1 + \widetilde{f}_2)/2 + \varphi(u)(\widetilde{f}_1 - \widetilde{f}_2)/2, \\ \dot{v} &= (\widetilde{g}_1 + \widetilde{g}_2)/2 + \varphi(u)(\widetilde{g}_1 - \widetilde{g}_2)/2, \\ \dot{\mu} &= 0. \end{aligned} \tag{4-14}$$

where the tilde over the function means that the respective function is given in such new coordinates. Taking μ as a parameter system (4-14) is clearly an explicit singular perturbation problem. Moreover, the parameter value $\mu = 0$ can be considered. In principle, the vector field Z_ε only makes sense for $\varepsilon > 0$. Note that $\beta : \mathbb{R}^2 \times \mathbb{R}^*_+ \to \mathbb{R}^2 \times \mathbb{R}^*_+$ is a diffeomorphism, so the vector field (4-13) is topologically conjugate to (4-14) on $\mathbb{R}^2 \times \mathbb{R}^*_+$. Notice that the singular limit $\varepsilon = 0$ in (4-13) leads us to the singular limit $\mu = 0$ in (4-14) via β. So to understand the behavior of (4-13) for $\varepsilon > 0$ sufficiently small, we only need to understand the behavior of (4-14) when $\mu > 0$ is sufficiently small. Since (4-14) is a singular perturbation problem we can consider the associated reduced and layer problem and then use Fenichel theory to obtain results on the behavior of the vector field dynamics.

It is important to note that the region of discontinuity Σ in the new coordinates is given by

$$\begin{aligned} u &= \tfrac{x}{\varepsilon}, \\ v &= y, \\ \mu &= \varepsilon. \end{aligned} \tag{4-15}$$

when $x = 0, y \in \mathbb{R}$ and $\varepsilon > 0$, i.e., $\widetilde{\Sigma} = \{(0, v, \mu);\ v \in \mathbb{R} \text{ and } \varepsilon > 0\}$. Note that for each fixed $\varepsilon > 0$, $\widetilde{\Sigma}$ is the axis v in the plan $\mu = \varepsilon$. When $\mu = 0$, from (4-15), we have $\varepsilon = 0$, $y = v$ and

$$x = \varepsilon u \ \Rightarrow \ x = 0 \cdot u = 0$$

for all $u \in \mathbb{R}$. Then the plane $u \times v$ is sent to Σ through β. Thus, by the continuity of β, it is reasonable to state that the dynamics of (4-14) in the plane $u \times v$ (when $\mu = 0$) corresponds to the dynamics of (4-13) along Σ in the singular limit $\varepsilon = 0$. Essentially, in the study of dynamics, we “blow up” a line into a plane.

Geometrically speaking, it is more convenient to consider the ***polar blow-up*** coordinates $\alpha : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} \times [0, \infty) \to \mathbb{R}^3$ given by $(x, y, \varepsilon) = \alpha(\theta, v, r) = (r \sin\theta, v, r\cos\theta)$. From α we have

$$\begin{aligned} \tan\theta &= \frac{x}{\varepsilon}, \\ v &= y, \\ r^2 &= x^2 + \varepsilon^2. \end{aligned} \tag{4-16}$$

Thus,

$$\begin{aligned} r^2\dot{\theta} &= \varepsilon\dot{x} - x\dot{\varepsilon}, \\ \dot{v} &= \dot{y}, \\ r\dot{r} &= x\dot{x} + \varepsilon\dot{\varepsilon}. \end{aligned} \tag{4-17}$$

From (4-17) we can write (4-13) in polar coordinates as

$$\begin{aligned} r\dot{\theta} &= \cos\theta[(\widetilde{f}_1 + \widetilde{f}_2)/2 + \varphi(\tan\theta)(\widetilde{f}_1 - \widetilde{f}_2)/2], \\ \dot{v} &= (\widetilde{g}_1 + \widetilde{g}_2)/2 + \varphi(\tan\theta)(\widetilde{g}_1 - \widetilde{g}_2)/2, \\ \dot{r} &= \sin\theta[(\widetilde{f}_1 + \widetilde{f}_2)/2 + \varphi(\tan\theta)(\widetilde{f}_1 - \widetilde{f}_2)/2]. \end{aligned} \tag{4-18}$$

Similarly the map β, note that the map $\alpha : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} \times (0,\infty) \to \mathbb{R}^2 \times \mathbb{R}^*_+$ is a diffeomorphism. So (4-13) and (4-18) are topologically conjugate. Furthermore the parameter value $\varepsilon = 0$ is now represented by $r = 0$. As discussed in the directional blow up, note that the region of discontinuity Σ in the polar coordinates denoted by $\widetilde{\Sigma}$ is given by (4-16) when

$$x = 0 \;\;\Rightarrow\;\; \theta = 0,$$

$$y \in \mathbb{R} \;\;\Rightarrow\;\; v \in \mathbb{R}$$

and

$$\varepsilon > 0 \;\;\Rightarrow\;\; r > 0.$$

Hence $\widetilde{\Sigma} = \{(0,v,r);\, v \in \mathbb{R} \text{ and } r > 0\}$. We also note that for every fixed $\varepsilon > 0$, $\widetilde{\Sigma}$ is the axis v in the plane $r = \varepsilon$. When $r = 0$ we have $\alpha(\theta, v, 0) = (0, v, 0)$ for all $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $v \in \mathbb{R}$. Then the set $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} \times \{0\}$ is sent to Σ through α. Thus, by the continuity of α, it is reasonable to state that the dynamics of (4-18) in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R} \times \{0\}$ corresponds to the dynamics of (4-13) along Σ in the singular limit $\varepsilon = 0$.

The reduced and layer problem from (4-18) are give, respectively, by

$$\begin{aligned} 0 &= \cos\theta[(\widetilde{f}_1 + \widetilde{f}_2)/2 + \varphi(\tan\theta)(\widetilde{f}_1 - \widetilde{f}_2)/2], \\ \dot{v} &= (\widetilde{g}_1 + \widetilde{g}_2)/2 + \varphi(\tan\theta)(\widetilde{g}_1 - \widetilde{g}_2)/2, \\ 0 &= \sin\theta[(\widetilde{f}_1 + \widetilde{f}_2)/2 + \varphi(\tan\theta)(\widetilde{f}_1 - \widetilde{f}_2)/2]. \end{aligned} \tag{4-19}$$

and

$$\begin{aligned} \dot{\theta} &= \cos\theta[(\widetilde{f}_1 + \widetilde{f}_2)/2 + \varphi(\tan\theta)(\widetilde{f}_1 - \widetilde{f}_2)/2], \\ \dot{v} &= 0, \\ r &= 0. \end{aligned} \tag{4-20}$$

The fast flow on $\mathbb{S}^1_+ \times \mathbb{R}$ is given by the solutions of system (4-20) and the slow flow is given by the solutions of (4-19). Note that the first and last equality of (4-19) are equivalent to

$$0 = (\widetilde{f}_1 + \widetilde{f}_2) + \varphi(\tan\theta)(\widetilde{f}_1 - \widetilde{f}_2)$$

so we can write (4-19) as

$$\begin{aligned} 0 &= (\widetilde{f}_1+\widetilde{f}_2)+\varphi(\tan\theta)(\widetilde{f}_1-\widetilde{f}_2),\\ \dot{v} &= (\widetilde{g}_1+\widetilde{g}_2)/2+\varphi(\tan\theta)(\widetilde{g}_1-\widetilde{g}_2)/2. \end{aligned}$$

For each $\varepsilon>0$, let us identify from α^{-1}(see (4-16)), who is the set $\widetilde{U}_\varepsilon$ (i.e. the set U_ε in the coordinates θ, v and r). We have that $U_\varepsilon=\{(x,y,\varepsilon);\ -\varepsilon<x<\varepsilon \text{ and } y\in\mathbb{R}\}$. As $\varepsilon\tan\theta=x$, follows

$$-\varepsilon<x<\varepsilon \ \Rightarrow\ -1<\tan\theta<1 \ \Rightarrow\ -\frac{\pi}{4}<\theta<\frac{\pi}{4};$$

as $v=y$ we have

$$y\in\mathbb{R} \ \Rightarrow\ v\in\mathbb{R};$$

and $\varepsilon=r\cos\theta$ leads us to

$$r=\frac{\varepsilon}{\cos\theta}.$$

Therefore $\widetilde{U}_\varepsilon=\{(\theta,v,r);\ -\pi/4<\theta<\pi/4, v\in\mathbb{R} \text{ and } r=\varepsilon/\cos\theta\}$. Similarly, we can conclude that the diffeomorphism α, for each $\varepsilon>0$, sends the plan $H_\varepsilon=\{(x,y,\varepsilon);\ x,y\in\mathbb{R}\}$ in the set $\widetilde{H}_\varepsilon=\{(\theta,v,r);\ -\pi/2<\theta<\pi/2, v\in\mathbb{R} \text{ and } r=\varepsilon/\cos\theta\}$. Figure 4.6 illustrates the sets H_ε and $\widetilde{H}_\varepsilon$. Note that the points of the plan $r=k$ (constant) are sent on a semi-cylinder by the diffeomorphism α. Figure 4.7 illustrates the sets $\widetilde{W}_\varepsilon=(\widetilde{H}_\varepsilon\backslash\widetilde{U}_\varepsilon)\cup\{(\theta,v,r);\ -\pi/4<\theta<\pi/4, v\in\mathbb{R} \text{ and } r=\sqrt{2}\varepsilon\}$ and $W_\varepsilon=\alpha(\widetilde{W}_\varepsilon)$. The set $\widetilde{W}_\varepsilon$ is essential for analyzing the behavior of (4-13) for small $\varepsilon>0$.

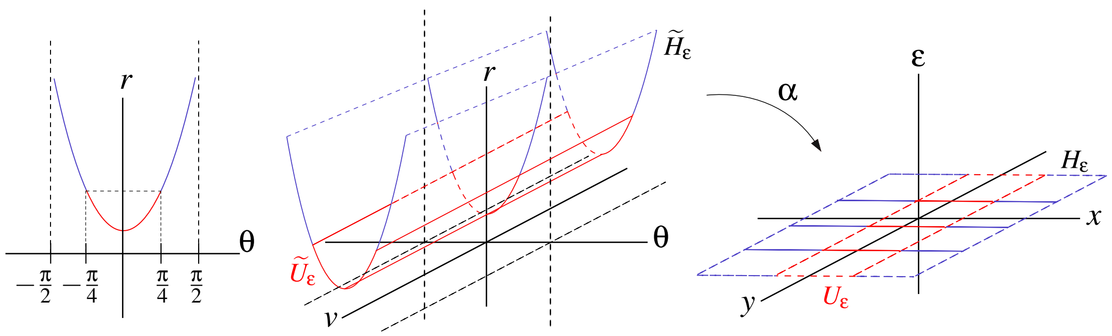

Figure 4.6: *Illustrate of the sets $\widetilde{H}_\varepsilon$, H_ε, $\widetilde{U}_\varepsilon$ and U_ε.*

We are considering that the region of discontinuity Σ is given by the function $f(x,y)=x$. In the most general case, assuming $\vec{0}\in\Sigma$, we can approximate Σ by the line $x=0$ in a neighborhood of the origin (they are locally diffeomorphic). So what has been discussed is still valid locally.

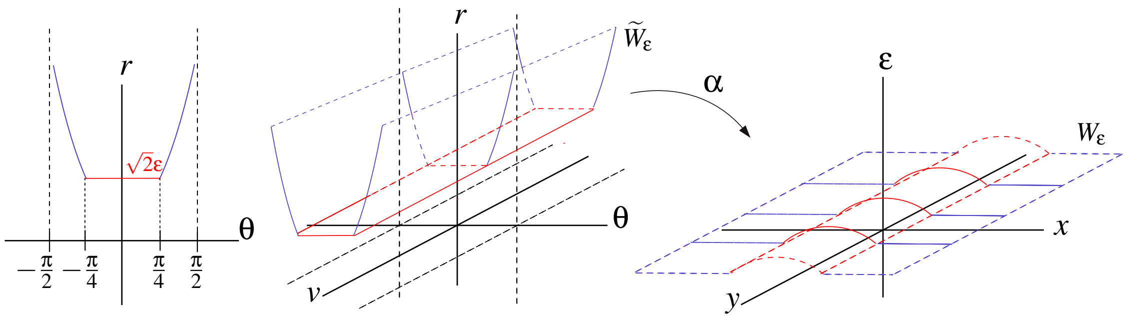

Figure 4.7: *Illustrate of the sets* W_ε *and* $\widetilde{W}_\varepsilon$.

Next we will present some results for a special family of discontinuous vector fields. We will then return to the discussion of the blow up method and present some applications. Let $\Omega_d^* \subset \Omega$ be the set of discontinuous vector fields (as in (4-12)) such that

$$f(x,y) = x, \qquad g_1 = g_2 = g.$$

Note that $\Omega_d^* \subset \Omega_d$.

For simplicity we assume that $\varphi(s) = s$ for $-1/2 < s < 1/2$.

Proposition 4.2.1 *Let $Z \in \Omega$ satisfying the hypotheses of Theorem 4.2.1 and Z_ε its φ-regularization written like a singular perturbation problem. We have:*

(i) if $p \in \Sigma^s$ then p is an attractor of Z_0 for the fast flow;

(ii) if $p \in \Sigma^e$ then p is a repeller of Z_0 for the fast flow;

(iii) if $p \in \Sigma$ is a regular point then either p is an attractor, or p is a repeller or p is a sewing point.

Proof: Let $p \in \Sigma$. Taking $\varepsilon = 0$, by Theorem 4.2.1, follows that the linear part of the corresponding singular problem at p has two eigenvalues: $\lambda_1 = 0$ and

$$\lambda_2 = (1/2)(X - Y)\nabla f(p) = \frac{f_1(p) - f_2(p)}{2}. \tag{4-21}$$

The eigenvalue λ_1 determines the slow manifold and the eigenvalue λ_2 determines if p is attractor ($\lambda_2 < 0$) or repeller ($\lambda_2 > 0$). If $p \in \Sigma^s$ then $Xf(p) = f_1(p) < 0$ and $Yf(p) = f_2(p) > 0$ and thus, from (4-21), follows that $\lambda_2 < 0$. If $p \in \Sigma^e$ then $Xf(p) = f_1(p) > 0$ and $Yf(p) = f_2(p) < 0$ and thus, from (4-21), follows that $\lambda_2 > 0$. If p is a regular point and supposing that p is not an attractor or a repeller, it means that $p \in (\Sigma^e \cup \Sigma^s)^C$. Furthermore, since p is regular, we have that $Xf(p)Yf(p) > 0$ and then p is a sewing point. ■

Corollary 4.2.1 *Consider $Z \in \Omega_d^*$. The trajectories of Z_ε given by (4-6), on U_ε are the solutions of the singular system*

$$\begin{aligned} \varepsilon\dot{x} &= \varepsilon(f_1 + f_2)/2 + x(f_1 - f_2)/2, \\ \dot{y} &= g. \end{aligned} \tag{4-22}$$

Proposition 4.2.2 *Consider $Z \in \Omega_d^*$. We have:*

(a) The critical set from (4-22) is given by

$$C_0 = \Sigma \cup \{(x,y);\, f_1(x,y) = f_2(x,y)\}.$$

(Remembering that if C_0 is a manifold we simply say that C_0 is the slow (or critical) manifold from (4-22)).

(b) $p \in \Sigma^c$ if and only if $f_1(p) f_2(p) > 0$.

(c) $p \in \Sigma^e$ if and only if $f_1(p) > 0$ and $f_2(p) < 0$.

(d) $p \in \Sigma^s$ if and only if $f_1(p) < 0$ and $f_2(p) > 0$.

(e) If $p \in \Sigma^c \cup \Sigma^e \cup \Sigma^s$ and $f_1(p) \neq f_2(p)$ then p is a normally hyperbolic point of (4-22).

Proof: The item (a) follows of the Definition 2.1.3. To prove (b)-(d) we compute $Xf(p)$ and $Yf(p)$:

$$Xf(p) = \langle (f_1, g), (1,0) \rangle = f_1(p),$$

$$Yf(p) = \langle (f_2, g), (1,0) \rangle = f_2(p).$$

So we use the Definition 3.2.3. For proof (e) we have just to compute the linear part of

$$\begin{aligned} \dot{x} &= x(f_1 - f_2)/2, \\ \dot{y} &= 0, \end{aligned}$$

at $p = (0,y) \in \Sigma^c \cup \Sigma^e \cup \Sigma^s$ which has the eigenvalue $\lambda = (f_1(0,y) - f_2(0,y))/2 \neq 0$ (since $f_1(p) \neq f_2(p)$). Therefore, from the Definition 2.1.4, follows that p is a normally hyperbolic point of (4-22). ■

Proposition 4.2.3 *Consider $Z \in \Omega_d^*$. The trajectories of the sliding vector field Z^Σ are solutions of the reduced problem from (4-22).*

Proof: The solutions of the reduced problem from (4-22) are given by

$$\begin{aligned} 0 &= x(f_1 - f_2), \\ \dot{y} &= g. \end{aligned} \tag{4-23}$$

The sliding vector field Z^Σ (defined in (3-7)) from $Z=(X,Y)$ is given by

$$\begin{aligned} \dot{x} &= 0, \\ \dot{y} &= g. \end{aligned} \tag{4-24}$$

Comparing (4-23) and (4-24) we arrive at the conclusion of the result. ■

Proposition 4.2.4 *Consider $Z\in\Omega_d^*$. We have that $p\in\Sigma$ is a fold point if and only if $f_1(p)=0$ (resp. $f_2(p)=0$) and the vector field X (resp. Y) is not tangent to the level 0 of the function $f1$ (resp. f_2) at p.*

Proof: Suppose that $p\in\Sigma$ is a fold point for X, then $Xf(p)=f_1(p)=0$ and $X^2f(p)\neq 0$. From the Definition 3.2.2, follows

$$X^2f(p)=\langle X(p),\nabla(Xf)(p)\rangle=\langle X(p),\nabla f_1(p)\rangle\neq 0.$$

Therefore vector field X is not tangent to the level 0 of the function f_1 at p. The conclusion is the same assuming that $p\in\Sigma$ is a fold point for Y. The reciprocal is immediate from the Definition 3.3.3. ■

Lemma 4.2.2 *Consider $Z\in\Omega_d^*$. We have that $p\in\Sigma$ is a regular point of Z if and only if $f_1(p)f_2(p)>0$ or $g(p)\neq 0$.*

Proof: The sliding vector field Z^Σ is given in (4-24). Supposing $p\in\Sigma$ is a regular point then $p\in\Sigma^c$ or, $p\in\Sigma^s\cup\Sigma^e$ and $Z^\Sigma(p)\neq 0$. If $p\in\Sigma^c$ follows that

$$(Xf(p))(Yf(p))>0 \ \Leftrightarrow \ f_1(p)f_2(p)>0;$$

if $p\in\Sigma^s\cup\Sigma^e$ and $Z^\Sigma(p)\neq 0$ follows that

$$Z^\Sigma(p)=(0,g(p))\neq 0 \ \Leftrightarrow \ g(p)\neq 0.$$

The reciprocal is immediate. ■

Lemma 4.2.3 *Let $p\in C_0$ (set critical) be a hyperbolic singular point of the slow flow from (2-1) with j^s-dimensional local stable manifold W^s and a J^u-dimensional local unstable manifold W^u. If Z is normally hyperbolic at p then there exists an ε-continuous family p_ε such that $p_0=p$ and p_ε has a (j^s+k^s)-dimensional local stable manifold W^s_ε and a (j^u+k^u)-dimensional local unstable manifold W^u_ε.*

For a proof see [4]. For a better understanding of the Lemma 4.2.3, the reader can also see the Theorem 1.4.2.

Theorem 4.2.3 *Consider $Z \in \Omega_d^*$ and the φ-regularized system Z_ε.*

(a) If $p \in \Sigma$ is a regular point of Z then there exist a neighborhood $V \subset \mathbb{R}^2$ with $p \in V$ and $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, Z_ε does not have singular points in V.

(b) If $p \in (\Sigma^s \cup \Sigma^e)$ is a hyperbolic singular point of Z^Σ then there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, Z_ε has a saddle point or a node point near p.

Proof: (a) Consider that $p \in \Sigma$ is a regular point. Lemma 4.2.2 implies that $f_1(p)f_2(p) > 0$ or $g(p) \neq 0$. Suppose that $f_1(p)f_2(p) > 0$. So $f_1(p)$ and $f_2(p)$ have the same sign and it implies that $Z_\varepsilon(p)$ is transversal to the line $x = 0$ at any q near p, for sufficiently small $\varepsilon > 0$. And we use the Flow Box Theorem (see Section 1.4). If $g(p) \neq 0$ then by the continuity of g, $g(q) \neq 0$ for any q near p and so $Z_\varepsilon(p) \neq \vec{0}$.

(b) Suppose $p \in (\Sigma^s \cup \Sigma^e)$. Without lost of generality assume that $p \in \Sigma^s$. Using Proposition 4.2.2 we have that p is normally hyperbolic. Lemma 4.2.3 implies that Z_ε has a singular point p_ε which approaches p when $\varepsilon \to 0$. If $f_1(p) > f_2(p)$ then $k^s = 0$ and $k^u = 1$ and finally p_ε is a repelling node if $j^u = 1$, $j^s = 0$ or a saddle if $j^u = 0$, $j^s = 1$. If $f_1(p) < f_2(p)$ then $k^s = 1$ and $k^u = 0$ and then p_ε is an attracting node if $j^u = 0$, $j^s = 1$ or a saddle if $j^u = 1$, $j^s = 0$. The proof for the case $p \in \Sigma^e$ is analogous. ■

Below are a few examples (from [2]) in which we use the blow up technique. For the following examples we consider a vector field $Z = (X,Y) \in \Omega_d^*$ and $\vec{0} \in \Sigma$. The approximation of Σ by the line $x = 0$ in a neighborhood of the origin is also considered. Furthermore, we do not impose any restrictions on the transition function φ. With the restriction $g_1 = g_2 = g$, singular problem in the blowing up locus (4-18) is

$$\begin{aligned} r\dot{\theta} &= \cos\theta[(\widetilde{f_1} + \widetilde{f_2})/2 + \varphi(\tan\theta)(\widetilde{f_1} - \widetilde{f_2})/2], \\ \dot{v} &= \widetilde{g}. \end{aligned} \tag{4-25}$$

We have not considered the $\dot{r}$ component in (4-25) because it is not relevant to the dynamics in the singular limit $r = 0$, as can be seen in (4-19) and (4-20).

Example 4.2.2 *(Sewing)*
Consider $X(x,y) = (1,1)$ and $Y(x,y) = (2,1)$.

Note that $\vec{0} \in \Sigma^c$ and (4-25) becomes

$$\begin{aligned} r\dot{\theta} &= \cos\theta[3/2 - \varphi(\tan\theta)/2], \\ \dot{v} &= 1. \end{aligned}$$

Since φ is non-decreasing in $\mathbb{R}$ and $\tan\theta$ is increasing in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, it follows that $\varphi(\tan\theta)$ is a non-decreasing function in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Note that $\lim_{\theta \to -\frac{\pi}{2}^+} \varphi(\tan\theta) = -1$ and $\lim_{\theta \to \frac{\pi}{2}^-} \varphi(\tan\theta) = 1$. More specifically, see that $\tan\theta \leq -1$ for all $\theta \in (-\frac{\pi}{2}, -\frac{\pi}{4}]$, so

$\varphi(\tan\theta) = -1$; $-1 < \tan\theta < 1$ for all $\theta \in (-\frac{\pi}{4}, \frac{\pi}{4})$, so $\varphi(\tan\theta) \in (-1,1)$); and $\tan\theta \geq 1$ for all $\theta \in [\frac{\pi}{4}, \frac{\pi}{2})$, so $\varphi(\tan\theta) = 1$. Of course $-1 \leq \varphi(\tan\theta) \leq 1$, then

$$0 < \cos\theta \leq \cos\theta[3/2 - \varphi(\tan\theta)/2] \leq 2\cos\theta,$$

i.e.

$$\cos\theta[3/2 - \varphi(\tan\theta)/2] > 0$$

for all $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In this case the critical manifold is an empty set.

The phase portrait of the fast and slow dynamics of the singular problem, for $r = 0$, and the phase portrait of the regularized vector field for $\varepsilon > 0$ sufficiently small are illustrated in Figure 4.8. ℵ

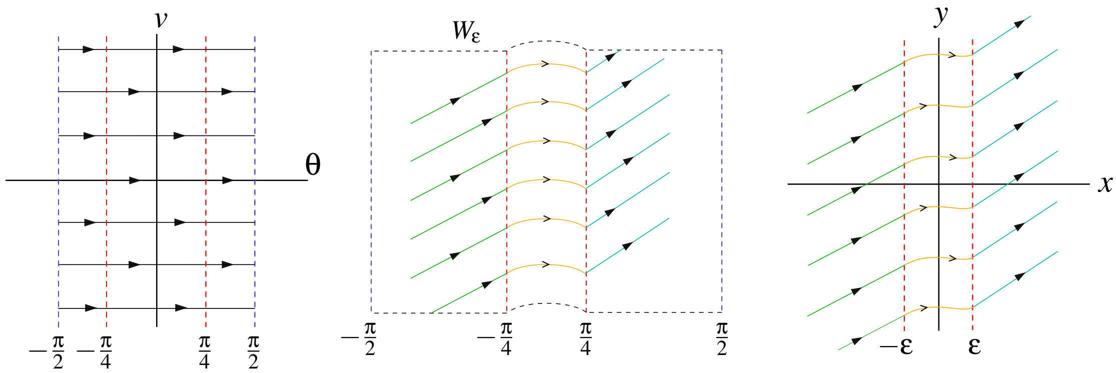

Figure 4.8: *Fast and slow dynamics of the singular perturbation problem corresponding to the sewing case and its regularization.*

Example 4.2.3 *(Escaping)*
Consider $X(x,y) = (1,1)$ and $Y(x,y) = (-1,1)$.

Note that $\vec{0} \in \Sigma^e$ and (4-25) becomes

$$\begin{aligned} r\dot{\theta} &= \cos\theta\varphi(\tan\theta), \\ \dot{v} &= 1. \end{aligned} \tag{4-26}$$

As we analyzed in the previous example, we know that $\varphi(\tan\theta)$ is a non-decreasing function in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $-1 \leq \varphi(\tan\theta) \leq 1$. From the Definition 4.1.1 (of φ), of course there is an unique $\theta_0 \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ such that $\varphi(\tan\theta_0) = 0$. Then the critical manifold form (4-26) is given by $C_0 = \{(\theta_0, v);\ v \in \mathbb{R}\}$. It is easy to see that

$$\frac{d[\cos\theta\varphi(\tan\theta)]}{d\theta}(\theta_0, v) = \cos\theta_0(1 + \tan^2\theta_0) \cdot \varphi'(\tan\theta_0) > 0$$

whatever $\theta_0 \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and $v \in \mathbb{R}$. Hence C_0 is normally hyperbolic. Note that

$$\cos\theta \cdot \varphi(\tan\theta) < 0$$

for all $\theta \in (-\pi/2, \theta_0)$, and

$$\cos\theta \cdot \varphi(\tan\theta) > 0$$

for all $\theta \in (\theta_0, \pi/2)$.

With the information presented, we can describe the fast and slow dynamics of (4-26). The phase portrait of the fast and slow dynamics of the singular problem, for $r = 0$, and the phase portrait of the regularized vector field for $\varepsilon > 0$ sufficiently small are illustrated in Figure 4.9. ℵ

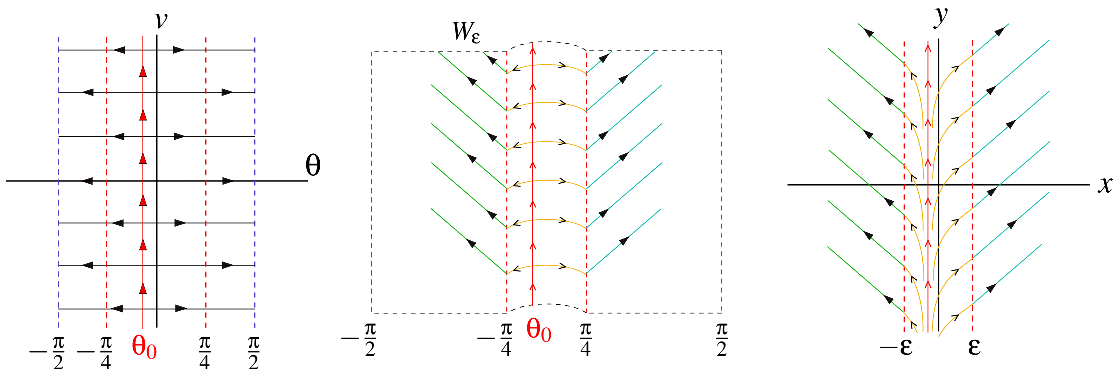

Figure 4.9: *Fast and slow dynamics of the singular perturbation problem corresponding to the escaping case and its regularization.*

An example for *sliding* can be considered in an entirely analogous way to the case of escaping: taking $X = (-1,-1)$ and $Y = (1,-1)$. The conclusions are the same as in the previous example inverting the orientation of the flow.

Example 4.2.4 *(Saddle)*
Consider $X(x,y) = (x+1,-y)$ and $Y(x,y) = (x-1,-y)$.

Note that $\vec{0} \in \Sigma^e$ and (4-25) becomes

$$\begin{aligned} r\dot{\theta} &= \cos\theta[r\sin\theta + \varphi(\tan\theta)], \\ \dot{v} &= -v. \end{aligned} \tag{4-27}$$

From (4-24) we have that $Z^\Sigma(x,y) = (0,g) = (0,-y)$, then $Z^\Sigma(\vec{0}) = \vec{0}$. Note that $\vec{0}$ is a singular point of Z^Σ and it is attractor. Then $\vec{0}$ is a saddle point. As in the previous example, it is easy to see that there is only one $\theta_0 \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ such that $\varphi(\tan\theta_0) = 0$. Thus the critical set from (4-27) is given by $C_0 = \{(\theta_0, v); v \in \mathbb{R}\}$. Furthermore C_0 is a normally hyperbolic critical manifold.

The phase portrait of the fast and slow dynamics of the singular problem, for $r = 0$, and the phase portrait of the regularized vector field for $\varepsilon > 0$ sufficiently small are illustrated in Figure 4.10. $\aleph$

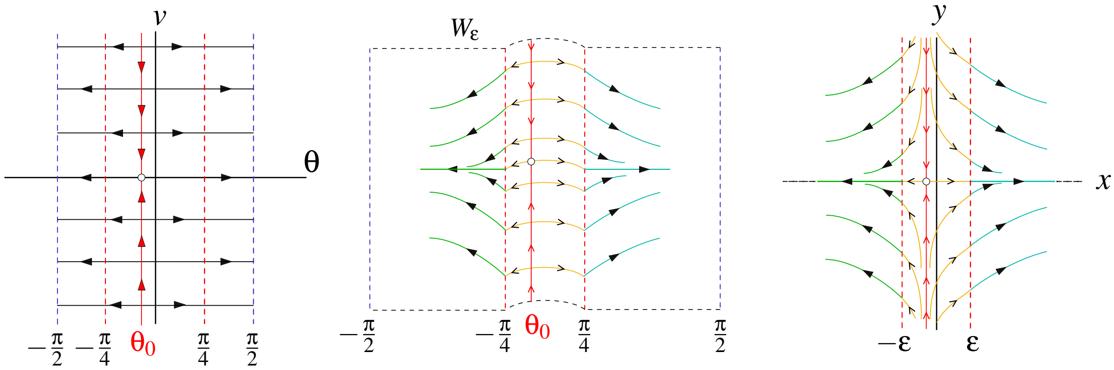

Figure 4.10: *Fast and slow dynamics of the singular perturbation problem corresponding to the saddle case and its regularization.*

Example 4.2.5 *(Fold)*
Consider $X(x,y) = (y,1)$ *and* $Y(x,y) = (1,1)$.

By the Proposition 4.2.4 follows that $\vec{0}$ is a fold point for X and (4-25) becomes

$$\begin{aligned} r\dot{\theta} &= \cos\theta[(v+1)/2 + \varphi(\tan\theta)(v-1)/2], \\ \dot{v} &= 1. \end{aligned} \tag{4-28}$$

The critical manifold is given by $C_0 = \{(\theta, v);\ v(\varphi(\tan\theta)+1) = \varphi(\tan\theta) - 1\}$. Note that the points of C_0 are the points on the graph of the function

$$v = v(\theta) = \frac{\varphi(\tan\theta) - 1}{\varphi(\tan\theta) + 1},$$

with $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$. It is easy to see that $v(\theta) \leq 0$ for all θ. We have that

$$v'(\theta) = 2\frac{\varphi'(\tan\theta)[1+\tan^2\theta]}{(\varphi(\tan\theta)+1)^2}.$$

Note that $v'(\theta) \geq 0$ for all $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{2}\right)$. More specifically, if $\theta \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, $v'(\theta) > 0$ then v is increasing and, if $\theta \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right)$, $v'(\theta) = 0$ then v is constant and equal to 0. To finish describing the points of C_0, we note that $\lim_{\theta \to -\frac{\pi}{4}^+} v(\theta) = -\infty$.

Now let is analyze which points of C_0 are normally hyperbolic. The layer problem from (4-28) is given by

$$\begin{aligned}\dot{\theta} &= \cos\theta[(v+1)/2+\varphi(\tan\theta)(v-1)/2],\\ \dot{v} &= 0.\end{aligned} \tag{4-29}$$

Denote by $h(\theta,v)=\cos\theta[(v+1)/2+\varphi(\tan\theta)(v-1)/2]$. For each $p=(\theta,v)\in C_0$, we have

$$\frac{\partial h}{\partial\theta}(p)=\cos\theta[\varphi'(\tan\theta)(1+\tan^2\theta)(v-1)/2].$$

We can see that $\frac{\partial h}{\partial\theta}(p)<0$ for all $\theta\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$ and $\frac{\partial h}{\partial\theta}(p)=0$ for all $\theta\in\left[\frac{\pi}{4},\frac{\pi}{2}\right)$. So the points on C_0 that are normally hyperbolic are the points on the graph of v such that $\theta\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$; furthermore they are attractor.

It is important to pay attention to the fact that in order to understand the behavior of our original system for $\varepsilon>0$ sufficiently small, we are always interested in understanding the dynamics of our system in the set V_ε. That said, in polar coordinates, we must understand the dynamics of the singular perturbation problem (4-28) when $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$. Given the transitive nature of the point $q=(\pi/4,0)$, let is introduce a change of variables in a neighborhood of q to study the dynamics of the system near it. Defines

$$\begin{aligned}\theta &= s\cos\psi+\tfrac{\pi}{4},\\ v &= s\sin\psi,\end{aligned}$$

where $\psi\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ and $s>0$. Writing (4-29) in the new coordinates gives us

$$\begin{aligned}\dot{s} &= \widetilde{h}\cos\psi,\\ s\dot{\psi} &= -\widetilde{h}\sin\psi,\end{aligned}$$

where

$$\widetilde{h}=\widetilde{h}(s,\psi)=\cos(s\cos\psi+\pi/4)[(s\sin\psi+1)/2+\varphi(\tan(s\cos\psi+\pi/4))(s\sin\psi-1)/2].$$

We have that

$$\dot{\psi}=-\frac{\sin\psi\cos(s\cos\psi+\pi/4)[(s\sin\psi+1)/2+\varphi(\tan(s\cos\psi+\pi/4))(s\sin\psi-1)/2]}{s}.$$

Applying L'Hôspital's method, we can see that

$$\lim_{s\to 0}\dot{\psi}=-\frac{\sqrt{2}\sin^2\psi}{2}.$$

It means that the angle component is decreasing for $\psi\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$ with a singular point at

$\psi = \pi$.

The phase portrait of the fast and slow dynamics of the singular problem, for $r = 0$, and the phase portrait of the regularized vector field for $\varepsilon > 0$ sufficiently small are illustrated in Figure 4.11. ℵ

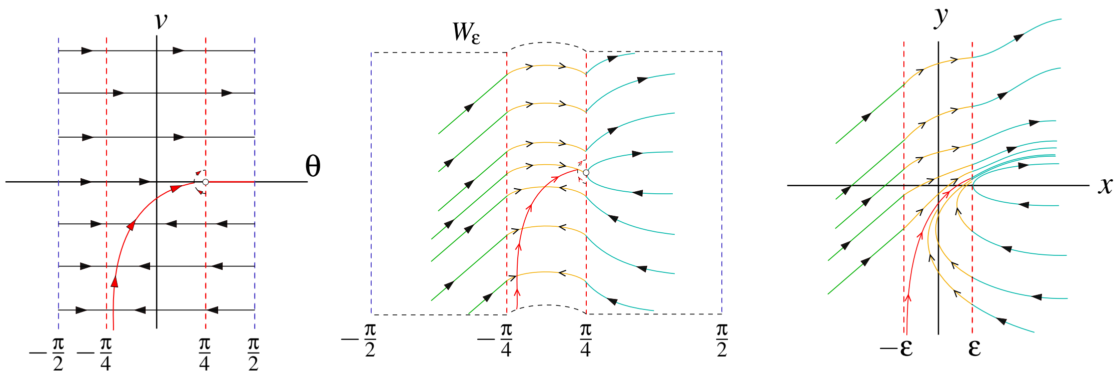

Figure 4.11: *Fast and slow dynamics of the singular perturbation problem corresponding to the fold case and its regularization.*

CHAPTER 5

A Conceptual Model of Glacial Cycles

This chapter presents a discontinuous system $Z = (X,Y)$, where X and Y are fast-slow vector fields defined under the same parameter $\varepsilon > 0$. The results presented in the previous chapters are not intended directly for fast-slow discontinuous vector fields. This is a different vector field structure from those presented throughout the dissertation. In order to deal with this problem, it is necessary to resort to analytical techniques that allow us to obtain information about the dynamics of the system. Furthermore, this problem is a strong motivation to develop a consistent theory that can provide information on the dynamics of fast-slow discontinuous vector fields.

Conceptual climate models provide an approach to understanding climate processes through a mathematical analysis of an approximation of reality. A new conceptual model of glacial cycles has been developed, consisting of a system of three ordinary differential equations defining a fast-slow discontinuous vector field. In this chapter we will present this model and some elements of its dynamics, according to Filippov's convention. For more details on the model, concepts and results presented in this chapter, see [21].

The model emphasized in this chapter is a system of three ordinary differential equations with a discontinuity boundary consisting of a plane in three-dimensional state space. Using techniques reminiscent of singular perturbation theory, it is possible to show the existence of a large periodic orbit crossing the discontinuity boundary.

5.1 Budyko's Equation and the Approximation of McGehee and Widiasih

Budyko's energy balance model concerns the average annual temperature in latitudinal zones in a world assumed to be symmetric about the equator. The Budyko's equation is given by

$$R\frac{\partial T(y,t)}{\partial t} = Qs(y)(1-\alpha(y)) - (A+BT) - C(T-\widetilde{T}), \tag{5-1}$$

where:

- y is the sine of the latitude. Due to symmetry considerations, $y \in [0,1]$, with $y = 0$ the equator and $y = 1$ the North Pole;
- $T = T(y,t)$ (°C) is the annual mean surface temperature on the circle of latitude at y. The lefthand side of (5-1) represents the change in energy stored in the Earth's surface at y (Wm^{-2});
- R ($\mathrm{J(m^2{}^\circ C)^{-1}}$) is the heat capacity of the Earth's surface;
- Q denotes the mean annual incoming solar radiation (or *insolation*), a parameter depending on the eccentricity of Earth's orbit;
- $s(y)$, which depends upon the obliquity of Earth's orbit, accounts for the distribution of insolation across latitude, and satisfies

$$\int_0^1 s(y)dy = 1;$$

- $\alpha(y)$ represents the planetary *albedo*. Albedo (sometimes called "reflection coefficient") is a measure of how reflective a surface is. It is a measure of the proportion of incoming solar radiation that is reflected back into the atmosphere and into space. Thus the $Qs(y)(1-\alpha(y))$-term represents the energy from the sun absorbed at the surface at latitude y;
- The empirically-derived $(A+BT)$-term models the *outgoing longwave radiation* (OLR) emitted by the Earth, while the transport of heat energy across latitudes is modeled by the $C(T-\widetilde{T})$-term, in which

$$\widetilde{T} = \int_0^1 T(y,t)dy$$

  is the global annual mean surface temperature;
- A, B and C are positive and empirical constants.

The equilibrium temperature profiles of (5-1) are shown to be given by

$$T^*(y) = \frac{1}{A+B}\left(Qs(y)(1-\alpha(y)) - A + \frac{C}{B}\left(Q(1-\widetilde{\alpha}) - A\right)\right) \tag{5-2}$$

where

$$\widetilde{\alpha} = \int_0^1 \alpha(y)s(y)dy. \tag{5-3}$$

Letting $p_0(y) = 1$ and $p_2(y) = \frac{1}{2}(3y^2-1)$ denote the first two even Legendre polynomials, we make use of the expression

$$s(y) = s_0p_0(y) + s_2p_2(y), \;\; s_0 = 1, \;\; s_2 = -0.482, \tag{5-4}$$

which is uniformly within 2% of the actual values of $s(y)$, in all that follows.

In Budyko's model one assumes the planet has an ice cap, with ice at all latitudes above a certain latitude $y = \eta$, and no ice south of $y = \eta$. The edge of the ice sheet η is called the ice line. The albedo function is then given by

$$\alpha_\eta(y) = \begin{cases} \alpha_1, & \text{if } y < \eta, \\ \alpha_2, & \text{if } y > \eta, \quad \alpha_1 < \alpha_2, \end{cases} \tag{5-5}$$

where α_1 and α_2 represent the albedos of the surface without ice cover and that with ice cover, respectively. From the definition of α_η and (5-4), it follows that (5-2) are even, piecewise quadratic functions having a discontinuity at $y = \eta$.

Defining $T^*_\eta(\eta) = \frac{1}{2}(\lim_{y\to\eta^-} T^*(y) + \lim_{y\to\eta^+} T^*(y))$, one finds the temperature at equilibrium at the ice line is given by

$$T^*_\eta(\eta) = \frac{1}{A+B}\left(Qs(\eta)(1-\alpha_0) - A + \frac{C}{B}\Big(Q(1-\widetilde{\alpha}(\eta) - A)\Big)\right), \tag{5-6}$$

with $\alpha_0 = \frac{1}{2}(\alpha_1 + \alpha_2)$. Budyko was interested in the existence of η-values for which $T^*_\eta(\eta) = T_c$, where T_c is a *critical temperature* above which ice melts and below which ice forms.

Budyko's motivation for studying the positive influence of albedo on ice was:

(i) If the temperature of the earth's surface decreased, the existing ice sheet would increase. There would be an increase in albedo and consequently an even greater decrease in temperatures, generating an increasingly larger ice layer.

(ii) Higher temperatures would result in a smaller ice layer and consequently a reduction in albedo. This would result in a increase in temperatures and a further reduction in the ice layer.

Budyko's model does not take into account the changes that can occur in the ice line η with changes in temperature. With adjustments from Widiasih implemented to Budyko's model, it was possible to reverse this limitation. Widiasih added an ODE to describe the changes of η over time, leading to the integral differential system

$$R\frac{\partial T}{\partial t} = Qs(y)(1-\alpha(y,\eta)) - (A+BT) - C(T-\widetilde{T}), \tag{5-7}$$

$$\frac{d\eta}{dt} = \rho(T(\eta,t) - T_c), \tag{5-8}$$

where $\rho > 0$ is a parameter governing the relaxation time of the ice sheet. $T(y,t)$ evolves according to equation (5-7), while the dynamics of η are determined by the temperature at the ice line, relative to the critical temperature. If $T(\eta,t) > T_c$, the ice sheet retreats toward the pole and if $T(\eta,t) < T_c$, moves equatorward .

McGehee and Widiasih provide an approximation for the system (5-7)(5-8) (see [21]) given by

$$\begin{aligned} \dot{w} &= -\tau(w - F(\eta)), \\ \dot{\eta} &= \rho(w - G(\eta)), \end{aligned} \tag{5-9}$$

where:

$$F(\eta) = \frac{1}{B}\left(Q(1-\alpha_0) - A + CL(\alpha_2 - \alpha_1)\left(\eta - \frac{1}{2} + s_2\widetilde{p}_2(\eta)\right)\right),$$

$$\widetilde{p}_2(\eta) = \int_0^{\eta} p_2(y)dy,$$

$$G(\eta) = -Ls_2(1-\alpha_0)p_2(\eta) + T_c,$$

$$L = \frac{Q}{B+C} \text{ and } \tau = \frac{B}{R}.$$

Note that F is a cubic polynomial and G a quadratic polynomial. The new variable w in (5-9) represents the global average temperature with the same physical meaning as the variable T in (5-7)(5-8).

5.2 Addition of a Snow Line

Ice accumulation corresponds to the set of all processes that cause a local increase in glacial mass. The accumulation of snow over time produces old, compact snow, which gives rise to ice. Ice ablation corresponds to the set of all processes that cause the local loss of glacial mass.

Considering the influence of the relative sizes of accumulation and ablation zones on glacial advance and retreat, the ‘snow line’ variable was introduced into the model, independent of the edge of the ice sheet, complementing the equations (5-9) .

Now, in the model, they implement independent snow and ice lines to consider accumulation and ablation zones. As discussed previously, the model continues to depend on latitude. First, they reformulate the role played by η, becoming the snow line variable. they denote by ξ the ice line variable, i.e., the edge of the ice sheet (see Figure 5.1). The ablation zone has size $\eta - \xi$ (when $\eta > \xi$), while the accumulation zone has size $1 - \eta$.

We continue considering the albedo function (5-5). The albedo of the ice sheet region in the ablation zone is lower than that located towards the pole η. We observed a reduction in albedo at the edges of the ice sheet due to superglacial forests. We also note that as ancient ice emerges and melts in the ablation zone, a surface layer of dust is created that was originally deposited with the snow on top of the ice sheet. The presence of wind currents produces even more dust and debris in the ablation zone. The value of the albedo α_1 in (5-5) can then be interpreted as an average of the albedos of the planet’s surface south of ξ and the darkened (relative to snow) ice surface at ablation zone. The

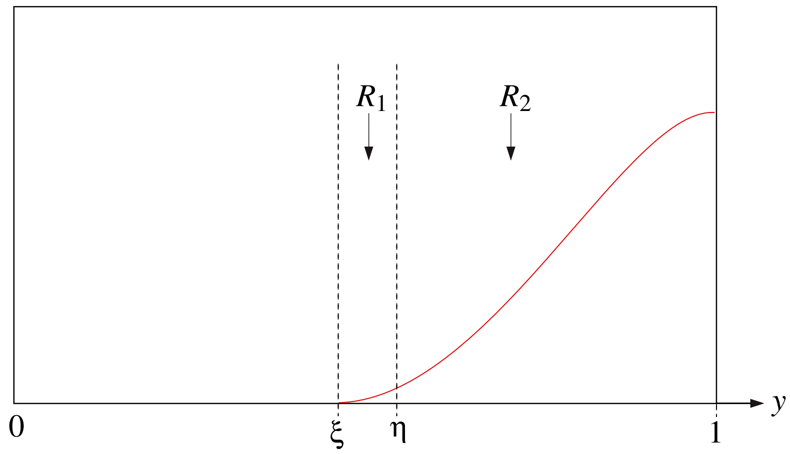

Figure 5.1: *Ablation zone R_1 and accumulation zone R_2. The model set-up. η is the snow line and ξ is the ice line. The shape of the glacier is for illustrative purposes only.*

model allows for different time scales for the snow line and the ice line (which moves more slowly).

5.3 Model of Glacial Cycles

The ***temperature-ice line-snow line model*** is a nonsmooth system with state space

$$U = \{(w,\eta,\xi);\ w \in \mathbb{R} \text{ and } \eta,\xi \in [0,1]\},$$

where ξ denotes the ice line, η denotes the snow line, and w is related to global average temperature. To define the model equations, pick parameters $b_0 < b < b_1$ representing ablation rates, and a parameter a denoting the accumulation rate. When $b(\eta-\xi)-a(1-\eta) < 0$ we assume the ice sheet advances, and we set

$$\begin{aligned} \dot{w} &= -\tau(w-F(\eta)) = f_1^-(w,\eta,\xi), \\ \dot{\eta} &= \rho(w-G_-(\eta)) = f_2^-(w,\eta,\xi), \\ \dot{\xi} &= \varepsilon(b_0(\eta-\xi)-a(1-\eta)) = \varepsilon f_3^-(w,\eta,\xi), \end{aligned} \tag{5-10}$$

when $b(\eta-\xi)-a(1-\eta) > 0$ we assume the ice sheet retreats, and we set

$$\begin{aligned} \dot{w} &= -\tau(w-F(\eta)) = f_1^+(w,\eta,\xi), \\ \dot{\eta} &= \rho(w-G_+(\eta)) = f_2^+(w,\eta,\xi), \\ \dot{\xi} &= \varepsilon(b_1(\eta-\xi)-a(1-\eta)) = \varepsilon f_3^+(w,\eta,\xi). \end{aligned} \tag{5-11}$$

The function $F(\eta)$ is given as in (5-9), $G_-(\eta) = G(\eta)$ with $T_c = T_c^-$, $G_+(\eta) = G(\eta)$ with $T_c = T_c^+$, and $\varepsilon > 0$ is a time constant for the movement of the ice line. The value of the parameters that correspond to the real problem are given in Table 5.1. We note most of these parameter values only serve to approximate complex aspects of the climate system. The ablation rates b_0 and b_1 and the accumulation rate a are not well-constrained. The values for time constants τ and ρ are somewhat arbitrary.

Table 5.1: *Parameter values.*

Parameter	Value	Units	Parameter	Value	Units
Q	343	Wm^{-2}	T_c^-	-5.5	°C
A	202	Wm^{-2}	b_0	1.5	Dimensionless
B	1.9	$Wm^{-2}(°C^{-1})$	b	1.75	Dimensionless
C	3.04	$Wm^{-2}(°C^{-1})$	b_1	5	Dimensionless
α_1	0.32	Dimensionless	a	1.05	Dimensionless
α_2	0.62	Dimensionless	τ	1	s^{-1}
T_c^+	-10	°C	ρ	0.1	$s^{-1}(°C)^{-1}$

Note: The time constant ε has units $(\text{seconds})^{-1}$.

Denote the vector field (5-10) by $Y = Y(w,\eta,\xi)$ and the vector field (5-11) by $X = X(w,\eta,\xi)$. We thus arrive at a piecewise-smooth three-dimensional vector field $Z = (X,Y)$ having a plane of discontinuity $\Sigma = \{(w,\eta,\xi) \in U;\, b(\eta-\xi) - a(1-\eta) = 0\} = \{(w,\eta,\xi) \in U;\, \xi = (1+\frac{a}{b})\eta - \frac{a}{b} := \gamma(\eta)\}$. Then

$$Z(p) = \begin{cases} X(p), & \text{for } p \in \Sigma^+, \\ Y(p), & \text{for } p \in \Sigma^-, \end{cases} \tag{5-12}$$

where $\Sigma^+ = \{(w,\eta,\xi) \in U;\, \xi \leq \gamma(\eta)\}$ and $\Sigma^- = \{(w,\eta,\xi) \in U;\, \xi \geq \gamma(\eta)\}$.

5.3.1 Equilibrium Points

Definition 5.3.1 *Let $x \in U$.*

(i) x is a ***regular equilibrium point*** *of (5-12) if either $X(x) = 0$ and $x \in \Sigma^+ \backslash \Sigma$, or if $Y(x) = 0$ and $x \in \Sigma^- \backslash \Sigma$.*

(ii) x is a ***virtual equilibrium point*** *of (5-12) if either $X(x) = 0$ and $x \in \Sigma^- \backslash \Sigma$, or if $Y(x) = 0$ and $x \in \Sigma^+ \backslash \Sigma$.*

(iii) x is a ***boundary equilibrium point*** *of (5-12) if $X(x) = 0$ or $Y(x) = 0$ and $x \in \Sigma$.*

An equilibrium point $P_+ = (w_+,\eta_+,\xi_+)$ for X satisfies $w_+ = F(\eta_+) = G_+(\eta_+)$ and $\xi_+ = (1+\frac{a}{b_1})\eta_+ - \frac{a}{b_1}$ (similarly for Y). With the parameters from the Table 5.1 follows

$$F(\eta) = G_+(\eta) \quad \Leftrightarrow \quad \omega_+(\eta) := -8.03203\eta^3 - 26.6061\eta^2 + 41.36\eta - 8.43211 = 0$$

and

$$F(\eta) = G_-(\eta) \quad \Leftrightarrow \quad \omega_-(\eta) := -8.03203\eta^3 - 26.6061\eta^2 + 41.36\eta - 12.9321 = 0.$$

Figure 5.2 illustrates the local graph of ω_+ and ω_-, in which we obtain an estimate for their roots in $[0,1]$.

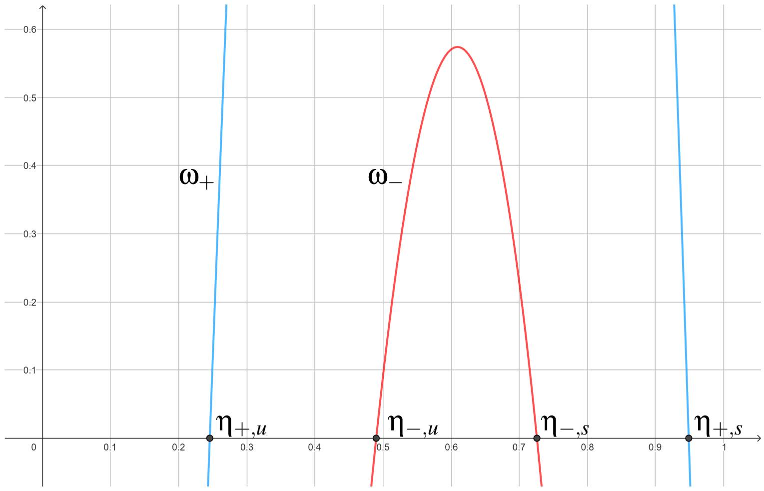

Figure 5.2: *Local graph of* ω_+ *and* ω_-.

We can see that X has three equilibrium points, two of which are U. The equilibrium points of X that are in U are

$$P_{+,s} = \left(w_{+,s}, \eta_{+,s}, \left(1+\frac{a}{b_1}\right)\eta_{+,s} - \frac{a}{b_1}\right) \text{ and } P_{+,u} = \left(w_{+,u}, \eta_{+,u}, \left(1+\frac{a}{b_1}\right)\eta_{+,u} - \frac{a}{b_1}\right),$$

where $(w_{+,s}, \eta_{+,s}, \xi_{+,s}) \approx (5.08, 0.95, 0.94)$ and $(w_{+,u}, \eta_{+,u}, \xi_{+,u}) \approx (-17.26, 0.25, 0.087)$.

From the fact that $b_0 < b < b_1$, we have that

$$\xi_{+,s} = \left(1+\frac{a}{b_1}\right)\eta_{+,s} - \frac{a}{b_1} > \left(1+\frac{a}{b}\right)\eta_{+,s} - \frac{a}{b} = \gamma(\eta_{+,s});$$

analogously, we note that $\xi_{+,u} > \gamma(\eta_{+,u})$. Hence $P_{+,s}, P_{+,u} \in \Sigma^-$. Therefore booth are virtual equilibrium points of Z. Furthermore $J(X)(P_{+,s})$ has three negative eigenvalues, while $J(X)(P_{+,u})$ has two negative eigenvalues and one positive eigenvalue.

Similarly, one can show Y has three equilibrium points, two of which are in U: a stable node $P_{-,s} = (w_{-,s}, \eta_{-,s}, \xi_{-,s}) \approx (-0.35, 0.72, 0.53)$ and a saddle $P_{-,u} = (w_{-,u}, \eta_{-,u}, \xi_{-,u}) \approx (-7.98, 0.49, 0.13)$ having a two-dimensional stable local manifold. We note $\xi_{-,s} < \gamma(\eta_{-,s})$ and $\xi_{-,u} < \gamma(\eta_{-,u})$, hence $P_{-,s}, P_{-,u} \in \Sigma^+$ and therefore booth are virtual equilibrium points of Z.

Assuming the parameters in Table 5.1, consider the line R_X given by $\xi = (1+\frac{a}{b_1})\eta - \frac{a}{b_1} = 1.21\eta - 0.21$ and R_Y given by $\xi = (1+\frac{a}{b_0})\eta - \frac{a}{b_0} = 1.7\eta - 0.7$. Figure 5.3 illustrates the projection of the equilibrium points of X and Y onto the plane $\eta \times \xi$, which are at R_X and R_Y, respectively; as well as the relative position of these points with respect to Σ, which in the plane $\eta \times \xi$, is given by $\xi = (1+\frac{a}{b})\eta - \frac{a}{b} = 1.6\eta - 0.6$.

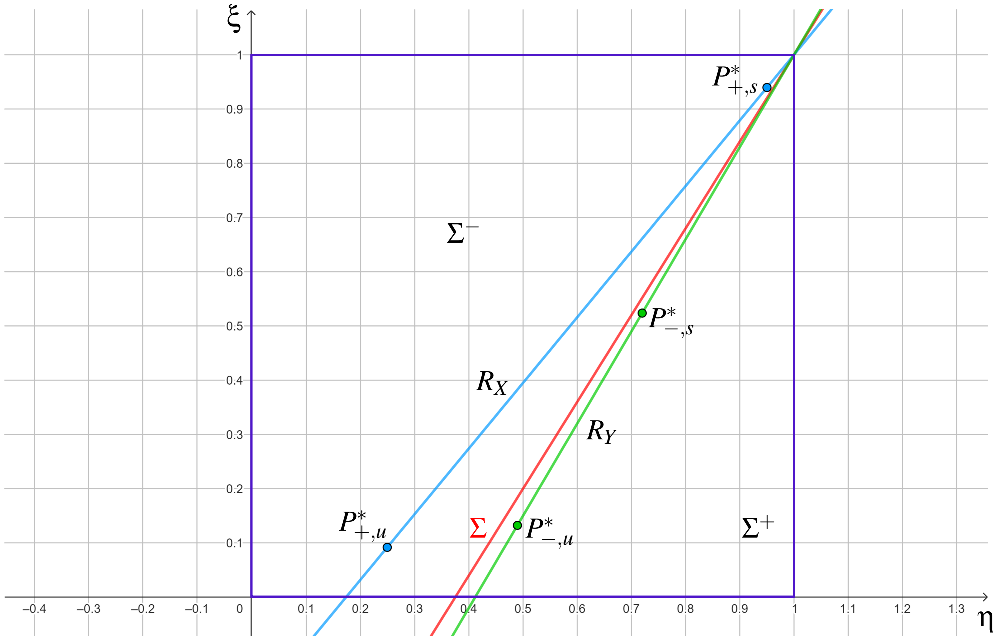

Figure 5.3: *Projection of the equilibrium points of X and Y onto the plane $\eta \times \xi$. The asterisk denotes the result of the projection, for example, $P^*_{+,s} = (w_{+,s}, \eta_{+,s})$; the same applies to the other points.*

5.3.2 Behavior on the Discontinuity Σ

Let is analyze the nature of the points in Σ. To do this, we will use the first-order Lie derivatives of the function $f(w,\eta,\xi) = b(\eta-\xi) - a(1-\eta)$ in the direction of X and Y. Note that $\Sigma = f^{-1}(0) \cap U = \{(w,\eta,\xi);\, w \in \mathbb{R},\, \frac{a}{a+b} \leq \eta \leq 1 \text{ and } \xi = \gamma(\eta)\}$; and $\nabla f(p) = (0, a+b, -b)$ for all $p \in \Sigma$. Note that $\omega : \mathbb{R} \times \left[\frac{a}{a+b}, 1\right] \to \Sigma$ given by $\omega(w,\eta) = (w,\eta,\gamma(\eta))$ is a parameterization of Σ.

Let $p = (w,\eta,\xi) \in \Sigma$. We have that

$$
\begin{aligned}
Xf(p) &= \langle X(p), \nabla f(p) \rangle \\
&= (a+b)\rho(w - G_+(\eta)) - b\varepsilon[b_1(\eta-\xi) - a(1-\eta)] \\
&= (a+b)\rho(w - G_+(\eta)) - b\varepsilon[b_1(\eta-\gamma(\eta)) - a(1-\eta)] \\
&= (a+b)\rho(w - G_+(\eta)) - \varepsilon a(1-\eta)(b_1 - b).
\end{aligned}
$$

Analogously, we can show that

$$Yf(p) = (a+b)\rho(w - G_{-}(\eta)) - \varepsilon a(1-\eta)(b_0 - b).$$

Note that

$$Xf(p) = 0 \;\Leftrightarrow\; w = G_{+}(\eta) + \frac{\varepsilon a(1-\eta)(b_1 - b)}{(a+b)\rho} := g_{+}(\eta),$$

with $\frac{a}{a+b} \le \eta \le 1$. The set of points $p \in \Sigma$ such that $Xf(p) = 0$ is given by $\Sigma_X^t = \{(w,\eta,\xi);\, w = g_{+}(\eta),\, \frac{a}{a+b} \le \eta \le 1 \text{ and } \xi = \gamma(\eta)\}$; in other words, $Xf(p)$ cancels out at the points on the curve $\alpha_{+} : \left[\frac{a}{a+b}, 1\right] \to \Sigma$ given by $\alpha_{+}(\eta) = (g_{+}(\eta), \eta, \gamma(\eta))$. Furthermore, it's easy to see that

$$Xf(p) > 0 \;\Leftrightarrow\; w > g_{+}(\eta)$$

and

$$Xf(p) < 0 \;\Leftrightarrow\; w < g_{+}(\eta).$$

Similarly, for Y, follows

$$Yf(p) = 0 \;\Leftrightarrow\; w = G_{-}(\eta) + \frac{\varepsilon a(1-\eta)(b_0 - b)}{(a+b)\rho} := g_{-}(\eta),$$

with $\frac{a}{a+b} \le \eta \le 1$, and the set of points $p \in \Sigma$ such that $Yf(p) = 0$ is given by $\Sigma_Y^t = \{(w,\eta,\xi);\, w = g_{-}(\eta),\, \frac{a}{a+b} \le \eta \le 1 \text{ and } \xi = \gamma(\eta)\}$; i.e., $Yf(p)$ cancels out at the points on the curve $\alpha_{-} : \left[\frac{a}{a+b}, 1\right] \to \Sigma$ given by $\alpha_{-}(\eta) = (g_{-}(\eta), \eta, \gamma(\eta))$. We have too that

$$Yf(p) > 0 \;\Leftrightarrow\; w > g_{-}(\eta)$$

and

$$Yf(p) < 0 \;\Leftrightarrow\; w < g_{-}(\eta).$$

Note that g_{+} and g_{-} are quadratic polynomials. The set of tangency points of Z is given by $\Sigma^t = \Sigma_X^t \cup \Sigma_Y^t$ (see Definition 3.2.4). Note that

$$q = (w,\eta,\xi) \in \Sigma_X^t \cap \Sigma_Y^t \;\Leftrightarrow\; g_{+}(\eta) = g_{-}(\eta) \text{ with } \eta \in \left[\frac{a}{a+b}, 1\right].$$

We note

$$g_{+}(\eta) = g_{-}(\eta) \;\Leftrightarrow\; \eta = \eta(\varepsilon) = 1 + \frac{(T_c^{+} - T_c^{-})(a+b)\rho}{\varepsilon a(b_1 - b_0)}.$$

As $\eta \in \left[\frac{a}{a+b}, 1\right]$ we should have

$$\frac{a}{a+b} \le \eta(\varepsilon) \le 1 \;\Leftrightarrow\; 0 \le \frac{1}{\varepsilon} \le \frac{(-b)a(b_1 - b_0)}{(T_c^{+} - T_c^{+})(a+b)^2\rho} \;\Leftrightarrow\; \varepsilon \ge \frac{(T_c^{-} - T_c^{+})(a+b)^2\rho}{ab(b_1 - b_0)}.$$

In the general case we are considering $T_c^- > T_c^+$. For the parameters in Table 5.1, we have

$$\frac{(T_c^- - T_c^+)(a+b)^2\rho}{ab(b_1 - b_0)} \approx 0.5486.$$

Assuming that the parameter $\varepsilon > 0$ is sufficiently small, we say that $\eta(\varepsilon) \notin \left[\frac{a}{a+b}, 1\right]$, then $\Sigma_X^t \cap \Sigma_Y^t = \emptyset$. Thus, we can conclude that

$$g_+(\eta) < g_-(\eta), \quad \forall \eta \in \left[\frac{a}{a+b}, 1\right],$$

as long as

$$\varepsilon < \frac{(T_c^- - T_c^+)(a+b)^2\rho}{ab(b_1 - b_0)}.$$

Hence

$$Xf(p) > 0 \text{ and } Yf(p) > 0 \;\Leftrightarrow\; w > g_-(\eta);$$

$$Xf(p) < 0 \text{ and } Yf(p) < 0 \;\Leftrightarrow\; w < g_+(\eta);$$

and

$$Xf(p) > 0 \text{ and } Yf(p) < 0 \;\Leftrightarrow\; g_+(\eta) < w < g_-(\eta).$$

Then, by the Definition 3.2.3, it follows
$\Sigma^c = \{(w,\eta,\xi) \in \Sigma;\, w > g_-(\eta) \text{ or } w < g_+(\eta)\}$,
$\Sigma^e = \{(w,\eta,\xi) \in \Sigma;\, g_+(\eta) < w < g_-(\eta)\}$
and
$\Sigma^s = \emptyset$.

From the characterization of the Σ points, considering Definition 3.2.7, we can understand how the solutions of Z behave through Σ.

We can calculate the second-order Lie derivative $X^2 f$ on α_+ and α_-, and evaluate for which values of $\eta \in \left[\frac{a}{a+b}, 1\right]$ it is negative, zero or positive; then we use Definitions 3.2.6 and 3.3.3. For the following analysis, there is no need to describe the nature of the points of tangency.

5.3.3 Section Maps for the Filippov's Flow

Let be M the two-dimensional stable manifold of the saddle point $P_{-,u}$ under the flow φ_Y (from Y). Due to the (w,ξ)-decoupling, the projection of M onto the (w,ξ)-plane is simply the (one-dimensional) stable manifold of $(w_{-,u}, \eta_{-,u})$ under the flow ψ_- given by subsystem $(\dot{w}, \dot{\eta})$ in (5-10). In particular, we note the surface M is independent of $\varepsilon > 0$.

M partitions Σ into two subsets, a 'top' (larger w) and a 'bottom' (smaller w). Denote this top subset by I . For any point $p = (w,\eta,\xi) \in I$, $\varphi_Y(p,t) = (w(t),\eta(t),\xi(t)) \to P_{-,s}$. This follows from the fact $(w(t),\eta(t)) \to (w_{-,s}, \eta_{-,s})$ as $t \to \infty$, and hence $\xi(t) \to$

$\xi_{-,s}$. Thus $I \subset W^s_-(P_{-,s})$ the stable manifold of the sink $P_{-,s}$ under the flow φ_Y (review Theorem 1.4.2). Furthermore $I \subset W^s_+(P_{+,s})$, the stable set of the sink $P_{+,s}$ under the flow φ_X. Again, all of this holds for any $\varepsilon > 0$.

Defines $\Sigma^c_+ = \{(w,\eta,\xi) \in \Sigma;\ w < g_+(\eta)\}$ and $\Sigma^c_- = \{(w,\eta,\xi) \in \Sigma;\ w > g_-(\eta)\}$. Let $\Gamma_+ = \Sigma^c_+ \cap I$, the set of points in Σ above M and below α_+. Let $\Gamma_- = \Sigma^c_- \cap I$, the set of points in Σ above M and above α_-.

Note that for any $\varepsilon > 0$, $\Gamma_+ \subset W^s_-(P_{-,s})$ and $Yf(p) < 0$ for all $p \in \Gamma_+$. Hence for any $\varepsilon > 0$, the φ_Y - trajectories starting at $p \in \Gamma_+$ enters Σ^- before necessarily intersecting Γ_- as it seeks to approach the (virtual) sink $P_{-,s}$. For any $\varepsilon > 0$ and for each $p \in \Gamma_+$, there is then a $t = t(p) > 0$ such that $\varphi_Y(p,t) \in \Gamma_-$. We write $t(p) = t(p,\varepsilon)$ to emphasize the dependence of this t-value on ε as well as on p. We note that $t(p,\varepsilon) \to \infty$ as $\varepsilon \to 0^+$.

This provides, for any $\varepsilon > 0$, a continuous map

$$r^-_\varepsilon : \Gamma_+ \to \Gamma_-, \qquad r^-_\varepsilon(p) = \varphi_Y(p,t(p,\varepsilon)). \tag{5-13}$$

Similarly, for any $\varepsilon > 0$, $\Gamma_- \subset W^s_+(P_{+,s})$ and $Xf(p) > 0$ for all $p \in \Gamma_-$. Hence of any $\varepsilon > 0$, the φ_X-trajectory starting at $p \in \Gamma_-$ enters Σ^+ before necessarily intersecting Γ_+ as it tries to approach the (virtual) sink $P_{+,s}$. For any $\varepsilon > 0$ and for each $p \in \Gamma_-$, there exists a $t = t(p,\varepsilon) > 0$ such that $\varphi_X(p,t) \in \Gamma_+$. This defines a continuous map

$$r^+_\varepsilon : \Gamma_- \to \Gamma_+, \qquad r^+_\varepsilon(p) = \varphi_X(p,t(p,\varepsilon)), \tag{5-14}$$

which exists for any $\varepsilon > 0$.

We then have a continuous section map $r^+_\varepsilon \circ r^-_\varepsilon : \Gamma_+ \to \Gamma_+$, defined for any $\varepsilon > 0$. We set $r_\varepsilon = r^+_\varepsilon \circ r^-_\varepsilon$ in all that follows.

5.3.4 Existence of an Attracting Periodic Orbit

It will be shown in this section that system (5-12), for $\varepsilon > 0$ sufficiently small, has a single attracting periodic orbit. To prove this fact, we will analyze the dynamics of Z in the case where $\varepsilon = 0$. We observe that the vector fields X and Y are singular perturbation problems. To make the explicit dependence of Z on the parameter ε, we write $Z = Z_\varepsilon = (X_\varepsilon, Y_\varepsilon)$. Taking $\varepsilon = 0$ we obtain the layer problem and reduced problem associated with Z, which are also discontinuous vector fields. The layer problem from Z is given by

$$\begin{aligned} \dot{w} &= -\tau(w - F(\eta)), \\ \dot{\eta} &= \rho(w - G_-(\eta)), \\ \dot{\xi} &= 0. \end{aligned}$$

if $b(\eta-\xi)-a(1-\eta)<0$; and

$$\begin{aligned} \dot{w} &= -\tau(w-F(\eta)), \\ \dot{\eta} &= \rho(w-G_{+}(\eta)), \\ \dot{\xi} &= 0. \end{aligned}$$

if $b(\eta-\xi)-a(1-\eta)>0$.
On the other hand, the reduced problem from Z is given by

$$\begin{aligned} 0 &= -\tau(w-F(\eta)), \\ 0 &= \rho(w-G_{-}(\eta)), \\ \dot{\xi} &= b_0(\eta-\xi)-a(1-\eta). \end{aligned}$$

if $b(\eta-\xi)-a(1-\eta)<0$; and

$$\begin{aligned} 0 &= -\tau(w-F(\eta)), \\ 0 &= \rho(w-G_{+}(\eta)), \\ \dot{\xi} &= b_1(\eta-\xi)-a(1-\eta). \end{aligned}$$

if $b(\eta-\xi)-a(1-\eta)>0$.

Each of the systems X_0 and Y_0 (seen as the layer problem) has an attracting line of equilibrium of the form $(w_{+,s},\eta_{+,s},\xi)$ and $(w_{-,s},\eta_{-,s},\xi)$, respectively (see Figure 5.4). The dynamics under these lines is described by observing whether the sign of the component $\dot{\xi}$ of X_0 and Y_0 (seen as the reduced problem). Let

$$Z_{+}=(w_{+,s},\eta_{+,s},\gamma(\eta_{+,s})) \text{ and } Z_{-}=(w_{-,s},\eta_{-,s},\gamma(\eta_{-,s})).$$

Note that each of these points lie in the discontinuity boundary Σ.

When $\varepsilon=0$, the trajectory starting at Z_+ converges to the regular equilibrium point $(w_{-,s},\eta_{-,s},\gamma(\eta_{+,s}))$ under the advancing flow φ_Y, while the trajectory starting at Z_- converges to the regular equilibrium point $(w_{+,s},\eta_{+,s},\gamma(\eta_{-,s}))$ under the retreating flow φ_X. For ε sufficiently small one might expect there to be a Filippov periodic orbit approximating the rectangular 'orbit' implicit in Figure 5.4.

Let us show that for any compact subset D_+ of Γ_+ containing Z_+ in its interior, and for all sufficiently small $\varepsilon>0$, the section map satisfies
(i) $r_\varepsilon(D_+)\subset D_+$,
and
(ii) r_ε is a contraction map on D_+.

Proposition 5.3.1 *(i) Given $c_1\in(0,1)$ and a compact set $D_+\subset\Gamma_+$, there exists $\varepsilon_1>0$ such that for all $\varepsilon\leq\varepsilon_1$ and for all $x_1,x_2\in D_+$, $|r_\varepsilon^-(x_2)-r_\varepsilon^-(x_1)|\leq c_1|x_2-x_1|$.*

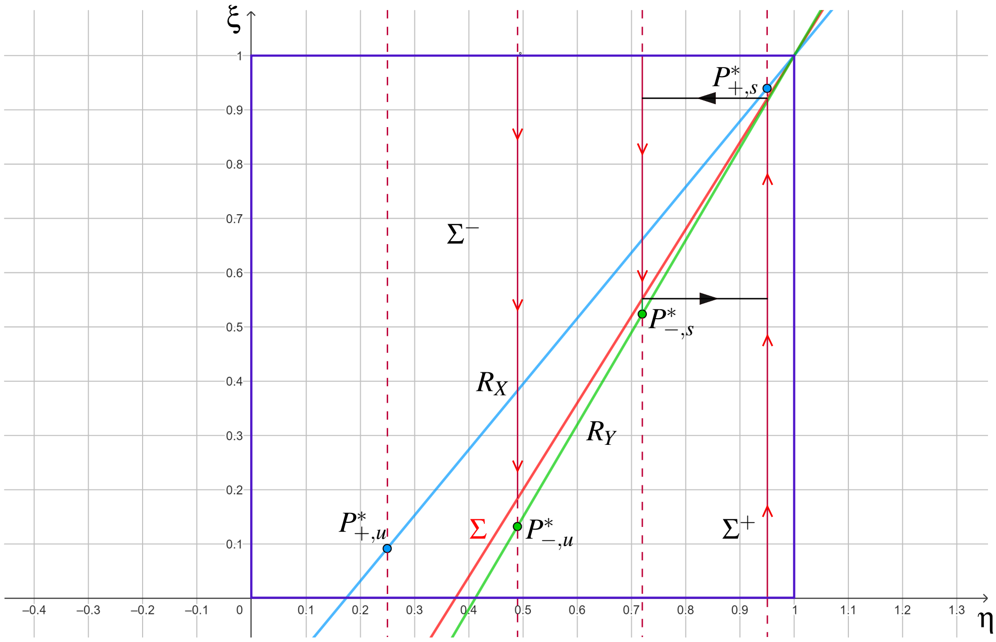

Figure 5.4: *The $\varepsilon = 0$ case projected onto the (η, ξ)-plane. The solid and dashed red vertical lines correspond to the critical manifold of Z (dashed-regular, dotted-virtual). A solution starting at $(w_{+,s}, \eta_{+,s}, \gamma(\eta_{+,s}))$ converges to $(w_{-,s}, \eta_{-,s}, \gamma(\eta_{+,s}))$ under φ_Y (top black curve). A solution starting at $(w_{-,s}, \eta_{-,s}, \gamma(\eta_{-,s}))$ converges to $(w_{+,s}, \eta_{+,s}, \gamma(\eta_{-,s}))$ under φ_X (bottom black curve). R_X and R_Y have the same meaning as in Figure 5.3.*

(ii) Given $c_2 \in (0,1)$ and a compact set $D_- \subset \Gamma_-$, there exists $\varepsilon_2 > 0$ such that for all $\varepsilon \leq \varepsilon_2$ and for all $y_1, y_2 \in D_-$, $|r^+_\varepsilon(y_2) - r^+_\varepsilon(y_1)| \leq c_2|y_2 - y_1|$.

Proof: We prove part (i). Let $c_1 \in (0,1)$ and let $D_+ \subset \Gamma_+$, D_+ compact. Recall ψ_- denotes the flow corresponding to subsystem $(\dot{w}, \dot{\eta})$ from (5-10). For any $x = (w_0, \eta_0, \xi_0) \in D_+$, $\psi_-((w_0, \eta_0), t) \to (w_{-,s}, \eta_{-,s})$ as $t \to \infty$. As $E = \{(w, \eta);\ (w, \eta, \xi) \in D_+\}$ is compact, there exists T_1 such that for all $t \geq T_1$ and for all $u, v \in E$,

$$|\psi_-(u,t) - \psi_-(v,t)| \leq c_1|u - v|. \tag{5-15}$$

Given $x \in D_+$, pick $\varepsilon(x) > 0$ such that $t(x, \varepsilon(x)) > T_1$ (where $t(x, \varepsilon(x))$ is as in (5-13)).

By the continuity of φ_Y with respect to initial conditions and time, there exists $\delta(x) > 0$ so that for all $y \in B_{\delta(x)}(x)$, $t(y, \varepsilon(x)) > T_1$, where $r^-_{\varepsilon(x)}(y) \in \Gamma_-$. Note for all $\varepsilon \leq \varepsilon(x)$, $t(y, \varepsilon) > T_1$.

Letting x vary, we get an open covering

$$D_+ \subset \bigcup_{x \in D_+} B_{\delta(x)}(x)$$

of the compact set D_+. Let $\{B_{\delta(x_i)(x_i)}; i = 1,\dots,N\}$ be a finite subcover, and set $\varepsilon_1 = \min\{\varepsilon(x_i); i = 1,\dots,N\}$. Then for any $\varepsilon \le \varepsilon_1$ and for all $x \in D_+$, $t(x,\varepsilon) > T_1$.

Now let $\varepsilon \le \varepsilon_1$, let $x_1 = (w_1,\eta_1,\gamma(\eta_1))$, $x_2 = (w_2,\eta_2,\gamma(\eta_2)) \in D_+$, and let $u = (w_1,\eta_1)$, $v = (w_2,\eta_2)$.

Let $r_\varepsilon^-(x_1) = (w_1',\eta_1',\gamma(\eta_1'))$, $r_\varepsilon^-(x_2) = (w_2',\eta_2',\gamma(\eta_2'))$, $u' = (w_1',\eta_1')$ and $v' = (w_2',\eta_2')$. By (5-15) and our choice of ε, $|v'-u'|^2 \le c_1^2|v-u|^2$. We have

$$\begin{aligned}
|r_\varepsilon^-(x_1) - r_\varepsilon^-(x_2)|^2 &= |v'-u'|^2 + (\gamma(\eta_2') - \gamma(\eta_1'))^2 \\
&= |v'-u'|^2 + \left(1+\frac{a}{b}\right)^2 (\eta_2' - \eta_1')^2 \\
&\le c_1^2|v-u|^2 + \left(1+\frac{a}{b}\right)^2 c_1^2(\eta_2 - \eta_1)^2 \\
&= c_1^2|v-u|^2 + c_1^2(\gamma(\eta_2) - \gamma(\eta_1))^2 \\
&= c_1^2|x_2 - x_1|^2.
\end{aligned}$$

Hence statement (i) holds. The proof of statement (ii) is similar. ■

The proof of the Proposition 5.3.1 was taken from [21].

Proposition 5.3.2 *(i) Let D_+ be any compact subset of Γ_+ with $Z_+ \in Int(D_+)$. Let D_- be any compact subset of Γ_-. There exists $\varepsilon_3 > 0$ such that for all $\varepsilon \le \varepsilon_3$, $r_\varepsilon^+(D_-) \subset D_+$.*

(ii) Let D_- be any compact subset of Γ_- with $Z_- \in Int(D_-)$. Let D_+ be any compact subset of Γ_+. There exists $\varepsilon_4 > 0$ such that for all $\varepsilon \le \varepsilon_4$, $r_\varepsilon^-(D_+) \subset D_-$.

Proof: We prove (i). Let $x = (w_0,\eta_0,\gamma(\eta_0)) \in D_-$. Pick $\delta > 0$ such that $U_\delta = B_{2\delta}(Z_+) \cap \Sigma \subset D_+$. Recall ψ_+ denotes the flow corresponding to subsystem $(\dot{w},\dot{\eta})$ from (5-11). Note $\psi_+((w_0,\eta_0),t) = (w(t),\eta(t)) \to (w_{+,s},\eta_{+,s})$ as $t \to \infty$. Additionally noting $\gamma(\eta)$ is continuous, there exists $T = T(x) > 0$ such that for all $t \ge T$,

$$|(w(t),\eta(t),\gamma(\eta(t))) - Z_+| < \delta.$$

Pick $\varepsilon(x) > 0$ such that $t(x,\varepsilon(x)) > T$ $(t(x,\varepsilon(x)))$ as in (5-14). Then for all $\varepsilon \le \varepsilon(x)$, $r_\varepsilon^+(x) \in B_\delta(Z_+) \cap \Sigma$.

Let $c_2 \in (0,1)$, $c_2 < \delta/\mathrm{diam}(D_-)$. Pick $\varepsilon_2 > 0$ as in Proposition 5.3.1 (ii). Let $\varepsilon_3 = \min\{\varepsilon(x),\varepsilon_2\}$. For $\varepsilon \le \varepsilon_3$ and $y \in D_-$,

$$|r_\varepsilon^+(y) - r_\varepsilon^+(x)| \le c_2|y-x| \le c_2 \cdot \mathrm{diam}(D_-) < \delta,$$

implying

$$|r_\varepsilon^+(y) - Z_+| \le |r_\varepsilon^+(y) - r_\varepsilon^+(x)| + |r_\varepsilon^+(x) - Z_+| < 2\delta.$$

Hence, $r_\varepsilon^+(D_-) \subset U_\delta \subset D_+$. The proof of (ii) is similar. ■

The proof of the Proposition 5.3.2 was taken from [21].

Theorem 5.3.1 *With parameters as in table 5.1, there exists* $\widetilde{\varepsilon} > 0$ *so that for all* $\varepsilon \leq \widetilde{\varepsilon}$*, system* (5-12) *admits an unique attracting periodic orbit.*

Proof: Let $D_+ \subset \Gamma_+$, with D_+ compact and $Z_+ \in \mathrm{Int}(D_+)$. Let $D_- \subset \Gamma_-$, with D_- compact and $Z_- \in \mathrm{Int}(D_-)$. Given $c_1, c_2 \in (0,1)$, pick $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ as in Proposition 5.3.1. Choose $\varepsilon_3 > 0$ and $\varepsilon_4 > 0$ as in Proposition 5.3.2.

For $\varepsilon \leq \widetilde{\varepsilon} = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$, $r_\varepsilon = r_\varepsilon^+ \circ r_\varepsilon^- : D_+ \to D_+$ is a contraction map with contraction factor $c_1 c_2 \in (0,1)$. Hence r_ε has an unique fixed point x^* to which all r_ε-orbits in D_+ converge. Flowing via φ_Y from x^* to $r_\varepsilon^-(x^*) = y^*$, and via φ_X from y^* to $r_\varepsilon^+(y^*) = x^*$, provides the desired periodic orbit. Every Filippov trajectory of system (5-12) passing through D_+ converges to this limit cycles. ∎

The proof of the Theorem 5.3.1 was taken from [21].

CHAPTER 6

Final Considerations

This dissertation presents a set of the main tools and perspectives for analyzing the dynamics of discontinuous vector fields. As highlighted throughout this dissertation, Filippov's convention is one of the main methodologies adopted in the study of the dynamics of discontinuous fields in the region of discontinuity, to the detriment of its geometric character; which allows us to develop a consistent mathematical theory. In addition, other techniques, such as the Sotomayor-Teixeira's regularization process, provide us with a method of describing the dynamics of a discontinuous vector field in the region of discontinuity, although it presents a limitation when we consider the transition function associated with the process.

The Sotomayor-Teixeira's regularization process applied to a Filippov's field $Z = (X,Y)$ (given in (3-2)) produces a family of smooth vector fields Z_ε (given in (4-2)), which depends explicitly on the regularization function φ associated. As was shown in Chapter 4 it is possible to associate the field Z_ε with a singular perturbation problem, and so we turn to Fenichel's theory (presented in Chapter 2) to obtain information on the dynamics of the regularized field (when $\varepsilon > 0$ is sufficiently small) and consequently of Filippov's field. Furthermore, in Chapter 4 we show some situations in which it is possible to obtain information about the dynamics of Z_ε regardless of the choice made for the regularization function, and under certain hypotheses, the validity of these results, locally, in even more general scenarios.

The mathematical model (5-12) presented in Chapter 5, which in turn is a Filippov's field $Z = (X,Y)$, has an analytical structure according to which we can identify it as a singularly perturbed Filippov's field, given that X and Y are singular perturbation problems. However, the Σ discontinuity region does not allow us to apply Fenichel's theory to the study of this model. In addition, there is currently no theory that allows us to obtain information on the dynamics of singularly perturbed piecewise-smooth fields, while having knowledge of the dynamics of this field in the singular limit (when $\varepsilon = 0$). Motivated by the relevance of the mathematical model (5-12), the next step will be to try to develop a mathematical theory with consistent results that allows us to describe the dynamics of singularly perturbed piecewise-smooth vector fields (for $\varepsilon > 0$ sufficiently small) having information about the dynamics of this field in the case where $\varepsilon = 0$.

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