Federal University of Goiás

Institute of Physics
Physics Postgraduate Program

Quantum Pequi Group

Detailed fluctuation theorem for quantum field theories in static curved spacetimes

Master's Dissertation in Physics

Rafael do Lago Souza Costa

Goiânia, 2026

Brazil

UFG

UNIVERSIDADE FEDERAL DE GOIÁS
INSTITUTO DE FÍSICA

TERMO DE CIÊNCIA E DE AUTORIZAÇÃO (TECA) PARA DISPONIBILIZAR VERSÕES ELETRÔNICAS DE TESES

E DISSERTAÇÕES NA BIBLIOTECA DIGITAL DA UFG

Na qualidade de titular dos direitos de autor, autorizo a Universidade Federal de Goiás (UFG) a disponibilizar, gratuitamente, por meio da Biblioteca Digital de Teses e Dissertações (BDTD/UFG), regulamentada pela Resolução CEPEC nº 832/2007, sem ressarcimento dos direitos autorais, de acordo com a Lei 9.610/98, o documento conforme permissões assinaladas abaixo, para fins de leitura, impressão e/ou download, a título de divulgação da produção científica brasileira, a partir desta data.

O conteúdo das Teses e Dissertações disponibilizado na BDTD/UFG é de responsabilidade exclusiva do autor. Ao encaminhar o produto final, o autor(a) e o(a) orientador(a) firmam o compromisso de que o trabalho não contém nenhuma violação de quaisquer direitos autorais ou outro direito de terceiros.

1. Identificação do material bibliográfico

[X] Dissertação [ ] Tese [ ] Outro*:_____________

*No caso de mestrado/doutorado profissional, indique o formato do Trabalho de Conclusão de Curso, permitido no documento de área, correspondente ao programa de pós-graduação, orientado pela legislação vigente da CAPES.

Exemplos: Estudo de caso ou Revisão sistemática ou outros formatos.

2. Nome completo do autor

Rafael do Lago Souza Costa

3. Título do trabalho

Detailed fluctuation theorem for quantum field theories in static curved spacetimes

4. Informações de acesso ao documento (este campo deve ser preenchido pelo orientador)

Concorda com a liberação total do documento [ ] SIM [X] NÃO[1]

[1] Neste caso o documento será embargado por até um ano a partir da data de defesa. Após esse período, a possível disponibilização ocorrerá apenas mediante:
a) consulta ao(à) autor(a) e ao(à) orientador(a);
b) novo Termo de Ciência e de Autorização (TECA) assinado e inserido no arquivo da tese ou dissertação.
O documento não será disponibilizado durante o período de embargo.
Casos de embargo:
- Solicitação de registro de patente;
- Submissão de artigo em revista científica;
- Publicação como capítulo de livro;
- Publicação da dissertação/tese em livro.

Obs. Este termo deverá ser assinado no SEI pelo orientador e pelo autor.

RAFAEL DO LAGO SOUZA COSTA

Teorema de flutuação detalhado para teorias quânticas de campos em espaços tempos curvos estáticos

Dissertation submitted to the Graduate Program in Physics, Institute of Physics, Federal University of Goiás (UFG), in partial fulfillment of the requirements for the degree of Master of Science in Physics.
Area of concentration: Physics
Research line: Quantum Optics and Quantum Information

Advisor: Professor Doctor Lucas Chibebe Céleri

GOIÂNIA
2026

Ficha de identificação da obra elaborada pelo autor, através do
Programa de Geração Automática do Sistema de Bibliotecas da UFG.

Costa, Rafael do Lago Souza
Detailed fluctuation theorem for quantum field theories in static curved spacetimes [manuscrito] / Rafael do Lago Souza Costa. - 2025.
112 f.: il.

Orientador: Prof. Dr. Lucas Chibebe Céleri.
Dissertação (Mestrado) - Universidade Federal de Goiás, Instituto de Física (IF), Programa de Pós-Graduação em Física, Goiânia, 2025.
Bibliografia. Apêndice.
Inclui lista de figuras.

1. Fluctuation Theorems. 2. Quantum Field Theory. 3. Curved Spacetimes. 4. Stochastic Thermodynamics. 5. Quantum Fluctuations. I. Céleri, Lucas Chibebe, orient. II. Título.

CDU 53

UFG

UNIVERSIDADE FEDERAL DE GOIÁS

INSTITUTO DE FÍSICA

ATA DE DEFESA DE DISSERTAÇÃO

Ata nº 229 da sessão de Defesa de Dissertação de Rafael do Lago Souza Costa, que confere o título de Mestre em Física, na área de concentração em Física.

Aos 05 dias do mês de agosto de 2025, a partir das 09h00min, por meio de videoconferência, realizou-se a sessão pública de Defesa de Dissertação intitulada "Detailed fluctuation theorem for quantum field theories in static curved spacetimes". Os trabalhos foram instalados pelo Orientador, Professor Doutor Lucas Chibebe Céleri (IF/UFG), com a participação dos demais membros da Banca Examinadora: Professor Doutor Daniel Augusto Turolla Vanzella (IFSC/USP), membro titular externo; e Professor Doutor Rômulo Cesar Rougemont Pereira (IF/UFG), membro titular interno. Durante a arguição, os membros da banca não fizeram sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Dissertação, tendo sido o candidato aprovado pelos seus membros. Proclamados os resultados pelo Professor Doutor Lucas Chibebe Céleri, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos membros da Banca Examinadora, aos 05 dias do mês de agosto de 2025.

TÍTULO SUGERIDO PELA BANCA

---

sei! assinatura eletrônica — Documento assinado eletronicamente por **Daniel Augusto Turolla Vanzela**, **Usuário Externo**, em 05/09/2025, às 10:35, conforme horário oficial de Brasília, com fundamento no § 3° do art. 4° do Decreto n° 10.543, de 13 de novembro de 2020.

sei! assinatura eletrônica — Documento assinado eletronicamente por **Lucas Chibebe Celeri**, **Professor do Magistério Superior**, em 05/09/2025, às 10:37, conforme horário oficial de Brasília, com fundamento no § 3° do art. 4° do Decreto n° 10.543, de 13 de novembro de 2020.

sei! assinatura eletrônica — Documento assinado eletronicamente por **Romulo Cesar Rougemont Pereira**, **Professor do Magistério Superior**, em 05/09/2025, às 12:32, conforme horário oficial de Brasília, com fundamento no § 3° do art. 4° do Decreto n° 10.543, de 13 de novembro de 2020.

A autenticidade deste documento pode ser conferida no site https://sei.ufg.br/sei/controlador_externo.php?acao=documento_conferir&id_orgao_acesso_externo=0, informando o código verificador **5633107** e o código CRC **7ADE31AA**.

Referência: Processo nº 23070.033840/2025-97 SEI nº 5633107

“The only true wisdom is knowing you know nothing.”

Socrates

"To my father, Carlos."

Acknowledgements

I extend my sincere gratitude to my advisor, Lucas Céleri, for his profound guidance, intellectual rigor, and unwavering belief in this work. Your mentorship not only shaped the theoretical framework of this research but also instilled in me a deeper appreciation for the elegance of quantum field theory in curved spacetimes.

I am equally grateful to Marcos Basso and Jonas Maziero, whose collaboration, critical insight, and constant support were fundamental throughout every stage of this master's journey. Their intellectual partnership was as impactful as that of my advisor, and their presence helped shape the direction, depth, and clarity of this work.

To my family, thank you for your unconditional support and patience throughout this journey. To my friends — Fábio Vaz, Pedro Nunes, João Pedro Lima, Luan Vieira, André Luiz, Guilherme André, Jordan Botelho and Gabriel Fortes — your unwavering moral support, shared laughter, and companionship provide the resilience needed to navigate the challenges of graduate studies.

I also gratefully acknowledge the financial support provided by CAPES (Coordenação de Aperfeiçoamento de Pessoal de Nível Superior), which made this research possible.

Abstract

This thesis presents a perturbative derivation of the Crooks fluctuation theorem for free quantum scalar fields in static curved spacetimes. By combining techniques from quantum field theory in curved spacetime (QFTCS), stochastic thermodynamics, and relativistic measurement theory, we construct an operational framework using Unruh-DeWitt detectors and Ramsey interferometry. This approach allows for a consistent and covariant formulation of quantum work distributions in gravitational backgrounds, resolving microcausality issues while maintaining the KMS condition as the source of time-asymmetric thermodynamic behavior. Our derivation confirms the Jarzynski equality and reveals how the arrow of time and detailed balance emerge from quantum field dynamics in curved geometries.

The analysis includes a systematic construction of quantum fluctuation relations, the definition of Shannon entropy for work distributions, and the renormalization of infrared divergences using spacetime smearing. All results are developed from the algebraic formulation of QFTCS but computed at the level of operator-valued distributions to simplify notation and maintain physical clarity. This work represents a step toward a covariant nonequilibrium thermodynamics of quantum fields, with perspectives for extension to dynamical spacetimes and interacting fermionic fields during the PhD phase. It also contributes tools applicable to relativistic quantum technologies and foundational studies in quantum field theory in curved spacetimes.

Keywords: Fluctuation Theorems, Quantum Field Theory, Curved Spacetimes, Stochastic Thermodynamics, Quantum Fluctuations.

Resumo

Esta dissertação apresenta uma derivação perturbativa do teorema de flutuação de Crooks para campos escalares quânticos livres em espaços-tempos curvos estáticos. Combinando técnicas da teoria quântica de campos em espaços-tempos curvos (QFTCS), termodinâmica estocástica e teoria de medidas relativísticas, construímos um formalismo operacional baseado em detectores de Unruh-DeWitt e interferometria de Ramsey. Essa abordagem permite uma formulação consistente e covariante de distribuições de trabalho em fundos gravitacionais, resolvendo conflitos de microcausalidade e mantendo a condição de KMS como origem da assimetria temporal dos processos termodinâmicos. A derivação confirma a igualdade de Jarzynski e mostra como a seta do tempo e a quebra de equilíbrio detalhado emergem da dinâmica quântica de campos em geometrias curvas.

A análise inclui uma construção sistemática das relações de flutuação quânticas, a definição de entropia de Shannon para distribuições de trabalho e a renormalização de divergências infravermelhas via espalhamento no espaço-tempo. Todos os resultados são desenvolvidos a partir da formulação algébrica da QFTCS, mas calculados no nível de distribuições operatoriais para simplificar a notação e manter a clareza física. Este trabalho representa um avanço em direção a uma termodinâmica fora do equilíbrio covariante para campos quânticos, com perspectivas de extensão para espaços-tempos dinâmicos e campos fermiônicos interagentes durante a etapa de doutorado. Também fornece ferramentas aplicáveis a tecnologias quânticas relativísticas e a estudos fundamentais em teoria quântica de campos em espaćos-tempos curvos.

Palavras-chave: Teoremas de Flutuação, Teoria Quântica de Campos, Espaços-Tempos Curvos, Termodinâmica Estocástica, Flutuações Quânticas.

Contents

List of Figures

Chapter 1

Introduction

The central goal of this dissertation is to derive the quantum Crooks fluctuation theorem for free quantum fields in static curved spacetimes, employing perturbative techniques in the weak coupling regime. This endeavor lies at the intersection of quantum field theory, thermodynamics, and general relativity. Fluctuation theorems provide exact relations for the statistical behavior of thermodynamic quantities like work, heat, and entropy production, even far from equilibrium. Extending these ideas to curved backgrounds introduces deep conceptual and technical challenges, particularly regarding time asymmetry, measurement, and covariance. By establishing a consistent formalism in static spacetimes, where a global timelike Killing vector allows well-defined energy measurements, we make progress toward a covariant theory of nonequilibrium thermodynamics for quantum fields.

Quantum field theory in curved spacetime (QFTCS) generalizes standard QFT by formulating quantum fields on nontrivial backgrounds, providing tools to describe phenomena such as Hawking radiation and particle production during inflation. However, QFTCS lacks a unique notion of vacuum or particle in generic geometries, complicating thermodynamic interpretations. In static spacetimes, the presence of a timelike Killing field permits the definition of conserved energy, positive-frequency modes, and equilibrium states via Kubo–Martin–Schwinger (KMS) conditions. These structures enable us to extend stochastic thermodynamics to curved geometries while preserving essential field-theoretic consistency. This dissertation builds upon the quantization of free fields in such settings to extract operationally meaningful thermodynamic quantities.

Fluctuation theorems have fundamentally reshaped our understanding of irreversibility and entropy production in microscopic systems. They provide probabilistic constraints that generalize the second law, including relations such as the Jarzynski equality and the Crooks theorem. In the context of QFTCS, these results acquire new physical significance, highlighting how spacetime geometry influences quantum fluctuations and thermodynamic asymmetries. Our approach relies on characterizing the statistics of work done on a quantum field via measurement protocols that respect relativistic causality. In static curved backgrounds, the necessary energy observables can be defined covariantly, allowing us to probe how geometric features affect nonequilibrium dynamics and entropy production.

Chapter 2 develops the canonical quantization of free scalar fields in globally hyperbolic and stationary spacetimes. Beginning with the classical Klein–Gordon field, we use a $3+1$ decomposition to define a foliation and construct the covariant phase space. This structure leads naturally to the introduction of a symplectic form and Poisson brackets, which underlie the field's quantization. We then construct Fock space representations of the canonical commutation relations via Weyl relations, discuss the role of Bogoliubov transformations, and address the renormalization of the energy-momentum tensor. Finally, the algebraic formulation of QFTCS is introduced to clarify the definition of vacuum and thermal states and the physical meaning of observables in curved spacetime.

Building on this foundation, Chapter 3 introduces particle detector models as operational probes for extracting physical predictions from QFTCS. We first review the geometric setup of Fermi normal coordinates and discuss the Fermi bound to ensure the causal consistency of measurements. The model of a localized nonrelativistic quantum system in a curved background is developed, and its coupling to a field is formalized. We then examine scalar Unruh–DeWitt (UDW) detectors, which function as effective thermometers and work meters. Particular attention is paid to the influence of acceleration and curvature on the detector's dynamics, which is crucial for interpreting work extraction and transition probabilities in the static curved spacetime setting.

Chapter 4 establishes the framework of stochastic thermodynamics for quantum fields, extending fluctuation-dissipation relations to relativistic settings. Classical results such as Einstein's relation and the Green–Kubo relations are revisited in the context of field theory, followed by a systematic construction of quantum fluctuation relations. Microreversibility is explored as the foundation for deriving the quantum Crooks theorem, which connects the probability distributions of forward and reverse work protocols. We identify the conditions under which time-reversal symmetry, initial thermal equilibrium, and projective energy measurements can be meaningfully defined for quantum fields in static curved spacetimes.

In Chapter 5, we formulate the Ramsey interferometry protocol as an alternative to the two-time measurement scheme. This approach ensures compatibility with causal structure and avoids inconsistencies in field measurements across spacelike surfaces. The characteristic function of work is constructed via detector-field interactions, and its Fourier transform yields the work distribution. We show how to compute the moments of this distribution perturbatively in the weak coupling limit. Crucially, we derive a Crooks-type relation connecting the forward and backward protocols, proving the quantum Crooks theorem perturbatively for free scalar fields in static spacetimes.

The Ramsey protocol also enables us to define information-theoretic quantities associated with the work statistics. In particular, we introduce the Shannon entropy of the work distribution as a measure of uncertainty and fluctuation. While this entropy is not thermodynamic in nature, it provides insight into the spread of energy changes induced by quantum processes. We discuss how this entropy reflects the structure of the field state, coherence, and the influence of curvature. The emergence of fluctuation relations and entropy-like quantities in static spacetimes thus enriches the operational content of QFTCS and connects it to broader principles of quantum thermodynamics.

Defining quantum work in field theory remains subtle, as work is not an observable in the standard sense. Instead, it emerges as a statistical quantity associated with transition probabilities under well-defined protocols. We adopt the energy difference paradigm, using projective measurements or interferometric methods to extract work distributions. Our perturbative treatment relies on localized detector-field couplings, which allow controlled expansions of the transition amplitudes and ensure that divergences can be handled through standard renormalization techniques. This makes our derivation of the Crooks theorem both physically interpretable and technically tractable.

This dissertation goes beyond theoretical development: the tools and methods we construct have potential applications in relativistic quantum technologies. Understanding how quantum fields behave under nonequilibrium protocols is essential for studying phenomena such as quantum thermometry, relativistic quantum information, and quantum energy teleportation. The operational framework developed here — combining QFTCS, stochastic thermodynamics, and measurement theory — could also inform the design of quantum systems that operate under gravitational or relativistic constraints.

In summary, this work establishes the quantum Crooks theorem for free scalar fields in static curved spacetimes through a perturbative analysis in the weak coupling regime. We rigorously construct the relevant theoretical framework, including field quantization, detector modeling, stochastic thermodynamics, and interferometric work extraction. The results presented here open the door to a covariant and operationally meaningful formulation of nonequilibrium thermodynamics in quantum field theory. This marks a significant step toward understanding irreversibility, information, and energy exchange in relativistic quantum systems interacting with curved backgrounds.

Chapter 2

Quantum fields in curved spacetimes

Following foundational formalisms [1, 2], this chapter explores the canonical quantization of fields as a cornerstone of quantum field theory. We introduce the formalism through the classical action of a minimally coupled Klein-Gordon scalar field in a globally hyperbolic spacetime, from which we derive the Klein-Gordon equation. The 3+1 decomposition is used to define a time coordinate in such spacetimes, facilitating the construction of the phase space and the rigorous analysis of the initial value problem, while introducing the symplectic structure and defining Poisson brackets, which are central to the quantization process. Throughout this chapter, we assume the reader is familiar with basic concepts of general relativity and differential geometry, as reviewed in Appendix A.

We then describe in detail the canonical quantization procedure, emphasizing the representation of Weyl relations in a Fock space and addressing the challenges posed by the selection of an inner product on the phase space and its impact on the uniqueness of the quantum theory. The transition to stationary spacetimes concludes this systematic approach, where conserved quantities arising from time isometries define a natural inner product, facilitating the quantization of fields in these spacetimes. This groundwork bridges to more specific applications discussed in later chapters.

Building on this foundation, we examine Bogoliubov transformations, which encode relationships between distinct quantization schemes in spacetimes with asymptotic stationarity. These transformations arise when comparing particle interpretations in regions separated by dynamical spacetime evolution, such as cosmological or black hole geometries. By connecting 'in' and 'out' vacuum states through time-dependent isometries, they provide a framework for analyzing particle creation effects and defining scattering processes in curved backgrounds, which is critical for understanding quantum state evolution under spacetime transformations.

Finally, we address the energy-momentum tensor of the quantum field, emphasizing renormalization challenges in curved spacetime. The semiclassical Einstein equation necessitates rigorous divergence treatment through Wald's axioms. Our transition to the algebraic formulation defines states and observables independently of specific Hilbert space representations, accommodating Hadamard states for finite stress-energy calculations and thermal (KMS) states that generalize equilibrium thermodynamics to curved spacetimes [1, 2].

2.1 Quantization in general spacetimes

The formalism of canonical quantization is usually discussed by starting with the classical action that describes the system of fields. In order to solve the problem presented in Chapter 1, we will use the classical action of a Klein-Gordon scalar field with mass $m > 0$ and minimal coupling, defined on the globally hyperbolic spacetime $(\mathbb{M}, g)$:

$$S[\phi] \equiv -\frac{1}{2}\int_{\mathbb{M}} \mathrm{d}^3\vec{x}\mathrm{d}t\sqrt{-g}\left[g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m^2\phi^2\right] , \tag{2.1}$$

where $\sqrt{-g}\mathrm{d}^3\vec{x}\mathrm{d}t$ denotes the spacetime volume element. Using the variational principle on S with respect to ϕ, we get

$$\nabla_\mu\nabla^\mu\phi - m^2\phi = 0 , \tag{2.2}$$

that is the Klein-Gordon equation.

To proceed with our study, we need to define a time coordinate t, but first we will introduce the so-called **3+1 decomposition**: consider a globally hyperbolic spacetime $(\mathbb{M}, g)$ with a good choice of foliation. We denote by n^μ a timelike and smooth vector field that points in the future direction of time such that at any point of spacetime, n^μ is a unit vector and normal to each hypersurface that lies in the chosen foliation. The metric tensor $h_{\mu\nu}$ induced on each Cauchy surface Σ_t is defined by:

$$h_{\mu\nu} \equiv g_{\mu\nu} + n_\mu n_\nu . \tag{2.3}$$

The definition of a time coordinate t on a globally hyperbolic spacetime is made by the choice of a foliation $\{\Sigma_t\}_{t\in\mathbb{R}}$. Once t is chosen, it is possible to define the conjugate momenta of the field ϕ by:

$$\pi \equiv \frac{\delta S[\phi]}{\delta(\partial_t\phi)} = \sqrt{h}n^\mu\nabla_\mu\phi , \tag{2.4}$$

where h denotes the determinant of the induced metric defined in Eq. (2.3).

Once the hypothesis that $(\mathbb{M}, g)$ is globally hyperbolic is satisfied, the initial value problem of the Klein-Gordon equation (2.2) is well-posed on the entire spacetime. More specifically, given two infinitely differentiable functions (ϕ_0, π_0), with domain in some Cauchy surface Σ_{t_0}, there exists a unique solution defined on the entire spacetime to the Klein-Gordon equation such that $\phi = \phi_0$ and $\pi = \pi_0$ when restricted to Σ_{t_0}. Moreover, for any closed set $S \subset \Sigma_{t_0}$ the solution ϕ, when restricted to the dependency domain[1] $D(S)$ of S, depends only on the initial values on S. In addition, the solution is smooth and depends continuously on these conditions.

In order to quantize the Klein-Gordon field, we first define the phase space (classical theory) of ϕ, that means to define the properties that must be satisfied by the initial-valued functions. For an arbitrary spacetime, it is convenient to choose the elements of the pair (ϕ_0, π_0) as elements of the set of smooth functions with compact support on Σ_{t_0}, denoted as $C_0^\infty(\Sigma_{t_0})$. The phase space is then defined as

$$\Gamma \equiv \{(\phi_0, \pi_0) |\ \phi_0 :\ \Sigma_{t_0} \to \mathbb{R}\ ,\ \pi_0 :\ \Sigma_{t_0} \to \mathbb{R}\ ;\ \phi_0, \pi_0\ \in C_0^\infty(\Sigma_{t_0})\}\ . \tag{2.5}$$

With this, we define $\mathscr{S}$ as the space of solutions to Eq. (2.2) that, when restricted to Σ_{t_0}, are elements of the phase space. From the fact that there exists a unique relation between the initial conditions and the solutions to the field equation, it follows that $\mathscr{S}$ can be identified with Γ. Furthermore, it is possible to define an antisymmetric bilinear structure Ω over $\mathscr{S}$ by

$$\Omega(\psi_1, \psi_2) \equiv \int_{\Sigma_t} (\pi_1\phi_2 - \pi_2\phi_2) \mathrm{d}^3\vec{x}\ , \tag{2.6}$$

where $\psi \equiv (\phi, \pi)$; Ω is called a **symplectic form** over $\mathscr{S}$ if it is non-degenerate. The right-hand side (RHS) of Eq. (2.6) is independent of the choice of Σ_t, once it is conserved by the dynamics. The Poisson Brackets of Ω for fixed ψ_1, ψ_2 is

$$\begin{aligned} \{\Omega(\psi_1, \psi_3), \Omega(\psi_2, \psi_3)\}_{\mathrm{PB}} &\equiv \int_{\Sigma_t} \mathrm{d}^3\vec{x} \left[\frac{\delta\Omega(\psi_1, \psi_3)}{\delta\phi_3}\frac{\delta\Omega(\psi_2, \psi_3)}{\delta\pi_3} - \frac{\delta\Omega(\psi_1, \psi_3)}{\delta\pi_3}\frac{\delta\Omega(\psi_2, \psi_3)}{\delta\phi_3}\right] \\ &= -\Omega(\psi_1, \psi_3)\ . \end{aligned} \tag{2.7}$$

Once the classical theory is defined, we proceed with the quantization of the Klein-Gordon scalar field. This process consists in the construction of an irreducible representation for the operators $\hat{W}(\psi)$, with $\psi\ \in \mathscr{S}$, such that they are unitary,

[1] The set $D(A)$ of points of the manifold such that every causal curve that is inextendable to the past or future, which passes through a point and intersects A only once.

vary continuously as a function of ψ in the strong topology[3] and satisfy the Weyl relations:

$$\begin{cases} \hat{W}(\psi_1)\hat{W}(\psi_2) = e^{\frac{i}{2}\Omega(\psi_1,\psi_2)}\hat{W}(\psi_1+\psi_2) \\ \\ \hat{W}^\dagger(\psi) = \hat{W}(-\psi) \end{cases} . \tag{2.8}$$

The Weyl relations presented in Eq. (2.8) are equivalent to the prescription of the canonical quantization applied to the Poisson Brackets defined in Eq. (2.7). In fact, if denoted by $\hat{\Omega}(\psi,\cdot)$, the generator of $\hat{W}(\psi)$ satisfies

$$[\hat{\Omega}(\psi_1,\cdot),\hat{\Omega}(\psi_2,\cdot)] = -i\Omega(\psi_1,\psi_2)\mathbb{1} \; . \tag{2.9}$$

The Weyl relations (2.8) are mathematically preferred over the commutator (2.9) because it is technically simpler to define the composition of bounded operators $\hat{W}$ than that of unbounded operators $\hat{\Omega}(\psi,\cdot)$. These generators $\hat{\Omega}(\psi,\cdot)$ can be identified with the usual field operators by

$$\hat{\Omega}(\psi,\cdot) = \int_{\Sigma_t} \mathrm{d}^3\vec{x}(\pi\hat{\phi} - \phi\hat{\pi}) \; , \tag{2.10}$$

if $\hat{\phi}$ and $\hat{\pi}$ satisfy the same dynamical equations that ϕ and π, respectively. More than this, we must have

$$\begin{cases} [\hat{\phi}(t,\vec{x}_p),\hat{\phi}(t,\vec{x}_q)] = [\hat{\pi}(t,\vec{x}_p),\hat{\pi}(t,\vec{x}_q)] = 0 \\ \\ [\hat{\phi}(t,\vec{x}_p),\hat{\pi}(t,\vec{x}_q)] = i\delta_t(\vec{x}_p-\vec{x}_q)\mathbb{1} \end{cases} , \tag{2.11}$$

where $\delta_t(\vec{x}_p - \vec{x}_q)$ is the spatial Dirac function defined over Σ_t.

In the context of fields, the representation for the Weyl relations is constructed in such a way that the operators $\hat{\Omega}(\psi,\cdot)$ are written in terms of creation and annihilation operators acting on a Fock space $\mathscr{F}(\mathscr{H})$. Given a Hilbert space $\mathscr{H}$, we define $\mathscr{F}(\mathscr{H})$ as

$$\mathscr{F}(\mathscr{H}) \equiv \mathbb{C} \oplus \mathscr{H}(\mathscr{H} \otimes_S \mathscr{H}) \oplus (\mathscr{H} \otimes_S \mathscr{H} \otimes_S \mathscr{H}) \oplus \ldots \; , \tag{2.12}$$

where $\oplus$ denotes the direct sum and $\otimes_S$ denotes the symmetrized tonsorial product [3]: if $u, v \in \mathscr{H}$, then $u \otimes_S v \equiv (u \otimes v + v \otimes u)/2$. The problem here is then how to define the Hilbert space $\mathscr{H}$. This can be done in the following way: suppose that there exists an inner product $\mu : \mathscr{S} \times \mathscr{S} \to \mathbb{R}$ over the phase space that satisfies, for every $\psi \in \mathscr{S}$,

$$\mu(\psi_1, \psi_2) = \sup_{\psi_2 \in \mathscr{S}} \frac{1}{4} \frac{\Omega^2(\psi_1, \psi_2)}{\mu(\psi_2, \psi_2)} , \tag{2.13}$$

where sup denotes the least real number that is greater than or equal to the possible values of the above ratio. Once we have μ, we define the real Hilbert space $\mathscr{S}_\mu$ through the complement of $\mathscr{S}$ by the norm defined by μ. The complement procedure consists in building a set $\mathscr{S}_\mu$ that contains all the elements of $\mathscr{S}$ and the limit of all the Cauchy sequences[2] that can be constructed with elements of $\mathscr{S}$[4]. Now we can extend μ and Ω to $\mathscr{S}_\mu$ and then define the inner product on $\mathscr{S}_\mu$ as $\langle \cdot, \cdot \rangle_{\mathscr{S}_\mu} \equiv 2\mu(\cdot, \cdot)$, where the factor 2 was inserted for convenience. Note that if we fix $\psi \in \mathscr{S}_\mu$, $\Omega(\cdot, \psi) : \mathscr{S}_\mu \to \mathbb{R}$ defines a linear functional and, because of (2.13), is bounded; let us denote it by $l_\psi(\sigma) \equiv \Omega(\sigma, \psi)$. By the theorem of Riesz representation (theorem 41.2.2.1 of Ref. [4]) there exists a unique vector $\chi \in \mathscr{S}_\mu$ such that $\langle \sigma, \chi \rangle_{\mathscr{S}_\mu} = \Omega(\sigma, \psi)$. This implies that Ω defines $\hat{J} : \mathscr{S}_\mu \to \mathscr{S}_\mu$, a bounded operator such that

$$\langle \sigma, \hat{J}\psi \rangle_{\mathscr{S}_\mu} = \Omega(\sigma, \psi) . \tag{2.14}$$

From Eq. (2.14) follows that $\hat{J}^\dagger = -\hat{J}$ and $\hat{J}^2 = -\mathbb{1}$ and because of this, $\hat{J}$ is called a **complex form**. The next step is to take $\mathscr{S}$ to the complex domain in order to obtain the complex Hilbert space $\mathscr{S}_\mu^{\mathbb{C}}$ and its extensions of μ, Ω and $\hat{J}$, that will be denoted by the same symbols. Once this procedure is done, we define the inner product $\langle \cdot, \cdot \rangle_{\mathscr{S}_\mu^{\mathbb{C}}} \equiv 2\mu(\cdot^*, \cdot)$, where the $*$ denotes the complex conjugation operation; here μ is already extended to $\mathscr{S}_\mu^{\mathbb{C}}$. In this Hilbert space, $i\hat{J}$ is a hermitian and limited operator that can be written as

$$i\hat{J} = \hat{P}_+ - \hat{P}_- , \tag{2.15}$$

by the usual Spectral Theorem. The operators $\hat{P}_\pm$ are the spectral projectors of $i\hat{J}$ and they play the role of projecting the vectors of $\mathscr{S}_\mu^{\mathbb{C}}$ onto the positive and negative parts of spec$\{i\hat{J}\}$, respectively. Finally, we define the Hilbert space $\mathscr{H} \subset \mathscr{S}_\mu^{\mathbb{C}}$ as the image of $\mathscr{S}_\mu^{\mathbb{C}}$ by the projector $\hat{P}_-$. We note that, for any two vectors of $\mathscr{H}$,

$$\begin{aligned} \langle \hat{P}_\pm\psi_1, \hat{P}_\pm\psi_2 \rangle_{\mathscr{S}_\mu^{\mathbb{C}}} &= \Omega((\hat{P}_\pm\psi_1)^*, \hat{J}^{-1}\hat{P}_\pm\psi_2) \\ &= -\Omega((\hat{P}_\pm\psi_1)^*, \hat{J}\hat{P}_\pm\psi_2) \\ &= \pm i\Omega((\hat{P}_\pm\psi_1)^*, \hat{P}_\pm\psi_2) , \end{aligned} \tag{2.16}$$

[2]This sequences must agree with the norm defined by μ

that is positive when $\psi_1 = \psi_2 \neq 0$. Furthermore

$$\langle (\hat{P}_-\psi)^*, \hat{P}_-\psi \rangle_{\mathscr{S}_\mu^\mathbb{C}} = -i\Omega(\hat{P}_-\psi, \hat{P}_-\psi) = 0 \,, \tag{2.17}$$

because of the anti-symmetry of the symplectic form, meaning that $(\hat{P}_-\psi)^\perp$, the vector orthogonal to $\hat{P}_-\psi$, satisfies

$$(\hat{P}_-\psi)^\perp = (\hat{P}_-\psi)^* \,. \tag{2.18}$$

Equation (2.16) implies that

$$\langle (\hat{P}_-\psi_1)^*, (\hat{P}_-\psi_2)^* \rangle_{\mathscr{S}_\mu^\mathbb{C}} = -\langle \hat{P}_-\psi_1, \hat{P}_-\psi_2 \rangle_{\mathscr{S}_\mu^\mathbb{C}} \,. \tag{2.19}$$

When $\psi_1, \psi_2 \in \mathscr{S}$, it is possible to show that

$$\langle \hat{P}_-\psi_1, \hat{P}_-\psi_2 \rangle_{\mathscr{S}_\mu^\mathbb{C}} = \mu(\psi_1, \psi_2) - \frac{i}{2}\Omega(\psi_1, \psi_2) \,. \tag{2.20}$$

We then define the Klein-Gordon inner product $(\cdot,\cdot)_{\mathrm{KG}} : \mathscr{H} \times \mathscr{H} \to \mathbb{C}$ as

$$(\cdot,\cdot)_{\mathrm{KG}} \equiv \langle \hat{P}_-\psi_1, \hat{P}_-\psi_2 \rangle_{\mathscr{S}_\mu^\mathbb{C}} = -i\Omega((\hat{P}_-\psi_1)^*, \hat{P}_-\psi_2) \,. \tag{2.21}$$

At this point, we made it clear that μ plays the role of giving us some way to split $\mathscr{S}_\mu^\mathbb{C}$ into two orthogonal subspaces, $\mathscr{H}$ and $\mathscr{H}^\perp$, this last being the image of $\mathscr{S}_\mu^\mathbb{C}$ by the projector $\hat{P}_+$. The result in (2.18) implies that there exists a unique relation between the complex conjugate of the elements of $\mathscr{H}$ and the elements of its orthogonal complement. This allows us to identify $\tilde{\mathscr{H}}$, the complex-conjugate space to $\mathscr{H}$, with $\mathscr{H}^\perp$ and, in this way, to write $\mathscr{S}_\mu^\mathbb{C} = \mathscr{H} \oplus \tilde{\mathscr{H}}$.

Once we have the Hilbert space $\mathscr{H}$, it is possible to proceed with the construction of the representation of the commutator (2.9) and, consequently, the Weyl relations (2.8). In this way, given $\Psi \in \mathscr{F}(\mathscr{H})$, denoted by

$$\Psi \equiv (c, \psi^{(1)}, \psi^{(2)}, \psi^{(3)}, \dots) \,, \tag{2.22}$$

we define, for $\sigma, \tau \in \mathscr{S}_\mu^\mathbb{C}$, the operators

$$\hat{a}\left((\hat{P}_-\sigma)^*\right)\Psi \equiv \left((\hat{P}_-\sigma, \psi^{(1)})_{KG}, \sqrt{2}\hat{P}_-\sigma \cdot \psi^{(2)}, \sqrt{3}\hat{P}_-\sigma \cdot \psi^{(3)}, \dots\right) \,, \tag{2.23}$$

called ***annihilation operator***, and

$$\hat{a}^\dagger(\hat{P}_-\tau)\Psi \equiv \left(0, c\hat{P}_-\tau, \sqrt{2}\hat{P}_-\tau \otimes_S \psi^{(1)}, \sqrt{3}\hat{P}_-\tau \otimes_S \psi^{(2)}, \dots\right) \,, \tag{2.24}$$

the ***creation operator***. Here, $\hat{P}_{-}\sigma \cdot \psi^{(n)}$ denotes the Klein Gordon inner product between $\hat{P}_{-}\sigma$ and the first Hilbert space that enters in each term of the symmetrized tonsorial product. For example, take $\psi^{(2)} = \psi_1 \otimes_S \psi_2$, then

$$\hat{P}_{-}\sigma \cdot \psi^{(2)} = \frac{(\hat{P}_{-}\sigma, \psi_1)_{KG}\psi_2 + (\hat{P}_{-}\sigma, \psi_2)_{KG}\psi_1}{2} \ .$$

Because of the way that we have defined, the creation and annihilation operators satisfy

$$\begin{cases} \left[\hat{a}\left((\hat{P}_{-}\sigma)^*\right), \hat{a}\left((\hat{P}_{-}\tau)^*\right)\right] = \left[\hat{a}^\dagger\left(\hat{P}_{-}\sigma\right), \hat{a}^\dagger\left(\hat{P}_{-}\tau\right)\right] = 0 \\ \left[\hat{a}\left((\hat{P}_{-}\sigma)^*\right), \hat{a}^\dagger\left(\hat{P}_{-}\tau\right)\right] = \mathbb{1}(\sigma, \tau)_{KG} \end{cases} . \tag{2.25}$$

Note that in both $\hat{W}(\psi)$ and $\hat{\Omega}(\psi, \cdot)$, $\psi \in \mathscr{S}$. We are interested in the projector $\hat{P}_{-}$ as a map between $\mathscr{S}$ and $\mathscr{H}$, which means that we want the ***restriction of*** $\hat{P}_{-}$ ***to the space spanned by real solutions of the field***, $K : \mathscr{S} \to \mathscr{H}$. Finally, $\forall \psi \in \mathscr{S}$ we write

$$\hat{\Omega}(\psi, \cdot) = i\hat{a}\left((K\psi)^*\right) - i\hat{a}^\dagger\left(K\psi\right) \ , \tag{2.26}$$

in such a way that, when we apply the commutation relations (2.25), we end up with (2.9).

As mentioned, there exists a relation between $\hat{\Omega}(\psi, \cdot)$ and the usual field operator $\hat{\phi}$. Let us see how this relation emerges in the formalism just constructed. In order to do this, we consider the non-homogeneous Klein-Gordon equation:

$$-\nabla_\mu \nabla^\mu \phi + m^2 \phi = f \ , \tag{2.27}$$

where $f \in C_0^\infty(\mathbb{M})$, the set of smooth functions of compact support over the manifold $\mathbb{M}$. The solutions to the Eq (2.27) can be written as the sum of the solutions to the homogeneous equation and the particular solution. The particular solution may be the advanced, Af or the retarded, Rf. Since both satisfy Eq. (2.27), which is linear, it follows that $Ef \equiv Af - Rf$ is the solution to the homogeneous equation with initial conditions of compact support on some Cauchy surface. Thus, E can be seen as a map between the space of functions of compact support on $\mathbb{M}$ and the space of solutions $\mathscr{S}$. It is possible to show for this map that (theorem 3.2.1 of Ref. [5])

- $\forall \psi \in \mathscr{S}\ \exists f \in C_0^\infty(\mathbb{M})$ such that $\psi = (Ef, \sqrt{h} n^\mu \nabla_\mu Ef) \equiv Ef$;

- $Ef = 0 \leftrightarrow f = \nabla_\mu \nabla^\mu y - m^2 y$, with $y \in C_0^\infty(\mathbb{M})$ and

- for $\psi \in \mathscr{S}$ and $f \in C_0^\infty(\mathbb{M})$ it is true that

$$\Omega(Ef, \psi) = \int_{\mathbb{M}} \mathrm{d}t \mathrm{d}^3\vec{x} \sqrt{-g} \phi f \ . \tag{2.28}$$

We then define

$$\hat{\Phi}(f) \equiv \hat{\Omega}(Ef, \cdot) = i\hat{a}\left((KEf)^*\right) - i\hat{a}^\dagger\left(KEf\right) \ . \tag{2.29}$$

In this way, $\hat{\Phi}$ satisfies the Klein-Gordon equation (2.2) and if we decompose it in terms of the Heisenberg operator $\hat{\phi}$ weighted by the function f, then $\hat{\phi}$ satisfies the field equation and the canonical commutation relations (2.11). Furthermore, it is possible to show that, for $x, y \in C_0^\infty(\mathbb{M})$,

$$[\hat{\Phi}(x), \hat{\Phi}(y)] = -i\Omega(Ex, Ey)\mathbb{1} = -i\mathbb{1} \int_{\mathbb{M}} \mathrm{d}t \mathrm{d}^3\vec{x} \sqrt{-g} x E y \ , \tag{2.30}$$

or, in terms of $\hat{\phi}$

$$[\hat{\phi}(x), \hat{\phi}(y)] = -iE(x, y)\mathbb{1} \ , \tag{2.31}$$

where $E(x, y)$ denotes the difference between the advanced and retarded Green functions of Eq (2.27) and it's known in the literature as ***Pauli-Jordan*** or ***Schwinger function*** [6]. Finally, for the vacuum state of our representation,

$$\Psi_0 = (c, 0, 0, 0, \dots) \ , \tag{2.32}$$

we have that [1]

$$\langle 0| \hat{\Phi}(x)\hat{\Phi}(y) |0\rangle \equiv \langle \Psi_0, \hat{\Phi}(x)\hat{\Phi}(y)\Psi_0 \rangle_{\mathscr{F}(\mathscr{H})} = \mu(Ex, Ey) - \frac{i}{2} \int_{\mathbb{M}} \mathrm{d}t \mathrm{d}^3\vec{x} \sqrt{-g} x E y \ , \tag{2.33}$$

where the equality can be shown using the properties of Eq. (2.20). The result of Eq. (2.33) shows us that the inner product μ assumed for $\mathscr{S}$ carries a relation with the real part of the so-called ***two point function*** associated with the quantum field.

Usually, the field quantization is implemented in a different way: we choose a complete set of solutions to Eq. (2.2) and decompose the field in terms of it. Let us see how this way of writing the field operator, maybe more familiar than Eq. (2.29), emerges. In order to do this, we consider a complete and orthonormal basis of $\mathscr{H}$,

$\{\varphi_j\}_{j\in I}$. Then, for $\psi \in \mathscr{S}$, we have that

$$K\psi = \sum_{j\in I} (\varphi_j, K\psi)_{KG} \varphi_j \ , \tag{2.34}$$

and because of the definition of the annihilation operator (2.23),

$$\hat{a}\left((K\psi)^*\right) = \sum_{j\in I} (K\psi, \varphi_j)_{KG} \hat{a}_j \ , \tag{2.35}$$

where $\hat{a}_j \equiv \hat{a}\left((\hat{P}_-\varphi_j)^*\right)$. Now, choosing $f \in C_0^\infty(\mathbb{M})$ in such a way that $\psi = Ef$ and using the definition of the Klein-Gordon inner product (2.21) and the result (2.28), it follows that

$$\hat{a}\left((K\psi)^*\right) = -i\sum_{j\in I} \int_{\mathbb{M}} \mathrm{d}t\mathrm{d}^3\vec{x}\sqrt{-g}f\varphi_j\hat{a}_j \ . \tag{2.36}$$

By doing the same for the creation operator we obtain

$$\hat{a}^\dagger(K\psi) = i\sum_{j\in I} \int_{\mathbb{M}} \mathrm{d}t\mathrm{d}^3\vec{x}\sqrt{-g}f\varphi_j^*\hat{a}_j^\dagger \ , \tag{2.37}$$

where $\hat{a}_j^\dagger \equiv \hat{a}^\dagger\left((\hat{P}_-\varphi_j)^*\right)$. Finally, when we put (2.36) and (2.37) into Eq. (2.29), we get

$$\hat{\Phi}(f) = \sum_{j\in I} \int_{\mathbb{M}} \mathrm{d}t\mathrm{d}^3\vec{x}\sqrt{-g}f\left(\varphi_j\hat{a}_j + \varphi_j^*\hat{a}_j^\dagger\right) \ . \tag{2.38}$$

This is an interesting way to write the field operator, because this suggests that every ***degree of freedom*** j of the field, described by the vector φ_j, behaves as a quantum harmonic oscillator. In fact, because of Eq. (2.25), it follows that

$$\begin{cases} [\hat{a}_j, \hat{a}_k] = [\hat{a}_j^\dagger, \hat{a}_k^\dagger] = 0 \\ [\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}\mathbb{1} \end{cases} . \tag{2.39}$$

The basis $\{\varphi_j\}_{j\in I}$ chosen for $\mathscr{H}$, however, is arbitrary and thus carries no physical information. In other words, the set of vectors $\{\varphi_j\}_{j\in I}$ is to $\mathscr{H}$ in quantum mechanics the same as the coordinate system is to $\mathbb{M}$ in general relativity.

The motivations behind the construction presented in the last paragraphs are as follows [7]. First, the construction of the representation of the Weyl relations (2.8) as a Fock space over a Hilbert space $\mathscr{H}$ tells what is the vision that quantum fields describe a set of particles: one element of $\mathscr{H}$ represents the quantum state of a particle while one element of $\mathscr{F}(\mathscr{H})$ describes some state of N particles, for different values of N. Secondly, the definition of $\mathscr{H}$ from the space of solutions to Eq. (2.2) is linked to the notion that in relativistic quantum mechanics of a particle the complex solutions of the Klein-Gordon equation would be wave functions. For the last, the definition of the field operator in terms of creation and annihilation operators reflects the intuitive idea that a field obeying a wave equation can be seen as a collection of quantum harmonic oscillators. However, at least with regard to the concept of particles, these motivations simply have a heuristic character, therefore not being the purpose of the above construction. In fact, contrary to what it may seem when, for example, the quantization of the electromagnetic field in flat space and its interaction with matter is studied, the concept of particle only emerges later, through the interpretation of certain processes and not as a fundamental element of the theory.

Note that until now the inner product μ, the key element in the construction of our formalism, was not defined. The question that emerges is then the existence and uniqueness of μ. In general globally-hyperbolic spacetimes it is possible to show that there always exists a μ satisfying Eq. (2.13) [8], however we don't know[3] a criterion, in general, to favor one among the different possible forms for μ. Worse than that, two different choices for μ can lead to unitarily non-equivalent representations for Weyl relations, that is, in principle, each representation can describe a different physical reality. Despite this, it was shown in [9], within the algebraic formalism for quantum fields [10], that the algebra defined by Weyl relations is independent of the choice made for the representation. Regarding the problem of this dissertation, we are not interested in this type of discussion since our focus is the asymptotically static space-time. Although there is no quantization prescription for privileged fields in general spacetimes, for those spacetimes that have temporal isometry, even if asymptotic, there is a natural choice for the inner product μ.

2.2 Quantization in stationary spacetimes

To show how the choice of μ is made in the stationary case, it is first necessary to define the energy-momentum tensor associated with the field ϕ. Given the action $S[\phi]$ that defines the classical field, we define the energy-momentum tensor $T_{\mu\nu}$ associated with this field by

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S[\phi]}{\delta g^{\mu\nu}} \ . \tag{2.40}$$

[3] And we believe that there not exists [5].

For the Klein-Gordon action (2.1) we have the following symmetric tensor:

$$T_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{g_{\mu\nu}}{2}\left[\nabla_\alpha \phi \nabla^\alpha \phi + m^2\phi^2\right] . \tag{2.41}$$

The energy-momentum tensor associated to the field is an important observable because it gives us local information about energy, momentum and stress of the field. Furthermore, this is a central quantity if we are interested in the gravitational role that the field can play, once $T_{\mu\nu}$ appears as a source term in the Einstein field equation. In particular, if ϕ satisfies the field equation (2.2), then it is possible to show that

$$\nabla_\mu T^\mu_{\ \nu} = 0 . \tag{2.42}$$

Consider a observer with 4-velocity u^μ. The scalar $u^\mu u^\nu T_{\mu\nu}$ has the interpretation of being the energy density attributed by the observer to the scalar field in the instant defined by the Cauchy surface that passes through the specific point. When the spacetime has a time isometry generated by the Killing field $\mathscr{X}^\mu$, when we contract both sides of Eq. (2.42) with $\mathscr{X}^\mu$ and integrate it over a region $\mathscr{N} \subset \mathbb{M}$ of spacetime[4] we end up with [1]

$$\int_{\Sigma_{t_1}} n^\mu \mathscr{X}^\nu T_{\mu\nu} d\Sigma = \int_{\Sigma_{t_2}} n^\mu \mathscr{X}^\nu T_{\mu\nu} d\Sigma , \tag{2.43}$$

where $d\Sigma = \sqrt{h} d^3\vec{x}$ denotes the volume element of the hypersurface Σ_t. To derive that result one just has to integrate by parts and make use of the Stokes theorem (theorem B.2.1 of Ref. [11]). That result implies that for a family of observables traveling on the integral curves of the field $\mathscr{X}^\mu$, the total energy associated to the field is conserved.

Consider $(\mathbb{M}, g)$ as a stationary and globally-hyperbolic spacetime and $\mathscr{X}^\mu$ the vector field that generates the time isometry $\alpha_t : \mathbb{M} \to \mathbb{M}$. The construction of μ in stationary spacetimes can be done in the following way. Consider the space $\mathscr{S}^{\mathbb{C}}$, obtained by extending the space of solutions $\mathscr{S}$ to the complex world. We define on that space of complex solutions the inner product $\langle \cdot, \cdot \rangle : \mathscr{S}^{\mathbb{C}} \times \mathscr{S}^{C} \to \mathbb{C}$ in some lucky way that

$$\langle \psi_1, \psi_2 \rangle \equiv \int_{\Sigma_t} \mathrm{d}\Sigma n^\mu \mathscr{X}^\nu T_{\mu\nu}(\psi_1, \psi_2) , \tag{2.44}$$

where

$$T_{\mu\nu}(\psi_1, \psi_2) \equiv \nabla_{(\mu}\phi_1^* \nabla_{\nu)}\phi_2 - \frac{g_{\mu\nu}}{2}[\nabla_\sigma \phi_1^* \nabla^\sigma \phi_2 + m^2 \phi_1^* \phi_2] , \tag{2.45}$$

with $\nabla_{(\mu}\phi_1^* \nabla_{\nu)}\phi_2 = [\nabla_\mu \phi_1^* \nabla_\nu \phi_2 + \nabla_\nu \phi_1^* \nabla_\mu \phi_2]/2$ being the symmetrization over

[4]This region is defined as that one that have two Cauchy surfaces, Σ_{t_1} and Σ_{t_2} $(t_2 > t_1)$, as border.

the ν, μ indices. Once ϕ_1 and ϕ_2 are complex solutions of the Klein-Gordon equation (2.2), we have that the RHS of Eq (2.44) is conserved and then it is independent of the chosen Cauchy surface. Over $\mathscr{S}^{\mathbb{C}}$ it is also possible to define the time translation map $\tau_t : \mathscr{S}^{\mathbb{C}} \to \mathscr{S}^{\mathbb{C}}$ such that

$$\tau_t(\psi) = (\phi \circ \alpha_{-t}, \pi \circ \alpha_{-t}) \ . \tag{2.46}$$

In fact, since, due to the symmetry of the background spacetime, if ϕ is a solution of Eq. (2.2), then its composition with the isometry α_t is also a solution. The action of the map τ_t consists in taking one solution ψ with initial conditions, say in the hypersurface Σ_{t_0}, and driving it to the solution $\tau_t(\psi)$, that has exactly the same initial conditions as ψ but in the hypersurface Σ_{t_0-t}. Due to the definition of isometry and the definition (2.46) it follows that $\tau_{t_1} \circ \tau_{t_2} = \tau_{t_1+t_2}$, which implies that isometry α_t defines a group in $\mathscr{S}^{\mathbb{C}}$. Furthermore, it is possible to show that

$$\langle \tau_t(\psi_1), \tau_t(\psi_2) \rangle = \langle \psi_1, \psi_2 \rangle \ , \tag{2.47}$$

so that, from the point of view of the inner product of Eq. (2.44), τ_t is a symmetry transformation of $\mathscr{S}^{\mathbb{C}}$.

We then define the Hilbert space $\mathscr{E}$ through the complement of $\mathscr{S}^{\mathbb{C}}$ on the norm defined by the inner product (2.44). In $\mathscr{S}^{\mathbb{C}}$, τ_t is linear, due to its definition in Eq. (2.46), and is limited, due to the property (2.47). Since, by construction, the domain of τ_t is dense in $\mathscr{E}$, it follows from the BLT theorem (theorem 42.1 of Ref. [4]) that there is a unique extension from τ_t to $\mathscr{E}$, $\hat{V}_t$, which is linear and bounded and whose norm is equal to that of τ_t in $\mathscr{S}^{\mathbb{C}}$. Since the property (2.47) is also satisfied when τ_t is replaced by $\hat{V}_t$ and α_t is a C^∞ map, then $\hat{V}_t$ forms a unitary group with a strongly continuous parameter acting on $\mathscr{E}$. By Stone's theorem, there is then a self-adjoint operator $\hat{h}$ such that $\hat{V}_t = e^{-it\hat{h}}$. Once $\hat{h}$ satisfies (theorem VIII.7 of Ref [3])

$$\lim_{t\to 0} \frac{\hat{V}_t\psi - \psi}{t} = -i\hat{h}\psi \ , \tag{2.48}$$

for $\psi \in \mathscr{S}^{\mathbb{C}}$, then by the definition of the Lie derivative, it follows that

$$\hat{h}\psi = (i\mathscr{D}_{\mathscr{X}}\phi, i\mathscr{D}_{\mathscr{X}}\pi) \ . \tag{2.49}$$

note that the domain of $\hat{h}$ contains the space $\mathscr{S}^{\mathbb{C}}$ because as the elements of $\mathscr{S}^{\mathbb{C}}$ are pairs of smooth functions, if the manifold is smooth, then their temporal derivatives are finite and smooth. Therefore, the image of $\hat{h}$ also contains $\mathscr{S}^{\mathbb{C}}$. Since $\mathscr{D}_{\mathscr{X}}\psi = 0$ for ψ with compact support on any hypersurface Σ_t if and only if $\psi = 0$, then $\hat{h}$ is bijective and there is $\hat{h}^{-1}$.

Suppose now that $m > 0$ and that there is a Cauchy surface on which $-\mathscr{X}^\mu \mathscr{X}_\mu \geq -\varepsilon \mathscr{X}^\mu n_\mu > \varepsilon^2$ for some $\varepsilon > 0$. In this case, it is possible to show that there is a constant $C(\varepsilon, m) > 0$ such that the symplectic form Ω, for $\psi_1, \psi_2 \in \mathscr{S}^{\mathbb{C}}$, satisfies the following inequality [12]

$$|\Omega(\psi_1^*, \psi_2)|^2 \leq C(\varepsilon, m)\langle \psi_1, \psi_1 \rangle \langle \psi_2, \psi_2 \rangle \ . \tag{2.50}$$

Thus, as Ω is a bilinear form and limited in the norm defined by $\langle \cdot, \cdot \rangle$, it follows that it is possible to extend it uniquely to all $\mathscr{E}$. Furthermore, it is possible to show that

$$\Omega(\psi_1^*, \hat{h}\psi_2) = 2i\langle \psi_1, \psi_2 \rangle \ . \tag{2.51}$$

The property (2.51) and the inequality given by Eq. (2.50) combined imply that $\hat{h}^{-1}$ is a bounded operator and that, therefore, zero does not belong to spec$\{\hat{h}\}$. Applying the spectral theorem to the operator $\hat{h}$, it is possible to define $\hat{E}^+ : \mathscr{E} \to \mathscr{E}$, the projector associated with the positive part of the spectrum of $\hat{h}$. Finally, given the projector $\hat{E}^+$, define $\mu : \mathscr{S} \times \mathscr{S} \to \mathbb{R}$ as

$$\mu(\psi_1, \psi_2) \equiv \text{Im}\left[\Omega(\hat{E}^+\psi_1, \hat{E}^+\psi_2)\right] = 2\text{Re}\left[\langle \hat{E}^+\psi_1, \hat{h}^{-1}\hat{E}^+\psi_2 \rangle\right] \tag{2.52}$$

Defined in this way, μ is an inner product over $\mathscr{S}$ and satisfies the condition in Eq. (2.13) [8]. Once you have μ, you can then implement the construction exposed in the previous section. The Hilbert space $\mathscr{H}$, however, can be obtained without further ado at this point by noting that the definition (2.52) is consistent with the complex form $\hat{J} = i\hat{h}|\hat{h}|^{-1}$, where $\hat{h} \equiv \sqrt{\hat{h}^\dagger \hat{h}}$. Thus, $\hat{P}_- = \hat{E}^+$ and $\mathscr{H}$ are the image of $\mathscr{E}$ by $\hat{E}^+$ completed through the Klein-Gordon inner product engendered by the projector $\hat{E}^+$.

The key idea behind this construction is to make that the "quantum states of a particle", i. e., the vectors of $\mathscr{H}$, can be written in terms of linear combinations of "positive energy eigenstates". In fact, if the chosen time coordinate t is that one defined by the isometry, we can write the equation (2.49) as

$$\hat{h}\psi = (i\partial_t \phi, i\partial_t \pi) \ . \tag{2.53}$$

Once that in quantum mechanics the operator $i\partial_t$ is interpreted as the energy operator, spec$\{\hat{h}\}$ encodes the possible values for this quantity[5] that can be obtained through a measure. Thus, in an energy measurement of a system whose

[5]From the point of view of observers following the integral lines of $\mathscr{X}^\mu$.

state is a vector of $\mathscr{H}$, the possible values that can be returned are, by construction, strictly positive. In this way, the prescription for the definition of μ[6] in stationary spacetimes is a version, in more precise terms, of the prescription of separation of solutions of the Klein-Gordon equation in two parts: positive and negative frequencies.

In static spacetimes, the model for this work, this idea can be phrased if we assume that the wavefunction of our system satisfies

$$i\partial_t\phi = \sqrt{\Delta}\phi \ . \tag{2.54}$$

The format of the Δ operator depends on the spatial derivatives, the mass of the field, and the scalar of curvature. If Δ is a positive operator in the norm of $\mathscr{H}$, then its root can be defined through the spectral theorem. The Eq. (2.54) makes it clear that spec$\{\hat{h}\}$ is related to the root of spec$\{\Delta\}$. In fact, once that

$$\hat{h}^2\phi = -\partial_t^2\phi = \Delta\phi \tag{2.55}$$

must mimic, in the chosen coordinate system, the field equation (2.2). Once Δ carries a relation with $\hat{h}$, it is possible to use the spectral projectors of Δ[7] to write the vectors of $\mathscr{H}$ given by $K\psi = (\phi_+, \pi_+)$ in the following way

$$\phi_+(t,\vec{r}) = \int_{\mathscr{I}} \frac{d\mu(j)}{\sqrt{2\omega_j}}\tilde{\phi}_+(j)F_j(\vec{x})e^{-i\omega_j t} \ , \tag{2.56}$$

and π_+ can be obtained using Eq. (2.4). On the above equation, $\mathscr{I}$ denotes a set of discrete or continuous indices that index the elements of spec$\{\Delta\}$ and $\mu(j)$ denotes a measure over this set while $\vec{x}$ denotes the spatial coordinates defined on the Cauchy surface Σ_t. The function $F_j : \Sigma_t \to \mathbb{C}$ is the eigenfunction of Δ with eigenvalue ω_j^2:

$$\Delta F_j = \omega_j^2 F_j \ , \tag{2.57}$$

where $\omega_j > 0$. These eigenfunctions of Δ must be chosen in the lucky way that

$$\int_{\Sigma_t} \frac{d\Sigma}{\|\mathscr{X}\|} F_j(\vec{x})F^*_{j'}(\vec{x}) = \delta_\mu(j-j') \ , \tag{2.58}$$

with δ_μ such that $\int_{\mathscr{I}} \delta_\mu(j-j')d\mu(j) = 1$. The Eq. (2.58) is usually denominated as the closure relation of the eigenfunctions of Δ and reflects the property that

[6]Or an complex form.

[7]When realized in terms of functions over the same Cauchy surface Σ_t

any pair of spectral projectors is orthogonal to each other in the case that they are associated to disjoint regions of the spectrum. The coefficient $\tilde{\phi}_+$ can be written in terms of the initial conditions that define ψ. For this, note that $\psi \in \mathscr{S}$ can be written as

$$\phi(t,\vec{x}) = \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\sqrt{2\omega_j}} \left[\tilde{\phi}_+(j)e^{-i\omega_j(t-t_0)} + \tilde{\phi}_-(j)e^{i\omega_j(t-t_0)}\right] F_j(\vec{x}) \,, \tag{2.59}$$

and

$$\pi(t,\vec{x}) = -i\frac{\sqrt{h}}{\|\mathscr{X}\|} \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\sqrt{2\omega_j}}\omega_j \left[\tilde{\phi}_+(j)e^{-i\omega_j(t-t_0)} - \tilde{\phi}_-(j)e^{i\omega_j(t-t_0)}\right] F_j(\vec{x}) \,. \tag{2.60}$$

By taking $t = t_0$ in Eqs. (2.59) and (2.60), we have that

$$\phi_0(\vec{x}) = \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\sqrt{2\omega_j}}[\tilde{\phi}_+(j) + \tilde{\phi}_-(j)]F_j(\vec{x}) \,, \tag{2.61}$$

and

$$\pi_0(\vec{x}) = -\frac{i\sqrt{h}}{\|\mathscr{X}\|} \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\sqrt{\omega_j}}\omega_j[\tilde{\phi}_+(j) - \tilde{\phi}_-(j)]F_j(\vec{x}) \,, \tag{2.62}$$

that drives us to the following system:

$$\begin{cases} \tilde{\phi}_+(j) + \tilde{\phi}_-(j) = \sqrt{2\omega_j} \displaystyle\int_{\Sigma_{t_0}} \frac{\mathrm{d}\Sigma}{\|\mathscr{X}\|}\phi_0(\vec{x})F_j^*(\vec{x}) \\ \\ \tilde{\phi}_+(j) - \tilde{\phi}_-(j) = i\dfrac{\sqrt{2\omega_j}}{\omega_j} \displaystyle\int_{\Sigma_{t_0}} \pi_0(\vec{x})F_j^*(\vec{x})d^3\vec{x} \end{cases} , \tag{2.63}$$

where we made use of the closure relation (2.58) in both (2.61) and (2.62) and $d\Sigma = \sqrt{h}d^3\vec{x}$. Now we can write

$$\tilde{\phi}_\pm(j) = \sqrt{\frac{\omega_j}{2}} \int_{\Sigma_{t_0}} \frac{\mathrm{d}\Sigma}{\|\mathscr{X}\|}\phi_0(\vec{x})F_j^*(\vec{x}) \pm i\frac{1}{\sqrt{2\omega_j}} \int_{\Sigma_{t_0}} \mathrm{d}^3\vec{x}\pi_0(\vec{x})F_j^*(\vec{x}) \,. \tag{2.64}$$

Now we can freely build the field operator in terms of the eigenfunctions F_j. In fact, using Eq. (2.38) for the operator and the shape of (2.56) for the vectors of $\mathscr{H}$, it follows that

$$\hat{\Phi}(f) = \sum_{k\in I} \int_{\mathbb{M}} \mathrm{d}t\mathrm{d}^3\vec{x}\sqrt{-g}f \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\sqrt{2\omega_j}}[\tilde{\varphi}_k(j)F_je^{-i\omega_jt}\hat{a}_k + \tilde{\varphi}_k^*(j)F_j^*e^{i\omega_jt}\hat{a}_k^\dagger] \,. \tag{2.65}$$

Expression (2.65) is called ***normal modes decomposition*** of the field and, in the Minkowski spacetime, it is nothing more than the ***plane wave expansion***. This decomposition will be useful in the calculations of the fluctuation theorem.

It is very common in the literature to call the Hilbert space $\mathscr{H}$ as ***the Hilbert space of a particle***. As mentioned at the end of **Sec.2.1**, the concept of a particle is not a fundamental element of the theory. However, in the case of stationary spacetimes, this denomination for $\mathscr{H}$ is justified by processes in which the field interacts for a finite time with other quantum systems that have well-defined energy states, as occurs with a quantum particle confined in a box or an atom free of external interactions. This quantum system that couples to the field is called ***the detector***, that will be our focus in the next chapter. In this scenario, the first order induced transitions between the states of the field and the states of the detector can be interpreted as the emission or absorption of particles of the field by the detector. In this sense, then, if a globally-hyperbolic spacetime has a time isometry, the vectors of $\mathscr{F}(\mathscr{H})$ naturally play the role of describing superpositions of occupation states of bosonic particles.

2.3 Bogoliubov transformations

Now consider a globally-hyperbolic spacetime $(\mathbb{M}, g)$ which is stationary in the past. This means that such spacetime contains a Cauchy surface Σ_t which past is related by the isometry α to the past of the surface $\Sigma_{t'}$, of a globally-hyperbolic and stationary spacetime $(\mathbb{M}', g')$. Then, the space of solutions $\mathscr{S}$ can be identified with the space of solutions $\mathscr{S}'$ in the following way. Consider an element of $\mathscr{S}$ with initial conditions (ϕ_0, π_0) on a Cauchy surface Σ_{t_0} of $(\mathbb{M}, g)$ which are in the past of Σ_t. Then, it is possible to associate the solution defined by solving the Cauchy initial value problem for $(\phi_0 \circ \alpha, \pi_0 \circ \alpha)$ in the spacetime $(\mathbb{M}', g')$ with the solution of spacetime $(\mathbb{M}, g)$. Done the correspondence between $\mathscr{S}$ and $\mathscr{S}'$, it is possible to induce an inner product μ on $\mathscr{S}$ through the inner product μ' defined on $\mathscr{S}'$, for instance, by the formalism presented in the previous section. The Hilbert space defined by that construction on $\mathscr{S}$ is usually denoted by $\mathscr{H}_{\text{in}}$. Thus the Fock space $\mathscr{F}(\mathscr{H}_{\text{in}})$ has that physical meaning of describing the occupation states of particles in the past of the hypersurface Σ_t, but not in the future. The same can be done when $(\mathbb{M}, g)$ is stationary in the future. In this case, the same procedure leaves us with the Hilbert space $\mathscr{H}_{\text{out}}$, in the lucky way that $\mathscr{F}(\mathscr{H}_{\text{out}})$ describes the occupation states of particles in the future of Σ_t, but not in the past. Analogous considerations apply even if the background spacetime is only asymptotically stationary in the past or future. It is possible to show that for spacetimes which have a stationary region in the past and another in the future, there exists a linear and unitary mapping, sometimes referred to as $S-$ ***matrix*** in the literature of Quantum Field

Theory on Curved Spacetimes (QFTCS). This matrix drives $\mathscr{F}(\mathscr{H}_{\text{in}})$ to $\mathscr{F}(\mathscr{H}_{\text{out}})$ and, consequently, codifies the spontaneous particle creation and scattering processes originated from the changes in the background geometry (theorem 4.4.1 of Ref. [5]).

This mapping, when implemented at the level of the Hilbert space $\mathscr{S}_{\mu}^{\mathbb{C}}$ is called ***Bogoliubov transformation***. Although these transformations can be used in a more general case, we will restrict ourselves to a globally-hyperbolic spacetime $(\mathbb{M}, g)$ that is asymptotically stationary both in the past and future. Consider $\{\varphi_k^{\text{in}}\}_{k\in I}$ a complete and orthonormal basis of $\mathscr{H}_{\text{in}}$ and $\{\varphi_l^{\text{out}}\}_{l\in O}$ a complete and orthonormal basis of $\mathscr{H}_{\text{out}}$. Supposing that the same Hilbert space spanned from the complex solutions of Eq. (2.2) can be written as $\mathscr{H}_{\text{out}} \otimes \tilde{\mathscr{H}}_{\text{out}}$ or $\mathscr{H}_{\text{in}} \otimes \tilde{\mathscr{H}}_{\text{in}}$, it is possible to write

$$\varphi_k^{\text{in}} = \sum_{l\in O} A_{kl}\varphi_l^{\text{out}} - B_{kl}^*(\varphi_l^{\text{out}})^* \,, \tag{2.66}$$

with $A_{kl}, B_{kl} \in \mathbb{C}\ \forall\ k \in I$ and $l \in O$. From the orthogonality relations of the basis in and out, follows that the coefficients of the linear combination Eq. (2.66) can be written in terms of the Klein-Gordon inner product:

$$A_{kl} = (\varphi_l^{\text{out}}, \varphi_k^{\text{in}})_{\text{KG}} \,, \tag{2.67}$$

and

$$B_{kl} = -(\varphi_l^{\text{out}}, (\varphi_k^{\text{in}})^*)_{\text{KG}} \,. \tag{2.68}$$

Furthermore, from equation (2.66) it is easy to show that the A_{kl} and B_{kl} have the following properties:

$$(\varphi_k^{\text{in}}, \varphi_{k'}^{\text{in}})_{\text{KG}} = \sum_{l\in O} A_{kl}^* A_{k'l} - B_{kl}B_{k'l}^* = \delta_{kk'} \,, \tag{2.69}$$

and

$$((\varphi_k^{\text{in}})^*, \varphi_{k'}^{\text{in}})_{\text{KG}} = \sum_{l\in O} A_{kl}B_{k'l}^* - A_{k'l}B_{kl}^* = 0 \,. \tag{2.70}$$

Also, for the field operator, written as Eq. (2.38) with Eq. (2.66) the expression becomes

$$\begin{aligned}
\hat{\Phi}(f) &= \sum_{k\in I}\int_{\mathbb{M}} \mathrm{d}t\mathrm{d}^3\vec{x}\sqrt{-g}f(\varphi_k^{\text{in}}\hat{a}_k + (\varphi_k^{\text{in}})^*\hat{a}_k^\dagger) \\
&= \sum_{k\in I}\int_{\mathbb{M}} \mathrm{d}t\mathrm{d}^3\vec{x}\sqrt{-g}f\sum_{l\in O}\left[\varphi_l^{\text{out}}(A_{kl}\hat{a}_k - B_{kl}\hat{a}_k^\dagger) + (\varphi_l^{\text{out}})^*(-B_{kl}^*\hat{a}_k + A_{kl}^*\hat{a}_k^\dagger)\right] \\
&= \sum_{l\in O}\int_{\mathbb{M}} \mathrm{d}t\mathrm{d}^3\vec{x}\sqrt{-g}f(\varphi_l^{\text{out}}\hat{b}_l + (\varphi_l^{\text{out}})^*\hat{b}_l^\dagger) \,.
\end{aligned} \tag{2.71}$$

From where follows the following relation between the creation and annihilation operators defined by each basis:

$$\begin{cases} \hat{b}_l = \sum_{k\in I} A_{kl}\hat{a}_k - B_{kl}\hat{a}_k^\dagger \\ \\ \hat{b}_l^\dagger = \sum_{k\in I} A_{kl}^*\hat{a}_k^\dagger - B_{kl}^*\hat{a}_k \end{cases} . \tag{2.72}$$

To the relations given in equation (2.72) we give the name of ***Bogoliubov transformation*** and to the coefficients A_{kl} and B_{kl} the ***Bogoliubov coefficients***.

Let us write the vectors φ_k^{in} and φ_l^{out}, respectively, in terms of the static positive-frequency modes in the past and future. This means that both in the past and future we can write the field equation (2.2) as

$$-\partial_t^2\phi = \Delta^{\text{in/out}}\phi . \tag{2.73}$$

The positive-frequency modes in the past are the spectral projectors of the operator Δ^{in}, when realized in terms of the functions u_j. In the same way, the positive-frequency modes in the future are the spectral projectors of the operator Δ^{out}, when realized in terms of the functions $v_{j'}$. Asymptotically in the past and future, respectively, we have

$$u_j(t,\vec{x}) = \frac{e^{-i\omega_j t}}{\sqrt{2\omega_j}}F_j(\vec{x}) , \tag{2.74}$$

and

$$v_{j'}(t,\vec{x}) = \frac{e^{-i\varsigma_{j'} t}}{\sqrt{2\varsigma_{j'}}}G_{j'}(\vec{x}) . \tag{2.75}$$

Unfortunately, without previous knowledge about the spacetime we cannot determine the shape of u_j and $v_{j'}$ in other regions of $(\mathbb{M}, g)$. Finally, we write the vectors $\varphi_{k,l}^{\text{in/out}}$:

$$\begin{cases} \varphi_k^{\text{in}} = \int_{\mathscr{I}} \mathrm{d}\mu(j)\tilde{\varphi}_k^{\text{in}}(j)u_j \\ \\ \varphi_l^{\text{out}} = \int_{\mathscr{O}} \mathrm{d}\xi(j')\tilde{\varphi}_l^{\text{out}}(j')v_{j'} . \end{cases} \tag{2.76}$$

With Eqs. (2.76) in hand it is possible to obtain a relation between the Bogoliubov coefficients, as defined in (2.67) and (2.68), and the modes. We can do that through the Klein-Gordon inner product (2.21). In fact, making the substitutions we are driven to

$$A_{kl} = \int_{\mathscr{I}} \mathrm{d}\mu(j) \int_{\mathscr{O}} \mathrm{d}\xi(j') \left(\tilde{\varphi}_l^{out}(j')\right)^* \tilde{\varphi}_k^{\text{in}}(j)\alpha_{jj'} , \tag{2.77}$$

and

$$B_{kl} = \int_{\mathscr{I}} \mathrm{d}\mu(j) \int_{\mathscr{O}} \mathrm{d}\xi(j') \left(\tilde{\varphi}_k^{\text{in}}(j)\tilde{\varphi}_l^{\text{out}}(j')\right)^* \beta_{jj'} \ , \tag{2.78}$$

where the coefficients $\alpha_{jj'}$ and $\beta_{jj'}$ are defined by

$$\alpha_{jj'} \equiv -i\Omega(v_{j'}^*, u_j) \ , \tag{2.79}$$

and

$$\beta_{jj'} \equiv i\Omega(v_{j'}^*, u_j^*) \ . \tag{2.80}$$

In literature it is common to see the name "Bogoliubov coefficient" applied to the $\alpha_{jj'}$ and $\beta_{jj'}$ coefficients, but we will make no difference in such term when applied to them or to A_{kl} and B_{kl}. Finally, using Eq. (2.66) with the expressions of Eq. (2.76) and the definition of $\alpha_{jj'}$ and $\beta_{jj'}$, it is also possible to obtain the following relation between the set of modes $\{u_j\}_{j\in\mathscr{I}}$ and $\{v_{j'}\}_{j'\in\mathscr{O}}$:

$$u_j = \int_{\mathscr{O}} \mathrm{d}\xi(j')(\alpha_{jj'}v_{j'} - \beta_{jj'}^* v_{jj'}^*) \ . \tag{2.81}$$

The equation (2.81) is useful for predicting the behavior of a set of modes in the past or future region of spacetime if one knows the behavior of the other set of modes in the future or past, respectively.

2.4 The energy-momentum tensor for the quantum field

Once we quantized the field and defined the system's space of states, we must proceed to the observables of the theory. As we will show in Chapter 4, the model that we are working on allows us to restrict our QFT systems to free field theories[8], but once one is interested in the gravitational role of the field, it is necessary to pay attention to the energy-momentum tensor operator, $\hat{T}_{\mu\nu}$. In QFTCS, when one considers free fields, the gravitational interaction comes to be the only way to observe the evolution of the quantum field. In literature it is pretty acceptable that at semiclassical level the quantum field reacts on the spacetime according to the semiclassical Einstein's equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi\langle\hat{T}_{\mu\nu}\rangle \ , \tag{2.82}$$

[8]Being physically more rigorous, these theories are not describing totally free fields once we are considering curvature effects of the spacetime and then the field will always interact with gravity. However, for mathematically means we can still calling them as free field theories by the reason that these theories describes fields that are non interacting with other systems.

where $\langle\hat{T}_{\mu\nu}\rangle$ is the expectation value of $\hat{T}_{\mu\nu}$ on some state of the field [5]. This equation is usually motivated by a parallel with the semiclassical approach of electrodynamics, but it can be also motivated through the formal deduction of the lower order correction for the classical gravitational field when the matter and perturbations over $g_{\mu\nu}$ are quantized. Anyway, we don't see in literature a final explanation for the Einstein's semiclassical equation from where it makes clear its validity.

The canonical quantization procedure tells us that, in order to obtain the $\hat{T}_{\mu\nu}$ operator corresponding to the classical observable $T_{\mu\nu}$, one has to make a change of ϕ by the corresponding Heisenberg operator $\hat{\phi}$ on the expression of $T_{\mu\nu}$ and symmetrize the result in such a way that one ends up with a Hermitian operator. However, Eq. (2.41) shows us that the energy-momentum tensor is an observable which depends on products of fields and its derivatives at the same point of spacetime. The fact that $\hat{\phi}$ is defined only in the distributional meaning leaves us to the conclusion that, in the context of canonical quantization, it is not possible to define $\hat{T}_{\mu\nu}$ as an operator acting on the space of states of the field or even give meaning to $\hat{T}_{\mu\nu}$ as a distribution. This kind of problem already appears in plane spacetime, and it is the trigger, both in Minkowski spacetime and in more general spacetimes, to the necessity of some renormalization procedure in order to give meaning to quantities such as the energy-momentum tensor.

For practical reasons, we can be satisfied only with defining the expectation value of quantities that are quadratic in the field operator, as $\hat{T}_{\mu\nu}$ or the field fluctuations, given by the expectation value of $\hat{\phi}^2$. For the energy-momentum tensor in particular, its expectation value on some state of the field can be defined through the four Wald axioms [5]. These axioms make fixed the shape of $\langle\hat{T}_{\mu\nu}\rangle$ apart from factors which depend locally on the background geometry, and thus it can be absorbed in constants such as Newton's gravitational constant and the Cosmological constant. Wald's axioms, however, do not give us a prescription for the construction of $\langle\hat{T}_{\mu\nu}\rangle$. In order to do so, we can make use of the formalism presented in this chapter. We already said that it is possible to construct $\hat{T}_{\mu\nu}$ from the classical expression for the energy-momentum tensor. For a field state $\Psi \in \mathscr{F}(\mathscr{H})$, the expectation value $\langle\Psi|\,\hat{T}_{\mu\nu}\,|\Psi\rangle \equiv \langle\Psi, \hat{T}_{\mu\nu}\Psi\rangle_{\mathscr{F}(\mathscr{H})}$ diverges; this divergence follows from the fact that the energy-momentum tensor operator is ill-defined. In order to get an expression for the expectation value of the energy-momentum tensor that has physical meaning, $\langle\hat{T}_{\mu\nu}\rangle$, it is then necessary to have some regularization and renormalization procedure that "removes the infinities" of $\langle\Psi|\,\hat{T}_{\mu\nu}\,|\Psi\rangle$. Broadly speaking, for the Wald axioms to be satisfied, the renormalization procedure adopted must

1. depend only on the local properties of the spacetime,

2. be such that the final result is covariant and

3. in the Minkowski spacetime it is equivalent to the normal ordering of the creation and annihilation operators.

For a special family of states, named Hadamard states, there exists an algorithm that allows one to build $\langle \hat{T}_{\mu\nu} \rangle$ satisfying the Wald axioms [5]. In the next section, we will define the KMS and Hadamard states of a quantum field.

2.5 States, observables and the algebraic formulation of QFTCS

In this section, we will develop our description while maintaining awareness that physics is, fundamentally, an experimental science. Although we will not elaborate on the implementation of concrete experiments, it is important to emphasize that the primary objective is to account for the outcomes of such experiments.

To ensure clarity, let us begin by considering what it means to speak of an experiment. In any laboratory setup, one can typically identify four relevant components:

1. the physical system under investigation, which is prepared in a particular manner and constitutes the subject of measurement;

2. the measurement apparatus, which interacts with the system to yield results;

3. the observer, who conducts the experiment;

4. the environment, encompassing all other factors.

The observer, when interpreted in the relativistic sense, should not influence the physical predictions. While quantum mechanics introduces subtleties regarding the observer's role—subject to ongoing interpretational debates—we shall disregard any such influence in our treatment. We also adopt the simplifying assumption that the environment does not significantly impact the measurement process. Consequently, we treat an experiment as being fully characterized by a physical system and a measurement apparatus[9]. To formalize this, we introduce the notation ω to denote the physical system along with its preparation, and Q to denote the measurement apparatus. Thus, ω_1 and ω_2 may correspond to either different systems or to the same system with distinct preparations. The same holds for Q_1 and Q_2, which may represent distinct apparatuses or a single apparatus with different preparations. It

[9] Including their specific preparations.

is crucial to include the preparation details, as varying preparations can lead to different measurement outcomes.

A measurement is performed by preparing isolated instances of the system ω and apparatus Q, including all preparation details, and allowing them to interact until the apparatus assumes a definite state—for example, until a pointer settles at a particular position—at which point the result is read off. We denote such possible outcomes by symbols p, q, which may represent numerical values or other labels (such as "up" and "down"). The experiment can be repeated by reproducing or effectively mimicking the same ω and Q. Some experiments yield identical outcomes upon repetition, while others exhibit varying results. Nevertheless, we postulate that the distribution of outcomes converges to an underlying probability distribution. This assumption is justified empirically—because it has consistently worked in practice—and philosophically, in that it is unclear how a coherent physical theory could be formulated otherwise. Absent such statistical regularity, physics as we know it might not be viable.

Suppose an experiment is repeated N times and the outcome p is observed n_p times. If a limiting distribution exists, we define:

$$w_\omega^Q(p) = \lim_{N\to\infty} \frac{n_p}{N} \tag{2.83}$$

The quantity $w_\omega^Q(p)$ is interpreted as the probability that the outcome of measuring system ω using apparatus Q is p. As expected, this satisfies $0 \leq w_\omega^Q(p) \leq 1$ and $\sum_p w_\omega^Q(p) = 1$.

Having introduced the probabilistic framework $w_\omega^Q(p)$, we can now begin to discuss the notions of states and observables.

Assume two preparations ω_1 and ω_2 satisfy

$$w_{\omega_1}^Q(p) = w_{\omega_2}^Q(p) , \tag{2.84}$$

for every apparatus Q and every possible outcome p. This condition expresses the fact that both preparations yield identical experimental statistics. It is thus natural to regard them as equivalent. Equation (2.84) defines an equivalence relation on the space of all system preparations. The corresponding quotient space will be referred to as the space of states. Accordingly, a ***state*** is defined as an equivalence class of system preparations that yield identical measurement statistics. From this point onward, ω will denote a state rather than a preparation.

Similarly, consider two apparatuses Q_1 and Q_2 for which

$$w_\omega^{Q_1}(p) = w_\omega^{Q_2}(p) , \tag{2.85}$$

for every state ω and every result p. Then Q_1 and Q_2 are empirically indistinguishable, and we introduce an equivalence relation on the space of all apparatuses. The corresponding quotient will be denoted by $\mathscr{A}$ and referred to as the space of ***observables***, with observables defined as equivalence classes of apparatuses yielding identical outcome statistics. We will henceforth use Q to denote observables.

There is no a priori restriction on the nature of measurement outcomes. These may range from abstract notions like "up" or "empty" to real numbers. However, it is standard and convenient to associate outcomes with real numbers. For example, when measuring the z-component of an electron's spin, one may assign outcomes as ± 1, $\pm\hbar/2$, 1 and 0, or any other pair of real values. These numerical assignments are conventions, and while they may represent the same physical observable up to scaling, the associated probability distributions differ. For example, an apparatus yielding ± 1 and another yielding $\pm\hbar/2$ are not equivalent under our earlier definition, even if they both measure the same physical quantity. To account for such rescaling, we introduce the notion of a ***function of an observable***.

Let $f : \mathbb{R} \to \mathbb{R}$ be any function. If $Q \in \mathscr{A}$, we define $f(Q) \in \mathscr{A}$ as the observable that returns $f(q)$ whenever Q would return q. Operationally, this may be implemented by relabeling the output scale of the apparatus. In the case of an analog device, one may simply replace the labels with the values given by f. If f is not injective, then $f(Q)$ distinguishes fewer outcomes than Q, resulting in a loss of information. In this sense, $f(Q)$ is coarser than Q.

A notable special case occurs when $f(q) = 1$ for all q, yielding $f(Q) = \mathbb{1}$, which is independent of the original observable Q. If $f(q) = c$ for some constant $c \in \mathbb{R}$, we denote $f(Q) = c\mathbb{1}$. While such observables are not empirically motivated, they are mathematically convenient.

Given a state ω and observable $Q \in \mathscr{A}$, we define the expectation value:

$$\omega(Q) \equiv \sum_p p w_\omega^Q(p) \; . \tag{2.86}$$

This expression captures the average outcome of measuring Q in state ω. Furthermore, the expectation value uniquely determines the distribution $w_\omega^Q(p)$ via:

$$w_\omega^Q(p) = \omega(\kappa_p(Q)) \; , \tag{2.87}$$

where the function $\kappa_p : \mathbb{R} \to \mathbb{R}$ is given by:

$$\kappa_p(q) = \begin{cases} 1, \text{ if } q = p, \\ 0, \text{ otherwise.} \end{cases} \tag{2.88}$$

Thus, the data $\omega(Q)$ and $w^Q_\omega(p)$ are mutually determined. Both may be viewed as complete coordinate systems on the space of states, since $\omega_1 = \omega_2$ if and only if $\omega_1(Q) = \omega_2(Q)$ for all Q, or equivalently, if $w^Q_{\omega_1}(p) = w^Q_{\omega_2}(p)$ for all Q, p.

Quantum mechanics frequently employs transition probabilities, which will be addressed in the following chapter. Nevertheless, all such quantities may ultimately be expressed in terms of expectation values—possibly involving distinct observables or states. In practice, all probabilities reduce to expressions of the form $\langle\psi|\phi\rangle$ and thus fall within the expectation value framework.

We now transition to the algebraic structure of observables. Instead of developing the full formalism, we highlight the essential components needed to characterize the important classes of states relevant to the fluctuation theorem mentioned in Chapter 1.

Given two observables Q_1 and Q_2, we define their sum $Q_1 + Q_2$ via:

$$\omega(Q_1) + \omega(Q_2) = \omega(Q_1 + Q_2) , \tag{2.89}$$

for every state ω. Though such an operation may lack direct experimental realization, it is a natural extension within the formalism. Scalar multiplication is defined by $c \cdot Q = f(Q)$ with $f(q) = cq$. Together, these operations endow $\mathscr{A}$ with the structure of a real vector space. Since functions of observables are defined, we can define powers such as Q^2 consistently.

We now define a bilinear product on $\mathscr{A}$ as:

$$Q_1 \circ Q_2 = \frac{1}{2}\left[(Q_1 + Q_2)^2 - Q_1^2 - Q_2^2\right] . \tag{2.90}$$

This is the ***Jordan Product***, which is manifestly commutative. Jordan, von Neumann, and Wigner showed that it satisfies weak associativity:

$$(Q_1^2 \circ Q_2) \circ Q_1 = Q_1^2 \circ (Q_2 \circ Q_1) . \tag{2.91}$$

If Q_1 and Q_2 are both functions of a common observable Q, then the Jordan product reduces to:

$$Q_1 \circ Q_2 = \frac{1}{2}(Q_1 Q_2 + Q_2 Q_1) , \tag{2.92}$$

where the product $Q_1 Q_2$ is interpreted pointwise as $f(Q)g(Q) = h(Q)$, with $h(q) = f(q)g(q)$. In this case, the product is homogeneous:

$$Q_1 \circ (\lambda Q_2) = (\lambda Q_1) \circ Q_2 = \lambda(Q_1 \circ Q_2) , \tag{2.93}$$

for all real λ. Assuming Eq. (2.93) holds for all $Q_1, Q_2 \in \mathscr{A}$, the vector space $\mathscr{A}$ satisfies:

1. it is a real vector space;
2. it is equipped with a bilinear product $\circ : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$;
3. $\circ$ is commutative;
4. $\circ$ is weakly associative.

Such a structure is referred to as a ***Jordan Algebra***. This space may also be endowed with a norm, but this will not be necessary for our purposes. For a more comprehensive account of the physical motivations underlying the algebraic formulation of quantum field theory, see [2]. As far as the treatment of states and observables is concerned, Jordan algebras provide a sufficient framework, allowing us to calculate the probabilities of interest.

However, quantum mechanics requires more from observables than merely assigning physical values — they must also serve as generators of symmetries. For instance, momentum not only corresponds to the physical notion of motion but also generates spatial translations. To accommodate this dual role, observables must form a Lie algebra, which the Jordan product alone does not support.

Suppose, instead, that we introduce an associative product $\cdot$ satisfying

$$Q_1 \circ Q_2 = \frac{1}{2}(Q_1 \cdot Q_2 + Q_2 \cdot Q_1) \, .$$

Then, the commutator

$$[Q_1, Q_2] = Q_1 \cdot Q_2 - Q_2 \cdot Q_1$$

naturally introduces a Lie algebra structure for the observables, resolving the limitation of the Jordan product. A practical realization of this idea is provided by the algebraic structure of linear operators on a Hilbert space. The technical term for such a structure is a ****-algebra***, defined by the following properties:

1. a complex vector space structure,
2. a bilinear associative product $\cdot : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$,
3. an antilinear involution $\dagger : \mathscr{A} \to \mathscr{A}$ satisfying $(Q_1 \cdot Q_2)^\dagger = Q_2^\dagger \cdot Q_1^\dagger$ and $(Q^\dagger)^\dagger = Q$.

The involution † generalizes the Hermitian conjugate in Hilbert space theory and is essential for recovering quantum mechanical structure. Reference [2] adopts this algebraic framework and recovers the results discussed in Section 2.1, including Eqs. (2.11) and (2.31), through what is known as the GNS construction [5].

Vacuum

The *-algebra we have constructed enables a description of quantum states in terms of correlation functions. Each element in the algebra is a linear combination of monomials of the form $\hat{\Phi}(f_1)\dots\hat{\Phi}(f_n)$ with $f_i \in C_0^\infty(\mathbb{M}, \mathbb{C})$. Thus, a state ω is completely characterized by its n-point functions:

$$W_n(f_1, f_2, \dots, f_n) \equiv \omega\big(\hat{\Phi}(f_1)\hat{\Phi}(f_2)\dots\hat{\Phi}(f_n)\big) \ . \tag{2.94}$$

In Lorentzian signature, these functions are known as ***Wightman functions***. We are now interested in formalizing the notion of the vacuum so that it aligns with familiar formulations in flat spacetime while being general enough for curved geometries.

In flat spacetime, the vacuum is typically defined as the unique Poincaré-invariant state, or alternatively, as the ground state of the Hamiltonian. However, these definitions become problematic in curved spacetimes where Poincaré symmetry may be absent, and the notion of a Hamiltonian is observer-dependent. Therefore, we seek a characterization of the vacuum that remains meaningful in general backgrounds.

One key property of the Minkowski vacuum is that it satisfies Wick's theorem, which underpins most perturbative calculations in quantum field theory. Motivated by this, we define a ***Gaussian*** (or *quasifree*) state as one whose correlation functions satisfy:

$$\begin{cases} W_{2n-1}(f_1, \dots, f_{2n-1}) = 0 \ , \\ \\ W_{2n}(f_1, \dots, f_{2n}) = \displaystyle\sum_{\text{pairings}} W_2(f_{i_1}, f_{i_2}) \cdots W_2(f_{i_{2n-1}}, f_{i_{2n}}) \ . \end{cases} \tag{2.95}$$

This is the familiar formulation of Wick's theorem, as described in [13], though it can also be expressed in terms of connected correlators.

A useful feature of Gaussian states is that their GNS representations are always Fock spaces, with the state being annihilated by all annihilation operators. However, not all such representations are irreducible. A state leads to an irreducible GNS representation if and only if it is *pure*. The Minkowski vacuum is both pure and Gaussian, and this characterization can be carried over to curved spacetime.

We thus define the vacuum to be a *pure Gaussian state*. This ensures that a Fock representation exists in which the vacuum corresponds to the absence of particles.

Returning to our earlier discussion, Ref. [2] shows that the Wald axioms guarantee the uniqueness of the renormalized energy-momentum tensor, but not its existence. To establish existence, we must confront the issue of defining products of distributions, as quantum fields are operator-valued distributions.

Multiplying distributions is subtle. For instance:

$$f(x)\delta(x) = f(0)\delta(x)$$

is well-defined for smooth functions f. Even some products of singular distributions can be meaningful, e.g.,

$$\delta(x)\delta(x-1) = 0\ .$$

These examples cleverly avoid overlapping singularities, but this condition is not strictly necessary. Consider the distributions

$$\varphi_\pm(x) = \mathrm{w}\lim_{\epsilon\to 0^+} \frac{1}{x \pm i\epsilon}\ ,$$

defined via weak limits. Applying these to the test function $b(x) = 1$ on $(-1, 1)$ yields:

$$\varphi_\pm(b) = \lim_{\epsilon\to 0^+} \int_{-1}^{1} \frac{1}{x \pm i\epsilon} dx = \mp i\pi\ .$$

Thus, these distributions are singular only in a distributional sense. Interestingly, they can still be squared:

$$\varphi_\pm^2(x) = \mathrm{w}\lim_{\epsilon\to 0^+} \frac{1}{(x \pm i\epsilon)^2} \quad \Rightarrow \quad \varphi_\pm^2(b) = -2\ .$$

However, not all such products are well-defined:

$$\varphi_+(x)\varphi_-(x) = \mathrm{w}\lim_{\epsilon\to 0^+} \frac{1}{x^2 + \epsilon^2}$$

diverges under integration:

$$\int_{-1}^{1} \frac{1}{x^2+\epsilon^2} dx = 2 \lim_{\epsilon\to 0^+} \frac{\tan^{-1}(1/\epsilon)}{\epsilon} = \infty\ .$$

The resolution lies in ***microlocal analysis***, which studies singularities using the Fourier transform. Distributions can be singular in specific directions in momentum space. If one distribution is singular in direction k, its product with another distribution is well-defined only if the second decays rapidly in direction $-k$. This is known as ***Hörmander's criterion***.

Using the identity

$$\mathbb{F}[fg](x) \propto \int \mathbb{F}[f](k)\mathbb{F}[g](x-k)d\vec{k} \ ,$$

we see that decay in $\mathbb{F}[f](k)$ must compensate for growth in $\mathbb{F}[g](x-k)$. In our earlier example:

$$\mathbb{F}[\varphi_\pm](k) \propto \mp i\Theta(\mp k) \ ,$$

so their multiplication fails Hörmander's criterion.

This framework generalizes to manifolds via localization: one multiplies the distribution by a compactly supported smooth function and analyzes it in locally flat coordinates using Fourier methods. The singularities are encoded in the *wavefront set* $\mathrm{WF}(\hat{\Phi}) \subseteq \tilde{T}\mathbb{M}$, a subset of the cotangent bundle with directionality.

As an example, the wavefront set of the Minkowski two-point function, a bi-distribution, is given by:

$$\mathrm{WF}(W_2) = \{(x, k_\mu; y, -k_\mu) \mid k_\mu \neq 0, k^\mu k_\mu = 0, (x-y)^\mu = \lambda k^\mu, \lambda \in \mathbb{R}, k_0 < 0\} \ . \tag{2.96}$$

If $y \to x$, the wavefront set includes both k_μ and $-k_\mu$, violating Hörmander's condition. Therefore, $W_2(x,x)$ is ill-defined. However, we can perform *normal ordering*:

$$:\hat{\Phi}(x)^2: \equiv \lim_{y\to x}\left[\hat{\Phi}(x)\hat{\Phi}(y) - W_2(x,y)\mathbb{1}\right] \ , \tag{2.97}$$

carefully removing singularities. This is well-defined away from the coincidence limit. For example,

$$\omega\left(:\hat{\Phi}(x)^2::\hat{\Phi}(y)^2:\right) = 2W_2(x,y)^2 \ .$$

To ensure that interacting fields can be constructed, we require the two-point function to be singular in a controlled manner. This motivates the ***microlocal spectrum condition*** (μSC). A state satisfies the μSC, or is a ***Hadamard state***, if its two-point function obeys [13]:

$$\mathrm{WF}(W_2) = \{(x, p_\mu; y, -q_\mu) \in \tilde{T}\mathbb{M} \times \tilde{T}\mathbb{M} \mid (x, p_\mu) \sim (y, q_\mu), \ p_\mu \rhd 0\} \ , \tag{2.98}$$

where $(x, p_\mu) \sim (y, q_\mu)$ indicates the existence of a null geodesic z connecting x and y, with p_μ and q_μ cotangent to z and related by parallel transport. The condition $p_\mu \rhd 0$ ensures that p_μ is non-zero and future-directed. Infinitely many such geodesics can exist, especially in coincidence limits, where the singularities span all future-directed null directions.

Hadamard states thus generalize the Minkowski vacuum to curved spacetimes and ensure that renormalized composite operators, such as the energy-momentum tensor, can be rigorously defined.

Thermal states

While the vacuum is perhaps the most important state in QFT in flat spacetime, another central class of states is those describing systems in thermal equilibrium. In the algebraic approach, thermal equilibrium is characterized by the Kubo-Martin-Schwinger (KMS) condition. To motivate this notion, consider a system with finitely many degrees of freedom in contact with a thermal reservoir at inverse temperature $\beta = 1/T$ (with Boltzmann's constant set to unity). Assuming thermal equilibrium, the system's state is described by the density matrix

$$\hat{\rho} = \frac{e^{-\beta \hat{H}}}{Z} \ , \tag{2.99}$$

where $Z = \mathrm{Tr}[e^{-\beta \hat{H}}]$ is the partition function and $\hat{H}$ is the system's Hamiltonian.

We use this finite-dimensional system as a source of physical intuition while avoiding the complications that arise in infinite-dimensional settings. In such finite systems, working with density matrices is sufficient. However, in quantum field theory, a privileged Hilbert space may not always be available. It is therefore desirable to formulate a purely algebraic condition for thermal equilibrium. Since thermal equilibrium is fundamentally tied to the time evolution of a system—for instance, it can be defined as the vanishing of entropy production rate[10]—it is natural to consider time evolution explicitly.

In the finite-dimensional setting, time evolution of an observable is governed by the Heisenberg equation. For a time-independent Hamiltonian, this gives

$$\hat{A}(t) = e^{it\hat{H}} \hat{A}(0) e^{-it\hat{H}} \ , \tag{2.100}$$

where $\hat{A}(t)$ is the observable at time t. This evolution can be recast as a one-parameter group of automorphisms acting on the algebra $\mathscr{A}$:

$$\theta_t(\hat{A}) = e^{it\hat{H}} \hat{A} e^{-it\hat{H}} \ . \tag{2.101}$$

Although this notation is not commonly used in standard quantum mechanics, it proves to be especially convenient in our context.

[10] That is, no net time evolution in entropy

The exponential in θ_t closely resembles that appearing in the definition of $\hat{\rho}$, differing only by the factor of i in the exponent. This similarity suggests the possibility of analytically continuing θ_t to complex values of t. If such a continuation exists, we find that, for any $\hat{A}, \hat{B} \in \mathscr{A}$,

$$\begin{aligned}\omega_{\hat{\rho}}(\hat{B}\hat{A}) &= \mathrm{Tr}[\hat{B}\hat{A}\hat{\rho}] \\ &= \frac{1}{Z}\mathrm{Tr}[\hat{B}\hat{A}e^{-\beta\hat{H}}] \\ &= \frac{1}{Z}\mathrm{Tr}[\hat{A}e^{-\beta\hat{H}}\hat{B}] \\ &= \frac{1}{Z}\mathrm{Tr}[\hat{A}e^{-\beta\hat{H}}\hat{B}e^{\beta\hat{H}}e^{-\beta\hat{H}}] \\ &= \frac{1}{Z}\mathrm{Tr}[\hat{A}\theta_{i\beta}(\hat{B})e^{-\beta\hat{H}}] \\ &= \mathrm{Tr}[\hat{A}\theta_{i\beta}(\hat{B})\hat{\rho}] \\ &= \omega_{\hat{\rho}}(\hat{A}\theta_{i\beta}(\hat{B}))\ .\end{aligned} \tag{2.102}$$

Here, $\omega_{\hat{\rho}}$ denotes the state defined via $\omega_{\hat{\rho}}(\hat{A}) = \mathrm{Tr}[\hat{A}\hat{\rho}]$. This calculation motivates the more general form

$$\omega(\hat{B}\hat{A}) = \omega(\hat{A}\theta_{i\beta}(\hat{B}))\ , \tag{2.103}$$

where θ_t is any one-parameter group of automorphisms on the algebra $\mathscr{A}$ that admits a suitable analytic extension. This property, known as the ***KMS condition***, was introduced by Haag, Hugenholtz, and Winnink [10] as a definition of thermal equilibrium in the algebraic framework.

While Eq. (2.103) is sufficient in the context of C^*-algebras, more general $*-$algebras require additional conditions for higher-order expectation values $\omega(\hat{A}_1 \ldots \hat{A}_n)$ with $n > 2$. However, since we focus primarily on Gaussian states—which are entirely characterized by their two-point functions—Eq. (2.103) will suffice for our purposes. A state that satisfies this condition is referred to as a KMS state for the one-parameter automorphism group θ_t at inverse temperature β.

There is a noteworthy distinction between the way we define vacuum states and KMS states. The vacuum is specified as a pure Gaussian state, a definition that depends solely on properties of the state and the observable algebra, independent of any external input. Given a state on an algebra, one can immediately determine whether it qualifies as a vacuum. In contrast, KMS states are defined relative to a chosen automorphism group—that is, a specific notion of time evolution. Thus, two distinct choices of time evolution may lead to different conclusions regarding whether a given state satisfies the KMS condition. As a result, thermal equilibrium requires not just a state but also a choice of dynamics.

This distinction becomes particularly important in QFT in curved spacetime, where different observers can adopt different time evolutions. Consequently, a state may be a KMS state for one observer but not for another. Even when two observers agree that a state satisfies the KMS condition, they may assign different temperatures to it. A striking illustration of this is provided by the Unruh effect [2], which demonstrates how observers undergoing different accelerations may perceive the same vacuum state as having different temperatures.

Chapter 3

General characterization of particle detector models

Particle detector models are essential in quantum field theory in curved spacetimes (QFTCS), enabling the study of phenomena like Hawking radiation and the Unruh effect. Traditional models, however, often assume inertial motion or static geometries, neglecting the interplay between detector dynamics, spacetime curvature, and relativistic acceleration. To address this, we implement the complete framework from [14] for localized non-relativistic quantum systems using Fermi normal coordinates (FNC). This approach adapts coordinates to a detector's timelike trajectory, constraining spatial extent via the Fermi bound to prevent geodesic crossings, while incorporating curvature-induced tidal forces (R_{0i0j}) that modify the detector's effective Hamiltonian.

The resulting model couples quantum systems to fields via a redshift-corrected Hamiltonian, including all relativistic corrections derived in [14] such as tidal deformations and acceleration noise. By recovering standard models (e.g., Unruh-DeWitt) through the established limiting procedures, this chapter applies the framework to probe quantum fields in black holes, cosmology, and analogue gravity experiments.

3.1 Fermi normal coordinates and the Fermi bound

This section has two main goals: to review Fermi normal coordinates, so that we can introduce the notation used throughout this chapter, and to define the Fermi bound. We study these in the next Subsections.

3.1.1 Fermi normal coordinates

Let $(\mathbb{M}, g)$ be a globally hyperbolic, four-dimensional spacetime equipped with a Lorentzian metric. Consider a timelike worldline $z(\tau)$ in $\mathbb{M}$, parametrized by proper time $\tau \in (\tau_{\min}, \tau_{\max})$. The Fermi normal coordinates (FNC) constructed around the trajectory $z(\tau)$ provide a coordinate system well-suited to describing physical quantities from the perspective of an observer moving along this path. These coordinates are particularly useful for expressing the metric in terms of spacetime curvature and the proper acceleration of the trajectory, within a neighborhood of the curve.

In FNC, the time coordinate is defined to be the proper time τ along the worldline. To construct the spatial coordinates $\vec{r}$, one first selects an orthonormal frame $e_\mu(\tau_0)$ at a point $z(\tau_0)$ on the curve, such that the frame spans the tangent space $T_{z(\tau_0)}\mathbb{M}$ and satisfies $e_0^\mu(\tau_0) = u^\mu(\tau_0)$, where u^μ is the four-velocity of the trajectory. This frame satisfies the orthonormality condition:

$$g(e_\mu, e_\nu) = \eta_{\mu\nu} \ , \tag{3.1}$$

with $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ denoting the Minkowski metric.

To extend the frame along the curve $z(\tau)$, we apply Fermi transport to the initial basis vectors $e_\mu(\tau_0)$. The Fermi transport equation is given by:

$$\frac{D(e_\mu)^\alpha}{\mathrm{d}\tau} + 2a^{[\alpha}u^{\beta]}(e_\mu)_\beta = 0 \ , \tag{3.2}$$

where $D/d\tau$ denotes the covariant derivative along $z(\tau)$, $a^\mu = Du^\mu/d\tau$ is the proper acceleration, and the antisymmetrization is defined as $a^{[\alpha}u^{\beta]} = (a^\alpha u^\beta - a^\beta u^\alpha)/2$. Fermi transport ensures that vectors are propagated along the curve in a manner that accounts for both the curvature of spacetime and the acceleration of the observer.

Because the acceleration is orthogonal to the velocity ($u_\mu a^\mu = 0$), the four-velocity u^μ remains Fermi transported along the curve. Consequently, the frame $e_\mu(\tau)$ obtained by Fermi transporting $e_\mu(\tau_0)$ satisfies $e_0^\mu(\tau) = u^\mu(\tau)$ for all τ. This resulting frame is known as the ***Fermi frame***.

We define the spacelike FNC $\vec{r} = (r^1, r^2, r^3)$ as follows. Let $\mathscr{N}_p$ denote the ***normal neighborhood*** of p, that is, the set of all points which can be connected to p by a unique geodesic. For a given τ, we define the rest surface $\Sigma_\tau \subset \mathscr{N}_{z(\tau)}$ as the set formed by all geodesics that originate at $z(\tau)$ with tangent vectors orthogonal to u^μ. The surfaces Σ_τ represent the ***local rest spaces*** around $z(\tau)$ and generate a local foliation of spacetime near the curve. Let $p \in \Sigma_\tau$ for some τ, then we assign coordinates $(\tau, \vec{r})$ to p if $p = \exp_{z(\tau)}(r^i e_i(\tau))$, where $\exp_{z(\tau)}$ denotes the exponential map at the point $z(\tau)$. The FNC are well defined in the world tube $\mathscr{T} \equiv \bigcup_\tau \Sigma_\tau$ surrounding the trajectory, so that any point $\times \in \mathscr{T}$ can be described as $\times = (\tau, \vec{r})$. A consequence of this definition is that the proper distance from a point $\times$ to the curve $z(\tau)$ is given by $r = \sqrt{\delta_{ij} r^i r^j}$, so proper distances from $z(\tau)$ can be computed using the Euclidean norm of the spatial FNC.

It is important to emphasize that although the time coordinate in the FNC corresponds to the proper time along the trajectory $z(\tau)$, in general it does not represent the proper time along the curves defined by $\vec{r} =$ constant. In a general curved spacetime, the vector ∂_τ is not orthogonal to the hypersurfaces Σ_τ and is not normalized when $\vec{r} \neq 0$. It is also convenient to define a local orthonormal frame associated with the FNC by extending the Fermi frame into the entire region $\mathscr{T}$. For a given event $\times \in \mathscr{T}$, we define the ***extended Fermi frame*** $e_\mu(\times)$ by parallel transporting the vectors $e_\mu(\tau)$ along the geodesic contained within Σ_τ that connects $z(\tau)$ to $\times$. This construction then provides an orthonormal frame at every point in the domain $\mathscr{T}$.

We can also express the components of the metric in FNC as a power series expansion in terms of the physical distance from a point to the reference curve, $r = \sqrt{\delta_{ij} r^i r^j}$. This expansion is given by [14]

$$\begin{cases} g_{\tau\tau} = -(1 + a_i(\tau) r^i)^2 - R_{0i0j}(\tau) r^i r^j + \mathscr{O}(r^3) \\ g_{\tau i} = -\dfrac{2}{3} R_{0jik}(\tau) r^j r^k + \mathscr{O}(r^3) \\ g_{ij} = \delta_{ij} - \dfrac{1}{3} R_{ikjl}(\tau) r^k r^l + \mathscr{O}(r^3) \end{cases}, \tag{3.3}$$

where $a_\mu(\tau)$ and $R_{\mu\nu\alpha\beta}(\tau)$ are the components of the acceleration and Riemann curvature tensor in FNC at $z(\tau)$, respectively. This expansion holds when $\|\vec{r}\|$ is much smaller than both the curvature radius of spacetime and $1/a$, where $a = \sqrt{a^\mu a_\mu}$ is the magnitude of the proper acceleration along the curve. Equation (3.3) has found a wide range of applications, including the analysis of extended bodies in general relativity [15, 16, 17], energy level shifts in hydrogen atoms due to spacetime curvature [18, 19, 20], the motion of point charges in curved spacetimes [21], and, more recently, the study of localized non-relativistic systems in curved backgrounds—which will be the focus of this chapter.

3.1.2 The Fermi bound

In this subsection we define, estimate, and discuss a length scale that we call the ***Fermi bound***. The Fermi bound sets the maximum radius that a system centered around the curve $z(\tau)$ can have while still being fully described by FNC. We begin with the definition of the τ-Fermi bound. Consider the set of spacelike geodesics connecting $z(\tau)$ to the boundary of Σ_τ. The τ-Fermi bound ℓ_τ is defined as the minimum proper length among the maximally extended geodesics in this set. In other words, it is the largest radius of a spacelike ball $B \subset T_{z(\tau)}\mathbb{M}$ orthogonal to $u^\mu(\tau)$ such that $\exp_{z(\tau)}(B) \subset \Sigma_\tau$. Any system defined within Σ_τ, centered at $z(\tau)$ and fully contained in a ball of radius smaller than ℓ_τ, can be entirely described using FNC.

Two factors influence the size of ℓ_τ: the acceleration of the curve and the curvature of spacetime. Acceleration causes the surfaces Σ_τ to bend such that geodesics may intersect after a distance of order $1/a$, even in flat spacetime. On the other hand, positive curvature causes geodesics to converge and intersect over a finite distance. Therefore, ℓ_τ depends on both the acceleration and the curvature. A schematic depiction of the region bounded by the τ-Fermi bound in each rest surface is shown in Fig. 3.1.

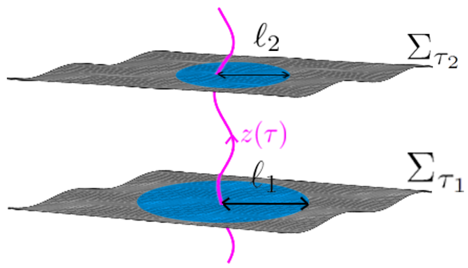

FIGURE 3.1: Schematic representation of the region delimited by the τ-Fermi bound (in blue) within each constant τ surface Σ_τ (in gray).

The overall ***Fermi bound*** ℓ is defined as the infimum over all τ of the τ-Fermi bounds:

$$\ell \equiv \inf_\tau \ell_\tau \ . \tag{3.4}$$

Each τ-Fermi bound sets a limit for the size of systems in Σ_τ that can be described using spacelike FNC. Therefore, the Fermi bound ℓ sets a global size limit for systems centered on $z(\tau)$ to be entirely described in FNC across all times. The Fermi bound also defines a world tube around the curve, spanned by all spacelike geodesics in each Σ_τ with proper length less than ℓ.

A concrete example of FNC and the Fermi bound can be illustrated using a uniformly accelerated observer in Minkowski spacetime. Let us consider inertial coordinates (t, x, y, z), and the trajectory

$$z(\tau) = \left(\frac{1}{a}\cosh(a\tau), \frac{1}{a}\sinh(a\tau), 0, 0\right) \ .$$

In this case, the FNC centered around $z(\tau)$ are the Rindler coordinates $(\tau, \vec{r})$ with $\vec{r} = (X, y, z)$. The metric becomes

$$\mathrm{d}s^2 = -(1 + aX)^2 \mathrm{d}\tau^2 + \mathrm{d}X^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,. \tag{3.5}$$

From Eq. (3.5), the metric degenerates at $X = -1/a$, corresponding to points $(0, 0, y, z)$ in inertial coordinates. These are precisely the events beyond which FNC cease to be valid. Since the proper spatial distance in FNC is Euclidean, the Fermi bound in this case is exactly $\ell = 1/a$. Figure 3.2 shows the FNC and Fermi bound for a uniformly accelerated trajectory.

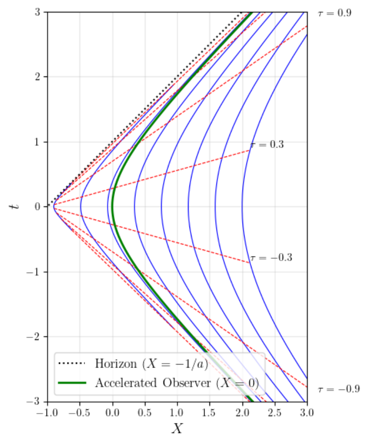

FIGURE 3.2: FNC for a uniformly accelerated trajectory in Minkowski spacetime.

More generally, it was shown in [14] that when the expansion of Eq. (3.3) is valid, a lower bound for the Fermi bound can be estimated as

$$\ell \gtrsim \inf_{\tau} \frac{1}{a + \sqrt{\lambda_R}} \,, \tag{3.6}$$

where $a = \sqrt{a_\mu a^\mu}$ is the norm of the 4-acceleration, and λ_R is the largest positive eigenvalue of the matrix $-R_{0i0j}$, when such eigenvalues exist. This estimate is particularly useful for establishing the range of validity of physical models built using FNC. In the case of uniformly accelerated motion in Rindler spacetime, the estimate is exact.

3.2 Non-relativistic quantum systems in curved spacetimes

In this section we will provide a framework that allows one to describe a localized non-relativistic quantum system in curved spacetimes. We will work with systems which can be described by a wavefunction in a 3-dimensional space and by internal degrees of freedom defined in a finite dimensional space.

3.2.1 A single particle in non-relativistic quantum mechanics

In order to obtain a generalization to curved spacetimes, let us first consider a system that can be described in terms of a wavefunction in a non-relativistic setup. We will assume that the system can be described in a Hilbert space $\mathscr{H} = \mathscr{H}_X \otimes \mathscr{H}_S$, where $\mathscr{H}_X$ is associated with the position degrees of freedom of the particle (its wavefunction), and $\mathscr{H}_S$ is a finite dimensional Hilbert space associated with its additional degrees of freedom (such as its spin). Then, the canonical variables associated with the position degrees of freedom in $\mathscr{H}_X$ are the position and momentum operators, $\hat{r}^i$ and $\hat{p}_j$. These satisfy the commutation relations

$$[\hat{r}^i, \hat{p}_j] = i\delta^i_j \mathbb{1} \ . \tag{3.7}$$

In order to translate this description to curved spacetimes, it will be useful to work in the position representation of such a system. Let $|\vec{r}\rangle$ denote the non-normalizable eigenstates of $\hat{\vec{r}} = (\hat{r}^1, \hat{r}^2, \hat{r}^3)$, and let $|s\rangle$ be any basis for $\mathscr{H}_S$. Then any state $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_s \int \mathrm{d}^3\vec{r} \, \langle \vec{r}, s|\psi\rangle \, |\vec{r}, s\rangle \ . \tag{3.8}$$

We define $\psi^s(\vec{r}) \equiv \langle \vec{r}, s|\psi\rangle$ as the wavefunction representation of $|\psi\rangle$ in the basis $|s\rangle$. The normalization of $|\psi\rangle$ then implies

$$\begin{aligned}
\langle\psi|\psi\rangle &= \sum_{s,s'} \int \mathrm{d}^3\vec{r}\mathrm{d}^3\vec{r}' (\psi^{s'}(\vec{r}'))^* \psi^s(\vec{r}) \, \langle \vec{r}'|\vec{r}\rangle \, \langle s'|s\rangle \\
&= \sum_{s,s'} \int \mathrm{d}^3\vec{r} (\psi^{s'}(\vec{r}))^* \psi^s(\vec{r}) \delta_{s's} \\
&= \int \mathrm{d}^3\vec{r} \psi^*_s(\vec{r}) \psi^s(\vec{r}) = 1 \ ,
\end{aligned} \tag{3.9}$$

where we denote $\psi_s(\vec{r}) = \delta_{s's}\psi^{s'}(\vec{r})$ and use Einstein's summation convention from the second to third lines of Eq. (3.9). That is, the components $\psi^s(\vec{r})$ can be seen as elements of $\mathbb{L}^2(\mathbb{R}^3)$. In fact, we have $\mathscr{H}_X \cong \mathbb{L}^2(\mathbb{R}^3)$, where the isomorphism is $|\psi\rangle \mapsto \psi(\vec{r}) = \langle\vec{r}|\psi\rangle$. In the space $\mathbb{L}^2(\mathbb{R}^3)$, the position operator acts in the wavefunctions as multiplication,

$$\langle\vec{r}|\,\hat{r}^i\,|\psi\rangle = r^i\psi(\vec{r}) \ , \tag{3.10}$$

and the momentum operator acts according to

$$\langle\vec{r}|\,\hat{p}_j\,|\psi\rangle = -i\partial_j\psi(\vec{r}) \ . \tag{3.11}$$

From Eqs. (3.10) and (3.11), it is clear that the commutation relations (3.7) are satisfied.

We assume that the dynamics of the system are prescribed by a self-adjoint Hamiltonian $\hat{H}(\hat{\vec{r}},\hat{\vec{p}},\{\hat{s}_i\},t)$, where $\{\hat{s}_i\}$ denotes any collection of operators acting in $\mathscr{H}_S$ and t denotes a possible external time dependence on the Hamiltonian. This Hamiltonian then generates unitary time evolution according to Schrödinger's equation,

$$i\partial_t\,|\psi(t)\rangle = \hat{H}(t)\,|\psi(t)\rangle \ , \tag{3.12}$$

where we have omitted the dependence of $\hat{H}$ in $\hat{\vec{r}}$, $\hat{\vec{p}}$ and $\{\hat{s}_i\}$ to lighten the notation. Equivalently, one can write the time-evolved state in terms of the time evolution operators $\hat{U}(t,t_0)$, defined by $|\psi\rangle = \hat{U}(t,t_0)\,|\psi(t_0)\rangle$, so that Schrödinger's equation gives

$$i\partial_t\hat{U}(t,t_0) = \hat{H}(t)\hat{U}(t,t_0) \ , \tag{3.13}$$

which can be shown to define the unitarity of $\hat{U}(t,t_0)$. In fact, the solution to Eq. (3.13) reads

$$\hat{U}(t,t_0) = \text{TEXP}\left(-i\int \mathrm{d}t\hat{H}(t)\right) \ , \tag{3.14}$$

where TEXP denotes the time-ordered exponential.

Finally, there is an important remark to be made regarding the system's Hamiltonian. Given that we will later provide a framework so that one could describe the system in a general relativistic setup, it will be important to consider the total energy of the system, which takes into consideration its rest mass. This essentially amounts to adding a term $Mc^2\mathbb{1}$ to the non-relativistic Hamiltonian (we set $c=1$), where M denotes the rest mass of the system. For instance, a particle of mass M under the influence of a potential $\hat{V}(\hat{\vec{r}})$ should be associated to the Hamiltonian

$$\hat{H} = M\mathbb{1} + \frac{\hat{\vec{p}}^2}{2M} + \hat{V}(\hat{\vec{r}}) \ . \tag{3.15}$$

Note that the introduction of the rest mass does not influence the dynamics of the system since it only amounts to an overall shift in the energy levels.

3.2.2 A localized non-relativistic quantum system in curved spacetimes

We now develop a framework to describe the localized quantum system from the previous subsection as it undergoes a timelike trajectory $z(\tau)$ in a given 4-dimensional background spacetime $(\mathbb{M}, g)$. We assume the system is spatially localized at each instant of time. This means there exists a timelike curve $z(\tau)$ around which the quantum system remains localized. For convenience, we refer to $z(\tau)$ as the ***trajectory of the system***. We assign Fermi Normal Coordinates (FNC) $\tau(\vec{r})$ around this curve, so that the system's local rest spaces correspond to surfaces Σ_τ, defined by constant values of the τ coordinate. Our goal is to define the Hilbert space of wavefunctions at each time coordinate τ as $\mathbb{L}^2(\Sigma_\tau)$ with a suitable integration measure. Some important considerations, however, must be addressed.

First, note that each Σ_τ is locally defined and does not extend beyond the normal neighborhood of $z(\tau)$. Consequently, to ensure wavefunctions are well-defined on all of $\mathbb{L}^2(\Sigma_\tau)$, we must restrict to functions defined within a finite-sized region or equivalently assume that the trapping potential becomes infinite outside a region centered at $\vec{r} = 0$ with radius smaller than the Fermi bound ℓ. This requirement can be relaxed if the potential confines the system strongly enough that the wavefunction is effectively localized within a radius ℓ around the trajectory. Under this relaxed assumption, some information about the "tails of the wavefunction" is lost. Nevertheless, if these tails are negligible compared to the wavefunction's amplitude in the region where Σ_τ is well defined, the description remains approximately valid. We refer to the assumption that the wavefunction is completely localized on each surface Σ_τ as ***Fermi localization***, and the condition that the wavefunction is mostly localized within Σ_τ as ***approximate Fermi localization***. Importantly, a physical (finite) trapping potential cannot achieve exact Fermi localization, only approximate Fermi localization.

Second, it is essential to emphasize that this formalism is not fundamental and does not apply to systems with arbitrarily high energy. A key assumption of our model is that the system's non-relativistic energy remains small compared to its rest energy. Equivalently, one may assume that $\sqrt{\langle\hat{\vec{r}}^2\rangle}$ is small compared to the rest mass M, or that the system's average velocity $\sqrt{\langle\hat{\vec{v}}^2\rangle}$ remains small relative to the speed of light. As we will show, this assumption ensures that the dynamical corrections due to motion and spacetime curvature reduce to known results from similar setups in the literature.

The first step in describing a non-relativistic system in curved spacetimes is to define the inner product on $\mathbb{L}^2(\Sigma_\tau)$. The natural choice is to integrate with respect to the induced measure on Σ_τ. Specifically, for wavefunctions $\psi(\vec{r})$ and $\phi(\vec{r})$ defined on Σ_τ,

$$(\psi, \phi)_\tau \equiv \int_{\Sigma_\tau} \mathrm{d}\Sigma \psi^*(\vec{r})\phi(\vec{r}) \ . \tag{3.16}$$

Fermi localization ensures that the wavefunctions are properly defined on Σ_τ and integrable in (3.16). Approximate Fermi localization allows small contributions from wavefunction tails to be neglected in the inner product. Beyond its geometrical motivation, the inner product $(\psi, \phi)_\tau$ also arises naturally in the reduction of Dirac spinors to wavefunctions on local rest spaces, as shown in [22]. Furthermore, it enables a consistent quantum description where canonically conjugate position and momentum operators are self-adjoint on $\mathbb{L}^2(\Sigma_\tau)$.

We define the position operator as $\hat{\vec{r}} = \hat{r}^i e_i$, where e_i denotes the Fermi frame, and the components $\hat{r}^i$ act on wavefunctions as

$$\hat{r}^i : \psi(\vec{r}) \mapsto r^i \psi(\vec{r}) \ . \tag{3.17}$$

This generalizes the standard position operator by assigning to each component a multiplicative action in FNC. Physically, this is justified by the fact that in FNC, $\|\vec{r}\| = \sqrt{\delta_{ij} r^i r^j}$ gives the proper distance from the point $\vec{r}$ to the origin of the curve. These operators are self-adjoint with respect to the inner product (3.16).

The momentum operator is similarly defined by its action on wavefunctions $\psi(\vec{r}) \in \Sigma_\tau$:

$$\hat{p}_j : \psi(\vec{r}) \mapsto -\frac{i}{h^{1/4}} \partial_j \left(h^{1/4} \psi(\vec{r}) \right) \ . \tag{3.18}$$

Though the appearance of the $1/4$ powers may seem arbitrary, they are essential to ensure that the operators $\hat{p}_j$ are self-adjoint with respect to the inner product.

Indeed,

$$
\begin{aligned}
(\psi, \hat{p}_j\phi)_\tau &= -\int \mathrm{d}^3\vec{r}\sqrt{h}\psi^*(\vec{r})\frac{i}{h^{1/4}}\partial_j(h^{1/4}\phi(\vec{r})) \\
&= -i\int \mathrm{d}^3\vec{r}h^{1/4}\psi^*(\vec{r})\partial_j(h^{1/4}\phi(\vec{r})) \\
&= i\int \mathrm{d}^3\vec{r}\left[\partial_j(h^{1/4}\psi^*(\vec{r}))\right]h^{1/4}\phi(\vec{r}) \\
&= \int \mathrm{d}^3\vec{r}\left[-i\partial_j(h^{1/4}\psi(\vec{r}))\right]^* h^{1/4}\phi(\vec{r}) \\
&= \int \mathrm{d}^3\vec{r}\sqrt{h}\left[-\frac{i}{h^{1/4}}\partial_j(h^{1/4}\psi(\vec{r}))\right]^*\phi(\vec{r}) \\
&= (\hat{p}_j\psi, \phi)_\tau \, ,
\end{aligned} \tag{3.19}
$$

where integration by parts is used in the third line, and the boundary terms vanish under the assumption of Fermi localization. Under approximate Fermi localization, boundary contributions are suppressed at the same order as other neglected tail effects, rendering the operators approximately self-adjoint[1]. In addition to being self-adjoint, the momentum operators satisfy the canonical commutation relations with the position operators of Eq. (3.17):

$$
\hat{r}^i\hat{p}_j - \hat{p}_j\hat{r}^i : \psi(\vec{r}) \mapsto i\delta^i_j\psi(\vec{r}) \, . \tag{3.20}
$$

Thus, the operators $\hat{r}^i$ and $\hat{p}_j$ defined above constitute a valid generalization of the standard quantum mechanical position and momentum operators. Given any non-relativistic quantum system characterized by position and momentum operators $\hat{\vec{r}}$ and $\hat{\vec{p}}$, its description around a trajectory $z(\tau)$ at fixed τ can be achieved by considering wavefunctions in Σ_τ and interpreting the operators via Eqs. (3.17) and (3.18). Since the pair of canonically conjugate operators fully determines the quantum theory at a given time, this construction suffices to describe the system's position degrees of freedom around $z(\tau)$.

To complete the description of the quantum system in curved spacetime, we must also include its internal degrees of freedom, represented by the set of operators $\{\hat{s}_i\}$. As these are internal to the particle, we continue to describe them using the same Hilbert space $\mathscr{H}_S$ with no changes to the inner product. However, while their representation remains unchanged, we may need to modify their dynamics in curved spacetime. We will address this point when we discuss time evolution in this formalism. Up to this point, we have constructed a description of a non-relativistic quantum system in curved spacetime on a fixed surface Σ_τ, but have yet to define

[1]For a fully consistent framework, one may truncate approximately Fermi localized wavefunctions, so that their evolution approximates that of compactly supported ones.

how the system evolves in time. That is, we still need to formulate a Hamiltonian framework within this setting.

First, notice that at each value of the time parameter τ, the wavefunctions are defined in a ***different*** Hilbert space $\mathbb{L}^2(\Sigma_\tau)$. This adds extra complications when writing the Schrödinger equation, as we cannot differentiate states with respect to time via a limit of infinitesimal differences, since $\psi(\tau_0+\delta\tau,\vec{r})$ and $\psi(\tau_0,\vec{r})$ are defined in different Hilbert spaces. In order to make sense of the Schrödinger equation in this setup, we must locally extend the wavefunctions defined in Σ_{τ_0}, so that we obtain a function $\psi(\tau,\vec{r})$ for $\tau\in[\tau_0,\tau_0+\epsilon)$ for a small $\epsilon>0$. It is then possible to compare its values at different τ's, so that differentiation can be performed. This essentially amounts to differentiation of a scalar function locally defined in spacetime with respect to the time parameter τ.

At this stage, one could naively think that given a Hamiltonian $\hat{H}(\hat{\vec{r}},\hat{\vec{p}},\{\hat{s}_i\},t)$ for a quantum particle in a non-relativistic setup, it is enough to replace its dependence on $\hat{\vec{r}},\hat{\vec{p}}$ and $\{\hat{s}_i\}$ as described previously, together with the replacement $t\mapsto\tau$ in order to write the Schrödinger equation. However, there is an important ingredient missing that also has to be considered: redshift. As already mentioned, the time parameter τ only corresponds to the proper time of an observer along the curve $z(\tau)$. This implies that the time evolution at each point of space should contain a redshift factor, associated with the time dilation of the foliation defined by the Σ_τ surfaces. In [14], T. Rick Perche computed the corresponding redshift factor. It is given by

$$\gamma(\tau,\vec{r})=\left|g_{\tau\tau}-g_{\tau i}g_{\tau j}h^{ij}\right|^{1/2}\ . \tag{3.21}$$

In a quantum setup, we would then be tempted to describe the Hamiltonian as $\gamma(\tau,\hat{\vec{r}})\hat{H}(\hat{\vec{r}},\hat{\vec{p}},\{\hat{s}_i\},\tau)$, promoting the space dependence in the redshift factor to the position operator $\hat{\vec{r}}$. However, this product will not necessarily be self-adjoint due to the dependence of $\hat{H}$ on $\hat{\vec{p}}$. In order to obtain a self-adjoint Hamiltonian, one could then use the Weyl quantization procedure of Chapter 2 for the Hamiltonian $\gamma(\tau,\hat{\vec{r}})\hat{H}(\hat{\vec{r}},\hat{\vec{p}},\{\hat{s}_i\},\tau)$ or use the Moyal product [23]. A simpler way to handle the self-adjoint problem is to define the effective Hamiltonian via a symmetrization as

$$\hat{\mathscr{H}}(\hat{\vec{r}},\hat{\vec{p}},\{\hat{s}_i\},\tau)=\frac{1}{2}[\gamma(\tau,\vec{r})\hat{H}(\hat{\vec{r}},\hat{\vec{p}},\{\hat{s}_i\},\tau)+\text{H.C.}]\ , \tag{3.22}$$

where H.C. denotes the Hermitian-conjugated of $\gamma(\tau,\vec{r})\hat{H}(\hat{\vec{r}},\hat{\vec{p}},\{\hat{s}_i\},\tau)$. Although different quantization methods might in principle give different Hamiltonians, in subsection 3.2.4 we will argue that the Hamiltonian can be well approximated by

$$\hat{H}(\hat{\vec{r}},\hat{\vec{p}},\{\hat{s}_i\},\tau)+Ma_i(\tau)\hat{r}^i+\frac{M}{2}R_{0i0j}(\tau)\hat{r}^i\hat{r}^j\ .$$

Moreover, this approximate correction is also independent of ordering ambiguities and yields the same result for any quantization prescription chosen.

We first write the Schrödinger equation for a system with no extra spin degrees of freedom. That is, in the case where the Hilbert space $\mathscr{H}_S$ is trivial, and the system can be entirely described by its wavefunction. In this case, Schrödinger equation can be written simply as

$$i\partial_\tau \psi(\tau, \vec{r}) = \hat{\mathscr{H}}(\hat{\vec{r}}, \hat{\vec{p}}, \tau)\psi(\tau, \vec{r}) \ , \tag{3.23}$$

where the position and momentum operators act in $\psi(\tau, \vec{r})$ according to Eqs. (3.17) and (3.18). We remark that although it might look like the newly introduced dynamics, and the extra factors in the differential operator $\hat{\vec{p}}$ give rise to a much more complicated differential equation, one can instead use the commutation relations between $\hat{\vec{r}}$ and $\hat{\vec{p}}$ in order to find the solutions to the Schrödinger equation.

In order to write the Schrödinger equation when the system also has internal degrees of freedom in $\mathscr{H}_S$, we write states in the Dirac notation, with $|\psi\rangle \in \mathscr{H}_X^{(\tau)} \otimes \mathscr{H}_S$, where $\mathscr{H}_X^{(\tau)} \cong \mathbb{L}^2(\Sigma_\tau)$ for each τ. In this context, the position eigenvectors are $|\vec{r}\rangle$ such that $\psi^s(\tau, \vec{r}) = \langle \vec{r}, s|\psi(\tau)\rangle$ and a decomposition of the identity in the position basis can be written as

$$\mathbb{1} = \sum_s \int \mathrm{d}\Sigma \, |\vec{r}, s\rangle\langle\vec{r}, s| \ . \tag{3.24}$$

In terms of Dirac's notation, we can then write the Schrödinger equation as

$$i\partial_\tau \, |\psi\rangle = \hat{\mathscr{H}}(\tau) \, |\psi\rangle \ , \tag{3.25}$$

where the τ differentiation in the position spaces is understood via the local extension of the wavefunctions, and we have omitted the dependence of $\hat{\mathscr{H}}$ in $\hat{\vec{r}}$, $\hat{\vec{p}}$ and $\{\hat{s}_i\}$ for simplicity. This equation also defines the unitary operators $\hat{U}(\tau, \tau_0)$ by $\hat{U}(\tau, \tau_0) \, |\psi(\tau_0)\rangle = |\psi(\tau)\rangle$. It is important to keep in mind that this family of unitary operators acts in different Hilbert spaces:

$$\hat{U}(\tau, \tau_0) : \mathscr{H}_X^{(\tau_0)} \otimes \mathscr{H}_S \to \mathscr{H}_X^{(\tau)} \otimes \mathscr{H}_S \ . \tag{3.26}$$

Finally, we comment on the possible need to perform additional changes to the Hamiltonian $\hat{H}$, apart from the redshift factor and the replacement of the position and momentum operators for their definitions of Eqs. (3.17) and (3.18). These additional changes could come from interactions of the other internal degrees of freedom of the system[2] with curvature and acceleration. Although it is not possible to give a general recipe for adapting general operators to curved spacetimes, the framework provided here can accommodate these changes in each case with minor modifications. For instance, in [22, 24], a fermionic particle in curved spacetimes is described, and the coupling of its spin with curvature is obtained. This could be implemented here by adding terms to the Hamiltonian of Eq. (3.22) corresponding to this interaction.

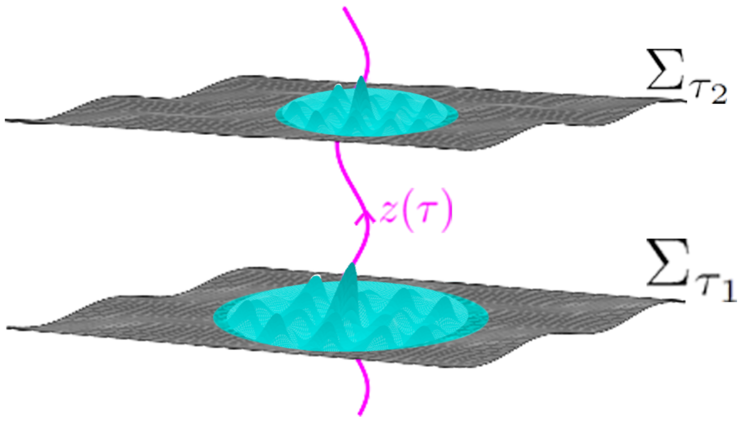

FIGURE 3.3: Schematic representation of the model for localized non-relativistic systems in curved spacetimes, with wavefunctions defined in the local rest spaces.

Overall, in this section we completed one of the main goals of this chapter: we provided a consistent description for a localized non-relativistic quantum system in curved spacetimes. A schematic representation of the model obtained can be found in Fig. 3.3. Throughout the remaining part of the chapter, we will discuss the consequences and applications of this formalism.

3.2.3 Validity of the model

In this section we discuss the regime of validity of the framework developed in this chapter and analyze its consistency with the principles of general relativity.

To begin with, we reiterate that the formalism introduced in Subsection 3.2.2 requires the non-relativistic quantum system to be sufficiently localized within the Fermi bound ℓ associated with the reference trajectory. For example, the framework is directly applicable to systems whose wavefunctions are compactly supported within a sphere of radius smaller than ℓ. While perfect localization would

[2] Encoded in the $\{\hat{s}_i\}$ operators.

require an infinite potential—which is unphysical—the formalism remains applicable when the trapping potential is finite but strong enough to approximately confine the system within this region. In such cases, the wavefunction tails extending beyond the radius ℓ can be neglected. Although this introduces an approximation, the resulting information loss can be controlled by analyzing the wavefunction tails for the relevant states of the system.

The regime of validity can be quantified using the estimate for the Fermi bound given in Eq. (3.6). Consider a non-relativistic quantum system that is to be described around a timelike trajectory $z(\tau)$ in curved spacetime. We assume that the system is *strongly supported* within a region of radius $R(\epsilon)$. This concept, introduced by Eduardo Martín-Martínez [25], characterizes quantum states whose wavefunctions decay sufficiently rapidly, such that the expectation values of a set of relevant observables can be computed to a desired precision ϵ using spatial integrals restricted to a ball of radius $R(\epsilon)$. The specific observables considered depend on the physical context and the desired predictions.

Under this assumption, our framework remains valid whenever $R(\epsilon) < \ell$, with the associated errors controlled by the parameter ϵ. If the system is sufficiently localized relative to the curvature of spacetime and the acceleration of the trajectory, Eq. (3.6) yields the approximate condition

$$R(\epsilon) < \frac{1}{a + \sqrt{\lambda_R}} \,, \tag{3.27}$$

where λ_R is the largest positive eigenvalue of the matrix $-R_{0i0j}$, and a denotes the maximal acceleration along the system's motion. If the system is confined by a sufficiently strong potential, its spatial extent can be estimated by $\sqrt{\langle \hat{\bar{r}}^2 \rangle}$, and the condition above becomes $\sqrt{\langle \hat{\bar{r}}^2 \rangle} < 1/(a + \sqrt{\lambda_R})$. Alternative methods of estimating localization may be required depending on the particular system and trapping potential.

Another key assumption is that the energy of the system is non-relativistic, i.e., much smaller than its rest energy. Although not strictly necessary for the construction of the formalism, this assumption ensures that the effective Hamiltonian in Eq. (3.22) accurately captures the dominant relativistic corrections to the system's internal dynamics. In Subsection 3.2.4, we will demonstrate that under this assumption, the framework recovers the leading relativistic corrections to wavefunction dynamics in curved spacetimes, consistent with the results of [22]. We also note that additional relativistic modifications to Eq. (3.22) may be necessary to fully capture the coupling between the system's internal degrees of freedom and spacetime geometry or trajectory-dependent effects.

Let us now consider the compatibility of our model with general relativity. The framework assumes that the spacetime geometry remains unaffected by the quantum system—an approximation that holds provided the system's stress-energy content is negligible. This semiclassical treatment is standard in relativistic quantum field theory in curved spacetimes and is expected to remain valid in our context. Furthermore, while some works argue that superpositions of localized quantum systems should induce superpositions of spacetime geometries [26], such effects lie beyond the scope of this analysis.

Finally, we address the issue of causality. The formalism allows curvature-dependent corrections to act locally on the system's position degree of freedom via the operator $\hat{\vec{r}}$. In a non-relativistic setting, these corrections are instantaneous with respect to the system's proper time τ, affecting the entire wavefunction at once. As a result, the model permits non-causal information propagation within the system, a known feature of non-relativistic quantum mechanics. These types of causality violations have been extensively studied [27, 28], and one of their observable consequences is the ability of systems to signal between spacelike-separated points over distances comparable to their size. This can be interpreted as an additional bound on the framework's validity, similar to what is encountered in other non-relativistic quantum models in curved spacetimes [29].

In summary, the framework developed in this chapter can be applied to describe non-relativistic quantum systems accurately, provided that:

1. the system is sufficiently localized within the Fermi bound,
2. its energy is much smaller than its rest mass, and
3. it is not employed in scenarios involving communication across spacelike-separated regions on length scales comparable to the system's size.

3.2.4 The coincidence limit and first order corrections

In this section we analyze the first-order corrections to the system's dynamics arising from acceleration and spacetime curvature. These corrections emerge from the redshift factor introduced in Eq. (3.21) and affect the system via modifications to the effective Hamiltonian.

The work of [22] developed a formalism to describe a localized fermionic particle in curved spacetime. By tracing over the spin degrees of freedom, the authors derived an effective Hamiltonian for the remaining non-relativistic wavefunction. The most significant corrections identified in this regime—assuming that the particle's non-relativistic energy is small compared to its rest mass—take the form $Ma_i\hat{r}^i$

and $(M/2)R_{0i0j}\hat{r}^i\hat{r}^j$. Similar results were reported in [30, 24]. We now demonstrate that, to first order in curvature and acceleration, these are precisely the corrections obtained within the formalism developed in Subsection 3.2.2.

These corrections originate from the modification of the original Hamiltonian $\hat{H}(\tau)$ by the redshift factor. Using the Fermi normal coordinate expansion in Eq. (3.3), the redshift factor $\gamma(\tau,\vec{r})$ can be expressed as

$$\gamma(\tau,\vec{r}) = 1 + a_i(\tau)r^i + \frac{1}{2}R_{0i0j}(\tau)r^ir^j + \mathscr{O}(r^3)\,. \tag{3.28}$$

Substituting this into Eq. (3.22), and keeping only terms linear in acceleration and quadratic in position (corresponding to second-order localization), the effective Hamiltonian becomes

$$\hat{\mathscr{H}}(\tau) \approx \hat{H}(\tau) + \frac{1}{2}\left[\left(a_i(\tau)\hat{r}^i + \frac{1}{2}R_{0i0j}(\tau)\hat{r}^i\hat{r}^j\right)\hat{H}(\tau) + \text{H.C.}\right]\,. \tag{3.29}$$

To simplify the analysis, we separate the Hamiltonian $\hat{H}(\tau)$ into its rest mass contribution and its non-relativistic energy component,

$$\hat{H}(\tau) = M\mathbb{1} + \hat{H}_{\text{NR}}(\tau)\,, \tag{3.30}$$

where $\hat{H}_{\text{NR}}(\tau)$ encodes the kinetic and potential energy of the system. The assumption of non-relativistic dynamics implies that $\langle\hat{H}_{\text{NR}}(\tau)\rangle \ll M$. As a result, in Eq. (3.29), the terms involving $\hat{H}_{\text{NR}}(\tau)$ are suppressed relative to those involving M.

Reintroducing units of c, the contribution from $a_i\hat{r}^i$ is scaled by $1/c^2$, and $R_{0i0j}\hat{r}^i\hat{r}^j$ is of order $(\text{size})^2/\mathscr{R}^2$, where $\mathscr{R}$ denotes a characteristic curvature radius. Therefore, these subleading contributions can be neglected in most physical setups, and we can approximate

$$\left(a_i\hat{r}^i + \frac{1}{2}R_{0i0j}\hat{r}^i\hat{r}^j\right)\hat{H}(\tau) \approx M\left(a_i\hat{r}^i + \frac{1}{2}R_{0i0j}\hat{r}^i\hat{r}^j\right)\,. \tag{3.31}$$

The effective Hamiltonian that governs evolution with respect to the proper time τ is then given by

$$\hat{\mathscr{H}}(\tau) \approx \hat{H}(\tau) + Ma_i(\tau)\hat{r}^i + \frac{M}{2}R_{0i0j}(\tau)\hat{r}^i\hat{r}^j\,. \tag{3.32}$$

These correction terms originate from the rest mass term $M\mathbb{1}$ in the Hamiltonian and the redshift factor $\gamma(\tau,\hat{\vec{r}})$. Since $M\mathbb{1}$ commutes with all operators and the redshift factor depends only on position, the leading-order corrections are unaffected by ordering ambiguities. Equation (3.32) matches the curvature-induced coupling used in [31, 32] and reproduces the dominant correction terms derived in [22]. However, Ref. [22] also includes additional terms that depend on the product of curvature or acceleration with the non-relativistic energy. These subleading terms are not automatically captured by the present formalism, but they can be manually incorporated into the effective Hamiltonian if higher precision is required.

Equation (3.32) provides a self-adjoint Hamiltonian under the non-relativistic approximation and yields a straightforward quadratic correction to $\hat{H}(\tau)$. This approximation is valid when the energy scale of the system is much smaller than Mc^2, and its spatial extent is much smaller than both the curvature radius and the characteristic acceleration scale c^2/a. These conditions define the regime of applicability of the corrections derived in this section.

3.3 Modeling the detector

In this section we relate the framework presented in section 3.2 to the formalism of particle detector models. We formulate a general notion of particle detector in the next subsection, and we analyze the consequences of the dynamics induced by the system's acceleration and the curvature of spacetime at the end.

3.3.1 General particle detector models

Broadly speaking, a particle detector model is any localized non-relativistic quantum system that couples locally to a quantum field. The first instance of a particle detector was introduced by Unruh [33], where he considered a particle in a box linearly coupled to a scalar quantum field. This model was later simplified by DeWitt [34], who considered a finite-dimensional system for the detector's internal degrees of freedom. Over the past years, more general particle detector models have been introduced in the literature, considering smeared detectors that could couple to different fields and operators in different quantum field theories [35, 36, 37]. In this section, we present a framework which generalizes these previous models.

In order to describe a particle detector model from the framework presented in section 3.2, we require two extra ingredients apart from the non-relativistic quantum system: a quantum field theory and an interaction between the quantum field and the localized system. In this section we refer to the non-relativistic quantum system as ***the detector***.

Consider a quantum field theory (QFT) for a field $\hat{\phi}^\sigma(\times)$ in the globally-hyperbolic spacetime $(\mathbb{M}, g)$, where σ denotes any collection of Lorentz indices, associated with the spin of the field. We then describe the QFT as an association of smooth compactly supported fields $f_\sigma(\times)$ to elements of a $*-$algebra $\mathscr{A}$. In general, we assume that $\mathscr{A}$ satisfies important properties related to causality, commutation relations, and conjugation [10], and a detailed description of these for fields of different spin can be found in [38]. For our purposes, one can think of a quantum field as an operator-valued distribution $\hat{\phi}^\sigma(\times)$ that acts in compactly supported fields $f_\sigma(\times)$ according to

$$\hat{\Phi}(f) = \int_{\mathbb{M}} \sqrt{-g}\, d\tau\, d^3\vec{r}\, f_\sigma(\times)\hat{\phi}^\sigma(\times)\,. \tag{3.33}$$

The derivatives and other operators acting on the field can then be obtained via functional differentiation and integration by parts. Regularized products of field operators at each point may then be introduced by extending the algebra $\mathscr{A}$ [13]. This extended set $\mathscr{A}$ then defines the set of valid operators in the QFT.

The interaction of the detector with the field is prescribed in terms of an interaction Hamiltonian which couples to an operator-valued distribution of the QFT, denoted $\hat{A}^\rho(\times)$, where ρ also stands for any collection of Lorentz indices[3]. In order to produce a scalar interaction Hamiltonian, one must have an operator in the detector's Hilbert space which is a tensor of the same rank as that of $\hat{A}^\rho(\times)$. We define the tensor operator as $\hat{\chi}(\tau) = \hat{\chi}^\rho(\tau)E_\rho$, where $\hat{\chi}^\rho(\tau)$ is an operator in $\mathscr{H}_X^{(\tau)} \otimes \mathscr{H}_S$ and E_ρ denotes the orthonormal frame for tensors of the same rank as $\hat{A}^\rho(\times)$ built from the extended Fermi frame e_μ. We further assume $\hat{\chi}^\rho(\tau)$ to be only a function of the operators $\hat{\vec{r}}$, $\hat{\vec{p}}$ and $\{\hat{s}_i\}$ and of the time parameter τ. For convenience, we work in the interaction picture from now on, so that $\hat{\chi}^\rho(\tau)$ includes the free time evolution associated with the detector's "free" Hamiltonian $\hat{\mathscr{H}}(\tau)$. Then, the interaction Hamiltonian is prescribed in the interaction picture as

$$\hat{H}_I(\tau) = \lambda\gamma(\tau,\hat{\vec{r}})\hat{\chi}^\dagger_\rho(\tau)\hat{A}^\rho(\tau,\hat{\vec{r}}) + \text{H.C.}\,, \tag{3.34}$$

where $\hat{A}^\rho(\tau,\hat{\vec{r}})$ denotes the components of the operator $\hat{A}^\rho(\times)$ in the frame E_ρ evaluated in FNC around the curve $z(\tau)$. Here we consider $\hat{A}^\rho(\times)$ and $\hat{\chi}^\rho(\tau)$ to be written in the orthonormal frame E_ρ in order to avoid unnecessary metric prescriptions in the contraction. Furthermore, the replacement of the dependence in the (classical) coordinates $\vec{r}$ by the quantum position operator $\hat{\vec{r}}$ formally means

$$\hat{A}^\rho(\tau,\hat{\vec{r}}) \equiv \int_{\Sigma_\tau} d\Sigma\, \hat{O}^\rho(\tau,\vec{r})\, |\vec{r}\rangle\!\langle\vec{r}|_\tau\,, \tag{3.35}$$

[3]Not necessarily the same as the field $\hat{\phi}^\sigma$.

where the subscript τ in $|\vec{r}\rangle\langle\vec{r}|_\tau$ denotes time evolution with respect to the detector's free Hamiltonian:

$$|\vec{r}\rangle\langle\vec{r}|_\tau \equiv \hat{U}(\tau)\,|\vec{r}\rangle\langle\vec{r}|\,\hat{U}^\dagger(\tau)\;, \tag{3.36}$$

with

$$\hat{U}(\tau) = \text{TEXP}\left(-i\int^\tau \hat{\mathscr{H}}(\tau')d\tau'\right)\;. \tag{3.37}$$

In Eq. (3.34), $\gamma(\tau, \hat{\vec{r}})$ denotes the redshift factor of Eq. (3.21) and $\hat{\chi}^\dagger_\rho(\tau)$ denotes the dual field to $\hat{\chi}^\rho(\tau)$.

Common examples of particle detector models found in the literature can be recovered when the Hamiltonian $\hat{\mathscr{H}}(\hat{\vec{r}}, \hat{\vec{p}}, \{\hat{s}_i\}, \tau)$ is independent of τ and $\hat{\chi}^\rho(\tau)$ can be written entirely in terms of the position operator, say $\hat{\chi}^\rho(\tau) = \chi^\rho(\tau, \hat{\vec{r}})$. For simplicity, we also assume that there are no internal degrees of freedom associated with $\mathscr{H}_S$. In this case, we can expand the Hamiltonian $\hat{H}_I(\tau)$ in terms of the position basis of the detector, $|\vec{r}\rangle$ via Eq. (3.24). We obtain

$$\hat{H}_I(\tau) = \lambda \int_{\Sigma_\tau} d^3\vec{r}\sqrt{-g}\,\chi^*_\rho(\times)\hat{A}^\rho(\times)\,|\vec{r}\rangle\langle\vec{r}|_\tau + \text{H.C.}\;, \tag{3.38}$$

where we use the fact that $\sqrt{-g} = \sqrt{h}\gamma(\times)$. We then assume that the detector's free Hamiltonian $\hat{\mathscr{H}}$ is independent of τ and has discrete energy eigenvectors, $|\psi_n\rangle$, with energy eigenvalues E_n: $\hat{\mathscr{H}}\,|\psi_n\rangle = E_n\,|\psi_n\rangle$. The eigenfunctions are defined as $\psi_n(\times) \equiv \langle\vec{r}|\psi_n(\tau)\rangle = e^{-iE_n\tau}\psi_n(\vec{r})$, where we write the wavefunction at $\tau = 0$ as $\psi_n(\vec{r})$. In the eigenbasis of the free Hamiltonian, the interaction Hamiltonian becomes:

$$\begin{aligned}
\hat{H}_I(\tau) &= \lambda\sum_{l,m}\int_{\Sigma_\tau} d^3\vec{r}\sqrt{-g}\,\chi^*_\rho(\times)\hat{A}^\rho(\times)\hat{U}\,|\psi_l\rangle\,\langle\psi_l|\vec{r}\rangle\,\langle\vec{r}|\psi_m\rangle\,\langle\psi_m|\,\hat{U}^\dagger + \text{H.C.}\\
&= \lambda\sum_{l,m}\int_{\Sigma_\tau} d^3\vec{r}\sqrt{-g}\,\chi^*_\rho(\times)\hat{A}^\rho(\times)e^{i(E_m-E_l)\tau}\psi^*_l(\vec{r})\psi_m(\vec{r})\,|\psi_l\rangle\langle\psi_m| + \text{H.C.}\\
&= \lambda\sum_{l,m}\int_{\Sigma_\tau} d^3\vec{r}\sqrt{-g}\,\chi^*_\rho(\times)\hat{A}^\rho(\times)e^{i\Xi_{lm}\tau}\psi^*_m(\vec{r})\psi_l(\vec{r})\,|\psi_m\rangle\langle\psi_l| + \text{H.C.}\;,
\end{aligned} \tag{3.39}$$

where $\Xi_{lm} \equiv E_l - E_m$ is the energy gap between the states labeled by l and m. Then, in order to draw a better comparison with the previous models in the literature (see [39, 40, 29]), we define the spacetime smearing tensors $(S_{lm})^\rho(\times) \equiv \psi_l(\vec{r})\psi^*_m(\vec{r})\chi^\rho(\times)$. We can then write the interaction Hamiltonian as

$$\hat{H}_I(\tau) = \lambda\sum_{l,m}\int_{\Sigma_\tau} d^3\vec{r}\sqrt{-g}\,(S_{lm})^*_\rho(\times)\hat{A}^\rho(\times)e^{i\Xi_{lm}\tau}\,|\psi_l\rangle\langle\psi_m| + \text{H.C.}\;, \tag{3.40}$$

from which we define the ***interaction Hamiltonian density*** as

$$\hat{h}_I(\times) \equiv \sum_{l,m} \hat{h}_I^{(lm)}(\times) = \lambda\sqrt{-g}\sum_{l,m}(S_{lm})^*_\rho(\times)\hat{A}^\rho(\times)e^{i\Xi_{lm}\tau}\,|\psi_l\rangle\langle\psi_m| + \text{H.C.}\,, \tag{3.41}$$

with

$$\hat{h}_I^{(lm)}(\times) \equiv \lambda\sqrt{-g}(S_{lm})^*_\rho(\times)\hat{A}^\rho(\times)e^{i\Xi_{lm}\tau}\,|\psi_l\rangle\langle\psi_m| + \text{H.C.} \tag{3.42}$$

In fact, most recent studies that consider finite-sized particle detectors in curved spacetimes [41, 42] prescribe the interaction of the detector with the field in terms of the Hamiltonian density in order to highlight the locality of the theory. In the approach presented here, locality is implemented in terms of the dependence on the position operator of the non-relativistic quantum system.

Overall, the model of Eq. (3.34) for the interaction of a localized non-relativistic quantum system with a quantum field represents the most general interaction between a non-relativistic quantum system localized around a trajectory and an operator in a QFT. The considerations about the description of non-relativistic quantum systems in curved spacetimes from section 3.2, including its regimes of validity and covariance of the model, also apply to the general particle detector models presented here, and naturally impose a limit for the regime of validity for these models.

Although the model of Eq. (3.34) is very general and can recover many models in the literature, it is not able to implement some features of specific models, especially when it comes to delocalization of the center of mass of detectors, which was considered in [43, 44], for instance. This delocalization would amount to describing the curve $z(\tau)$ quantum mechanically, which would require significant changes in our formalism and is out of the scope of this work.

The previous particle detector models used in the literature can be recovered from our model by choosing an appropriate quantum system, together with the detector and field operator that mediate the interaction. Let us recover an important model in the next subsection.

3.3.2 The scalar UDW model

The simplest scalar UDW model found in the literature consists of a two-level system coupled to a real scalar quantum field $\hat{\phi}(\times)$, according to the interaction Hamiltonian density.

$$\hat{h}_I(\times) = \lambda\sqrt{-g}S(\times)(e^{i\Xi\tau}\hat{\xi}^+ + e^{-i\Xi\tau}\hat{\xi}^-)\hat{\phi}(\times)\,, \tag{3.43}$$

where λ is the coupling constant, $S(\times)$ is the real spacetime smearing function, Ξ is the positive energy gap between the two levels of the system, and $\xi^{\pm}$ are the raising and lowering operators of the detector.

Now we show how to recover Eq. (3.43) from the general model (3.34) from any non-relativistic quantum system by restricting it to two-levels and neglecting the terms of the interaction which commute with the detector's free Hamiltonian. Consider a localized quantum system that is entirely described by its position degrees of freedom and has τ-independent discrete energy levels $E_{1,2}$ with eigenstates $|\psi_{1,2}\rangle$. We prescribe the interaction with the quantum field by the Hamiltonian

$$\hat{H}_I(\tau) = \lambda\gamma(\tau,\hat{\vec{r}})\chi(\tau,\hat{\vec{r}})\hat{\phi}(\times)\ , \tag{3.44}$$

where $\chi(\times) = \chi(\tau,\vec{r})$ is any real scalar function evaluated in FNC. We then identify the operator $\hat{\chi}(\tau) = \chi(\tau,\hat{\vec{r}})$ and the field operator $\hat{A}(\times) = \hat{\phi}(\times)$, and then

$$\hat{h}_I^{(lm)}(\times) = \lambda\sqrt{-g}(S_{lm})^*(\times)\hat{\phi}(\times)e^{i\Xi_{lm}\tau}\,|\psi_l\rangle\!\langle\psi_m|\ , \tag{3.45}$$

with $S_{lm}(\times) = \psi_l(\vec{r})\psi_m(\vec{r})\chi(\times)$. We restrict our system to $S_{lm}(\times) = \Lambda_{ml}(\times)$, for $l, m = 1, 2$ being the two levels of the detector, and define the energy gap as

$$\Xi \equiv E_2 - E_1 > 0\ , \tag{3.46}$$

which gives

$$\hat{h}_I^{(lm)}(\times) = \lambda\sqrt{-g}S^*_{lm}(\times)\hat{\phi}(\times)e^{i\Xi_{lm}\tau}\,|\psi_l\rangle\!\langle\psi_m|\ , \tag{3.47}$$

$$\begin{aligned}
\Rightarrow\ \hat{h}_I(\times) &= \sum_{m=1}^{2}[\hat{h}_I^{(1m)}(\times) + \hat{h}_I^{(2m)}(\times)] \\
&= \hat{h}_I^{(11)}(\times) + \hat{h}_I^{(12)}(\times) + \hat{h}_I^{(21)}(\times) + \hat{h}_I^{(22)}(\times) \\
&= \lambda\sqrt{-g}\hat{\phi}(\times)[S^*_{11}(\times)\,|\psi_1\rangle\!\langle\psi_1| + S^*_{22}(\times)\,|\psi_2\rangle\!\langle\psi_2| \\
&+ S^*_{12}(\times)e^{-i\Xi\tau}\,|\psi_1\rangle\!\langle\psi_2| + S^*_{21}(\times)e^{i\Xi\tau}\,|\psi_2\rangle\!\langle\psi_1|] \\
&= \lambda\sqrt{-g}\hat{\phi}(\times)[S_{11}(\times)\,|\psi_1\rangle\!\langle\psi_1| + S_{22}(\times)\,|\psi_2\rangle\!\langle\psi_2| \\
&+ S_{21}(\times)e^{-i\Xi\tau}\,|\psi_1\rangle\!\langle\psi_2| + S_{12}(\times)e^{i\Xi\tau}\,|\psi_2\rangle\!\langle\psi_1|]\ ,
\end{aligned} \tag{3.48}$$

where we have used the fact[4] that $S^*_{lm}(\times) = S_{ml}(\times)$.

[4] $S^*_{lm}(\times) = [\psi_l(\vec{r})\psi^*_m(\vec{r})\chi(\times)]^* = \psi_m(\vec{r})\psi^*_l(\vec{r})\chi(\times)\ \Rightarrow\ S^*_{lm}(\times) = S_{ml}(\times)$.

The next step is to write $\hat{\mathscr{H}} = \sum_{k=1}^{2} E_k \left|\psi_k\middle\rangle\!\middle\langle\psi_k\right|$ and compute the following commutator:

$$\begin{aligned}
[\hat{\mathscr{H}}, S_{ll} \left|\psi_l\middle\rangle\!\middle\langle\psi_l\right|] &= \sum_{k=1}^{2} E_k [\left|\psi_k\middle\rangle\!\middle\langle\psi_k\right|, \Lambda_{ll} \left|\psi_l\middle\rangle\!\middle\langle\psi_l\right|] \\
&= \sum_{k=1}^{2} E_k \left(S_{ll} \left|\psi_k\right\rangle \left\langle\psi_k|\psi_l\right\rangle \left\langle\psi_l\right| - S_{ll} \left|\psi_l\right\rangle \left\langle\psi_l|\psi_k\right\rangle \left\langle\psi_k\right|\right) \\
&= S_{ll} \left(\sum_{k=1}^{2} E_k \delta_{kl} \left|\psi_k\middle\rangle\!\middle\langle\psi_l\right| - \sum_{k=1}^{2} E_k \delta_{lk} \left|\psi_l\middle\rangle\!\middle\langle\psi_k\right|\right) \\
&= 0\,.
\end{aligned} \tag{3.49}$$

The result of Eq. (3.49) is important because terms that commute with the free Hamiltonian can be neglected, since they produce a global phase shift in the state. Then, we have

$$\hat{h}_I(\times) = \lambda\sqrt{-g}\hat{\phi}(\times)S(\times)\left[e^{i\Xi\tau} \left|\psi_2\middle\rangle\!\middle\langle\psi_1\right| + e^{-i\Xi\tau} \left|\psi_1\middle\rangle\!\middle\langle\psi_2\right|\right]\,, \tag{3.50}$$

from which we define $\hat{\xi}^+ \equiv \left|\psi_2\middle\rangle\!\middle\langle\psi_1\right|$ and $\hat{\xi}^- \equiv \left|\psi_1\middle\rangle\!\middle\langle\psi_2\right|$, and finally recover Eq. (3.43) from (3.50):

$$\hat{h}_I(\times) = \lambda\sqrt{-g}S(\times)(e^{i\Xi\tau}\hat{\xi}^+ + e^{-i\Xi\tau}\hat{\xi}^-)\hat{\phi}(\times). \tag{3.51}$$

Thus, we have shown how the general particle detector model from Eq. (3.34) can be used to recover one of the most significant particle detector models in the literature by picking a quantum system which yields the corresponding spacetime smearing function $S(\times)$. Finally, we mention that one could choose any other scalar operator for $\hat{A}(\times)$, such as $:\hat{\phi}^2(\times):$, which would give detector models studied in [45, 46, 47], for instance. Moreover, the QFT that this detector couples to can be more general than a scalar field theory. For instance, for a spinor field $\hat{\psi}(\times)$, the operator $\hat{A}(\times)$ can be chosen as $\hat{A}(\times) =: \hat{\overline{\psi}}(\times)\hat{\psi}(\times):$, which would recover other models studied in [45, 46, 47]. A generalization of the reduction presented here can also be carried out naturally for the case of complex scalar fields, recovering the scalar models of [42, 48, 49]. Overall, the model of Eq. (3.34) can be used to recover any coupling with a scalar quantum field, or with a scalar operator in a more general QFT.

3.3.3 Consequences of the acceleration-curvature dynamics for particle detector models

As discussed in subsections 3.2.2 and 3.2.4, when one considers a non-relativistic quantum system undergoing a timelike trajectory in curved spacetimes, the acceleration of the trajectory and the curvature of spacetime influence the dynamics of the particle according to the redshift factor in Eq. (3.21). Under the assumptions that (1) the kinetic plus potential energy of the system is sufficiently small compared to its rest energy and (2) the system is sufficiently localized with respect to the curvature radius of spacetime and its inverse acceleration, the dynamics introduced by the system's motion and the background spacetime are associated with the Hamiltonian

$$\hat{H}_{\text{rel}}(\tau) = M a_i(\tau)\hat{r}^i + \frac{M}{2}R_{0i0j}(\tau)\hat{r}^i\hat{r}^j \;, \tag{3.52}$$

from Eq. (3.32). These dynamics can by themselves promote energy level transitions within a particle detector, which generate an extra source of noise when probing a quantum field. That is, in order to be able to use a particle detector model to infer properties of a quantum field, one must make sure that the dynamics introduced by the system's trajectory and curvature of spacetime are sufficiently smaller than the effect of the interaction with the field. In this section, we discuss this regime in detail and show how we can keep track of these new internal dynamics of particle detectors.

When one considers particle detector models, it is usual to assume that the Hamiltonian $\hat{H}$ in Eq. (3.22) is time independent, so that one can apply perturbative techniques and use the interaction picture by picking the free Hamiltonian to be $\hat{H}$. While $\hat{H}$ is picked as the free Hamiltonian, the dynamics of the detector's internal degree of freedom are governed by the Hamiltonian $\hat{\mathscr{H}}(\tau) = \hat{H} + \hat{H}_{\text{rel}}(\tau)$. Notice that for general motion in a curved spacetime, the Hamiltonian $\hat{H}_{\text{rel}}(\tau)$ in Eq. (3.52) will depend on the time parameter τ. In most cases, this forces us to treat the effect from the curve's acceleration and the curvature of spacetime as a perturbation. Thus, in order to apply perturbative techniques and to use the interaction picture with free Hamiltonian $\hat{H}$, we would obtain an effective interaction Hamiltonian.

$$\hat{\mathscr{H}}_I(\tau) = \hat{H}_I(\tau) + \hat{H}_{\text{rel}}(\tau) \;, \tag{3.53}$$

where $\hat{H}_I(\tau)$ is the interaction with the quantum field from Eq. (3.34). Thus, as expected, the resulting excitations of the detector will be both due to the interaction with the field and due to the relativistic corrections in $\hat{H}_{\text{rel}}(\tau)$. The interaction unitary time evolution operator for the detector-field system can be written as

$$\hat{U}_I = \text{TEXP}\left(-i\int^{\tau} \hat{\mathscr{H}}_I(\tau')d\tau'\right) . \tag{3.54}$$

Note that due to the non-relativistic nature of the detector, different choices of time parameter will in general result in different time evolution operators. This has been discussed in detail in [29], and the discussion naturally carries over to the general models presented here. In essence, this unitary time evolution operator promotes time evolution of the field and detector with respect to the proper time of the detector's trajectory. This is justified because the interaction with the field is also prescribed in this frame.

The time evolution operator $\hat{U}_I$ in equation (3.54) is well defined, and can be used to describe the unitary time evolution of the detector and field states. Using perturbation theory in the Hamiltonian $\hat{\mathscr{H}}_I(\tau)$, we can then apply the Dyson expansion to the time evolution operator, so that

$$\hat{U}_I \approx \mathbb{1} + \hat{U}_I^{(1)} + \hat{U}_I^{(2)} , \tag{3.55}$$

with

$$\hat{U}_I^{(1)} = -i\int^{\tau} \hat{\mathscr{H}}_I(\tau')d\tau' , \tag{3.56}$$

and

$$\hat{U}_I^{(2)} = -\int^{\tau}\int^{\tau} \hat{\mathscr{H}}_I(\tau')\hat{\mathscr{H}}_I(\tau'')\theta(\tau'-\tau'')d\tau'd\tau'' , \tag{3.57}$$

where $\theta(\tau'-\tau'')$ denotes the Heaviside step function, which introduces time ordering with respect to τ. By writing $\hat{\mathscr{H}}_I(\tau) = \hat{H}_I(\tau) + \hat{H}_{\text{rel}}(\tau)$, we can split the unitary time evolution due to the interaction with the field, $\hat{H}_I(\tau)$, and due to the acceleration of the trajectory and curvature of spacetime, $\hat{H}_{\text{rel}}(\tau)$ as

$$\hat{U}_I^{(1)} = \hat{U}_{\text{field}}^{(1)} + \hat{U}_{\text{rel}}^{(1)} , \tag{3.58}$$

and

$$\hat{U}_I^{(2)} = \hat{U}_{\text{field}}^{(2)} + \hat{U}_{\text{mix}}^{(2)} + \hat{U}_{\text{rel}}^{(2)} , \tag{3.59}$$

where

$$\begin{cases} \hat{U}^{(1)}_{\text{field}} = -i \int^{\tau} \hat{H}_I(\tau')d\tau' \\ \hat{U}^{(1)}_{\text{rel}} = -i \int^{\tau} \hat{H}_{\text{rel}}(\tau')d\tau' \\ \hat{U}^{(2)}_{\text{field}} = - \int^{\tau} \int^{\tau} \hat{H}_I(\tau')\hat{H}_I(\tau'')\theta(\tau' - \tau'')d\tau' d\tau'' \\ \hat{U}^{(2)}_{\text{mix}} = - \int^{\tau} \int^{\tau} \left[\hat{H}_I(\tau')\hat{H}_{\text{rel}}(\tau'') + \hat{H}_{\text{rel}}(\tau')\hat{H}_I(\tau'')\right] \theta(\tau' - \tau'')d\tau' d\tau'' \\ \hat{U}^{(2)}_{\text{rel}} = - \int^{\tau} \int^{\tau} \hat{H}_{\text{rel}}(\tau')\hat{H}_{\text{rel}}(\tau'')\theta(\tau' - \tau'')d\tau' d\tau'' \end{cases} . \tag{3.60}$$

We remark again that when considering particle detector models in the literature, one usually neglects the change in the internal dynamics of the detector due to its motion and the spacetime curvature. This corresponds to setting $\hat{H}_{\text{rel}} = 0$, so that the only contributions up to third order in Dyson expansion of the interaction are due to $\hat{U}^{(1)}_{\text{field}}$ and $\hat{U}^{(2)}_{\text{field}}$. The remaining terms are additional corrections predicted by the formalism exposed in this chapter.

In order to provide an explicit example, we consider the excitation probability for the detector to transition from a state $|\psi_l\rangle$ to an orthogonal state $|\psi_m\rangle$ when the field is in a Hadamard state ω, so that its odd n-point functions vanish. Then, the excitation probability can be written as

$$p_{l\to m} = p^{\text{field}}_{l\to m} + p^{\text{rel}}_{l\to m} , \tag{3.61}$$

where

$$p^{\text{field}}_{l\to m} \equiv \lambda^2 \int_{\mathbb{M}^2} d\times\, d\times' \, (S_{lm})^{\rho}(\times)(S_{lm})^*_{\rho'}(\times')e^{-i\Xi_{lm}(\tau-\tau')}\langle \hat{A}^{\dagger}_{\rho}(\times)\hat{A}^{\rho'}(\times')\rangle_{\omega} \tag{3.62}$$

is the excitation probability due to the interaction with the field, $d\times = \sqrt{-g}d\tau\, d^3\vec{r}$ is the invariant spacetime volume element and $\Xi_{lm} = E_l - E_m$ is the energy gap between the states $|\psi_l\rangle$ and $|\psi_m\rangle$. The term $p^{\text{rel}}_{l\to m}$ is

$$p^{\text{rel}}_{l\to m} \equiv \int_{\mathbb{M}^2} d\times\, d\times' \, \psi^*_l(\vec{r})\psi_m(\vec{r})\psi_l(\vec{r}')\psi^*_m(\vec{r}')e^{-i\Xi_{lm}(\tau-\tau')}\left(Ma_i(\tau)r^i + \frac{M}{2}R_{0i0j}(\tau)r^ir^j\right) , \tag{3.63}$$

and corresponds to the effect of the acceleration of the detector and the curvature of spacetime in the excitation probability. Thus, if the particle detector gets excited after its interaction, one can only claim that this excitation was due to the interaction with the field if $p^{\text{rel}}_{l\to m} \ll p^{\text{field}}_{l\to m}$.

At last, we comment on a scenario where $\hat{H}_{\text{rel}}$ is time independent, where the excitation of a detector can be completely associated with the quantum field, with a change to Ξ_{lm} due to its acceleration and to the curvature of spacetime. Consider the situation where the spacetime is locally static, so that it possesses a locally defined timelike Killing vector field $\mathscr{X}^\mu$. Then, if the detector moves along the flux of the Killing field and its localization is within the region where $\mathscr{X}^\mu$ is Killing, we have that a_i = constant and R_{0i0j} = constant, so that $\hat{H}_{\text{rel}}$ is independent of τ. This happens for a static detector in Schwarzschild spacetime, for instance. In this case, the Hamiltonian $\hat{\mathscr{H}}$ in Eq. (3.32) is time independent, so that one can find time independent eigenstates $|\psi'_n\rangle$ and energy values E'_n for $\hat{\mathscr{H}}$. Notice that $|\psi'_n\rangle$ and E'_n will depend on $\hat{H}_{\text{rel}}$, and implicitly depend on a_i and R_{0i0j}. In this case, the full $\hat{\mathscr{H}} = \hat{H} + \hat{H}_{\text{rel}}$ Hamiltonian can be considered the free Hamiltonian in the interaction picture, and one only obtains contributions to the time evolution operator arising from the interaction with the quantum field.

Overall, we conclude that one must be careful when using particle detector models in curved spacetimes in order to probe quantum fields, because a time-varying proper acceleration and spacetime curvature may create an effective noise which blurs the effect of the field. However, if these terms are small enough compared to the effect of the quantum field in the detector, most of the excitation probability (and other observables of the theory) can be associated with the field. We also identified regimes where the detector's acceleration and curvature of spacetime can be implemented in the description of the detector, allowing one to associate detector excitations entirely with the quantum field.

3.4 Conclusions

In this chapter, we provided a general recipe for describing a localized non-relativistic quantum system undergoing a timelike trajectory in curved spacetimes. The framework presented can be applied to describe any system with one position degree of freedom that is sufficiently localized in space around the trajectory. If the system fulfills this condition, we showed how to consistently define position and momentum operators in the rest spaces of the trajectory and identified the necessary relativistic modifications to the Hamiltonian in order to implement time evolution.

We also defined the Fermi bound ℓ associated with a given timelike trajectory in a curved background as the maximum proper length around the trajectory such that the FNC can be used to fully describe a system with radius smaller than ℓ. The Fermi bound then defines a size limit for quantum systems which can be described within this framework. Moreover, we estimated the Fermi bound in terms of the spacetime curvature and the proper acceleration of the curve, providing a lower bound for the regime of validity of the model.

Finally, we applied our description of non-relativistic quantum systems to particle detector models. This allowed us to define a general particle detector model: a sufficiently localized non-relativistic quantum system which couples to ***any*** operator in a given QFT. In particular, the model obtained can be reduced to other previously considered particle detector models studied in the literature, such as the UDW. We also identified the necessary modifications to the usual perturbative treatment of particle detectors, which take into account their motion and the curvature of spacetime. Overall, the formalism here presented is the first step towards a full characterization of localized non-relativistic quantum systems used to probe quantum fields in curved spacetimes.

Chapter 4

Fundamentals of Stochastic Thermodynamics

The study of fluctuation relations has significantly transformed our comprehension of nonequilibrium thermodynamics, forging a connection between microscopic dynamics and macroscopic irreversibility. Although classical and quantum systems have been thoroughly investigated, extending these principles to quantum field theories (QFTs) introduces both challenges and novel perspectives. QFTs, which describe systems with infinitely many degrees of freedom and obey relativistic invariance, require a careful framework to merge stochastic thermodynamics with renormalization and processes involving particle creation and annihilation. This chapter develops such a framework, centering on the Jarzynski equality and Crooks fluctuation theorem in the realm of scalar QFTs within Minkowski spacetime. Utilizing the LSZ reduction formula alongside perturbative renormalization, we derive the work distribution function for a time-dependent interaction, demonstrating how energy transfer emerges from both particle dynamics and vacuum fluctuations. This formulation not only extends stochastic thermodynamics into relativistic domains but also sheds light on the interplay between quantum coherence, renormalization group flow, and thermalization in externally driven field theories.

A cornerstone of this study is the verification that fluctuation theorems remain valid in QFTs, even in the presence of renormalization and second quantization. By examining both tree-level and loop-level contributions to the work distribution, we find that quantum corrections—though modifying the scale of fluctuations—preserve the exponential symmetry expressed in the Crooks relation,

$$\mathscr{P}[w;\Lambda]/\mathscr{P}[-w;\tilde{\Lambda}] \;=\; e^{\beta w-\Delta F}.$$

Notably, the Jarzynski equality proves to be remarkably robust, enduring even when divergences in vacuum energy demand precise regularization, emphasizing its universal applicability across scales. These findings illustrate how thermodynamic irreversibility is encoded within the correlation functions of QFTs, providing valuable insights into energy transfer mechanisms in high-energy collisions and early-universe phase transitions. By integrating stochastic thermodynamics with QFT, this chapter establishes a foundation for investigating nonequilibrium phenomena in gauge theories, curved spacetime backgrounds, and other contexts where relativistic and quantum effects are essential.

4.1 Fluctuation-dissipation relations

On microscopic scales, matter constantly undergoes thermal and quantum fluctuations. Statistical Mechanics enables the interpretation and quantification of these fluctuating processes.

A quintessential example is a dilute gas in thermal equilibrium, classically characterized by the Maxwell-Boltzmann velocity distribution. This distribution is derived under the assumptions that the microscopic constituents follow Hamiltonian dynamics and that the gas atoms interact through negligible short-range forces. Importantly, the Maxwell-Boltzmann distribution describes a state of thermal equilibrium. But what becomes of other fluctuating quantities when the system is driven out of equilibrium? In this section, we are primarily concerned with the fluctuations of the work exchanged during nonequilibrium processes. Two main elements are central to our analysis:

- The system's initial state is thermal and represented by the Gibbs canonical distribution:
$$\rho_\beta = \frac{e^{-\beta H_0}}{Z_0}\,, \tag{4.1}$$
where H_0 denotes the initial Hamiltonian, and Z_0 is the corresponding partition function;

- The dynamics at the microscopic level are reversible.

The first assumption pertains to the statistical nature of the initial state, assuming a well-defined probability distribution. The second refers to the inherently Hamiltonian structure of the microscopic evolution.

We now explore how the system responds when it is externally driven out of equilibrium, possibly in a non-adiabatic manner.

4.1.1 Einstein's relation

The origins of fluctuation relations can be traced to Einstein's seminal work [50]. In 1905, he demonstrated that the linear response of a thermally equilibrated system to an external perturbation is directly related to its equilibrium fluctuations. Considering a Brownian particle immersed in a fluid, Einstein established a relation between the particle's mobility μ and the diffusion constant D:

$$\mu = \frac{D}{k_B T} \,, \tag{4.2}$$

where k_B is Boltzmann's constant (set to unity in our discussion), and T denotes the equilibrium temperature of the fluid. In dissipative media, mobility quantifies the ratio between the terminal drift velocity v_d of a particle and an applied external force F:

$$\mu = \frac{v_d}{F} \,. \tag{4.3}$$

Equation (4.2) links a nonequilibrium quantity —mobility μ— to the equilibrium temperature T. A brief derivation follows. Suppose the external force is conservative, such that $F(x) = -\nabla U(x)$ for a smooth potential $U : \mathbb{R}^3 \to \mathbb{R}$. The drift velocity at position x then reads

$$\begin{aligned} v_d(x) &= \mu(x) F(x) \\ &= -\mu(x) \nabla U(x) \,. \end{aligned} \tag{4.4}$$

Assuming equilibrium, the particle concentration follows Maxwell-Boltzmann statistics:

$$\rho(x) = A e^{-\frac{U(x)}{k_B T}} \,, \tag{4.5}$$

where A is a normalization constant. The drift current density becomes

$$J_{\text{drift}}(x) = \rho(x) v_d(x) = -\rho(x)\mu(x)\nabla U(x) \,. \tag{4.6}$$

Diffusion contributes a second component to the current, governed by Fick's law:

$$J_{\text{diffusion}}(x) = -D(x)\nabla\rho(x) \,. \tag{4.7}$$

At equilibrium, the total current vanishes: $J_{\text{drift}} + J_{\text{diffusion}} = 0$. Taking the gradient of Eq. (4.5), we obtain

$$\nabla\rho(x) = -\frac{\nabla U(x)}{k_B T}\rho(x) \,. \tag{4.8}$$

Substituting into the current balance yields

$$-\rho(x)\nabla U(x)\left[\mu(x)-\frac{D(x)}{k_BT}\right]=0 \;\Leftrightarrow\; \mu=\frac{D}{K_BT}\;. \tag{4.9}$$

This derivation shows that Eq. (4.2) is approximate, as it relies on Fick's law. Einstein's relation is the earliest example of a fluctuation-dissipation theorem—connecting thermal fluctuations with linear response in systems that respect detailed balance. It implies that a system's response to a small external perturbation mirrors its response to spontaneous fluctuations in equilibrium.

A further instance is provided by Johnson-Nyquist noise [51, 52], which arises from thermal agitation of electrons in an equilibrium conductor. This phenomenon manifests as measurable voltage fluctuations across an isolated resistor, characterized by a power spectral density dependent on the conductor's temperature. The mean square value of the fluctuating voltage reads:

$$\langle V^2\rangle = 4R\Delta\nu k_BT\;, \tag{4.10}$$

where $\Delta\nu$ is the bandwidth over which the voltage is measured.

4.1.2 Green-Kubo relations

Let us now briefly review the framework of fluctuation-dissipation theorems in quantum systems, as encapsulated by the Green-Kubo relations (GKR).

Consider an isolated quantum system with a self-adjoint Hamiltonian operator $\hat{H}_0$ on a Hilbert space $\mathscr{H}$. Suppose the system is in thermal equilibrium at inverse temperature β, so that its state is

$$\hat{\rho}_\beta=\frac{e^{-\beta\hat{H}_0}}{Z_0}\;. \tag{4.11}$$

Now, imagine that an external time-dependent force perturbs the system, modifying the Hamiltonian to

$$\hat{H}(\Lambda_t)=\hat{H}_0-\Lambda_t\hat{Q}\;, \tag{4.12}$$

where $\Lambda_t : [0,\tau]\to\mathbb{R}$ is a time-dependent scalar function, and $\hat{Q}=\hat{Q}^\dagger$ is the observable conjugate to the perturbing force. Assuming weak perturbations, we confine ourselves to linear response. The response is measured by the time-dependent change in the expectation value of a bounded observable $\hat{B}$:

$$\Delta\hat{B}(t)\equiv \mathrm{Tr}\left[\hat{B}\hat{\rho}_t\right]-\mathrm{Tr}\left[\hat{B}\hat{\rho}_\beta\right]=\int_0^t K_{\hat{B}\hat{Q}}(t-s)\Lambda_s ds\;, \tag{4.13}$$

where $\hat{\rho}_t$ is the time-evolved state under the perturbed Hamiltonian (4.12). The kernel $K_{\hat{B}\hat{Q}}(t)$, known as the *response function*, is given by

$$K_{\hat{B}\hat{Q}}(t) \equiv \frac{\omega_{\hat{\rho}}\left([\hat{Q}, \hat{B}(t)]\right)}{i} , \tag{4.14}$$

with $\hat{B}(t) = e^{it\hat{H}_0}\hat{B}e^{-it\hat{H}_0}$ denoting the Heisenberg picture evolution of $\hat{B}$.

The fluctuation properties are captured by the correlation function $\mathscr{C}_{\hat{B}\hat{Q}}(t)$:

$$\mathscr{C}_{\hat{B}\hat{Q}}(t) \equiv \frac{\omega_{\hat{\rho}}\left(\{\hat{Q}, \hat{B}(t)\}\right)}{2} , \tag{4.15}$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ is the anticommutator. The quantum fluctuation-dissipation theorem [53] relates the two functions:

$$\tilde{\mathscr{C}}_{\hat{B}\hat{Q}}(j) = \frac{j}{2}\coth\left(\frac{\beta j}{2}\right)\tilde{K}_{\hat{B}\hat{Q}}(j) , \tag{4.16}$$

where $\tilde{b}(j)$ denotes the Fourier transform: $\tilde{b}(j) = \int_{\mathbb{R}} e^{-ijt}f(t)dt$. Reintroducing $\hbar$ for the classical limit yields:

$$\tilde{\mathscr{C}}_{\hat{B}\hat{Q}}(j) = \frac{\hbar j}{2}\coth\left(\frac{\beta\hbar j}{2}\right)\tilde{K}_{\hat{B}\hat{Q}}(j) , \tag{4.17}$$

and in the limit $\hbar \to 0^+$, where $\coth(x) \to 1/x$, we recover

$$\tilde{K}(j) = \beta\tilde{\mathscr{C}}(j) . \tag{4.18}$$

This classical limit aligns with Einstein's relation (4.2). The GKR represents the inception of a broader class of fluctuation-dissipation relations, extending beyond linear response into nonlinear regimes.

4.1.3 Classical fluctuation relations

Let us consider a fluctuating quantity x—for instance, the work performed during a non-equilibrium transformation—and denote by $p_F(x)$ the probability density function associated with the forward process, and by $p_B(x)$ the corresponding density in the time-reversed (backward) process. Accordingly, the subscripts "F" and "B" refer to the "Forward" and "Backward" realizations. Owing to microreversibility, fluctuation relations generally emerge in the following form:

$$p_F(x) = e^{\beta(x-a)} p_B(-x) \,, \tag{4.19}$$

where the constant a encapsulates equilibrium properties related to the initial conditions of both forward and backward transformations. Equation (4.19) establishes a connection between non-equilibrium properties—represented by the distributions p_F and p_B—and equilibrium parameters β and a. It implies that due to microreversibility, forward trajectories are exponentially favored over their backward counterparts at the macroscopic level. As an illustration, consider the rare event of spontaneous freezing of water in a glass: although thermodynamically permissible, Eq. (4.19) quantifies such an event as overwhelmingly improbable.

Let us now examine the statistical fluctuations of work in a classical system driven by an external, time-dependent parameter. Consider a classical Hamiltonian system governed by

$$H(z, \lambda) = H_0(z) - \lambda Q(z) \,, \tag{4.20}$$

where $z = (q, p)$ denotes a point in the phase space Γ, $H_0(z)$ is the autonomous (unperturbed) Hamiltonian, and $\lambda \in \mathbb{R}$ is a scalar external control parameter coupled to an observable $Q(z)$. We assume that both H_0 and Q are smooth over Γ. The canonical Gibbs distribution corresponding to this Hamiltonian reads

$$\rho_\beta^\lambda(z) = \frac{e^{-\beta H(z,\lambda)}}{Z(\lambda)} = \frac{e^{-\beta(H_0(z) - \lambda Q(z))}}{Z(\lambda)} \,, \tag{4.21}$$

where $Z(\lambda) = \int_\Gamma e^{-\beta H(z,\lambda)}\, dz$ denotes the classical partition function. Its logarithm defines the Helmholtz free energy,

$$F(\lambda) = -\frac{1}{\beta} \ln Z(\lambda) \,. \tag{4.22}$$

In the case where $\Gamma = \mathbb{R}^{2d}$, the canonical measure in Eq. (4.21) remains well-defined provided that $H(z, \lambda) \to \infty$ sufficiently fast as $|z| \to \infty$, for all $\lambda \in \mathbb{R}$; i.e., the Hamiltonian must be confining.

Now, let us introduce the time-reversal operation defined on the phase space as:

$$\theta : \Gamma \to \Gamma \,, \quad (q, p) \mapsto (q, -p) \,. \tag{4.23}$$

We impose two assumptions: 1) H_0 is invariant under time reversal, i.e., $H_0(\theta(z)) = H_0(z)\ \forall z \in \Gamma$; and 2) Q possesses a definite parity under time reversal, namely

$Q(\theta(z)) = \eta_Q Q(z)$ with $\eta_Q = \pm 1$. It follows that

$$H(\theta(z), \lambda) = H(z, \eta_Q \lambda) \quad \forall z \in \Gamma,\ \lambda \in \mathbb{R}\ . \tag{4.24}$$

Indeed, using the definitions:

$$\begin{aligned} H(\theta(z), \lambda) &= H_0(\theta(z)) - \lambda Q(\theta(z)) \\ &= H_0(z) - \lambda \eta_Q Q(z) \\ &= H(z, \eta_Q \lambda)\ . \end{aligned} \tag{4.25}$$

Therefore, condition (4.24) is equivalent to the assumptions above. Now, consider a smooth function $\Lambda : [0, \tau] \to \mathbb{R}$ describing the external driving protocol in time. Under this protocol, the Hamiltonian becomes time-dependent: $H(z, \Lambda_t)$, and the perturbation term is $-\Lambda_t Q(z)$.

To formulate a time-reversed evolution (since reversing time itself is infeasible), we define the *backward protocol* as

$$\tilde{\Lambda} : [0, \tau] \to \mathbb{R}\ , \quad \tilde{\Lambda}_t \equiv \eta_Q \Lambda_{\tau - t}\ . \tag{4.26}$$

In this setup, the backward protocol effectively retraces the evolution of the external force from $\tilde{\Lambda}_0 = \eta_Q \Lambda_\tau$ to $\tilde{\Lambda}_\tau = \eta_Q \Lambda_0$. Denote by $\varphi_{t,0}[z_0; \Lambda]$ the Hamiltonian flow generated by $H(z, \Lambda_t)$ with initial condition $z_0 \in \Gamma$, that is,

$$\begin{cases} \dot{q}(t) = \nabla_p H(z(t), \Lambda_t) \\ \dot{p}(t) = -\nabla_q H(z(t), \Lambda_t) \end{cases}, \tag{4.27}$$

where $z(0) = z_0$ and $z(t) = \varphi_{t,0}[z_0; \Lambda]$. The flows corresponding to Λ and $\tilde{\Lambda}$ satisfy:

$$\varphi_{t,0}[z_0; \Lambda] = \theta(\varphi_{\tau - t,0}[\theta(z(\tau)); \tilde{\Lambda}])\ , \tag{4.28}$$

with $z(\tau) = \varphi_{\tau,0}[z_0; \Lambda]$, so that $\theta(z(\tau))$ serves as the initial condition for the backward protocol.

The initial state is assumed to be sampled from the canonical distribution at $t = 0$, given by

$$\rho_\beta^{\Lambda_0} = \frac{e^{-\beta H(z, \Lambda_0)}}{Z(\Lambda_0)}\ . \tag{4.29}$$

Letting the system evolve under the protocol Λ, the total work done on the system is

$$W[z_0; \Lambda] = H(z(\tau), \Lambda_\tau) - H(z_0, \Lambda_0)\ , \tag{4.30}$$

with $z(\tau) = \varphi_{\tau,0}[z_0; \Lambda]$. The work may equivalently be expressed as

$$W[z_0; \Lambda] = -\int_0^\tau \dot{\Lambda}_t Q(z(t))\, dt \;, \tag{4.31}$$

where $\dot{\Lambda}_t = \frac{d\Lambda_t}{dt}$. To verify this:

$$\begin{aligned} W[z_0; \Lambda] &= \int_0^\tau \frac{d}{dt} H(z(t), \Lambda_t)\, dt \\ &= \int_0^\tau \left[\nabla_z H \cdot \dot{z}(t) + \frac{\partial H}{\partial \lambda} \dot{\Lambda}_t \right] dt \\ &= \int_0^\tau \left[-Q(z(t)) \dot{\Lambda}_t \right] dt \;, \end{aligned} \tag{4.32}$$

where the first term vanishes due to symplectic orthogonality.

Jarzynski equality

Our objective now is to compute the average value of $e^{-\beta W[z_0;\Lambda]}$ over the ensemble of initial conditions drawn from the canonical distribution (4.29). This average is defined as:

$$\langle e^{-\beta W} \rangle_\Lambda \equiv \int_\Gamma e^{-\beta W[z_0;\Lambda]} \rho_\beta^{\Lambda_0}(z_0)\, dz_0 \;. \tag{4.33}$$

The Jarzynski equality asserts that this average satisfies the identity:

$$\langle e^{-\beta W} \rangle_\Lambda = e^{-\beta(F(\Lambda_\tau) - F(\Lambda_0))} \;, \tag{4.34}$$

where $F(\lambda)$ is the Helmholtz free energy defined in Eq. (4.22).

This constitutes a fluctuation relation, linking a non-equilibrium observable on the left-hand side to an equilibrium quantity on the right-hand side. Importantly, the Jarzynski identity (4.34) holds exactly, regardless of the amplitude of the perturbation Λ, or how far the system is driven out of its initial equilibrium configuration $\rho_\beta^{\Lambda_0}$. Let us derive it from the definition (4.33):

$$\begin{aligned} \langle e^{-\beta W} \rangle_\Lambda &= \int_\Gamma e^{-\beta W[z_0;\Lambda]} \rho_\beta^{\Lambda_0}(z_0)\, dz_0 \\ &= \frac{1}{Z(\Lambda_0)} \int_\Gamma e^{-\beta(H(z(\tau),\Lambda_\tau) - H(z_0,\Lambda_0))} e^{-\beta H(z_0,\Lambda_0)}\, dz_0 \\ &= \frac{1}{Z(\Lambda_0)} \int_\Gamma e^{-\beta H(z(\tau),\Lambda_\tau)}\, dz_0 \;. \end{aligned} \tag{4.35}$$

Applying Liouville's theorem [54], we recognize that the Hamiltonian flow $\varphi_{\tau,0}[\cdot;\Lambda]$ is volume-preserving on Γ, with Jacobian determinant equal to one:

$$\begin{cases} z = z(\tau) = \varphi_{\tau,0}[z_0;\Lambda] \\ \\ \left|\dfrac{\partial z}{\partial z_0}\right| = 1 \end{cases} . \tag{4.36}$$

Hence, we may change variables in Eq. (4.35) to obtain:

$$\begin{aligned} \langle e^{-\beta W}\rangle_\Lambda &= \frac{1}{Z(\Lambda_0)}\int_\Gamma e^{-\beta H(z,\Lambda_\tau)}\,dz \\ &= \frac{Z(\Lambda_\tau)}{Z(\Lambda_0)} \\ &= e^{-\beta(F(\Lambda_\tau)-F(\Lambda_0))} , \end{aligned} \tag{4.37}$$

recovering the Jarzynski equality.

From Eq. (4.34), we can now derive a fundamental thermodynamic inequality. Jensen's inequality,

$$f(\langle x\rangle) \le \langle f(x)\rangle , \tag{4.38}$$

valid for convex functions f, yields:

$$\langle e^{-\beta W}\rangle_\Lambda \ge e^{-\beta\langle W\rangle_\Lambda} , \tag{4.39}$$

and thus,

$$\langle W\rangle_\Lambda \ge F(\Lambda_\tau) - F(\Lambda_0) \equiv \Delta F . \tag{4.40}$$

This is a manifestation of the second law of thermodynamics in terms of average work. Introducing the notion of dissipated work [55]:

$$W_{\text{diss}} \equiv W - \Delta F , \tag{4.41}$$

we deduce from the inequality above that

$$\langle W_{\text{diss}}\rangle_\Lambda \ge 0 , \tag{4.42}$$

indicating that, on average, the dissipated work is non-negative—regardless of the specific driving protocol Λ.

Crooks fluctuation theorem

Let us now consider Crooks' fluctuation theorem, from which the Jarzynski equality naturally follows. To proceed, we introduce the probability density function (PDF) of work performed under the protocol Λ:

$$p[w;\Lambda] = \langle \delta(w - W) \rangle_\Lambda \ , \tag{4.43}$$

where the average is taken with respect to the canonical ensemble (4.29):

$$p[w;\Lambda] = \int_\Gamma \frac{e^{-\beta H(z_0,\Lambda_0)}}{Z(\Lambda_0)} \delta(w - W[z_0;\Lambda])\, dz_0 \ . \tag{4.44}$$

From this PDF, one can recover the average of any continuous function f. Indeed:

$$\begin{aligned} \langle f(W) \rangle_\Lambda &= \int_\Gamma f(W[z_0;\Lambda]) \frac{e^{-\beta H(z_0,\Lambda_0)}}{Z(\Lambda_0)}\, dz_0 \\ &= \int_\mathbb{R} f(w)\, p[w;\Lambda]\, dw \ . \end{aligned} \tag{4.45}$$

We now aim to establish the Crooks fluctuation theorem:

$$p[w;\Lambda] = e^{\beta(w-\Delta F)} p[-w;\tilde{\Lambda}] \ , \tag{4.46}$$

with

$$\Delta F = F(\Lambda_\tau) - F(\Lambda_0) = -\frac{1}{\beta} \ln\left(\frac{Z(\Lambda_\tau)}{Z(\Lambda_0)}\right) \ . \tag{4.47}$$

This relation expresses that the probability of performing work w under Λ is exponentially more likely than the probability of extracting work w under the reversed protocol $\tilde{\Lambda}$.

We start by rewriting the definition (4.43):

$$\begin{aligned} p[w;\Lambda] &= \int_\Gamma \frac{e^{-\beta H(z_0,\Lambda_0)}}{Z(\Lambda_0)} \delta(w - W[z_0;\Lambda])\, dz_0 \\ &= \frac{e^{\beta w}}{Z(\Lambda_0)} \int_\Gamma e^{-\beta H(z(\tau),\Lambda_\tau)} \delta(-w - H(z_0,\Lambda_0) + H(z(\tau),\Lambda_\tau))\, dz_0 \\ &= \frac{Z(\Lambda_\tau)}{Z(\Lambda_0)} e^{\beta w} \int_\Gamma \frac{e^{-\beta H(z(\tau),\Lambda_\tau)}}{Z(\Lambda_\tau)} \delta(-w - H(z_0,\Lambda_0) + H(z(\tau),\Lambda_\tau))\, dz_0 \\ &= e^{\beta(w-\Delta F)} \int_\Gamma \rho_\beta^{\Lambda_\tau}(z(\tau)) \delta(-w - H(z_0,\Lambda_0) + H(z(\tau),\Lambda_\tau))\, dz_0 \ . \end{aligned} \tag{4.48}$$

To proceed, we invoke microreversibility. Recall from Eq. (4.28) that:

$$z_0 = \theta(\varphi_{\tau,0}[\theta(z(\tau));\tilde{\Lambda}]) \ . \tag{4.49}$$

Then, using Eq. (4.24), we have:

$$
\begin{aligned}
H(z_0, \Lambda_0) &= H(\theta(\varphi_{\tau,0}[\theta(z(\tau)); \tilde{\Lambda}]), \Lambda_0) \\
&= H(\varphi_{\tau,0}[\theta(z(\tau)); \tilde{\Lambda}], \eta_Q \Lambda_0) = H(\varphi_{\tau,0}[\theta(z(\tau)); \tilde{\Lambda}], \tilde{\Lambda}_\tau) \;, \qquad (4.50)
\end{aligned}
$$

and likewise,

$$
H(z(\tau), \Lambda_\tau) = H(\theta(z(\tau)), \tilde{\Lambda}_0) \;. \qquad (4.51)
$$

Substituting these into Eq. (4.48), and making the change of variables $z = \theta(z(\tau))$, whose Jacobian equals one, we find:

$$
\begin{aligned}
\frac{p[w; \Lambda]}{e^{\beta(w-\Delta F)}} &= \int_\Gamma \rho_\beta^{\Lambda_\tau}(z(\tau)) \delta(-w - H(\varphi_{\tau,0}[\theta(z(\tau)); \tilde{\Lambda}], \tilde{\Lambda}_\tau) + H(\theta(z(\tau)), \tilde{\Lambda}_0))\, dz_0 \\
&= \int_\Gamma \rho_\beta^{\tilde{\Lambda}_0}(z) \delta(-w - H(\varphi_{\tau,0}[z; \tilde{\Lambda}], \tilde{\Lambda}_\tau) + H(z, \tilde{\Lambda}_0))\, dz \\
&= p[-w; \tilde{\Lambda}] \;. \qquad (4.52)
\end{aligned}
$$

Here we used that $\rho_\beta^{\Lambda_\tau}(\theta(z)) = \rho_\beta^{\tilde{\Lambda}_0}(z)$, completing the derivation.

The Crooks fluctuation theorem thus quantifies the asymmetry in work fluctuations between forward and backward driving. For $w > \Delta F$, the ratio $p[w; \Lambda]/p[-w; \tilde{\Lambda}]$ grows exponentially, indicating that dissipative events are exponentially more probable than those that would reduce the system's energy.

Finally, let us recover the Jarzynski equality from the Crooks theorem. From Eqs. (4.45) and (4.46), we write:

$$
\begin{aligned}
\langle e^{-\beta W} \rangle_\Lambda &= \int_\mathbb{R} e^{-\beta w} p[w; \Lambda]\, dw \\
&= \int_\mathbb{R} e^{-\beta w} e^{\beta(w-\Delta F)} p[-w; \tilde{\Lambda}]\, dw \\
&= e^{-\beta \Delta F} \int_\mathbb{R} p[-w; \tilde{\Lambda}]\, dw = e^{-\beta \Delta F} \;. \qquad (4.53)
\end{aligned}
$$

As a further generalization, consider an observable $B : \Gamma \to \mathbb{R}$ with definite time-reversal parity, $B(\theta(z)) = \eta_B B(z)$, $\eta_B = \pm 1$. Then, for any test function $u : [0, \tau] \to \mathbb{R}$, the following generalized Jarzynski equality holds:

$$
\left\langle \exp\left(-\beta W + \int_0^\tau u_t B(t)\, dt \right) \right\rangle_\Lambda = e^{-\beta \Delta F} \left\langle \exp\left(\int_0^\tau \tilde{u}_t B(t)\, dt \right) \right\rangle_{\tilde{\Lambda}} \;, \qquad (4.54)
$$

where $\tilde{u}_t = \eta_B u_{\tau - t}$. This functional identity is the generating relation from which one derives fluctuation-dissipation theorems at all orders via functional differentiation with respect to Λ and u at the origin.

4.2 Quantum fluctuation relations

Let us consider a Hamiltonian operator acting on a Hilbert space $\mathscr{H}$, defined by

$$\hat{H}(\lambda) = \hat{H}_0 - \lambda\hat{Q} \ , \tag{4.55}$$

where $\hat{H}_0$ and $\hat{Q}$ are Hermitian operators, and λ is a real parameter associated with an external force. The quantum canonical Gibbs state is described by the density matrix

$$\hat{\rho}^{\lambda}_{\beta} = \frac{1}{Z(\lambda)} e^{-\beta\hat{H}(\lambda)} \ , \tag{4.56}$$

where

$$Z(\lambda) = \mathrm{Tr}\Big[e^{-\beta\hat{H}(\lambda)}\Big] \tag{4.57}$$

is the partition function. The Helmholtz free energy is defined, as in the classical case, in terms of $Z(\lambda)$.

Assume now that $\{\hat{H}(\lambda)\}_{\lambda\in\mathbb{R}}$ is a family of self-adjoint, possibly unbounded operators sharing a common dense domain D. We also suppose that $\hat{\rho}^{\lambda}_{\beta}$ is trace-class for all $\lambda \in \mathbb{R}$ and $\beta > 0$. Under these conditions, the Hamiltonian has a discrete spectrum with finite multiplicity:

$$\hat{H}(\lambda) = \sum_m E^{\lambda}_m \hat{\Pi}^{\lambda}_m \ , \tag{4.58}$$

where $\{E^{\lambda}_m\}$ are distinct eigenvalues ($E^{\lambda}_m \neq E^{\lambda}_n$ for $m \neq n$), and the corresponding eigenprojectors $\{\hat{\Pi}^{\lambda}_m\}$ have finite rank, satisfying $\mathrm{Tr}\Big[\hat{\Pi}^{\lambda}_m\Big] < \infty$ for all m. If $\mathscr{H}$ is infinite-dimensional, we further assume $E^{\lambda}_m \to \infty$ as $m \to \infty$.

Let $t \in [0,\tau] \mapsto \Lambda_t \in \mathbb{R}$ be a prescribed protocol governing the external force, making the Hamiltonian time-dependent: $t \mapsto \hat{H}(\Lambda_t)$. The quantum evolution in the interval $[t_0, t]$ follows Schrödinger's equation, which reads:

$$\begin{cases} i\partial_t \hat{U}_{t,t_0}[\Lambda] \left|\psi\right\rangle = \hat{H}(\Lambda_t)\hat{U}_{t,t_0}[\Lambda] \left|\psi\right\rangle \\ \hat{U}_{t_0,t_0}[\Lambda] = \mathbb{1} \end{cases} , \ (t \geq t_0) \ \forall \ \left|\psi\right\rangle \in \mathscr{H}\|_D \ , \tag{4.59}$$

where $\mathscr{H}\|_D$ denotes the restriction of $\mathscr{H}$ to domain D. Thus, an initial state $\left|\psi\right\rangle \in \mathscr{H}\|_D$ evolves to $\hat{U}_{t,t_0}[\Lambda]\left|\psi\right\rangle \in \mathscr{H}\|_D$ at time t. Alternatively, differentiating with respect to t_0, we obtain the final value problem:

$$\begin{cases} i\partial_{t_0} \hat{U}_{t,t_0}[\Lambda] \left|\psi\right\rangle = \hat{U}_{t,t_0}[\Lambda]\hat{H}(\Lambda_{t_0}) \left|\psi\right\rangle \\ \hat{U}_{t,t}[\Lambda] = \mathbb{1} \end{cases} , \ (t \geq t_0) \ \forall \ \left|\psi\right\rangle \in \mathscr{H}\|_D \ , \tag{4.60}$$

and the common solution to both equations is expressed as

$$\begin{aligned}\hat{U}_{t,t_0}[\Lambda] &= \text{TEXP}\left(-i\int_{t_0}^{t}\hat{H}(\Lambda_{t'})\mathrm{d}t'\right) \\ &= \text{TEXP}\left(-i\int_{t_0}^{t}(\hat{H}_0-\Lambda_{t'}\hat{Q})\mathrm{d}t'\right) .\end{aligned} \tag{4.61}$$

Analogously to the classical scenario, one might attempt to define the quantum work operator as the difference between the final and initial Hamiltonians in the Heisenberg picture:

$$\hat{W}[\Lambda] \equiv \hat{U}^{\dagger}_{\tau,0}[\Lambda]\hat{H}(\Lambda_\tau)\hat{U}_{\tau,0}[\Lambda] - \hat{H}(\Lambda_0) . \tag{4.62}$$

However, trying to follow the classical derivation of the Jarzynski equality fails here, due to the general non-commutativity of the Hamiltonians at different times:

$$[\hat{H}(\Lambda_t), \hat{H}(\Lambda_{t_0})] = (\Lambda_t - \Lambda_{t_0})[\hat{H}_0, \hat{Q}] . \tag{4.63}$$

It turns out that the relation

$$\langle e^{-\beta\hat{W}}\rangle_\Lambda \equiv \mathrm{Tr}\left[\hat{\rho}_\beta^{\Lambda_0} e^{-\beta\hat{W}[\Lambda]}\right] = e^{-\beta\Delta F} \tag{4.64}$$

holds if and only if $[\hat{H}(\Lambda_t), \hat{H}(\Lambda_{t_0})] = 0$ for all $t_0, t \in [0,\tau]$. This situation occurs either under constant protocols (implying $\Delta F = 0$) or when $[\hat{H}_0, \hat{Q}] = 0$, which corresponds to the classical limit.

While this might suggest no quantum analogue of Jarzynski's identity exists, the issue lies in the definition of work in Eq. (4.62). Since work reflects a process rather than a state, it cannot generally be associated with a single self-adjoint operator. Instead, it must be defined via a two-time energy measurement: one at $t = 0$ and another at $t = \tau$, with the difference in measurement outcomes representing the realized work.

This procedure, called the two-time measurement protocol (TTMP), provides the quantum version of the classical work definition from single trajectories. Unlike in classical physics, however, the act of measurement introduces intrinsic quantum fluctuations in addition to thermal fluctuations. Therefore, the difference of measurement outcomes is not equivalent to measuring the operator difference in (4.62), which lacks a clear operational interpretation.

The TTMP proceeds as follows:

1. The system is prepared in the initial Gibbs state:

$$\hat{\rho}_\beta^{\Lambda_0} = \frac{e^{-\beta \hat{H}(\Lambda_0)}}{Z(\Lambda_0)} \sum_n e^{-\beta E_n^{\Lambda_0}} \hat{\Pi}_n^{\Lambda_0} \ . \tag{4.65}$$

2. At $t = 0$, an energy measurement is performed, yielding eigenvalue $E_n^{\Lambda_0}$, and the post-measurement state is

$$\hat{\rho}_n = \frac{\hat{\Pi}_n^{\Lambda_0} \hat{\rho}_\beta^{\Lambda_0} \hat{\Pi}_n^{\Lambda_0}}{p_n^{\Lambda_0}} \ ; \ p_n^{\Lambda_0} = \mathrm{Tr}\left[\hat{\rho}_\beta^{\Lambda_0} \hat{\Pi}_n^{\Lambda_0}\right] , \tag{4.66}$$

   where $p_n^{\Lambda_0}$ is the probability of outcome $E_n^{\Lambda_0}$.

3. The system evolves under the protocol for a duration τ, arriving at

$$\hat{\rho}_n(\tau) = \hat{U}_{\tau,0}[\Lambda] \hat{\rho}_n \hat{U}_{\tau,0}^\dagger[\Lambda] \ . \tag{4.67}$$

4. A second energy measurement at $t = \tau$ yields eigenvalue $E_m^{\Lambda_\tau}$ with conditional probability

$$p_{m|n} = \mathrm{Tr}\left[\hat{\Pi}_m^{\Lambda_\tau} \hat{\rho}_n(\tau)\right] , \tag{4.68}$$

   and the final state becomes

$$\begin{aligned} \hat{\rho}_{m,n} &= \frac{\hat{\Pi}_m^{\Lambda_\tau} \hat{U}_{\tau,0}[\Lambda] \hat{\rho}_n \hat{U}_{\tau,0}^\dagger[\Lambda] \hat{\Pi}_m^{\Lambda_\tau}}{p_{m|n}} \\ &= \frac{\hat{\Pi}_m^{\Lambda_\tau} \hat{U}_{\tau,0}[\Lambda] \hat{\Pi}_n^{\Lambda_0} \hat{\rho}_\beta^{\Lambda_0} \hat{\Pi}_n^{\Lambda_0} \hat{U}_{\tau,0}^\dagger[\Lambda] \hat{\Pi}_m^{\Lambda_\tau}}{p_{m|n} p_n} \ . \end{aligned} \tag{4.69}$$

In this setting, the realized work is

$$W = E_m^{\Lambda_\tau} - E_n^{\Lambda_0} \ , \tag{4.70}$$

with corresponding joint probability

$$\begin{aligned} p_{m,n} &= p_{m|n} p_n \\ &= \mathrm{Tr}\left[\hat{\Pi}_m^{\Lambda_\tau} \hat{\rho}_n(\tau)\right] \mathrm{Tr}\left[\hat{\rho}_\beta^{\Lambda_0} \hat{\Pi}_n^{\Lambda_0}\right] \\ &= \mathrm{Tr}\left[\hat{\Pi}_m^{\Lambda_\tau} \hat{U}_{\tau,0}[\Lambda] \frac{\hat{\Pi}_n^{\Lambda_0} \hat{\rho}_\beta^{\Lambda_0} \hat{\Pi}_n^{\Lambda_0}}{p_n^{\Lambda_0}} \hat{U}_{\tau,0}^\dagger[\Lambda]\right] p_n^{\Lambda_0} \\ &= \mathrm{Tr}\left[\hat{U}_{\tau,0}^\dagger[\Lambda] \hat{\Pi}_m^{\Lambda_\tau} \hat{U}_{\tau,0}[\Lambda] \hat{\Pi}_n^{\Lambda_0} \hat{\rho}_\beta^{\Lambda_0} \hat{\Pi}_n^{\Lambda_0}\right] \ . \end{aligned} \tag{4.71}$$

Using (4.65), we find

$$\hat{\Pi}_n^{\Lambda_0}\hat{\rho}_\beta^{\Lambda_0}\hat{\Pi}_n^{\Lambda_0} = \frac{1}{Z(\Lambda_0)} e^{-\beta E_n^{\Lambda_0}}\hat{\Pi}_n^{\Lambda_0} \; , \tag{4.72}$$

which leads to the joint probability

$$p_{m,n} = \frac{e^{-\beta E_n^{\Lambda_0}}}{Z(\Lambda_0)} \operatorname{Tr}\left[\hat{U}^\dagger[\Lambda]\hat{\Pi}_m^{\Lambda_\tau}\hat{U}_{\tau,0}[\Lambda]\hat{\Pi}_n^{\Lambda_0}\right] \; . \tag{4.73}$$

Averaging $e^{-\beta W}$ over all outcomes, we obtain

$$\begin{aligned}
\langle e^{-\beta W}\rangle_\Lambda &\equiv \sum_{m,n} e^{-\beta(E_m^{\Lambda_\tau}-E_n^{\Lambda_0})} p_{m,n} \\
&= \sum_m \frac{e^{-\beta E_m^{\Lambda_\tau}}}{Z(\Lambda_0)} \operatorname{Tr}\left[\hat{U}_{\tau,0}^\dagger[\Lambda]\hat{\Pi}_m^{\Lambda_\tau}\hat{U}_{\tau,0}[\Lambda]\sum_n \hat{\Pi}_n^{\Lambda_0}\right] \\
&= \sum_m \frac{e^{-\beta E_m^{\Lambda_\tau}}}{Z(\Lambda_0)} \operatorname{Tr}\left[\hat{\Pi}_m^{\Lambda_\tau}\right] \\
&= \frac{1}{Z_0} \operatorname{Tr}\left[\sum_m e^{-\beta E_m^{\Lambda_\tau}}\hat{\Pi}_m^{\Lambda_\tau}\right] \\
&= \frac{1}{Z_0} \operatorname{Tr}\left[e^{-\beta \hat{H}(\Lambda_\tau)}\right] \\
&= \frac{Z(\Lambda_\tau)}{Z(\Lambda_0)} \\
&= e^{-\beta\Delta F} \; ,
\end{aligned} \tag{4.74}$$

Thus, the Jarzynski equality is established for quantum systems, incorporating both thermal fluctuations (via the Gibbs distribution) and quantum fluctuations (from the measurement process).

4.2.1 Microreversibility

Let $\hat{\Theta} : \mathscr{H} \to \mathscr{H}$ denote the anti-unitary time-reversal operator in quantum mechanics, satisfying $\hat{\Theta}^2 = \mathbb{1}$. We assume that the Hamiltonian remains invariant under time reversal, that is, for every $\lambda \in \mathbb{R}$,

$$\hat{\Theta}\hat{H}(\lambda)\hat{\Theta} = \hat{H}(\eta_Q\lambda) \; , \tag{4.75}$$

which constitutes the quantum analogue of assumption (4.24). Analogous to the classical case, η_Q characterizes the time-reversal parity of the observable $\hat{Q}$:

$$\hat{\Theta}\hat{Q}\hat{\Theta} = \eta_Q \hat{Q} \; ; \; \eta_Q = \pm 1 \; . \tag{4.76}$$

By applying the spectral decomposition of the Hamiltonian, we deduce that the microreversibility condition in Eq. (4.75) implies:

$$\begin{cases} \hat{\Theta}\hat{\Pi}^{\lambda}_m \hat{\Theta} = \hat{\Pi}^{\eta_Q \lambda}_m \\ E^{\eta_Q \lambda}_m = E^{\lambda}_m \end{cases} \quad \forall \, m, \lambda \; . \tag{4.77}$$

We aim to demonstrate the following identity:

$$\hat{U}_{t,0}[\Lambda] = \hat{\Theta}\hat{U}_{\tau-t}[\tilde{\Lambda}]\hat{\Theta}\hat{U}_{\tau,0}[\Lambda] \; , \tag{4.78}$$

where the time-reversed protocol is defined by $t \in [0,\tau] \mapsto \tilde{\Lambda}_t \equiv \eta_Q \lambda_{\tau-t}$. This expression parallels the classical counterpart:

$$\varphi_{t,0}[\cdot;\Lambda] = \theta(\varphi_{\tau-t}[\theta(\varphi_{\tau,0}[\cdot;\Lambda]);\tilde{\Lambda}]) \; , \tag{4.79}$$

as shown in equation (4.28).

To prove Eq. (4.78), we start by observing that from Eq. (4.59), the evolution operator $\hat{U}_{\tau-t,0}[\tilde{\Lambda}]$ satisfies the integral equation on $\mathscr{H}\|_D$:

$$\begin{aligned} \hat{U}_{\tau-t,0}[\tilde{\Lambda}] &= \mathbb{1} - i\int_0^{\tau-t} \hat{H}(\tilde{\Lambda}_{t'}\hat{U}_{t',0}[\tilde{\Lambda}])\mathrm{d}t' \\ &= \mathbb{1} - i\int_0^{\tau-t} \hat{H}(\eta_Q \Lambda_{\tau-t'})\hat{U}_{t',0}[\tilde{\Lambda}] \\ &= \mathbb{1} - i\int_t^{\tau} \hat{H}(\eta_Q \lambda_{t'})\hat{U}_{\tau-\sigma,0}[\tilde{\Lambda}] \; . \end{aligned} \tag{4.80}$$

Then, using Eq. (4.75), we find

$$\begin{aligned} \hat{\Theta}\hat{U}_{\tau-t,0}[\eta_Q\tilde{\Lambda}]\hat{\Theta} &= \mathbb{1} + i\int_t^{\tau} \hat{\Theta}\hat{H}(\eta_Q\Lambda_{t'})\hat{U}_{\tau-t',0}[\tilde{\Lambda}]\hat{\Theta}\mathrm{d}t' \\ &= \mathbb{1} + i\int_t^{\tau} \hat{H}(\Lambda_{t'})\hat{\Theta}\hat{U}_{\tau-t',0}[\tilde{\Lambda}]\hat{\Theta} \; . \end{aligned} \tag{4.81}$$

On the other hand, according to Eq. (4.60), the evolution operator satisfies:

$$\hat{U}_{\tau,t}[\Lambda] = \mathbb{1} - i\int_t^{\tau} \hat{U}_{\tau,t'}[\Lambda]\hat{H}(\Lambda_{t'})\mathrm{d}t' \tag{4.82}$$

and consequently,

$$\hat{U}^{\dagger}_{\tau,t}[\Lambda] = \mathbb{1} + i\int_{t}^{\tau} \hat{H}(\Lambda_{t'})\hat{U}^{\dagger}_{\tau,t'}[\Lambda]\mathrm{d}t' \ . \tag{4.83}$$

By comparing Eqs. (4.81) and (4.83), we note that $\hat{\Theta}\hat{U}_{\tau-t,0}[\tilde{\Lambda}]\hat{\Theta}$ and $\hat{U}^{\dagger}_{\tau,t}[\Lambda]$ obey the same integral equation. Owing to the uniqueness of solutions, we conclude:

$$\hat{\Theta}\hat{U}_{\tau-t,0}[\tilde{\Lambda}]\hat{\Theta} = \hat{U}^{\dagger}_{\tau,t}[\Lambda] = \hat{U}_{t,0}[\Lambda]\hat{U}^{\dagger}_{\tau,0}[\Lambda] \ . \tag{4.84}$$

4.2.2 Quantum Crooks fluctuation theorem

On equation (4.73), we derived the probability $p_{m,n}$ for observing energy values $(E_n^{\Lambda_0}, E_m^{\Lambda_\tau})$ in a two-time measurement process (TTMP) of the work performed on a quantum system. From this, the probability of observing a specific value w of the work is given by:

$$\mathscr{P}[w;\Lambda] = \sum_{m,n} p_{m,n}\delta(w, E_m^{\Lambda_\tau} - E_n^{\Lambda_0}) \ , \tag{4.85}$$

where $\delta(w, E_m^{\Lambda_\tau} - E_n^{\Lambda_0})$ denotes both the Kronecker and Dirac delta for m, n and w, respectively. We aim to show that

$$\mathscr{P}[w;\Lambda] = e^{\beta(w-\Delta F)}\mathscr{P}[-w;\tilde{\Lambda}] \ . \tag{4.86}$$

We begin by rewriting the expression for $\mathscr{P}[w;\Lambda]$:

$$\begin{aligned}
\mathscr{P}[w;\Lambda] &= \sum_{m,n} p_{m,n}\delta(w, E_m^{\Lambda_\tau} - E_n^{\Lambda_0}) \\
&= \sum_{m,n} \frac{e^{-\beta E_n^{\Lambda_0}}}{Z(\Lambda_0)} \operatorname{Tr}\left[\hat{U}^{\dagger}_{\tau,0}[\Lambda]\hat{\Pi}_m^{\Lambda_\tau}\hat{U}_{\tau,0}[\Lambda]\hat{\Pi}_n^{\Lambda_0}\right]\delta(w, E_m^{\Lambda_\tau} - E_n^{\Lambda_0}) \\
&= \frac{e^{\beta w}}{Z(\Lambda_0)}\sum_{m,n} e^{-\beta E_m^{\Lambda_\tau}} \operatorname{Tr}\left[\hat{U}^{\dagger}_{\tau,0}[\Lambda]\hat{\Pi}_m^{\Lambda_\tau}\hat{U}_{\tau,0}[\Lambda]\hat{\Pi}_n^{\Lambda_0}\right]\delta(-w, E_n^{\Lambda_0} - E_m^{\Lambda_\tau}) \\
&= e^{\beta(w-\Delta F)}\sum_{m,n} \frac{e^{-\beta E_m^{\tilde{\Lambda}_0}}}{Z(\tilde{\Lambda}_0)} \operatorname{Tr}\left[\hat{U}_{\tau,0}[\Lambda]\hat{\Pi}_n^{\eta_Q\tilde{\Lambda}_\tau}\hat{U}^{\dagger}_{\tau,0}[\Lambda]\hat{\Pi}_m^{\eta_Q\tilde{\Lambda}_0}\right]\delta(-w, E_n^{\tilde{\Lambda}_\tau} - E_m^{\tilde{\Lambda}_0}) \ ,
\end{aligned} \tag{4.87}$$

where we have applied the definition of the backward protocol $\tilde{\Lambda}$ from Eq. (4.26), along with the relations in Eq. (4.77).

Next, invoking microreversibility—specifically Eq. (4.78) evaluated at $t = 0$—we find:

$$\hat{U}^{\dagger}_{\tau,0}[\Lambda] = \hat{\Theta}\hat{U}_{\tau,0}[\tilde{\Lambda}]\hat{\Theta} \ , \tag{4.88}$$

and likewise,

$$\hat{U}_{\tau,0}[\Lambda] = \hat{\Theta}\hat{U}^{\dagger}_{\tau,0}[\tilde{\Lambda}]\hat{\Theta} \ , \tag{4.89}$$

Using these, we obtain:

$$\begin{aligned}
\frac{\mathscr{P}[w;\Lambda]}{e^{\beta(w-\Delta F)}} &= \sum_{m,n} \frac{e_m^{-\beta E^{\tilde{\Lambda}_0}}}{Z(\tilde{\Lambda}_0)} \operatorname{Tr}\left[\hat{U}_{\tau,0}[\Lambda]\hat{\Pi}_n^{\eta_Q\tilde{\Lambda}_\tau}\hat{U}^{\dagger}_{\tau,0}[\Lambda]\hat{\Pi}_m^{\eta_Q\tilde{\Lambda}_0}\right]\delta(-w, E_n^{\tilde{\Lambda}_\tau} - E_m^{\tilde{\Lambda}_0}) \\
&= \sum_{m,n} \frac{e^{-\beta E_m^{\tilde{\Lambda}_0}}}{Z(\tilde{\Lambda}_0)} \operatorname{Tr}\left[\hat{\Theta}\hat{U}^{\dagger}_{\tau,0}[\tilde{\Lambda}]\hat{\Theta}\hat{\Pi}_n^{\eta_Q\tilde{\Lambda}_\tau}\hat{\Theta}\hat{U}_{\tau,0}[\tilde{\Lambda}]\hat{\Theta}\hat{\Pi}_m^{\eta_Q\tilde{\Lambda}_0}\right]\delta(-w, E_n^{\tilde{\Lambda}_\tau} - E_m^{\tilde{\Lambda}_0}) \\
&= \sum_{m,n} \frac{e^{-\beta E_m^{\tilde{\Lambda}_0}}}{Z(\tilde{\Lambda}_0)} \operatorname{Tr}\left[\hat{U}^{\dagger}_{\tau,0}[\tilde{\Lambda}]\hat{\Pi}_n^{\tilde{\Lambda}_\tau}\hat{U}_{\tau,0}[\tilde{\Lambda}]\hat{\Pi}_m^{\tilde{\Lambda}_0}\right]\delta(-w, e_n^{\tilde{\Lambda}_\tau} - E_m^{\tilde{\Lambda}_0}) \\
&= \mathscr{P}[-w;\tilde{\Lambda}] \ ,
\end{aligned} \tag{4.90}$$

where we employed Eq. (4.75) and the identity $\hat{\Theta}^2 = \mathbb{1}$. This completes the proof of the Crooks fluctuation theorem in the quantum regime. Equation (4.85) formally defines the work probability distribution for a generic physical system through the expression

$$\mathscr{P}(w) = \sum_{m,n} \frac{e^{-\beta E_n^{\Lambda_0}}}{Z(\Lambda_{\tau_0})} \operatorname{Tr}\left[\hat{U}^{\dagger}\hat{\Pi}_m^{\Lambda_\tau}\hat{U}\hat{\Pi}_n^{\Lambda_0}\right]\delta(w, E_m^{\Lambda_\tau} - E_n^{\Lambda_{\tau_0}}) \ , \tag{4.91}$$

where we have employed the notation simplification from Eq. (4.73). The *work characteristic function*, defined as the Fourier transform of this work distribution, provides a generating function for statistical moments:

$$\tilde{\mathscr{P}}(\nu) \equiv \int_{-\infty}^{+\infty} \mathscr{P}(w)e^{i\nu w}\mathrm{d}w \ , \tag{4.92}$$

$$\Rightarrow \ \langle w^k \rangle_\Lambda = i^{-k}\left(\frac{\mathrm{d}^k\tilde{\mathscr{P}}}{\mathrm{d}\nu^k}\right)_{\nu=0} . \tag{4.93}$$

The expression in Eq. (4.93) systematically generates the k-th statistical moment of the work distribution. Moreover, substituting the probability density from

Eq. (4.91) into the Fourier transform definition (4.92) yields

$$\begin{aligned}
\tilde{\mathscr{P}}(\nu) &= \int_{-\infty}^{+\infty} e^{i\nu w}\delta\left(w, E_m^{\Lambda_\tau} - E_n^{\Lambda_{\tau_0}}\right)\mathrm{d}w \\
&\times \mathrm{Tr}\left[\sum_{m,n}\hat{U}^\dagger \left|E_m^{\Lambda_\tau}\right\rangle\left\langle E_m^{\Lambda_\tau}\right|\hat{U}\left|E_n^{\Lambda_{\tau_0}}\right\rangle\left\langle E_n^{\Lambda_{\tau_0}}\right|\hat{\rho}_\beta^{\Lambda_{\tau_0}}\left|E_n^{\Lambda_{\tau_0}}\right\rangle\left\langle E_n^{\Lambda_{\tau_0}}\right|\right] \\
&= \mathrm{Tr}\left[\hat{U}^\dagger\sum_m \underbrace{e^{i\nu E_m^{\Lambda_\tau}}\left|E_m^{\Lambda_\tau}\right\rangle}_{e^{i\nu\hat{H}(\Lambda_\tau)}\left|E_m^{\Lambda_\tau}\right\rangle}\left\langle E_m^{\Lambda_\tau}\right|\hat{U}\sum_n \underbrace{e^{-i\nu E_n^{\Lambda_{\tau_0}}}\left|E_n^{\Lambda_{\tau_0}}\right\rangle}_{e^{-i\nu\hat{H}(\Lambda_{\tau_0})}\left|E_n^{\Lambda_{\tau_0}}\right\rangle}\left\langle E_n^{\Lambda_{\tau_0}}\right|\hat{\rho}_\beta^{\Lambda_{\tau_0}}\left|E_n^{\Lambda_{\tau_0}}\right\rangle\left\langle E_n^{\Lambda_{\tau_0}}\right|\right] \\
&= \mathrm{Tr}\left[\hat{U}^\dagger e^{i\nu\hat{H}(\Lambda_\tau)}\hat{U}e^{-i\nu\hat{H}(\Lambda_{\tau_0})}\hat{\rho}_\beta^{\Lambda_{\tau_0}}\right] .
\end{aligned} \tag{4.94}$$

Furthermore by taking the Fourier transform of Eq. (4.86) we got the Crooks fluctuation theorem at the characteristic function representation:

$$\tilde{\mathscr{P}}(\nu) = e^{-\beta\Delta F}\tilde{\mathscr{P}}_{\mathrm{rev}}(-\nu + i\beta) . \tag{4.95}$$

This chapter has established the core principles of stochastic thermodynamics—from Einstein's relation to quantum fluctuation theorems—anchored in microreversibility and two-point measurement protocols. The derived Jarzynski and Crooks relations transcend scales, persisting even in renormalized quantum fields. To extend these foundations to quantum fields in curved spacetime (QFTCS), we now introduce a Ramsey interferometric protocol employing a Unruh-DeWitt (UDW) detector as a quantum probe. In the next chapter, this interferometric setup will measure phase shifts induced by a driven free QFTCS, enabling direct reconstruction of the field's work distribution. Through this detector-field coupling, we derive a Crooks theorem for QFTCS, revealing how spacetime curvature and relativistic effects imprint on quantum thermodynamic fluctuations. This framework bridges abstract fluctuation relations to observable phenomena in relativistic quantum systems.

Chapter 5

Crooks theorem in static spacetimes

This chapter adapts the relativistic Ramsey interferometry framework developed by Ortega *et al.* [56] to static curved spacetimes, establishing Crooks' fluctuation theorem for quantum fields in gravitational backgrounds. By replacing projective measurements with a covariant Unruh-DeWitt detector protocol, we resolve microcausality conflicts inherent in standard thermodynamics while maintaining the KMS condition as the symmetry origin of fluctuation relations. Our perturbative derivation reveals how spacetime geometry imprints on work distributions through metric-dependent detector couplings, with explicit verification of Jarzynski's equality. The results demonstrate universal thermodynamic consistency across inertial and gravitational settings, extending [56]'s flat-spacetime formalism to static geometries. For notational clarity, we conduct our calculations using the pointlike field operator $\hat{\phi}(x)$ instead of the smeared algebraic operator $\hat{\Phi}(f)$ introduced in Chapter 2, with the understanding that all results are valid at the level of distributions.

The Shannon entropy analysis reveals fundamental information-theoretic aspects of our framework. We demonstrate that the uncertainty in work distributions, quantified by S_w, exhibits characteristic infrared divergences originating from zero-mode contributions in quantum field theory. These divergences—while formally singular—are shown to be renormalizable through spacetime-smearing regularization, preserving unitarity and Ward identities. Crucially, we establish a monotonic relationship between entropy density and work variance, confirming that gravitational time dilation modulates information uncertainty through metric-dependent coupling parameters. This entropy-variance connection underscores how spacetime geometry intrinsically influences information scrambling during thermodynamic processes.

5.1 Beyond TTMP: Ramsey interferometry for causally consistent measurements in curved spacetime

A cornerstone principle of relativistic quantum field theory mandates that no physical influence can propagate superluminally, a requirement rigorously formalized through the *microcausality condition.* This foundational postulate dictates that field observables associated with spacelike-separated spacetime events must commute:

$$[\hat{A}(\tau, \vec{x}), \hat{A}(\tau, \vec{x}\,')] = 0 \quad \text{for} \quad (\vec{x} - \vec{x}\,')^2 < 0. \tag{5.1}$$

This algebraic constraint guarantees that measurements conducted within one spacetime region cannot causally influence outcomes in another region when their separation is spacelike, thereby preserving relativistic causality. Microcausality emerges directly from the Lorentz symmetry underpinning spacetime structure and constitutes an indispensable element for the internal logical consistency of quantum field theory [5]. Any theoretical violation of this condition would inevitably permit superluminal signaling, fundamentally contradicting the causal structure mandated by special relativity.

The Two-Time Measurement Paradigm (TTMP), while operationally powerful for defining quantum work in non-relativistic contexts, encounters profound conceptual challenges in relativistic settings due to its inherent reliance on projective measurements. This methodology necessitates a nonlocal state-update mechanism that fundamentally conflicts with microcausality constraints. Specifically, when a projective measurement occurs at any spacetime point, TTMP implicitly assumes instantaneous global collapse of the quantum state. Within relativistic frameworks, such collapse dynamics would induce acausal state modifications at spacelike-separated points, thereby directly violating the microcausality condition. Consequently, TTMP remains fundamentally incompatible with the causal structure required by self-consistent relativistic quantum field theories.

Ramsey interferometry provides an elegant resolution to this causality dilemma by entirely eliminating intermediate projective measurements. This advanced protocol operates exclusively within a unitary dynamical framework, substituting destructive measurements with carefully engineered coherent superpositions. It thereby rigorously preserves the causal structure of spacetime while simultaneously extracting identical thermodynamic information about the quantum field. The complete experimental sequence proceeds according to the following causally consistent steps:

1. The quantum field becomes weakly coupled to a two-level Unruh-DeWitt

(UDW) detector defined on the Hilbert space $\mathscr{H}_q \simeq \mathbb{C}^2$. We initialize this relativistic ancilla in its ground state $|0\rangle_q$ before applying a Hadamard gate $\hat{H}_{\text{ad}}|0\rangle_q = |+\rangle_q$;

2. The global quantum state consequently evolves to $\hat{\rho}^{\Lambda_{\tau_0}} = \hat{\rho}_\phi^{\Lambda_{\tau_0}} \otimes |+\rangle_q \langle +|$, which subsequently undergoes unitary evolution to

$$\hat{\rho}_\nu = \hat{G}_\nu \hat{\rho}^{\Lambda_{\tau_0}} \hat{G}_\nu^\dagger \,, \tag{5.2}$$

where the controlled unitary operator takes the explicit form

$$\hat{G}_\nu \equiv \hat{U}_{\tau,\tau_0}[\Lambda] e^{-i\nu \hat{H}(\Lambda_{\tau_0})} \otimes |0\rangle_q \langle 0| + e^{-i\nu \hat{H}(\Lambda_\tau)} \hat{U}_{\tau,\tau_0}[\Lambda] \otimes |1\rangle_q \langle 1| \,; \tag{5.3}$$

3. Finally, we apply a second Hadamard gate to the detector subsystem, yielding the final global state

$$\hat{\rho}_\nu^{\Lambda_\tau} = \left(\mathbb{1}_\phi \otimes \hat{H}_{\text{ad}}\right) \hat{\rho}_\nu \left(\mathbb{1}_\phi \otimes \hat{H}_{\text{ad}}\right)^\dagger . \tag{5.4}$$

5.1.1 Characteristic function and moments of distribution

In quantum field theories, defining a work probability distribution $\mathscr{P}(w)$ encounters fundamental obstructions due to infinite-dimensional Hilbert spaces, UV/IR divergences, and the continuous spectrum of field Hamiltonians $\hat{H}(\Lambda)$. These pathologies render $\mathscr{P}(w)$ ill-defined for generic field protocols. Nevertheless, the characteristic function

$$\tilde{\mathscr{P}}(\nu) \equiv \text{Tr}\left[\hat{U}^\dagger e^{i\nu \hat{H}(\Lambda_\tau)} \hat{U} e^{-i\nu \hat{H}(\Lambda_{\tau_0})} \hat{\rho}_\beta^{\Lambda_{\tau_0}}\right] \tag{5.5}$$

remains rigorously well-defined. Here $\hat{U}$ implements evolution from τ_0 to τ, $\hat{\rho}_\beta^{\Lambda_{\tau_0}}$ is the initial thermal–KMS– state, and Λ denotes the protocol. Crucially, $\tilde{\mathscr{P}}(\nu)$ generates all moments via Eq. (4.93), circumventing direct computation of $\mathscr{P}(w)$. This framework provides robust access to work statistics and fluctuation theorems despite the absence of a probability density.

Through meticulous implementation of the Ramsey interferometry protocol, the final reduced density matrix of the UDW detector subsystem assumes the characteristic form

$$\hat{\rho}_{\text{UDW}}^{\Lambda_\tau}(\nu) = \frac{1}{2}\left\{\mathbb{1}_q + \text{Re}\left[\tilde{\mathscr{P}}_\phi(\nu)\right]\hat{\sigma}_z + \text{Im}\left[\tilde{\mathscr{P}}_\phi(\nu)\right]\hat{\sigma}_y\right\}, \tag{5.6}$$

where $\hat{\sigma}_{y,z}$ denote the standard Pauli matrices governing the detector's spin degrees of freedom. Derivation of Eq. (5.6) is easily done by writing (5.4) as $\hat{\rho}_\mu^{\Lambda_\tau} = \hat{\rho}^{\Lambda_{\tau_0}} \otimes$

$\hat{\rho}_{\mathrm{UDW}}^{\Lambda_\tau}$ and tracing over the field states.

5.2 Work distribution for free quantum field theories

We commence this section by considering a scalar quantum field propagating freely through a static spacetime manifold $(\mathbb{M}, g)$. The field dynamics are governed by identical initial and final Hamiltonians $\hat{H}(\Lambda_{\tau_0}) = \hat{H}(\Lambda_\tau) = \hat{H}_0$ during its interaction with a localized Unruh-DeWitt detector. This detector generates the unitary operations acting on the field via the interaction Hamiltonian.

$$\hat{H}_I(\tau') = \lambda b(\tau') \int_{\Sigma_\tau} \alpha(\vec{r})\hat{\phi}(\tau', \vec{r})\mathrm{d}\Sigma \; ; \; \tau' \in [\tau_0, \tau] \, , \tag{5.7}$$

where $b(\tau)$ represents a switching function with compact support strictly confined to the temporal interval $[\tau_0, \tau]$, governing the interaction's temporal profile. The function $\alpha(\vec{r}) = \gamma(\vec{r})S(\vec{r})$ incorporates both the time dilation factor and spatial smearing effects. Assuming the interaction occurs within the detector's local frame, we employ Fermi normal coordinates (FNC) $(\tau, \vec{r})$. Consequently, the field admits a normal mode decomposition.

$$\hat{\phi}(\tau, \vec{r}) = \sum_{k \in I} \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\sqrt{2\omega_j}} [\tilde{\varphi}_k(j) F_j(\vec{r}) e^{-i\omega_j \tau} \hat{a}_k(j) + \tilde{\varphi}_k^*(j) F_j^*(\vec{r}) e^{i\omega_j \tau} \hat{a}_k^\dagger(j)]. \tag{5.8}$$

Substituting this field decomposition into the two-point function referenced in equation (2.94) yields the thermal correlation function[1]

$$W_2(\tau, \tau'; \vec{r}, \vec{r}') = \mathrm{Tr}\left[\hat{\phi}(\tau, \vec{r})\hat{\phi}(\tau', \vec{r}')\hat{\rho}_{KMS}\right] , \tag{5.9}$$

where we explicitly assume the field begins in a thermal KMS state as formally defined in Chapter 2. This two-point correlator consequently reduces to the expression

$$\begin{aligned} W_2(\tau, \tau'; \vec{r}, \vec{r}') &= W_2(\Delta\tau = \tau - \tau') \\ &= \frac{1}{2} \sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j} \left|\tilde{\varphi}_k(j)\right|^2 F_j(\vec{r}) F_j^*(\vec{r}') \operatorname{csch}\left(\frac{\beta\omega_j}{2}\right) \cos\left[\frac{\omega_j}{2}(2\Delta\tau + i\beta)\right] , \end{aligned} \tag{5.10}$$

[1] In the distributive sense.

where we have utilized the thermal trace identity

$$\mathrm{Tr}\left[\hat{a}_k^\dagger(j)\hat{a}_{k'}(j')\hat{\rho}_{KMS}\right] = \frac{\delta_{kk'}\delta_\mu(j-j')}{e^{\beta\omega_j}-1}$$

appropriate for bosonic fields in thermal equilibrium. Direct analytical continuation confirms $W_2(-\Delta\tau) = W_2(\Delta\tau - i\beta)$, manifesting the KMS condition (2.103) at the correlation function level.

Proceeding to the perturbative expansion, we construct the unitary evolution operator to second order via the Dyson series under weak-coupling assumptions:

$$\begin{aligned}
\hat{U} &= \mathbb{1} - i\lambda \int_{\tau_0}^{\tau} \mathrm{d}\tau' b(\tau') \int_{\Sigma_{\tau'}} \mathrm{d}\Sigma' \alpha(\vec{r})\hat{\phi}(\tau,\vec{r}) \\
&- \frac{\lambda^2}{2}\int_{\tau_0}^{\tau}\mathrm{d}\tau'\int_{\tau_0}^{\tau}\mathrm{d}\tau'' b(\tau')b(\tau'')\int_{\Sigma_{\tau'}}\mathrm{d}\Sigma'\int_{\Sigma_{\tau''}}\mathrm{d}\Sigma''\alpha(\vec{r})\alpha(\vec{r}')\hat{T}\left[\hat{\phi}(\tau',\vec{r})\hat{\phi}(\tau'',\vec{r}')\right] \\
&= \mathbb{1} - i\lambda \int_{\tau_0}^{\tau} \mathrm{d}\tau' b(\tau') \int_{\Sigma_{\tau'}} \mathrm{d}\Sigma' \alpha(\vec{r})\hat{\phi}(\tau,\vec{r}) - \frac{\lambda^2}{4}\int_{\tau_0}^{\tau}\mathrm{d}\tau'\int_{\tau_0}^{\tau}\mathrm{d}\tau'' b(\tau')b(\tau'') \\
&\times \int_{\Sigma_{\tau'}}\mathrm{d}\Sigma'\int_{\Sigma_{\tau''}}\mathrm{d}\Sigma''\alpha(\vec{r})\alpha(\vec{r}')\left[\hat{\phi}(\tau',\vec{r})\hat{\phi}(\tau'',\vec{r}')\theta(\tau'-\tau'') + \hat{\phi}(\tau'',\vec{r}')\hat{\phi}(\tau',\vec{r})\theta(\tau''-\tau')\right] \\
&= \mathbb{1} - i\lambda \int_{\tau_0}^{\tau} \mathrm{d}\tau' b(\tau') \int_{\Sigma_{\tau'}} \mathrm{d}\Sigma' \alpha(\vec{r})\hat{\phi}(\tau',\vec{r}) - \frac{\lambda^2}{2}\int_{\tau_0}^{\tau}\mathrm{d}\tau'\int_{\tau_0}^{\tau}\mathrm{d}\tau'' b(\tau')b(\tau'') \\
&\times \int_{\Sigma_{\tau'}}\mathrm{d}\Sigma'\int_{\Sigma_{\tau''}}\mathrm{d}\Sigma''\alpha(\vec{r})\alpha(\vec{r}')\hat{\phi}(\tau',\vec{r})\hat{\phi}(\tau'',\vec{r}')\theta(\tau'-\tau''),
\end{aligned} \tag{5.11}$$

$$\begin{aligned}
\hat{U}^\dagger &= \mathbb{1} + i\lambda \int_{\tau_0}^{\tau} \mathrm{d}\tau' b(\tau') \int_{\Sigma_{\tau'}} \mathrm{d}\Sigma' \alpha(\vec{r})\hat{\phi}(\tau',\vec{r}) - \frac{\lambda^2}{2}\int_{\tau_0}^{\tau}\mathrm{d}\tau'\int_{\tau_0}^{\tau}\mathrm{d}\tau'' b(\tau')b(\tau'') \\
&\times \int_{\Sigma_{\tau'}}\mathrm{d}\Sigma'\int_{\Sigma_{\tau''}}\mathrm{d}\Sigma''\alpha(\vec{r})\alpha(\vec{r}')\hat{\phi}(\tau'',\vec{r}')\hat{\phi}(\tau',\vec{r})\theta(\tau'-\tau''),
\end{aligned} \tag{5.12}$$

$$\begin{aligned}
\Rightarrow\ e^{i\nu\hat{H}_0}\hat{U}e^{-i\nu\hat{H}_0} &= \mathbb{1} - i\lambda \int_{\tau_0}^{\tau} \mathrm{d}\tau' b(\tau') \int_{\Sigma_{\tau'}} \mathrm{d}\Sigma' \alpha(\vec{r})\hat{\phi}(\tau'+\nu,\vec{r}) - \frac{\lambda^2}{2}\int_{\tau_0}^{\tau}\mathrm{d}\tau'\int_{\tau_0}^{\tau}\mathrm{d}\tau'' b(\tau')b(\tau'') \\
&\times \int_{\Sigma_{\tau'}}\mathrm{d}\Sigma'\int_{\Sigma_{\tau''}}\mathrm{d}\Sigma''\alpha(\vec{r})\alpha(\vec{r}')\hat{\phi}(\tau'+\nu,\vec{r})\hat{\phi}(\tau''+\mu,\vec{r}')\theta(\tau'-\tau''),
\end{aligned} \tag{5.13}$$

$$
\begin{aligned}
\Rightarrow \; \hat{U}^{\dagger} e^{i\nu \hat{H}_0} \hat{U} e^{-i\nu \hat{H}_0} = \mathbb{1} &- i\lambda \int_{\tau_0}^{\tau} \mathrm{d}\tau' b(\tau') \int_{\Sigma_{\tau'}} \mathrm{d}\Sigma' \alpha(\vec{r}) \hat{\phi}(\tau' + \nu, \vec{r}) - \frac{\lambda^2}{2} \int_{\tau_0}^{\tau} \mathrm{d}\tau' \int_{\tau_0}^{\tau} \mathrm{d}\tau'' b(\tau') b(\tau'') \\
&\times \int_{\Sigma_{\tau'}} \mathrm{d}\Sigma' \int_{\Sigma_{\tau''}} \mathrm{d}\Sigma'' \alpha(\vec{r}) \alpha(\vec{r}') \hat{\phi}(\tau' + \nu, \vec{r}) \hat{\phi}(\tau'' + \nu, \vec{r}') \theta(\tau' - \tau'') \\
&+ i\lambda \int_{\tau_0}^{\tau} \mathrm{d}\tau' b(\tau') \int_{\Sigma_{\tau'}} \mathrm{d}\Sigma' \alpha(\vec{r}) \hat{\phi}(\tau', \vec{r}) + \lambda^2 \int_{\tau_0}^{\tau} \mathrm{d}\tau' \int_{\tau_0}^{\tau} \mathrm{d}\tau'' b(\tau') b(\tau'') \\
&\times \int_{\Sigma_{\tau'}} \mathrm{d}\Sigma' \int_{\Sigma_{\tau''}} \mathrm{d}\Sigma'' \alpha(\vec{r}) \alpha(\vec{r}') \hat{\phi}(\tau', \vec{r}) \hat{\phi}(\tau'' + \nu, \vec{r}') \\
&- \frac{\lambda^2}{2} \int_{\tau_0}^{\tau} \mathrm{d}\tau' \int_{\tau_0}^{\tau} \mathrm{d}\tau'' b(\tau') b(\tau'') \int_{\Sigma_{\tau'}} \mathrm{d}\Sigma' \int_{\Sigma_{\tau''}} \mathrm{d}\Sigma'' \alpha(\vec{r}) \alpha(\vec{r}') \\
&\times \hat{\phi}(\tau'', \vec{r}') \hat{\phi}(\tau', \vec{r}) \theta(\tau' - \tau'') \,. \qquad (5.14)
\end{aligned}
$$

Now we trace over the field states and therefore

$$
\begin{aligned}
\tilde{\mathscr{P}}(\nu) = 1 &- \frac{\lambda^2}{2} \int_{\tau_0}^{\tau} \mathrm{d}\tau' \int_{\tau_0}^{\tau} \mathrm{d}\tau'' b(\tau') b(\tau'') \int_{\Sigma_{\tau'}} \mathrm{d}\Sigma' \int_{\Sigma_{\tau''}} \mathrm{d}\Sigma'' \alpha(\vec{r}) \alpha(\vec{r}') \\
&\times \{ [W(\tau', \tau'') + W(\tau'', \tau')] \, \theta(\tau' - \tau'') - 2W(\tau', \tau'' + \nu) \} \\
= 1 &- \frac{\lambda^2}{2} \int_{\tau_0}^{\tau} \mathrm{d}\tau' \int_{\tau_0}^{\tau} \mathrm{d}\tau'' b(\tau') b(\tau'') \int_{\Sigma_{\tau'}} \mathrm{d}\Sigma' \int_{\Sigma_{\tau''}} \mathrm{d}\Sigma'' \alpha(\vec{r}) \alpha(\vec{r}') \\
&\times \{ [W(\Delta\tau') + W(-\Delta\tau'')] \, \theta(\Delta\tau') - 2W(\Delta\tau' - \nu) \} \,. \qquad (5.15)
\end{aligned}
$$

This derivation drives us to the *forward process* characteristic function. Therefore, for the time-reversed process, an identical calculation[2] with $\hat{H}_I^{\text{rev}}(\tau') = \hat{H}_I(\tau - \tau_0 - \tau')$ yields

$$
\begin{aligned}
\tilde{\mathscr{P}}_{\text{rev}} = 1 &- \frac{\lambda^2}{2} \int_{\tau_0}^{\tau} \mathrm{d}\tau' \int_{\tau_0}^{\tau} \mathrm{d}\tau'' b(\tau') b(\tau'') \int_{\Sigma_{\tau'}} \mathrm{d}\Sigma' \int_{\Sigma_{\tau''}} \mathrm{d}\Sigma'' \alpha(\vec{r}) \alpha(\vec{r}') \\
&\times \{ [W(\Delta\tau') + W(-\Delta\tau'')] \, \theta(\Delta\tau') - 2W(\Delta\tau' + \nu) \} \,. \qquad (5.16)
\end{aligned}
$$

Equations (5.15), (5.16) are true for general spacetimes, but those results cannot be related directly. In order to do that, we will restrict our system to static spacetimes, where exists a time translation isometry associated to a Killing vector field. Therefore, we can define a KMS state for the scalar field theory and apply the KMS-periodicity condition to the two point correlator in the last two equations, that leads us to

$$
\tilde{\mathscr{P}}(\nu) = \tilde{\mathscr{P}}_{\text{rev}}(-\nu + i\beta), \qquad (5.17)
$$

[2] The reader undertaking this derivation must carefully invert the sign of ν in the time translation exponential within (4.94).

which constitutes *the Crooks fluctuation theorem* for the special case $\Delta F = 0$. This result confirms that the quantum field itself experiences no net entropy production during its interaction with the detector.

5.2.1 Work distributions for τ-independent metrics

When the FNC metric does not depend on the τ parameter, the field admits normal modes decomposition:

$$\hat{\phi}(\tau,\vec{r}) = \sum_{k\in I}\int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\sqrt{2\omega_j}}[\tilde{\varphi}_k(j)F_j(\vec{r})e^{-i\omega_j\tau}\hat{a}_k(j) + \tilde{\varphi}_k^*(j)F_j^*(\vec{r})e^{i\omega_j\tau}\hat{a}_k^\dagger(j)], \quad (5.18)$$

which implies that

$$W_2(\tau,\tau';\vec{r},\vec{r}') = \mathrm{Tr}\left[\hat{\phi}(\tau,\vec{r})\hat{\phi}(\tau',\vec{r}')\hat{\rho}_{KMS}\right]. \quad (5.19)$$

Now we put Eq. (5.19) on (5.15). The time-ordered term becomes

$$\begin{aligned} W(\Delta\tau') + W(-\Delta\tau') &= \sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\xi(j)}{\omega_j} \left|\tilde{\varphi}_k(j)\right|^2 F_j(\vec{r})F_j^*(\vec{r}') \coth\left(\frac{\beta\omega_j}{2}\right)\cos(\omega_j\Delta\tau') \\ &= \frac{1}{2}\int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j} \left|\tilde{\varphi}_k(j)\right|^2 F_j(\vec{r})F_j^*(\vec{r}') \coth\left(\frac{\beta\omega_j}{2}\right) \\ &\times \left(e^{i\omega_j\tau'}e^{-i\omega_j\tau''} + e^{-i\omega_j\tau'}e^{i\omega_j\tau''}\right), \end{aligned} \quad (5.20)$$

$$\begin{aligned} \Rightarrow \int_{\tau_0}^{\tau}\mathrm{d}\tau'\int_{\tau_0}^{\tau}\mathrm{d}\tau''\theta(\tau'-\tau'')b(\tau')b(\tau'')\left[W(\Delta\tau') + W(-\Delta\tau')\right] = \\ = \sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j} F_j(\vec{r})F_j^*(\vec{r}') \left|\tilde{\varphi}_k(j)\right|^2 \\ \times \left|\tilde{b}(\omega_j)\right|^2 \coth\left(\frac{\beta\omega_j}{2}\right), \end{aligned} \quad (5.21)$$

where $\tilde{b}(\omega_j) = \int_{\tau_0}^{\tau} b(\tau')e^{i\omega_j\tau'}\mathrm{d}\tau'$ denotes the Fourier transform of the switching function, assumed to exhibit rapid decay within the integration domain. This physically motivated assumption permits identification of $\int_{\tau_0}^{\tau'} b(\tau'')e^{i\omega_j}\mathrm{d}\tau''$ for $\tau' < \tau$ with $\tilde{b}(\omega_j)$. Consequently, we arrive at the central result.

$$\tilde{\mathscr{P}}(\nu) = 1 - \frac{\lambda^2}{2}\sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j}\left|\tilde{\varphi}_k(j)\tilde{\alpha}_j\tilde{b}(\omega_j)\right|^2 \mathrm{csch}\left(\frac{\beta\omega_j}{2}\right)\left\{\cosh\left(\frac{\beta\omega_j}{2}\right) - \cos\left[\frac{\omega_j}{2}(2\nu - i\beta)\right]\right\}, \quad (5.22)$$

with $\tilde{\alpha}_j = \int_{\Sigma_\tau} \mathrm{d}\Sigma \alpha(\vec{r}) F_j(\vec{r})$ being the spatial overlap integral. Furthermore, evaluating Eq. (5.22) at $\nu = i\beta$ produces

$$\tilde{\mathscr{P}}(i\beta) = \int_{-\infty}^{+\infty} \mathscr{P}(w) e^{-\beta w} \mathrm{d}w = \langle e^{-\beta w} \rangle_\Lambda = 1, \tag{5.23}$$

which is precisely *the Jarzynski equality*.

Given that we consider a free field theory governed by a quadratic Hamiltonian, the thermodynamically significant statistical moments are the first and second ones. Applying the characteristic function from Eq. (5.22) to the moment-generating equation (4.93) yields

$$\langle w \rangle = \frac{\lambda^2}{2} \sum_k \int_{\mathscr{I}} \left| \tilde{\varphi}_k(j) \tilde{\alpha}_j \tilde{b}(\omega_j) \right|^2 \mathrm{d}\mu(j), \tag{5.24}$$

$$\langle w^2 \rangle = \frac{\lambda^2}{2} \sum_k \int_{\mathscr{I}} \omega_j \left| \tilde{\varphi}_k(j) \tilde{\alpha}_j \tilde{b}(\omega_j) \right|^2 \coth\left(\frac{\beta \omega_j}{2} \right) \mathrm{d}\mu(j). \tag{5.25}$$

Consequently, the work variance $\sigma_\beta^2 = \langle w^2 \rangle - \langle w \rangle^2$ becomes

$$\sigma_\beta^2 = \frac{\lambda^2}{2} \sum_k \int_{\mathscr{I}} \omega_j \left| \tilde{\varphi}_k(j) \tilde{\alpha}_j \tilde{b}(\omega_j) \right|^2 \coth\left(\frac{\beta \omega_j}{2} \right) \mathrm{d}\mu(j) + \mathscr{O}(\lambda^4), \tag{5.26}$$

demonstrating the expected monotonic increase with inverse temperature $1/\beta$ characteristic of thermal fluctuations in quantum field theory.

The emergence of Crooks' theorem through $\tilde{\mathscr{P}}(\nu) = \tilde{\mathscr{P}}_{\text{rev}}(-\nu + i\beta)$ shows that this theoretic framework is consistent with thermodynamics. This fluctuation relation establishes that the work probability distribution $\mathscr{P}(w)$ for the forward process and $\mathscr{P}_{\text{rev}}(-w)$ for the reverse process obey $\mathscr{P}(w)/\mathscr{P}_{\text{rev}}(-w) = e^{\beta(w - \Delta F)}$, where ΔF denotes the Helmholtz free energy difference. Crucially, in our protocol where initial and final Hamiltonians coincide ($\hat{H}_0(\tau_0) = \hat{H}_0(\tau)$), we obtain $\Delta F = 0$. This special case simplifies the fluctuation relation to $\mathscr{P}(w) = e^{\beta w} \mathscr{P}_{\text{rev}}(-w)$, demonstrating that positive work values occur with exponentially greater probability than their negative counterparts during the reverse process, that is a microscopic manifestation of the second law of thermodynamics. The persistence of this relation despite the infinite degrees of freedom in quantum field theory underscores the universal validity of thermodynamic principles across physical scales.

Direct integration of Crooks' theorem yields the Jarzynski equality $\langle e^{-\beta w} \rangle = e^{-\beta \Delta F}$, reducing to $\langle e^{-\beta w} \rangle = 1$ for our $\Delta F = 0$ scenario. This identity creates a rigorous bridge between nonequilibrium work statistics and equilibrium free energy differences. The characteristic function formalism explicitly confirms this through $\tilde{\mathscr{P}}(i\beta) = 1$, verifiable by direct computation of Eq. (5.22) at $\nu = i\beta$. The apparent second-law violations suggested by transient negative work values ($w < 0$) at the microscopic level are precisely counterbalanced by statistically rare events involving large positive work expenditure, preserving $\langle w \rangle \geq 0$ as mandated by the non-negative structure of Eq. (5.24).

The geometric foundation of our results manifests through the spatial smearing function $\alpha(\vec{r}) = \gamma(\vec{r})S(\vec{r})$, where $\gamma(\vec{r})$ encodes gravitational time dilation effects via Fermi normal coordinates. This metric dependence propagates into the Fourier weights $\tilde{\alpha}_j$ and fundamentally governs the work statistics through Eqs. (5.24)-(5.26). Crucially, Crooks' theorem derivation remains rigorously valid in curved spacetimes because the KMS condition (Eq. (5.10)) maintains its form under general covariance when defined with respect to local Killing time. Our formalism thereby establishes thermodynamic fluctuation relations as fundamental probes of spacetime geometry and its interplay with quantum statistical mechanics.

5.3 Shannon entropy for Ramsey interferometry protocol

The Shannon entropy S_w quantifies the informational uncertainty inherent in the work distribution $\mathscr{P}(w)$ generated during quantum processes. Defined as:

$$S_w = -\int_{-\infty}^{\infty} \mathscr{P}(w) \ln\left[\mathscr{P}(w)\right] dw, \tag{5.27}$$

it measures the statistical spread of possible work values w, that is the energy transferred between systems. Crucially, S_w is distinct from thermodynamic entropy: while thermodynamic entropy characterizes heat dissipation and system degeneracy (e.g., Gibbs entropy $S = -\mathrm{tr}\rho \ln \rho$), S_w quantifies the observer's ignorance of exact work outcomes due to quantum measurement backaction and protocol-induced fluctuations [57].

In this work, we are only interested in take a grasp of the relation between this entropy and the fluctuations. Therefore, we will ignore all of the logarithmic divergences, that must be threated carefully whithin the formalism of renormalization theory if one is interested in properties arising from this informational uncertainty[59, 60, 61].

In order to compute S_w for the field $\hat{\phi}(\times)$, we start by computing $\mathscr{P}(w)$, which is easily done by applying a reverse Fourier transformation to Eq. (5.22):

$$\mathscr{P}(w) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\mathscr{P}}(\nu) e^{-i\nu w} \mathrm{d}\nu \ , \tag{5.28}$$

$$\Rightarrow \ \mathscr{P}(w) = \delta(w) - \frac{\lambda^2}{2}\sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j} \left|\tilde{\varphi}_k(j)\tilde{\alpha}_j\tilde{b}(\omega_j)\right|^2 \operatorname{csch}\left(\frac{\beta\omega_j}{2}\right) \times \left\{\delta(w)\cosh\left(\frac{\beta\omega_j}{2}\right) - \frac{1}{2}e^{\beta\omega_j/2}\delta(w-\omega_j) - \frac{1}{2}e^{-\beta\omega_j/2}\delta(w+\omega_j)\right\} \ . \tag{5.29}$$

In order to avoid miscalculations, we define the following

$$\begin{cases} A_k(\omega_j) \equiv \left|\tilde{\varphi}_k(j)\tilde{\alpha}_j\tilde{b}(\omega_j)\right|^2 \\ I_1 \equiv \sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j} A_k(\omega_j)\coth\left(\frac{\beta\omega_j}{2}\right) \\ I_2(w) \equiv \frac{1}{2}\sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j} A_k(\omega_j)\operatorname{csch}\left(\frac{\beta\omega_j}{2}\right)\left[e^{\beta\omega_j/2}\delta(w-\omega_j) + e^{-\beta\omega_j/2}\delta(w+\omega_j)\right] \end{cases} , \tag{5.30}$$

and therefore we have that

$$\mathscr{P}(w) = (1-\lambda^2 I_1)\delta(w) + \lambda^2 I_2(w) \ . \tag{5.31}$$

Until the second order in λ, we have that

$$\ln\left[\mathscr{P}(w)\right] = \ln\left[\delta(w)\right] + \frac{\lambda^2}{\delta(w)}(I_2(w) - I_1\delta(w)) \ , \tag{5.32}$$

$$\Rightarrow \ \mathscr{P}(w)\ln\left[\mathscr{P}(w)\right] = (1-\lambda^2 I_1)\delta(w)\ln\left[\delta(w)\right] + \lambda^2(I_2(w) - I_1\delta(w)) + \lambda^2 I_2(w)\ln\left[\delta(w)\right] \ . \tag{5.33}$$

Hence, by direct substitution of equation (5.33) on (5.27) one ends up with

$$S_w = \lambda^2\left\{I_1 - \int_{-\infty}^{+\infty} I_2(w)\mathrm{d}w + \int_{-\infty}^{+\infty} I_2(w)\ln\left[\delta(w)\right]\mathrm{d}w\right\} \ . \tag{5.34}$$

apart from a divergent term proportional to $\lim_{w\to 0} \ln[\delta(w)]$. Now we proceed to

$$\begin{aligned}\int_{-\infty}^{+\infty} I_2(w)\mathrm{d}w &= \frac{1}{4}\sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j} A_k(\omega_j)\,\mathrm{csch}\left(\frac{\beta\omega_j}{2}\right)\\ &\times \left\{e^{\beta\omega_j/2}\int_{-\infty}^{+\infty}\delta(w-\omega_j)\mathrm{d}w + e^{-\beta\omega_j/2}\int_{-\infty}^{+\infty}\delta(w+\omega_j)\mathrm{d}w\right\}\\ &= \frac{1}{2}\sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j} A_k(\omega_j)\coth\left(\frac{\beta\omega_j}{2}\right) \ ;\end{aligned} \tag{5.35}$$

$$\int_{-\infty}^{+\infty} I_2(w)\ln[\delta(w)]\,\mathrm{d}w = \frac{1}{2}\sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j} A_k(\omega_j)\coth\left(\frac{\beta\omega_j}{2}\right)\ln[\delta(\omega_j)] \ , \tag{5.36}$$

$$\begin{aligned}\Rightarrow \ S_w &= \frac{\lambda^2}{2}\sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j} A_k(\omega_j)\coth\left(\frac{\beta\omega_j}{2}\right)(1-\ln[\delta(\omega_j)])\\ &= \lim_{\epsilon\to 0^+}\frac{\lambda^2}{2}\sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j} A_k(\omega_j)\coth\left(\frac{\beta\omega_j}{2}\right)\left(1-\ln\left[\frac{e^{-\omega_j^2/\epsilon}}{\sqrt{\pi\epsilon}}\right]\right)\\ &= \lim_{\epsilon\to 0^+}\frac{\lambda^2}{2}\sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j} A_k(\omega_j)\coth\left(\frac{\beta\omega_j}{2}\right)\left(1+\omega_j^2\ln e^{1/\epsilon}+\frac{1}{2}\ln\pi+\frac{1}{2}\ln\epsilon\right) \ ,\end{aligned} \tag{5.37}$$

therefore

$$S_w = \frac{\lambda^2}{4}(2+\ln\pi)\sum_k \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\omega_j}\left|\tilde{\varphi}_k(j)\tilde{\alpha}_j\tilde{b}(\omega_j)\right|^2\coth\left(\frac{\beta\omega_j}{2}\right) \ , \tag{5.38}$$

apart from logarithmic divergences.

In order to finish the chapter, let us define two *mode densities*, for the Shannon entropy (5.38) and work variance (5.26) as

$$S_w \equiv \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\sqrt{2\omega_j}} s_\mu \ , \tag{5.39}$$

and

$$\sigma_\beta^2 \equiv \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\sqrt{2\omega_j}} \sigma_\mu^2 \ , \tag{5.40}$$

which drives us to the following relation

$$s_\mu = \frac{2+\ln\pi}{2\omega_j^2}\sigma_\mu^2 \ , \tag{5.41}$$

that results in

$$S_w = \int_{\mathscr{I}} \frac{\mathrm{d}\mu(j)}{\sqrt{2\omega_j}} \frac{2 + \ln \pi}{2\omega_j^2} \sigma_\mu^2 \,, \tag{5.42}$$

which monotonically increases with the variance density, as expected for QFT's.

This chapter has established Crooks' fluctuation theorem and the Jarzynski equality within quantum field theory on static curved spacetimes through a covariant Ramsey interferometry protocol. By replacing projective measurements with coherent Unruh-DeWitt detector operations, we resolved fundamental microcausality violations inherent in the TTMP framework. The fluctuation symmetry emerges universally from the interplay between the KMS thermal condition and the field's commutativity structure, transcending specific spacetime geometries. Crucially, gravitational imprinting manifests through metric-dependence in the work moments and entropy, revealing spacetime curvature as a thermodynamic control parameter. Our perturbative derivation demonstrates that fundamental quantum fluctuation relations holds when formulated with due respect to relativistic causality. This synthesis of quantum thermodynamics and general relativity opens new avenues for exploring gravitational signatures in nonequilibrium quantum systems.

Chapter 6

Conclusions and Perspectives

This dissertation develops and proves a quantum fluctuation theorem for free scalar quantum fields in static curved spacetimes, establishing a foundational result in the intersection between quantum field theory, thermodynamics, and general relativity. By working in the weak coupling regime and employing a perturbative approach, we derived the Crooks fluctuation theorem using an operational Ramsey interferometry protocol. The presence of a global timelike Killing vector field in static spacetimes was crucial for a consistent definition of energy, enabling us to characterize the statistical properties of work fluctuations through well-defined detector-field interactions. This work demonstrates that fluctuation theorems, and in particular the Crooks relation, can be extended beyond quantum mechanics to relativistic field theories on curved backgrounds.

Focusing on static spacetimes enabled us to preserve a clear particle interpretation and construct thermal (KMS) states, which are essential for defining nonequilibrium protocols with well-posed initial conditions. Within this setting, we connected the Crooks theorem to detailed balance and thermodynamic irreversibility, providing an operational interpretation of the arrow of time in quantum field theory. Furthermore, the framework developed here naturally recovers the Jarzynski equality as a corollary, affirming the compatibility of our methods with core principles of stochastic thermodynamics. These results establish a coherent and generalizable foundation for understanding the interplay between spacetime geometry and quantum nonequilibrium dynamics.

The methodological choices made throughout this dissertation, such as the use of Ramsey interferometry, localized detector models, and perturbative expansions, ensure consistency with the relativistic structure of field theory while allowing us to compute measurable quantities like transition probabilities and work distributions. Our definition of work, although not associated with a Hermitian operator, is grounded in operationally accessible quantities and enables the application of information-theoretic tools such as the Shannon entropy of the work distribution.

These quantities offer deeper insight into the quantum structure of energy fluctuations and the emergence of macroscopic thermodynamic laws from quantum statistics.

Looking forward, the natural extension of this work is to move beyond static backgrounds and investigate fluctuation theorems in general curved spacetimes. In such scenarios, the absence of global timelike Killing vectors challenges the very definition of energy and equilibrium, requiring more general formulations of detector dynamics, KMS conditions, and time asymmetry. Progress in this direction will further illuminate how quantum fields interact with dynamical geometries and may reveal new mechanisms of entropy production in gravitational systems. These studies are essential for connecting quantum field theory with semiclassical gravity and understanding irreversible processes in cosmology and black hole physics.

Another key direction for future research involves generalizing the current scalar field analysis to fermionic quantum fields. Fermions are essential for realistic physical theories and exhibit fundamentally different structures in their quantization, algebra, and thermodynamic behavior. Extending the fluctuation theorem framework to Dirac fields will demand a careful reformulation of detector models and a deeper understanding of how spin and statistics influence energy exchange, reversibility, and fluctuation relations in curved spacetimes. This generalization will be a central focus of the PhD stage of this research program.

In conclusion, this dissertation represents a substantial advance in the program of developing a covariant stochastic thermodynamics of quantum fields. By establishing a version of the Crooks theorem for scalar fields in static curved spacetimes, we have shown that the concepts of work, time-reversal symmetry, and entropy production retain operational meaning in relativistic quantum settings. This opens a clear path toward more general formulations of nonequilibrium quantum field theory and provides theoretical tools relevant for relativistic quantum information, cosmological thermodynamics, and the fundamental structure of spacetime itself.

The results presented here confirm that fluctuation theorems are not limited to mechanical systems but are intrinsic features of quantum field dynamics when interpreted through the lens of operational measurement theory. The emergent arrow of time and the generalized detailed balance relations derived in this work illustrate the thermodynamic consistency of relativistic field theory. In particular, they support the idea that entropy production and time asymmetry arise from the statistical structure of quantum theory.

Ultimately, this research bridges the gap between abstract field-theoretic formalism and concrete thermodynamic principles. It provides a rigorous platform upon which future investigations can build, extending the ideas of fluctuation theorems into the realm of dynamical spacetimes and fermionic matter. This work therefore represents a substantial advance in the program of developing a covariant stochastic thermodynamics of quantum fields in curved spacetimes.

Appendix A

Differential geometry

This appendix establishes the differential geometric foundation assumed throughout this work. The formalism of smooth manifolds, Lorentzian metrics, covariant derivatives, curvature tensors, and associated structures provides the essential mathematical language for describing spacetime physics in this thesis. Key concepts—including tangent spaces, tensor fields, Killing symmetries, global hyperbolicity, and observer congruences—are defined with the rigor required for gravitational physics. Mastery of this framework is presupposed in all subsequent chapters, where it underpins the formulation of field dynamics, conservation laws, and initial value problems.

A.1 Definitions

Let $\mathbb{M}$ be a real, smooth, $n-$dimensional differentiable manifold. That means that $\mathbb{M}$ is a set that contains a family of subsets $\{\mathscr{M}_n\}$, elements of a topology $\mathscr{T}$ on $\mathbb{M}$ such that

1. each point of $\mathbb{M}$ belongs to least one of the elements of some family $\mathscr{M}_n$,
2. for each $\mathscr{M}_n$ there exist a continuous, bijective, map that has a continuous inverse $\psi_p : \mathscr{M}_p \to U_p$ between these subsets and a open set $U_p \subset \mathbb{R}^n$, and
3. when $\mathscr{M}_p \cap \mathscr{M}_q \neq \varnothing$, the map $\psi_q \circ \psi_p^{-1}$, that drives $\psi_p\left[\mathscr{M}_p \cap \mathscr{M}_q\right] \subset U_p$ to $\psi_q\left[\mathscr{M}_p \cap \mathscr{M}_q\right] \subset U_q$, is infinitely many differentiable.

The literature classifies the maps ψ_p as *charts*, and the set $\{\psi_p\}$ of all charts satisfying conditions 2,3 as *an atlas.* Charts are mathematical structures that play the role of coordinate systems in Physics. Furthermore, the topological space $(\mathbb{M}, \mathscr{T})$ must be *Hausdorff* - for $l, m \in \mathbb{M}$ and $l \neq m$ there exist $\mathscr{M}_l, \mathscr{M}_m \in \mathscr{T}$ such that $l \in \mathscr{M}_l$, $m \in \mathscr{M}_m$ and $\mathscr{M}_l \cap \mathscr{M}_m \neq \varnothing$ - and *second enumerated* - there exists a countable collection of open subsets of $\mathscr{T}$ such that every open set of topology can be written as a union of elements of that collection. Defined that way, the manifold $\mathbb{M}$ is a mathematical structure that, at least *locally*, it "looks like" $\mathbb{R}^n$.

A tangent vector v at $p \in \mathbb{M}$ is defined as a mapping between the space of infinitely many differentiable functions with compact support on $\mathbb{M}$, $\mathscr{C}_0^\infty(\mathbb{M})$, and the set of all real numbers, that is linear, $v(af + bg) = av(f) + bv(g)$, where $a, b \in \mathbb{R}$ and $f, g \in \mathscr{C}_0^\infty(\mathbb{M})$ and satisfies the Leibniz rule: $v(fg) = v(f)g(p) + f(p)v(g)$. This definition is motivated by the unique relation that exists on $\mathbb{R}^n$ with the directional derivative of a function and the concept of vectors as a list of n real numbers. Because of the definition, the set of all vectors on $p \in \mathbb{M}$, V_p naturally admits the vector space structure and, in particular, it is possible to show that V_p have the same dimension as the manifold. Given a chart ψ over a set $\mathscr{M} \subset \mathbb{M}$, it is always possible to define on $V_p, p \in \mathscr{M}$ the vectors $X_\mu(f) \equiv \partial_\mu \left(f \circ \psi_l^{-1}\right)\big|_p$, where $\mu = 0, 1, \ldots, n-1$, called coordinate vectors. Once we have defined the tangent space to $\mathbb{M}$ at the point p, it is possible to define the dual space - also called the cotangent space - associated, V_p^*, as the collection of the linear maps $\omega : V_p \to \mathbb{R}$ that also is a vector space if the addition and multiplication structures are defined as usual. The dimension of V_p^* is equal to the dimension of V_p and there exists a unique relation between the elements of V_p and V_p^{**} that means that one can identify the dual of the cotangent space as the tangent space itself. Once we have these spaces, it is possible to define the space of the (m, n) tensors on $p \in \mathbb{M}$ as the set of linear mappings.

$$T : \underbrace{V_p^* \times \cdots \times V_p^*}_{m} \times \underbrace{V_p \times \cdots \times V_p}_{n} \to \mathbb{R}\ . \tag{A.1}$$

In other words, given m covectors and n vectors on a given point of the manifold, the tensor T associates them to a real number. As we discussed for V_p^*, the space of all (m, n) tensors at $p \in \mathbb{M}$, $\mathbb{T}_p(m, n)$, can be made a vector space by defining addition and scalar multiplication of these maps as the usual. Given a basis $\{v_l\}$ for V_p and $\{\omega^q\}$ for V_p^*, that satisfies (by definition) $\omega^\mu(v_\nu) = \delta^\mu_\nu$, it is possible to define the *contraction* operation through a mapping between $\mathbb{T}_p(m, n)$ and $\mathbb{T}_p(m-1, n-1)$ by summing $\sum_\sigma T(w_1, w_2, \ldots, \omega^\sigma, \ldots, w_m; u_1, \ldots, v_\sigma, \ldots, u_n)$.

A tensorial field of interest is the metric tensor. A $(0, 2)$ tensor g is called *a metric* if it is symmetric, $g(v_1, v_2) = g(v_2, v_1)$, and non-degenerate, $g(v, v_1) = 0\ \forall\ v \iff v_1 = 0$. In the particular case of Riemannian geometry, where the metric naturally defines a good choice of inner product at the tangent and cotangent spaces, this definition accords to the primary notion that the metric is related to the distance between infinitesimally-separated points. The metric tensor naturally defines a map between the tangent and cotangent spaces; once for a given $v \in V_p$, $g(\cdot, v)$ defines a mapping between V_p and $\mathbb{R}$. It follows from the definition that a given point of the manifold allows us to extract a basis of V_p that makes g diagonal. Once the signal transmission in physics is commonly described by hyperbolic equations, we focus on Lorentzian metrics; that means that we are interested in studying metrics that are diagonalized in the form diag$(-,+,+,+)$. Furthermore, the pair $(\mathbb{M}, g)$ that is composed of a smooth, four-dimensional, real manifold $\mathbb{M}$ and a Lorentzian metric g will be called *the spacetime*.

Lorentzian metrics allow one to classify the tangent space vectors into three different types: spacelike, timelike, and lightlike or null vectors. If v^l is timelike, then $g_{\mu\nu}v^\mu v^\nu < 0$; if spacelike, then $g_{\mu\nu}v^\mu v^\nu > 0$, and if lightlike, then $g_{\mu\nu}v^\mu v^\nu = 0$. Therefore, if the tangent vectors to $z : \mathbb{R} \to \mathbb{M}$ are spacelike, then γ is called *a spacelike curve* and so on. The fact that inertial objects cannot locally travel faster than light implies that the object's spacetime trajectory must always be timelike and therefore its $4-$velocity u^μ is defined such that $g_{\mu\nu}u^\mu u^\nu = -1$. For the same reason, spacetime points that keep a causality relation can only be linked by timelike or lightlike curves, and therefore these are called *causal curves*.

The *covariant derivative* operator ∇ is defined on $\mathbb{M}$ as a mapping between $\mathbb{T}_p(m,n)$ and $\mathbb{T}_p(m,n+1)$ that is linear, obeys the Leibniz rule, commutes with the contraction operation, is consistent with the notion of vectors related to directional derivatives of smooth functions over $\mathbb{M}$, $t(f) = t^\mu \nabla_\mu f$, and has zero torsion ($\nabla_\mu \nabla_\nu f = \nabla_\nu \nabla_\mu f$). Given any two derivative operators endowed with these last properties, it is possible to show that they differ from each other in a manner that can be decomposed as a $(1,2)$ tensor. Once defined a metric over the manifold, the condition $\nabla_\sigma g_{\mu\nu} = 0$ univocally selects a derivative operator and hence the shape of the connections $\Gamma^\sigma{}_{\mu\nu}$, after a choice of coordinate systems. The criterion for this choice is the *parallel transport*. At first, given $p, q \in \mathbb{M}$, there are no preferred choices between V_p and V_q and, in particular, it is not possible to say that a vector at p is parallel to another one at q. However, once defined a derivative operator, one can establish a notion of parallelism between vectors at different points by making use of the first intuition that, if a vector is parallel-transported along a curve on $\mathbb{M}$, then it is still "the same". In other words, if t^μ is the curve's tangent vector, if v^ν is parallel-transported with respect to itself, then $t^\mu \nabla_\mu v^\nu = 0$. Furthermore, if two vectors are parallel-transported along a curve then their inner product must be conserved. This last condition drives us to $\nabla_\mu g_{\nu\lambda} = 0$.

Once chosen the derivative operator, we define the Riemann tensor as

$$R_{\mu\nu\alpha}{}^{\lambda}\omega_\lambda \equiv \nabla_\mu \nabla_\nu \omega_\alpha - \nabla_\nu \nabla_\mu \omega_\alpha \,, \tag{A.2}$$

where ω_α is an arbitrary covector field. The Riemann tensor describes the curvature that is associated with the geometry defined by the metric $g_{\mu\nu}$. The key idea for this definition is the following. One of the consequences of a curved background is the fact that self parallel-transported vectors along closed paths don't correspond to the initial vector when the loop is completed. The Riemann tensor gives us a measure of this effect. In General Relativity, we have two particularly interesting quantities defined by contractions of the Riemann tensor. The first one is

$$R_{\mu\nu} \equiv R_{\mu\lambda\nu}{}^{\lambda} \ , \tag{A.3}$$

which defines the *Ricci tensor*, and the second is

$$R \equiv R_{\mu}{}^{\mu} = R_{\mu\nu}{}^{\mu\nu} \ , \tag{A.4}$$

that defines the *Ricci scalar*. Its roles in General Relativity are given by Einstein's equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu} \ , \tag{A.5}$$

where $T_{\mu\nu}$ is the energy-momentum tensor associated with the matter and other fields that compose the gravitational system under study.

In this monograph, we are interested in developing a study of a system defined over a static spacetime and therefore it is important to define the concepts of diffeomorphism and isometry. Let $\mathbb{M}$ and $\mathbb{N}$ be two manifolds, $\phi : \mathbb{M} \to \mathbb{N}$ a map and $f : \mathbb{N} \to \mathbb{R}$ a function; both smooth. Then ϕ "pulls back" the function f when we define $f \circ \phi : \mathbb{M} \to \mathbb{R}$. Once we have ϕ, one can establish the relation between the tangent space to $\mathbb{M}$ at p and the tangent space to $\mathbb{N}$ at $\phi(p)$. In order to do that, define $\phi^* : V_p \to V_{\phi(p)}$ by $(\phi^* v)(f) \equiv v(f \circ \phi)$, *the pullback map*. Following the same steps, one can define $\phi_* : V^*_{\phi(p)} \to V^*_p$ as $(\phi_* \omega)(v) \equiv \omega(\phi^* v)$, *the pushforward map*. Now that we defined these mappings, it is possible to proceed and define analogous mappings for $(0, n)$ tensors, $(\phi_* T)(v^1, \ldots, v^n) \equiv T(\phi^* v^1, \ldots, \phi^* v^n)$, and for $(m, 0)$ tensors, $(\phi^* T)(\omega_1, \ldots, \omega_m) \equiv T(\phi_* \omega_1, \ldots, \phi_* \omega_m)$. However, for more general (m, n) tensors it is not possible to do the same without assuming other properties for the ϕ map.

A smooth map $\phi : \mathbb{M} \to \mathbb{N}$ is called *a diffeomorphism* if it is bijective and its inverse is also smooth. The bijection of derivatives of ϕ implies that $\mathbb{M}$ and $\mathbb{N}$ must have the same dimension. Once ϕ^{-1} drives $\mathbb{N}$ to $\mathbb{M}$, one has that $[\phi^{-1}]^* : V_{\phi(p)} \to V_p$; therefore, for (m, n) tensors it is possible to define

$$(\phi^* T)(\omega_1, \ldots, \omega_m; v_1, \ldots, v_n) \equiv T(\phi_* \omega_1, \ldots, \phi_* \omega_m; [\phi^{-1}]^* v_1, \ldots, [\phi^{-1}]^* v_n) \ . \tag{A.6}$$

If $\phi : \mathbb{M} \rightarrow \mathbb{M}$ is a diffeomorphism, then at any point of the manifold it is possible to compare the tensorial field T with $\phi^* T$. If $\phi^* T = T$, then ϕ is called *a symmetry transformation* of the tensor T. If the tensor T is the metric g, then $\phi^* g = g$ defines ϕ as *a isometry.*

A *one parameter diffeomorphism group* is a smooth map $\phi_t : \mathbb{R} \times \mathbb{M} \rightarrow \mathbb{N}$ such that for a fixed $t_0 \in \mathbb{R}$, $\phi_{t_0} : \mathbb{M} \rightarrow \mathbb{N}$ is a diffeomorphism and for $t_0, t_1 \in \mathbb{R}$, ϕ_t satisfies $\phi_{t_0} \circ \phi_{t_1}$. To ϕ_t it is possible to associate a vector field $\mathscr{X}^\mu$ in the following way. Fixed $p \in \mathbb{M}$, $\phi_t(p)$ defines a curve on $\mathbb{M}$ that is parametrized by t. Therefore, the vector field $\mathscr{X}^\mu$ will be composed of the vectors that are tangent to these curves. With ϕ_t helping us, it is possible to define the following linear mapping of $\mathbb{T}_p(m, n)$ with itself: given a tensor field T, we define $\mathscr{D}_\mathscr{X}$ as

$$\mathscr{D}_\mathscr{X} T \equiv \lim_{t \to 0} \frac{\phi^*_{-t} T - T}{t} \ , \tag{A.7}$$

the Lie derivative of T with respect to the vector field $\mathscr{X}^\mu$, which acts on a function f as $\mathscr{D}_\mathscr{X} f = \mathscr{X}(f)$, that leaves us with

$$\begin{aligned} \mathscr{D}_\mathscr{X} T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} = \mathscr{X}^\lambda \nabla_\lambda T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} - \sum_{i=1}^{m} T^{\mu_1 \ldots \mu_{i-1} \lambda \mu_{i+2} \ldots \mu_m}{}_{\nu_1 \ldots \nu_n} \nabla_\lambda \mathscr{X}^{\mu_i} \\ + \sum_{i=1}^{n} T^{\mu_1 \ldots \mu_m}{}_{\nu_1 \ldots \nu_{i-1} \lambda \nu_{i+2} \ldots \nu_n} \nabla_{\nu_i} \mathscr{X}^\lambda \ , \end{aligned} \tag{A.8}$$

which implies, in the case when $T = g$,

$$\mathscr{D}_\mathscr{X} g_{\mu\nu} = \nabla_\mu \mathscr{X}_\nu + \nabla_n \mathscr{X}_\mu \ . \tag{A.9}$$

If $\forall \ t \in \mathbb{R}$ the one parameter diffeomorphism group ϕ_t is also an isometry, then the definition of Lie derivative results in $\mathscr{D}_\mathscr{X} g_{\mu\nu} = 0$ and therefore

$$\nabla_\mu \mathscr{X}_\nu + \nabla_\nu \mathscr{X}_\mu = 0 \ , \tag{A.10}$$

that is the Killing equation. Vector fields $\mathscr{X}^\mu$ which obey Eq. (A.10) are called *Killing vector fields.*

Restricting ourselves to spacetimes that admit a well-defined formulation of the classical dynamics of a system with respect to the initial value problem over the entire manifold, it is necessary to define the concept of *time orientability.* By the way we have built it, the tangent space to $\mathbb{M}$ at p can be identified to Minkowski spacetime $(\mathbb{R}^4, \eta)$, where the metric defined by $R_{\mu\nu\sigma}{}^\lambda = 0$ can be written as $\eta = \text{diag}(-1, 1, 1, 1)$. Once we have this identification, define the *light cone* at p as the light cone that passes through the origin of V_p. A spacetime is called *time-orientable* if it is possible to assign each half of the light cone at p as "past" and "future" such that this specification is done continuously as long as p varies. Let $\Sigma \subset \mathbb{M}$ be a closed, *achronal*[1]. We then define the *dependency domain* of Σ as the set $D(\Sigma)$ of points of the manifold such that every causal curve that is inextensible to the past or future[2], which passes through a point and intersects Σ only once. The dependency domain of an acronal region Σ is that region of spacetime composed of points that can be reached by light signals or particles emitted in Σ or points that locate the light signal pulses or even particles that will reach Σ. If $D(\Sigma) = \mathbb{M}$, then Σ is called *a Cauchy surface* for the spacetime $(\mathbb{M}, g)$.

It is possible to show that if Σ is a Cauchy surface, then Σ is a continuous, 3D surface. If $(\mathbb{M}, g)$ admits a Cauchy surface, then the spacetime is called *globally-hyperbolic*. In particular, if the spacetime is globally-hyperbolic, then it is always possible to choose a smooth "time function" t and a foliation $\{\Sigma_t\}_{t\in\mathbb{R}} \equiv \mathbb{R} \times \Sigma$ in such a way that each point of the manifold is an element of a single surface Σ_t and that, for constant t, $\Sigma_t \equiv \{t\} \times \Sigma$ is a Cauchy surface. Therefore, a globally-hyperbolic spacetime is one that allows one to tell the entire physical system history from the initial value problem posed on $\mathbb{M}$.

To finish this appendix, we define *an observer* as a timelike curve z whose $4-$ velocity is future-oriented at every point of z. Finally, *a family o observers* is composed of a collection of curves future-directed such that at each point of the manifold we have only one curve passing through.

[1]A set is achronal when no pair $p, q \in \Sigma$ can be connected by a timelike curve.

[2]This means that an observer describing a lightlike or timelike trajectory will not be at rest for any point in the neighbourhood of $\mathbb{M}$

Bibliography

[1] W. C. C. Lima. "Do mecanismo de despertar do vácuo (ou de como fazer o vácuo pesado)". Tese de Doutorado. PhD thesis. Instituto de Física de São Carlos, Universidade de São Paulo, 2012. DOI: 10.11606/T.76.2012.tde-01062012-182200. URL: https://www.teses.usp.br (visited on 02/21/2025).

[2] Níckolas de Aguiar Alves. Nonperturbative Aspects of Quantum Field Theory in Curved Spacetime. In: *arXiv* (2023). eprint: 2305.17453 (gr-qc).

[3] Michael Reed and Barry Simon. *Methods of Modern Mathematical Physics, Volume 1: Functional Analysis.* Revised and Enlarged Edition. Academic Press, 1981. ISBN: 9780125850506, 0125850506.

[4] João Carlos Alves Barata. *Notas para um Curso de Física-Matemática.* Versão de 19 de dezembro de 2024, 3000 páginas. 2024. URL: https://denebola.if.usp.br/~jbarata/Notas_de_aula/capitulos.html.

[5] Robert M. Wald. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics.* 1st ed. Chicago Lectures in Physics. University of Chicago Press, 1994. ISBN: 0226870278, 9780226870274, 0226870251, 9780226870250.

[6] N. D. Birrell and P. C. W. Davies. *Quantum Fields in Curved Space.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1984. ISBN: 0521278589, 9780521278584.

[7] A. Ashtekar and Anne Magnon. Quantum Fields in Curved Space-Times. In: *Proc. R. Soc. A* 300 (1975), pp. 197–217. DOI: 10.1098/rspa.1975.0181.

[8] Piotr Chmielowski. States of Scalar Field on Spacetimes with Two Isometries with Timelike Orbits. In: *Class. Quantum Gravity* 11.1 (1994), p. 41. DOI: 10.1088/0264-9381/11/1/008.

[9] J. Dimock. Algebras of Local Observables on a Manifold. In: *Commun. Math. Phys.* 77 (1980), pp. 219–228. DOI: 10.1007/BF01269921.

[10] Rudolf Haag. *Local Quantum Physics: Fields, Particles, Algebras.* Second. Theoretical and Mathematical Physics. Springer, 1996. ISBN: 3540610516, 9783540610517.

[11] Robert M. Wald. *General Relativity.* First. University of Chicago Press, 1984. ISBN: 9780226870335, 0226870332, 0226870324, 9780226870328.

[12] B. S. Kay. Linear Spin-Zero Quantum Fields in External Gravitational and Scalar Fields. In: *Commun. Math. Phys.* 71 (1980), pp. 29–46. DOI: 10.1007/BF01230084.

[13] Igor Khavkine and Valter Moretti. "Algebraic QFT in Curved Spacetime and Quasifree Hadamard States". In: *Advances in Algebraic Quantum Field Theory.* Ed. by Romeo Brunetti, Claudio Dappiaggi, Klaus Fredenhagen, and Jakob Yngvason. Springer, 2015, pp. 191–251. DOI: 10.1007/978-3-319-21353-8_5.

[14] T. Rick Perche. Localized Non-Relativistic Quantum Systems in Curved Spacetimes. In: *Phys. Rev. D* 106 (2022), p. 025018. DOI: 10.1103/PhysRevD.106.025018. arXiv: 2206.01225 [quant-ph].

[15] W. G. Dixon. Dynamics of extended bodies in general relativity. I. Momentum and angular momentum. In: *Proc. R. Soc. A* 314 (1970), pp. 499–527. DOI: 10.1098/rspa.1970.0020.

[16] W. G. Dixon. Dynamics of extended bodies in general relativity - II. Moments of the charge-current vector. In: *Proc. R. Soc. A* 319 (1970), pp. 509–547. DOI: 10.1098/rspa.1970.0189.

[17] W. G. Dixon. Dynamics of extended bodies in general relativity III. Equations of motion. In: *Philos. Trans. R. Soc. A* 277 (1974), pp. 59–119. DOI: 10.1098/rsta.1974.0038.

[18] L. Parker. One-electron atom as a probe of spacetime curvature. In: *Phys. Rev. D* 22 (1980), pp. 1922–1934. DOI: 10.1103/PhysRevD.22.1922.

[19] L. Parker. One-Electron Atom in Curved Space-Time. In: *Phys. Rev. Lett.* 44 (1980), p. 1559. DOI: 10.1103/PhysRevLett.44.1559.

[20] L. Parker and L. O. Pimentel. Gravitational Perturbation of the Hydrogen Spectrum. In: *Phys. Rev. D* 25 (1982), pp. 3180–3190. DOI: 10.1103/PhysRevD.25.3180.

[21] E. Poisson. The motion of point particles in curved spacetime. In: *Living Rev. Relativ.* 7 (2004), p. 6. DOI: 10.12942/lrr-2004-6.

[22] T. R. Perche and J. Neuser. A wavefunction description for a localized quantum particle in curved spacetimes. In: *Class. Quantum Gravity* 38 (2021), p. 175002. DOI: 10.1088/1361-6382/ac0f4c.

[23] J. E. Moyal. Quantum mechanics as a statistical theory. In: *Math. Proc. Camb. Philos. Soc.* 45 (1949), pp. 99–124. DOI: 10.1017/S0305004100000487.

[24] A. Alibabaei. Geometric post-Newtonian description of spin-half particles in curved spacetime. In: *arXiv* (2022). eprint: 2204.05997 (gr-qc).

[25] A. Sahu, I. Melgarejo-Lermas, and E. Martin-Martinez. Sabotaging the harvesting of correlations from quantum fields. In: *Phys. Rev. D* 105 (2022), p. 065011. DOI: 10.1103/PhysRevD.105.065011.

[26] F. Giacomini and Č. Brukner. Quantum superposition of spacetimes obeys Einstein's equivalence principle. In: *AVS Quantum Sci.* 4 (2022), p. 015601. DOI: 10.1116/5.0070018.

[27] J. de Ramon, M. Papageorgiou, and E. Martin-Martinez. Relativistic causality in particle detector models: Faster-than-light signaling and impossible measurements. In: *Phys. Rev. D* 103 (2021), p. 085002. DOI: 10.1103/PhysRevD.103.085002.

[28] D. Grimmer, B. de S. L. Torres, and E. Martin-Martinez. Measurements in QFT: Weakly coupled local particle detectors and entanglement harvesting. In: *Phys. Rev. D* 104 (2021), p. 085014. DOI: 10.1103/PhysRevD.104.085014.

[29] E. Martin-Martinez, T. R. Perche, and B. de S. L. Torres. Broken covariance of particle detector models in relativistic quantum information. In: *Phys. Rev. D* 103 (2021), p. 025007. DOI: 10.1103/PhysRevD.103.025007.

[30] A. Roura. Gravitational redshift in quantum-clock interferometry. In: *Phys. Rev. X* 10 (2020), p. 021014. DOI: 10.1103/PhysRevX.10.021014.

[31] S. Boughn and T. Rothman. Aspects of graviton detection: Graviton emission and absorption by atomic hydrogen. In: *Class. Quantum Gravity* 23 (2006), pp. 5839–5852. DOI: 10.1088/0264-9381/23/20/013.

[32] J. P. M. Pitelli and T. R. Perche. Angular momentum based graviton detector. In: *Phys. Rev. D* 104 (2021), p. 065016. DOI: 10.1103/PhysRevD.104.065016.

[33] W. G. Unruh. Notes on black-hole evaporation. In: *Phys. Rev. D* 14 (1976), pp. 870–892. DOI: 10.1103/PhysRevD.14.870.

[34] B. DeWitt. *General Relativity; an Einstein Centenary Survey*. Cambridge: Cambridge University Press, 1980.

[35] A. Pozas-Kerstjens and E. Martin-Martinez. Harvesting correlations from the quantum vacuum. In: *Phys. Rev. D* 92 (2015), p. 064042. DOI: 10.1103/PhysRevD.92.064042.

[36] N. Funai, J. Louko, and E. Martin-Martinez. $\hat{v} \cdot \hat{a}$ vs $\hat{x} \cdot \hat{e}$: Gauge invariance in quantum optics and quantum field theory. In: *Phys. Rev. D* 99 (2019), p. 065014. DOI: 10.1103/PhysRevD.99.065014.

[37] R. Lopp and E. Martin-Martinez. Quantum delocalization, gauge, and quantum optics: Light-matter interaction in relativistic quantum information. In: *Phys. Rev. A* 103 (2021), p. 013703. DOI: 10.1103/PhysRevA.103.013703.

[38] M. Benini and C. Dappiaggi. "Models of free quantum field theories on curved backgrounds". In: *Advances in Algebraic Quantum Field Theory*. Springer, 2015, pp. 75–124. DOI: 10.1007/978-3-319-21353-8_3.

[39] T. R. Perche. General features of the thermalization of particle detectors and the Unruh effect. In: *Phys. Rev. D* 104 (2021), p. 065001. DOI: 10.1103/PhysRevD.104.065001.

[40] E. Martin-Martinez, T. R. Perche, and B. de S. L. Torres. General relativistic quantum optics: Finite-size particle detector models in curved spacetimes. In: *Phys. Rev. D* 101 (2020), p. 045017. DOI: 10.1103/PhysRevD.101.045017.

[41] E. Tjoa and R. B. Mann. Harvesting correlations in Schwarzschild and collapsing shell spacetimes. In: *J. High Energy Phys.* 2020 (2020), p. 155. DOI: 10.1007/JHEP08(2020)155.

[42] T. R. Perche, C. Lima, and E. Martin-Martinez. Harvesting entanglement from complex scalar and fermionic fields with linearly coupled particle detectors. In: *Phys. Rev. D* 105 (2022), p. 065016. DOI: 10.1103/PhysRevD.105.065016.

[43] V. Sudhir, N. Stritzelberger, and A. Kempf. Unruh effect of detectors with quantized center of mass. In: *Phys. Rev. D* 103 (2021), p. 105023. DOI: 10.1103/PhysRevD.103.105023.

[44] N. Stritzelberger, L. J. Henderson, V. Baccetti, N. C. Menicucci, and A. Kempf. Entanglement harvesting with coherently delocalized matter. In: *Phys. Rev. D* 103 (2021), p. 016007. DOI: 10.1103/PhysRevD.103.016007.

[45] A. Sachs, R. B. Mann, and E. Martin-Martinez. Entanglement harvesting and divergences in quadratic Unruh-deWitt detector pairs. In: *Phys. Rev. D* 96 (2017), p. 085012. DOI: 10.1103/PhysRevD.96.085012.

[46] A. M. Sachs, R. B. Mann, and E. Martin-Martinez. Entanglement harvesting from multiple massless scalar fields and divergences in Unruh-deWitt detector models. In: *arXiv* (2018). eprint: 1808.05980 (quant-ph).

[47] D. Hummer, E. Martin-Martinez, and A. Kempf. Renormalized Unruh-deWitt particle detector models for boson and fermion fields. In: *Phys. Rev. D* 93 (2016), p. 024019. DOI: 10.1103/PhysRevD.93.024019.

[48] B. de S. L. Torres, T. R. Perche, A. G. S. Landulfo, and G. E. A. Matsas. Neutrino flavor oscillations without flavor states. In: *Phys. Rev. D* 102 (2020), p. 093003. DOI: 10.1103/PhysRevD.102.093003.

[49] T. R. Perche and E. Martin-Martinez. Antiparticle detector models in QFT. In: *Phys. Rev. D* 104 (2021), p. 105021. DOI: 10.1103/PhysRevD.104.105021.

[50] A. Einstein. Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt. German. In: *Ann. Phys. (Leipzig)* 322 (1905), pp. 549–557. DOI: 10.1002/andp.19053220607.

[51] J. Johnson. [Title to be confirmed]. In: *Phys. Rev.* 32 (1928), p. 97. DOI: 10.1103/PhysRev.32.97.

[52] H. Nyquist. Thermal Agitation of Electric Charge in Conductors. In: *Phys. Rev.* 32 (1 1928), pp. 110–113. DOI: 10.1103/PhysRev.32.110. URL: https://link.aps.org/doi/10.1103/PhysRev.32.110.

[53] Herbert B. Callen and Theodore A. Welton. Irreversibility and Generalized Noise. In: *Phys. Rev.* 83 (1 1951), pp. 34–40. DOI: 10.1103/PhysRev.83.34. URL: https://link.aps.org/doi/10.1103/PhysRev.83.34.

[54] Nivaldo A. Lemos. *Mecânica Analítica*. 2nd ed. Livraria da Física, 2007. ISBN: 8588325241, 9788588325241.

[55] L. D. Landau and E. M. Lifshitz. *Statistical Physics, Third Edition, Part 1: Volume 5*. 3rd ed. Course of Theoretical Physics, Volume 5. Butterworth-Heinemann, 1980. ISBN: 0750633727, 9780750633727.

[56] Alvaro Ortega, Emma McKay, Alvaro M. Alhambra, and Eduardo Martin-Martinez. Work Distributions on Quantum Fields. In: *Phys. Rev. Lett.* 122 (24 June 2019). Provided as ortega2019.pdf, p. 240604. DOI: 10.1103/PhysRevLett.122.240604. arXiv: 1802.00995 [quant-ph]. URL: https://link.aps.org/doi/10.1103/PhysRevLett.122.240604.

[57] Anthony Kiely, Eoin O'Connor, Thomás Fogarty, Gabriel T. Landi, and Steve Campbell. Entropy of the quantum work distribution. In: *Phys. Rev. Research* 5 (2023). arXiv:2210.07896 [quant-ph], p. L022010. DOI: 10.1103/PhysRevResearch.5.L022010.

[58] Donald Marolf, Mukund Rangamani, and Toby Wiseman. Holographic thermal field theory on curved spacetimes. In: *JHEP* 02 (2014), p. 060. DOI: 10.1007/JHEP02(2014)060. arXiv: 1312.0612 [hep-th].

[59] John C. Collins. *Renormalization: An Introduction to Renormalization, the Renormalization Group and the Operator-Product Expansion.* Cambridge University Press, 1986. ISBN: 978-0521311779.

[60] Timothy J. Hollowood. *Renormalization Group and Fixed Points in Quantum Field Theory.* Springer, 2013. DOI: 10.1007/978-3-642-36312-2.

[61] Mikko Laine and Aleksi Vuorinen. *Basics of Thermal Field Theory.* Vol. 925. Lecture Notes in Physics. Springer, 2016. ISBN: 978-3-319-31931-5. DOI: 10.1007/978-3-319-31933-9.