Federal University of the Goiás
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Melnikov Theory for Filippov systems with an algebraic switching curve

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GERARDO HOMERO ANACONA ERAZO

Melnikov Theory for Filippov systems with an algebraic switching curve

Dissertation submitted to the graduate program in mathematics of the Mathematics and Statistics Institute Federal University of the Goiás, as part of the requirements for obtaining the degree of Master in mathematics.

Specialized field: Dynamical Systems.

Advisor: Prof. Otávio Marçal Leandro Gomide

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ATA DE DEFESA DE DISSERTAÇÃO

Ata nº 02 da sessão de Defesa de Dissertação de **Gerardo Homero Anacona Erazo**, que confere o título de Mestre em Matemática, **na área de Sistemas Dinâmicos.**

Ao vigésimo quinto dia do mês de maio do ano de dois mil e vinte e dois, a partir das quatorze horas, de forma híbrida, realizou-se a sessão pública de Defesa de Dissertação intitulada **"Melnikov Theory for Filippov systems with an algebraic switching curve".** Os trabalhos foram instalados pelo presidente da banca, Professor Doutor **Otávio Marçal Leandro Gomide - IME/UFG** com a participação dos demais membros da Banca Examinadora: Professora Doutora Kamila da Silva Andrade - IME/UFG membro titular interno e o Professor Doutor **Oscar Alexander Ramírez Cespedes - MAT/UFV** membro titular externo. Ressaltamos que apenas o professor doutor Oscar Alexander Ramírez Cespedes - MAT/UFV participou da defesa de dissertação via web videoconferência. Durante a arguição os membros da banca **não fizeram** sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Dissertação, tendo sido o candidato **aprovado** pelos seus membros. Proclamados os resultados pelo Professor Doutor **Otávio Marçal Leandro Gomide - IME/UFG**, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos Membros da Banca Examinadora, ao vigésimo quinto dia do mês de maio do ano de dois mil e vinte e dois.

TÍTULO SUGERIDO PELA BANCA

Melnikov Theory for Filippov systems with an algebraic switching curve

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Gerardo Homero Anacona Erazo

Licensed in mathematics by the University of Cauca (UNICAUCA) in 2018 and Mathematics by the University of Cauca in 2019, during the Master's degree was awared scholarship by CAPES and work in dynamical system.

This work is dedicated to all those friends who could not study due their socioeconomic conditions.

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"Anyone who stops learning is old, whether at twenty or eighty. Anyone who keeps learning stays young. The greatest thing in life is to keep your mind young."

Henry Ford,
1863-1947.

Resumo

Anacona Erazo, Gerardo. **Melnikov Theory for Filippov systems with an algebraic switching curve**. Goiânia, 2022. 63p. Dissertação de Mestrado. Graduate program in mathematics, Mathematics and Statistics Institute , Federal University of the Goiás.

A existência de ciclos limite em sistemas dinâmicos é um tópico de pesquisa muito atrativo devido a suas amplas aplicações. A importância do desenvolvimento deste tema pode ser justificada por um dos itens da famosa lista de problemas proposta por David Hilbert que permanece em aberto e consiste em encontrar o número máximo de ciclos limite de um sistema planar polinomial. Em virtude de sua relevância, atualmente há um grande interesse em estender a busca de ciclos limite a outros tipos de sistemas, como os famigerados sistemas de Filippov.

Neste trabalho, estudamos uma técnica para encontrar ciclos limite em sistemas dinâmicos suaves por partes, chamada Teoria de Melnikov, e aplicamos tal técnica para encontrar cotas inferiores para o número máximo de ciclos limite de sistemas polinomiais por partes da forma

$$Z(x,y) = \begin{cases} \begin{pmatrix} y + \varepsilon P^{+}(x,y) \\ -x + \varepsilon Q^{+}(x,y) \end{pmatrix}, & \text{if} \quad y \geq x^{n}, \\ \\ \begin{pmatrix} y + \varepsilon P^{-}(x,y) \\ -x + \varepsilon Q^{-}(x,y) \end{pmatrix}, & \text{if} \quad y \leq x^{n}, \end{cases}$$

onde $P^{\pm}$ e $Q^{\pm}$, são polinômios de grau 2, $n \in \mathbb{N}$.

Palavras–chave

Campo de vetores suave por partes, Teoria de Melnikov, ciclos limite.

Abstract

Anacona Erazo, Gerardo. **Melnikov Theory for Filippov systems with an algebraic switching curve**. Goiânia, 2022. 63p. MSc. Dissertation. Graduate program in mathematics, Mathematics and Statistics Institute , Federal University of the Goiás.

The existence of limit cycles in dynamical systems is a very attractive research topic due to its extensive range of applications. The importance of the development of such theme can be seen in one item of the famous list of problems provided by David Hilbert which is still open and consists in finding the maximum number of limit cycles of a planar polynomial system. In light of this, nowadays there is great interest in extend the search of limit cycles to another types of systems, as the so-called Filippov systems.

In this work we study a technique to find limit cycles in piecewise smooth dynamical systems called Melnikov Theory and we apply it to provide lower bounds for the maximum number of limit cycles of piecewise polynomial systems of the form

$$Z(x,y) = \begin{cases} \begin{pmatrix} y + \varepsilon P^{+}(x,y) \\ -x + \varepsilon Q^{+}(x,y) \end{pmatrix}, & \text{if} \quad y \geq x^n, \\ \\ \begin{pmatrix} y + \varepsilon P^{-}(x,y) \\ -x + \varepsilon Q^{-}(x,y) \end{pmatrix}, & \text{if} \quad y \leq x^n, \end{cases}$$

where $P^{\pm}$ and $Q^{\pm}$ are polynomials of degree 2, and $n \in \mathbb{N}$.

Keywords

Piecewise smooth vector field, Melnikov Theory, Limit cycles.

Contents

Introduction

The Qualitative Theory of Differential Equations derives from the work of many mathematicians, but the origin of this is recognized to H. J. Poincaré (1854-1912). This theory provides tools for understanding the dynamics of differential systems without knowing their explicit solutions. It is based on recognizing certain elements, one of these type of elements are the so-called periodic orbits, which are closed curves in the phase space. In particular, it is crucial to detect isolated periodic orbits, known as limit cycles, to describe the behavior of a system.

In this work we are interested in detecting limit cycles in piecewise smooth vector fields. Roughly speaking, piecewise smooth vector fields are vector fields which looses smoothness somewhere, i.e., they are smooth in its domain with exception of some measure zero set called switching set. There are many approaches to define the trajectories of a piecewise smooth vector field, but we will adopt in this work the one proposed by Filippov due to its applicability in real problems, see for instance [4, 8, 26]. We refer such vector fields as Filippov systems.

One useful tool to find limit cycles is the Melnikov Theory which converts the search for cycles in finding zeros of certain functions called Melnikov functions. More specically, the main idea of Melnikov Theory is to write a Poincaré map Δ in terms of Melnikov functions, $\{\Delta_i\}_{i=1}^k$, which control the zeros of Δ. In this case, k is known as the order of the method.

In this work, we apply the Melnikov Theory to detect bifurcating limit cycles from a piecewise polynomial perturbation of a planar linear center. More specifically, we consider the linear center

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \tag{0-1}$$

(see Figure 1) and we split the plane into two regions separated by the curve $y = x^n$, $n \in \mathbb{N}$. We consider a small polynomial perturbation of degree 2 of (0-1) in each region (see Figure 2), which gives rise to the Filippov system

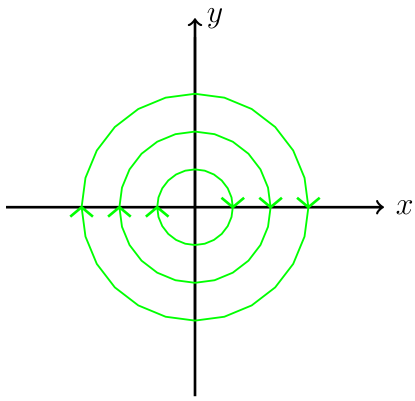


Figure 1: *Phase portrait of system* (0-1)

.

$$Z_{\varepsilon,n}(x,y) = \begin{cases} \begin{pmatrix} y + \varepsilon P^{+}(x,y) \\ -x + \varepsilon Q^{+}(x,y) \end{pmatrix}, & \text{if} \quad y \geq x^n, \\ \\ \begin{pmatrix} y + \varepsilon P^{-}(x,y) \\ -x + \varepsilon Q^{-}(x,y) \end{pmatrix}, & \text{if} \quad y \leq x^n, \end{cases} \tag{0-2}$$

where $P^{\pm}, Q^{\pm}$ are polynomials of degree 2.

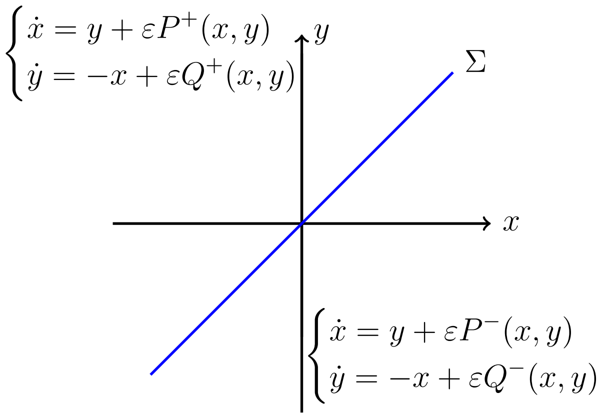


Figure 2: *Filippov system with switching curve* $y = x$.

Calling $m_1(n)$ as the maximum number of zeros of the first Melnikov function, Δ_1, of (0-2), as a result of this work, we obtain the following values of $m_1(n)$ described in the Table 1. As a consequence, we obtain that the maximum number of limit cycles $H(n)$ of (0-2) satisfies that $H(1) \geq 2$, $H(2) \geq 4$, $H(3) \geq 5$, $H(n) \geq 6$ if $n \geq 6$ is even and $H(n) \geq 7$ if $n \geq 7$ is odd.

We highlight that such kind of problem was previously approached in the literature in different scenarios. In [6], the authors consider Filippov systems in the form given by (0-2) where $n = 3$ and $P^{\pm}, Q^{\pm}$ are polynomials of degree 1, and

n	$m_1(n)$
1	2
2	4
3	5
$n > 2$, n even	$6 \leq m_1(n) \leq 7$
$n > 3$, n odd	7

Table 1: *Values of* $m_1(n)$.

using Melnikov Theory up to order 2, they obtain lower bounds for $H(3)$. In [3], the authors also consider the same scenario approached in [6] but they extend the results to switching curves of the form $y = x^n$, $n \in \mathbb{N}$, and using Melnikov Theory up to order 6, they obtained sharpened lower bound for $H(n)$, for each $n \in \mathbb{N}$. In the preprint [37], the authors consider Filippov system in the form (0-2) with switching curve given by $y^n = x^m$, with $m, n \in \mathbb{N}$, and $P^{\pm}, Q^{\pm}$ are polynomials of degree 1, and using Melnikov Theory up to order 3, they obtained lower bounds for $H(n)$. Finally, in the preprint [39] the authors provides lower bounds $Z(m, n)$ for the general case of system (0-2), where $P^{\pm}$ and $Q^{\pm}$ are polynomials of degree n and the switching curve is given by $y = x^m$, $n, m \in \mathbb{N}$.

The problem approached in this work is a special case of the problem approached in [39], in fact, we provide lower bounds for $Z(m, 2)$, $m \in \mathbb{N}$, nevertheless our techniques allow us to provide sharpened lower bounds in comparison with the bounds obtained in [39]. In Table 2, we can see a comparison between our results and the results of [39] in this special case.

m	$Z(m, 2)$ in this work	$Z(m, 2)$ in [39].
1	2	2
2	4	4
3	5	$5 \leq Z(3, 2) \leq 6$
$m > 2$, m even	$6 \leq m_1 \leq 7$	$5 \leq Z(m, 2) \leq 7$
$m > 3$, m odd	7	$6 \leq Z(m, 2) \leq 7$

Table 2: *Comparison between* $Z(m, 2)$ *obtained in this work and in [39].*

The structure of this work is described below:

- Chaprter 1. We introduce Filippov systems, which are piecewise smooth vector fields with trajectories given by the Filippov's convention. Also, we give some general results on Chebyshev Theory, which shall be used to find zeros of Melnikov functions.

- Chapter 2. In this chapter we provide the Melnikov Theory up to order 2 for piecewise smooth vector fields. In particular, we establish formulas for the Melnikov Functions of orders 1 and 2, and we prove that such functions allow us to find limit cycles of original system.
- Chapter 3. In this chapter we apply Melnikov Theory to detect limit cycles in a class of Filippov systems having a switching curve of the form $y = x^n$, $n \in \mathbb{N}$.
- Chapter 4. In this chapter we highlight the principal topics of this work and we give some further directions this work can take.

CHAPTER 1

Preliminaries

In this chapter we define some topics which are needed to develop this work, among these topics, we briefly discuss Existence and Uniqueness Theorem For ordinary differential equations and Landau symbols. More details can be found in [32] and [36], respectively. Later we discuss autonomous and non-autonomous Filippov systems and Chebyshev systems.

1.1 Existence and Uniqueness Theorem For ordinary differential equations

In this chapter we study nonlinear systems of differential equations

$$\dot{x} = f(x), \tag{1-1}$$

where $f : E \to \mathbb{R}^n$ is a smooth function and E is an open subset of $\mathbb{R}^n$.

Remark 1. *When the function f is continuous in E, we denote $f \in C(E)$, and if the derivatives of f are continuous we denote $f \in C^1(E)$.*

Before stating the Fundamental Existence and Uniqueness Theorem for the system (1-1), it is necessary to define the solution of (1-1).

Definition 1.1.1. *Suppose that $f \in C(E)$, where E is an open susbset of $\mathbb{R}^n$ then $\varphi(t)$ is a solution of the differential equation (1-1) on an interval I if, only if, $\varphi(t)$ is differentiable on I and if for all $t \in I$, $\varphi(t) \in E$ and*

$$x'(t) = f(x(t)).$$

and given $x_0 \in E$, $\varphi(t)$ is a solution of the initial value problem

$$\begin{aligned} \dot{x} =& f(x) \\ x(t_0) =& x_0 \end{aligned} \tag{1-2}$$

on an interval I if, and only if, $t_0 \in I$, $\varphi(t_0) = x_0$ and $\varphi(t)$ is a solution of the differential equation (1-1) on the interval I.

We define, for $\delta > 0$ and $x \in \mathbb{R}^n$

$$B_\delta(x) = \{y \in \mathbb{R}^n : \ \|x - y\| < \delta\},$$

where $\|\cdot\|$ denotes the Euclidean norm.

Theorem 1.1.1. (The Fundamental Existence and uniqueness Theorem). *Let E be an open subset of $\mathbb{R}^n$ containing x_0 and assume that $f \in C^1(E)$. Then there exists an $a > 0$ such that the initial value problem*

$$\begin{aligned} x' &= f(x), \\ x(t_0) &= x_0 \end{aligned}$$

has a unique solution $\varphi(t)$ in the interval $[-a.a]$.

Now, we study the dependence of the solution of the initial value problem

$$\begin{aligned} \dot{x} =& f(x) \\ x(0) =& y \end{aligned} \tag{1-3}$$

on the initial condition $x(0) = y$.

Theorem 1.1.2. (Dependence on Initial Conditions). *Let E be an open subset of $\mathbb{R}^n$ containing x_0 and assume that $f \in C^1(E)$. then there exists an $a > 0$ and $\delta > 0$ such that for all $y \in B_\delta(x_0)$ the initial value problem*

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= y \end{aligned}$$

has a unique solution $\varphi(t, y)$ with $\varphi \in C(G)$ where $G = [-a, a] \times B_\delta(x_0) \subset \mathbb{R}^{n+1}$. Furthermore, for each $y \in B_\delta(x_0)$, $\varphi(t)$ is a twice continuously differentiable function of t for $t \in [-a, a]$.

If the differential equation depends on a parameter $\mu \in \mathbb{R}^n$, i.e., if the function $f(x)$ in (1-3) is replaced by $f(x, \mu)$, then the solution $\varphi(t, y, \mu)$ also depends on the parameter μ.

Theorem 1.1.3. (Dependence on Parameters) *Let E be an open subset of $\mathbb{R}^{n+m}$ containing the point (x_0, μ_0) where $x_0 \in \mathbb{R}^n$ and $\mu_0 \in \mathbb{R}^m$ and assume that*

$f \in C^1(E)$. It follows that there exist $a > 0$ and $\delta > 0$ such that for all $y \in B_\delta(x_0)$ and $\mu \in B_\delta(\mu_0)$, the initial value problem

$$\dot{x} = f(x, \mu)$$
$$x(0) = y$$

has unique solution $\varphi(t, y, \mu)$ with $\varphi \in C^1(G)$ where $G = [-a, a] \times B_\delta(x_0) \times B_\delta(\mu_0)$.

Finally, we define the following concepts associated to the solutions of an ODE.

Definition 1.1.2. (Orbit). *If we denote $I(x_0)$ the maximal interval, where the solution $\varphi(t, x_0)$ of the problem value problem (1-3) is defined. We define the set*

$$\gamma(p) = \{\varphi(t, x_0) : \ t \in I(x_0)\}$$

as the orbit of f through x_0.

Remark 2. *$I(x_0)$ is the maximal interval of φ if, for any open interval J where the function $\varphi(t, x_0)$ is defined, we have $J \subset I(x_0)$, see [32] for more details.*

Remark 3. *Fixing an initial point x_0 and considering $I = I(x_0)$ then the mapping $\varphi(\cdot, x_0) : I \to E$ defines a solution curve or trajectory of the system (1-1). As usual, the mapping $\varphi(\cdot, x_0)$ defines a curve Γ in $E$and the trajectory can be seen as the motion along such curve Γ through the point x_0 in the subset $E \subset \mathbb{R}^n$.*

Definition 1.1.3. (Periodic function). *A function $f(t, x)$ is periodic in the variable t if, and only if, there exists $L > 0$ such that $f(t_0, x) = f(t_0 + L, x)$ for all $x \in E$ and for all $t_0 \in \mathbb{R}$: In this case, we define the period by*

$$T = \inf\{L \in (0, \infty) : f(t_0, x) = f(t_0 + L, x) \ \text{for all} \ x \in D\} \tag{1-4}$$

and we say that f is a T-periodic function in the variable t.

We can also consider nonlinear systems of differential equations of the form

$$\dot{x} = f(t, x), \tag{1-5}$$

where the function f depends on the independent variable t, which are called nonautonomous systems. Nevertheless, all nonautonomous system (1-5) with $x \in \mathbb{R}^n$ can be written as an autonomous system of the form (1-1) with $x \in \mathbb{R}^{n+1}$, where the $x_{n+1} = t$ and $\dot{x}_{n+1} = 1$ (see the Subsection 1.4 below for more details). In light of this, all definitions and results in this section can be stated for nonautonomous systems.

1.2 Big O notation: Landau symbols

Big O notation (with a capital letter O, not zero), also called Landau's symbol, is a symbolism used in Mathematics to describe the asymptotic behavior of functions. In short, it tells how fast a function grows or declines. It will appear frequently when we expand a function in Taylor series, procedure widely used in the work, so this notation is introduced for completeness.

Definition 1.2.1. (Order function) *A function $\delta(\varepsilon)$ is an order function if, and only if, $\delta(\varepsilon)$ is positive, continuous and $\lim \delta(\varepsilon)$ when $\varepsilon \to 0$ exists.*

Let $\varphi(t, \varepsilon)$ be a function that assumes values in $\mathbb{R}^n$ defined for $\varepsilon \geq 0$ and $t \in I_\varepsilon$, where I_ε is a real interval that depends on ε. The expression $\varepsilon \downarrow 0$ means that there exists ε_0 such as the statement in the definition above is satisfied for each $\varepsilon \in (0, \varepsilon_0]$.

Definition 1.2.2. (Big O) *We say that $\varphi(t, \varepsilon)$ is of order $\delta(\varepsilon)$ for $\varepsilon \downarrow 0$ if, and only if, there exist constants $\varepsilon_0 > 0$ and $K > 0$ such that:*

$$\|\varphi(t, \varepsilon)\| \leq K\delta(\varepsilon)$$

for each $t \in I_\varepsilon$ and $\varepsilon \in (0, \varepsilon_0)$. When the function $\varphi(t, \varepsilon)$ is of order $\delta(\varepsilon)$ for $\varepsilon \downarrow 0$ we denote $\varphi(t, \varepsilon) = \mathcal{O}(\delta(\varepsilon))$

Remark 4. *Usually with the Landau symbols we will work in the neighborhood of $\varepsilon = 0$, then the notation $\varepsilon \downarrow 0$ can be omitted.*

In the following proposition we omit the argument of functions $\varphi_i(t, \varepsilon)$, $i = 1, 2$.

Proposition 1.2.1. *If φ_1 and φ_2 are function such that $\varphi_1 = \mathcal{O}(\delta_1(\varepsilon))$ and $\varphi_2 = \mathcal{O}(\delta_2(\varepsilon))$, then*

1. $\varphi_1 \cdot \varphi_2 = \mathcal{O}(\delta_1(\varepsilon) \cdot \delta_2(\varepsilon))$
2. $\varphi_1 = \mathcal{O}(\delta_1(\varepsilon))$ *and* $\varphi_2 = \mathcal{O}(\delta_2(\varepsilon))$, *then* $\varphi_1 + \varphi_2 = \mathcal{O}(max(\delta_1, \delta_2))$.
3. *If k is a nonzero constant, then* $\mathcal{O}(|k| \cdot \varphi_1) = \mathcal{O}(\varphi_1)$.

1.3 Filippov systems

Since in real life there are many phenomena that can not be properly modeled by smooth systems, it is necessary to consider piecewise smooth systems to model certain phenomena. Nevertheless, it is not trivial to define the solution of a piecewise smooth system, and among all the possibilities, the Filippov's convention to define the solution of a piecewise smooth system was introduced by a russian mathematician called Aleksei Fedorovich Filippov, and it gave rise to the so-called Filippov systems. Such systems are very useful to describe non-smooth phenomena and for this reason, they appear frequently in control systems see [26], impact and friction mechanics [8], and nonlinear oscillations [4] for instance.

In this section we discuss the Filippov's convention to define the solutions of a piecewise smooth system.

Definition 1.3.1. (Piecewice smooth vector field) *The vector field $X : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is said to be piecewise smooth if, and only if the set D can be partitioned in open and connected sets $\{D_i\}_{i=1}^k$ such that the restriction of the vector field X to any $\overline{D}_i$ is smooth. More specifically, there is a collection $\{D_i\}_{i=1}^k$ of open, connected and disjoint sets such that*

$$D = \bigcup_{i=1}^{k} \overline{D}_i,$$

and the vector field $X_i : \overline{D_i} \to \mathbb{R}^n$ defined by

$$X_i := X|_{D_i} : \overline{D}_i \to \mathbb{R}^n,$$

is smooth for each $i \in \{1, 2, \ldots, n\}$.

Let us denote $\Sigma \subset \bigcup_{i=1}^k \partial D_i$ the set where the vector field is not smooth, it's worth saying that $\mu(\Sigma) = 0$. Here the symbol ∂ means boundary of a set and μ denote the Lebesgue measure for more details about the measure theory see [38].

Now, we study a definition of trajectories for this kind of system which is know as Filippov's convention. For simplicity, we restrict ourselves to the case where D is an open subset of $\mathbb{R}^n$ and Σ is a hypersurface in $\mathbb{R}^n$ of dimension 1. More specifically, let $h : D \to \mathbb{R}^n$ be a C^1 function having $0 \in \mathbb{R}$ as a regular value, and let $\Sigma = h^{-1}(0)$. So the set D splits into two regions namely $\Sigma^+ = \{x \in D : h(x) > 0\}$ and $\Sigma^- = \{x \in D : h(x) < 0\}$, separated by Σ.
Let $X, Y : D \to \mathbb{R}^n$ be two smooth vector fields and consider the piecewise smooth vector field given by

$$Z(x) = \begin{cases} X(x), & \text{if} \quad h(x) > 0, \\ \\ Y(x), & \text{if} \quad h(x) < 0. \end{cases} \tag{1-6}$$

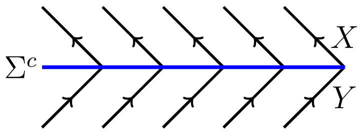
We concisely denote $Z = (X, Y)$.

Remark 5. *In this case Σ is referred as the switching manifold of* (1-6).

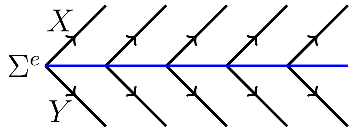
In order to establish the concept of trajectory for a systems defined in (1-6) we distinguish some regions of Σ, see Figure 1.1.

- **Crossing region.** $\Sigma^c = \{p \in \Sigma : \ \ Xh(p) \cdot Yh(p) > 0\}$,
- **Escaping region.** $\Sigma^e = \{p \in \Sigma : \ \ Xh(p) > 0, \ \ Yh(p) < 0\}$,
- **Sliding region.** $\Sigma^s = \{p \in \Sigma : \ \ Xh(p) < 0, \ \ Yh(p) > 0\}$.

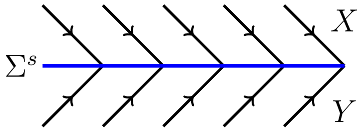
where $Xh(p)$ (respectively $Yh(p)$) is given by $X(p) \cdot \nabla h(p)$ (respectively $X(p) \cdot \nabla h(p)$) and it is called Lie derivative of h in the direction of X (respectively Y) at p.



(a) *Crossing region.*



(b) *Escaping region.*



(c) *Sliding region.*

Figure 1.1: *Different regions in the switching curve* Σ.

Remark 6. *The regions above are open subsets of* Σ *which can have more than one connected component.*

In this work, we study Filippov systems having only crossing regions, and for this reason, we define the trajectories of a Filippov system through points in $\Sigma^c \cup \Sigma^{\pm}$. The definition of trajectory through the remaining points of the phase space can be found in [13]. If $p \in \Sigma$ and $Z(p) = 0$ then the trajectory is stationary. In the other points we have

- **Case 1.** If $p \notin \Sigma$, then the trajectory is given by X or Y in the usual way.
- **Case 2.** If $p \in \Sigma^c$, then the vector fields X and Y point to the same side Σ^+ or Σ^-, so it is sufficient to connect the trajectories of X and Y through p.

Remark 7. *The rules used to define the local trajectories of* (1-6) *are known as Filippov's convention.*

Now, we formalize the concept of trajectory in Filippov systems. Here we suppose that $\varphi_X(t, p)$ is the trajectory of X through p in the region Σ^+ and $\varphi_Y(t, p)$ is the trajectory of Y through p in the region Σ^-, which are both defined in an interval I containing p in its interior.

Definition 1.3.2. *The trajectory of a Filippov system* (1-6) *through a point* p *is defined in the following way*

- *If* $p \in \Sigma^+$ *or* $p \in \Sigma^-$, *is such that* $X(p) \neq 0$ *or* $Y(p) \neq 0$, *respectively, then the trajectory is given by* $\varphi_Z(t,p) = \varphi_X(t,p)$ *or* $\varphi_Z(t,p) = \varphi_Y(t,p)$, *respectively, for* $t \in I \subset \mathbb{R}$.
- *If* $p \in \Sigma^c$ *and* $Xf(p), Yf(p) > 0$, *we define the local trajectory of* (1-6) *by*

$$\varphi_Z(t,p) = \begin{cases} \varphi_X(t,p), & t \in I \cap \{t \geq 0\} \\ \varphi_Y(t,p), & t \in I \cap \{t \leq 0\}. \end{cases}$$

- *If* $p \in \Sigma^c$ *and* $Xf(p), Yf(p) > 0$,*we define the local trajectory of* (1-6) *by*

$$\varphi_Z(t,p) = \begin{cases} \varphi_Y(t,p), & t \in I \cap \{t \geq 0\} \\ \varphi_X(t,p), & t \in I \cap \{t \leq 0\}. \end{cases}$$

Considering the goals of our work, we distinguish some special orbits which play an important role in the dynamics of Filippov systems.

Definition 1.3.3. (Local orbit) *The local orbit of a point* $p \in D$ *is given by* $\gamma(p) = \{\varphi_Z(t,p) : t \in I\}$.

Definition 1.3.4. (Regular periodic orbit) *An orbit* $\gamma(p) = \{\varphi_Z(t,p) : t \in I\}$ *is a regular periodic orbit if, and only if,* $\gamma \subset \Sigma^+ \cup \Sigma^- \cup \overline{\Sigma}^c$ *and* $\varphi_Z(t+T,p) = \varphi_Z(t,p)$ *for some* $T > 0$, *for all* $t \in \mathbb{R}$. $T_0 = \inf\{T > 0 : \quad \varphi_Z(t+T,p) = \varphi_Z(t,p), \quad t \in \mathbb{R}\}$ *is called the period of* $\gamma(p)$.

Definition 1.3.5. (Regular crossing periodic orbit) *The set* $\gamma(p) = \{\varphi_Z(t,p) : t \in I\}$ *is regular crossing periodic orbit if, and only if,* $\gamma \cap \Sigma \subset \overline{\Sigma}^c$, $\gamma \cap \Sigma \neq \emptyset$, *see Figure 1.2.*

Definition 1.3.6. (Limit cycle). *An orbit* $\gamma(p) = \{\varphi_Z(t,p) : t \in I\}$ *is a limit cycle of* (1-6) *if, and only if, there exists a neighborhood* V *of* γ *in* D *such that it is the unique periodic orbit of* (1-6) *in this neighborhood. (See figure 1.2).*

1.4 Non-autonomous and autonomous systems

We have studied Filippov systems $Z = (X, Y)$ where X and Y are autonomous vector fields, i.e, X and Y depend only on the spatial variable x and their associated systems are of the form

$$\dot{x} = X(x) \quad \text{and} \quad \dot{y} = Y(x). \tag{1-7}$$

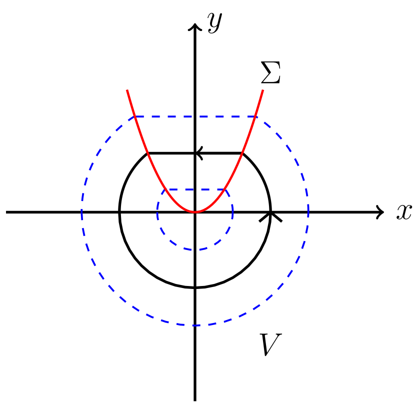


Figure 1.2: *Crossing orbit and limit cycle.*

For our purposes, we need to consider the case where X and Y depend on the time variable t i.e. they are non-autonomous and can be written as

$$\dot{x} = X(t, x) \quad \text{and} \quad \dot{y} = Y(t, x). \tag{1-8}$$

Example 1. *An example of non-autonomous system is*

$$\dot{x} = t. \tag{1-9}$$

and an example of autonomous system is

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 1. \end{cases} \tag{1-10}$$

So we need to adapt the theory developed in the last section to non-autonomous Filippov systems. In order to do this, we relate a non-autonomous system to an autonomous system in the following way.

Definition 1.4.1. *Given a non-autonomous system $\dot{x} = X(t, x)$ where $X : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is smooth, the autonomous system associate to $\dot{x} = X(t, x)$ is defined by*

$$\begin{cases} \dot{x} = X(x_{n+1}, x), \\ \dot{x}_{n+1} = 1. \end{cases} \tag{1-11}$$

Example 2. *In Example 1 we can associate the system*

$$\dot{x} = t,$$

to the autonomous system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = 1. \end{cases}$$

Now we relate the solution of the non-autonomous initial value problem

$$\begin{cases} \dot{x}(t) = X(t, x), \\ x(t_0) = x_0, \end{cases} \tag{1-12}$$

to the solution of the related autonomous initial value problem

$$\begin{cases} \dot{x} = X(x_{n+1}, x), \\ \dot{x}_{n+1} = 1, \\ x(0) = (x_0, t_0). \end{cases} \tag{1-13}$$

Denote $\gamma : I_{t_0} \to \mathbb{R}^n$ and $\varphi : I_0 \to \mathbb{R}^{n+1}$ by the solutions of systems (1-12) and (1-13), respectively, where the intervals $I_{t_0} = (t_0 - \varepsilon, t_0 + \varepsilon)$ and $I_0 = (\varepsilon, \varepsilon)$ are given by the Existence-Uniqueness Theorem 1.1.1. In the next result, we prove that φ and γ are related through the following expression

$$\varphi(t) = (\gamma(t + t_0), t + t_0) \quad \text{for all} \quad t \in I_0. \tag{1-14}$$

Proposition 1.4.1. *If γ and φ are functions that satisfy the expression (1-14), then γ is the solution of the initial value problem (1-12) if, and only if, φ is the solution of the initial value problem (1-13).*

Proof. Let $\gamma(t)$ be the solution of (1-12) defined in I_{t_0} and consider $\varphi(t)$ given by (1-14). From (1-12) and (1-14) we have that

$$\dot{\varphi}(t) = (\dot{\gamma}(t + t_0), 1) = (f(t + t_0, \gamma(t + t_0)), 1) = (f(\varphi(t)), 1), \quad \text{for all} \quad t \in (-\varepsilon, \varepsilon). \tag{1-15}$$

Also we have from (1-14) and (1-12) that

$$\varphi(0) = (\gamma(0), t_0) = (x_0, t_0).$$

Therefore $\varphi(t) : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ is the solution of (1-13).

On the other hand, let φ be the solution of (1-13) defined in $(-\varepsilon, \varepsilon)$ and denote the *j-th* component of φ by φ_j.

Define $\gamma(t) = (\varphi_1(t - t_0), \ldots, \varphi_n(t - t_0))$ for each $t \in I_{t_0}$ and notice that $\varphi_{n+1}(t) = t + t_0$.

Since $\dot{\varphi}(t) = (f(\varphi_{n+1}(t), \varphi_1(t), \cdots, \varphi_n(t)), 1)$, it follows that $\dot{\varphi}(t-t_0) = (f(\varphi_{n+1}(t-t_0), \varphi_1(t-t_0), \ldots, \varphi_n(t-t_0)), 1) = (f(t, \gamma(t)), 1)$, which means that $\dot{\varphi}(t-t_0) = (\dot{\gamma}(t), 1)$. It follows that $\gamma(t)$ is the solution of (1-12). □

Example 3. *The solution of system* (1-9) *with initial condition $\gamma(t_0) = x_0$ is given by*

$$\gamma(t) = \frac{t^2}{2} + x_0 - \frac{t_0^2}{2} \tag{1-16}$$

and the solution of (1-10) *with initial condition $\varphi(0) = (x_0, t_0)$ is given by*

$$\varphi(t) = \left(\frac{(t+t_0)^2}{2} + x_0 - \frac{t_0^2}{2}, t + t_0 \right). \tag{1-17}$$

Moreover the phase portrait can be seen in Figure 1.3.

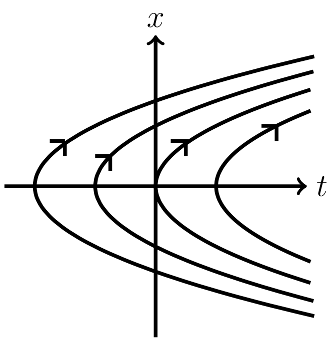


Figure 1.3: *phase portrait of* (1-17)

Since we have only defined Filippov solutions for autonomous systems, when we are working with non-autonomous Fillipov systems we can relate them with an autonomous systems through Definition 1.4.1 and Proposition 1.4.1. Thus we can extend the concept of solution developed in Section 1.3 to non-autonomous Fillipov systems.

In fact, assume that we have a non-autonomous Filippov system

$$Z(t,x) = \begin{cases} X_1(t,x), & \text{if } \theta(t,x) > 0, \\ \\ X_2(t,x), & \text{if } \theta(t,x) < 0 \end{cases}. \tag{1-18}$$

Then, we set $t = x_{n+1}$ and $(x_{n+1}, x) = \mathbf{x}$, and consider its associated autonomous

Filippov system

$$Z(\mathbf{x}) = \begin{cases} X = \begin{pmatrix} 1 \\ X_1(\mathbf{x}) \end{pmatrix}, & \text{if} \quad \theta(\mathbf{x}) > 0, \\ \\ Y = \begin{pmatrix} 1 \\ Y_1(\mathbf{x}) \end{pmatrix}, & \text{if} \quad \theta(\mathbf{x}) < 0. \end{cases} \tag{1-19}$$

Since the solutions of X_1 and X (respectively Y_1 and Y) are related through Proposition 1.4.1 we can construct the solutions (1-18) through solutions of (1-19) obtained by Filippov's convention.

Note that if $\theta : \mathbb{R} \times D \to \mathbb{R}^n$ is given by $\theta(x_{n+1}, x) = x_{n+1} - \tilde{\theta}(x)$, where $\tilde{\theta} : D \to \mathbb{R}$ is a smooth function, then computing the product $X\theta(p)Y\theta(p)$ (where $X\theta(p)$ and $Y\theta(p)$ are the Lie derivatives defined in the previous section) we get

$$\begin{aligned} X\theta(p)Y\theta(p) &= \langle (1, \nabla\tilde{\theta}), (1, X_1) \rangle \langle (1, \nabla\tilde{\theta}), (1, X_2) \rangle \\ &= (1 + \langle \nabla\tilde{\theta}, X_1 \rangle)(1 + \langle \nabla\tilde{\theta}, X_2 \rangle) \\ &= 1 + \langle \nabla\tilde{\theta}, X_2 \rangle + \langle \nabla\tilde{\theta}, X_1 \rangle + \langle \nabla\tilde{\theta}, X_1 \rangle \langle \nabla\tilde{\theta}, X_2 \rangle. \end{aligned} \tag{1-20}$$

Hence, in the case where the term $\langle \nabla(\theta), X_2 \rangle + \langle \nabla(\theta), X_1 \rangle + \langle \nabla(\theta), X_1 \rangle \langle \nabla(\theta), X_2 \rangle$ is sufficiently small we have that $X\theta(p)Y\theta(p) > 0$, which means that we have only crossing points in the switching manifold.

In particular, when X_1 and X_2 are of the form $X_1(\mathbf{x}) = \sum_{i=1}^{k} \varepsilon^i F_i^-(\mathbf{x})$ and $X_2(\mathbf{x}) = \sum_{i=1}^{k} \varepsilon^i F_i^+(\mathbf{x})$, respectively, where $F_i^{\pm}$ are smooth vector fields we have that, for ε sufficiently small, all points of the switching manifold Σ are crossing points, i.e. $\Sigma = \Sigma^c$.

1.5 Chebyshev systems

Another important theory for the development of this work is the Theory of Chebyshev systems. In general words, this theory can be used to find the number of zeros of some function which lives in the set spanned by a finite family of functions, under some conditions.

We start with some necessary definitions and agreements. Let $F = [u_0, \dots, u_n]$ be an ordered set of smooth functions defined on the closed interval $[a, b]$, we denote the vector space spanned by the family F by $Span(F)$, and the maximum number of zeros, counting multiplicity, that any nontrivial function in $Span(F)$ can have will be denoted by $Z(F)$. A classical tool to study $Z(F)$ is the Theory of Chebyshev. we shall establish below.

Definition 1.5.1. *The set $F = [u_0, \ldots, u_n]$ is said to be an Extended Chebyshev System (ET-system) on $[a, b]$ if, and only if, the number $Z(F) \leq n$.*

Remark 8. *When $Z(F) = n + k$, $k \in \mathbb{Z}$, then the set F is called Chebyshev system with positive accuracy k in $[a, b]$. (see [30] for more details).*

Definition 1.5.2. *We say that $F = [u_0, \ldots, u_n]$ is an extended complete Chebyshev system (ECT-system) on a closed interval $[a, b]$ if, and only if, for all $k \in \{1, 2, \ldots, n\}$ the set $[u_0, u_1, \ldots, u_k]$ is an Extended Chebyshev system.*

Remark 9. *The interval $[a, b]$ in the Definitions 1.5.1 and 1.5.2 can be replaced the interval (a, b), since $(a, b) = \bigcup_{i=1}^{\infty}[a + \frac{1}{n}, b - \frac{1}{n}]$.*

There exists a result that can help us to verify when the set F is an extended complete Chebyshev system of a finite set of functions F. In order to establish such result, the next definition is needed.

Definition 1.5.3. *The Wronskian of the ordered set of functions $\{u_0, u_1, \cdots, u_k\}$ is defined by:*

$$W_k(x) = W(u_0, u_1, \ldots, u_k)(x) = \det(M(u_0, u_1, \ldots, u_k)(x))$$

where

$$M(u_0, u_1, \ldots, u_n)(x) = \begin{pmatrix} u_0(x) & u_1(x) & \ldots & u_k(x) \\ u_0'(x) & u_1'(x) & \ldots & u_k'(x) \\ . & & \ldots & . \\ . & & \ldots & . \\ . & & \ldots & . \\ u_0^n(x) & u_1^n(x) & \ldots & u_k^k(x) \end{pmatrix}$$

The following result is stated and proved in [35].

Proposition 1.5.1. *The set $F = [u_0, \ldots, u_n]$ is an Extended complete Chebyshev system on the interval $[a, b]$ if, and only if for all $k \in \{1, 2, \ldots, n\}$ we have that $W(u_0, u_1, \ldots, u_k)(x) \neq 0$ for all $x \in [a, b]$.*

Remark 10. *Notice that every ECT-system is an EC-system, since putting $k = n$ in the Definition 1.5.2 we obtain that $[u_0, u_1, \cdots, u_n]$ is an EC-system. However, there exists sets which are EC-systems, but do not satisfy the definition of an ECT-system. In fact, consider $F = [1, x^2, x]$ in $[-2, 2]$. This ordered family in an EC-system since any element in Span(F) has the form $\alpha_1 + \alpha_2 x^2 + \alpha_2 x$, and thus $Z(F) \leq 2$. But, if we consider the set $[1, x^2]$ we have that $p(x) = x^2 - 1$ is in Span$(1, x^2)$ and has two zeroes in $[-2, 2]$, so $Z([1, x^2]) > 1$, and thus F does not satisfy Definition 1.5.2.*

In Qualitative Theory of Differential Equations, ECT-systems can be used to study the number of isolated periodic orbits (limit cycles) bifurcating from a period annulus, see [34]. More concretely, this technique is useful to get upper bounds for the number of zeros of the Poincaré-Pontryaguin-Melnikov function and to provide lower bounds for the so called weak Hilbert 16th problem, see [40] and [10]. One said this, we enunciate the following theorems which allow us to know the behavior of the isolated zeroes of the functions in $Span(F)$. We say that a zero of a function is simple if the Jacobian matrix of f evaluated at the this point is a non-singular matrix. The following two theorems can be found in [35] and [30] respectively.

Theorem 1.5.2. *Let $F = [u_0, u_1, \dots, u_n]$ be an ECT-system on a closed interval $[a, b]$. Then, the number of isolated zeros for every element of $Span(F)$ does not exceed n. Moreover, for each configuration of $m \leq n$ zeros, taking into account their multiplicity, there exists $f \in Span(F)$ with this configuration of zeros.*

The following theorem extend the theorem above to the Chebyshev systems with positive accuracy (see Remark 8).

Theorem 1.5.3. *Let $F = [u_0, u_1, \dots, u_n]$ be an ordered set of functions on $[a, b]$. Assume that all the Wronskians $W_k(x), k = 0, \dots, n-1$, are non vanishing except $W_n(x)$, which has exactly one zero on (a, b) and this zero is simple. Then, the number of isolated zeros for every element of $Span(F)$ do not exceed $n + 1$. Moreover, for any configuration of $m \leq n + 1$ zeros there exists $f \in Span(F)$ realizing it.*

Finally, the following theorem improves the bounds given by Theorems 1.5.2 and 1.5.3, however this theorem do not ensure that the bound can be reached. For more details, see [30].

Theorem 1.5.4. *Let $F = [u_0, u_1, \dots, u_n]$ be an ordered set of analytic functions in $[a, b]$. Assume that all the v_i zeros of the Wronskian W_i are simple for $i \in \{0, 1, \dots, n\}$. Then the number of isolated zeros for every element of Span(F) does not exceed*

$$n + v_n + v_{n-1} + 2(v_{n-2} + \cdots + v_0) + \mu_{n-1} + \cdots + \mu_3 \tag{1-21}$$

where

$$\mu_i = min(2v_i, v_{i-3} + \cdots + v_0) \quad i \in \{3, 4, \dots, n-1\}. \tag{1-22}$$

CHAPTER 2

Melnikov Funtions for Filippov system with nonlinear switching manifold

This section is devoted to develop the main theory given in [6]. In this work, the authors develop the Melnikov Theory to find lower bounds to the maximum number of limits cycles bifurcating from a piecewise smooth perturbation of a linear center with switching curve given by $y = x^3$. The results in this paper generalizes the results obtained in [9, 21].

Let D be an open set of $\mathbb{R}^d$, $\mathbb{S}^1 = \mathbb{R}/T$ for some period $T > 0$, and N a positive integer. We consider a finite set of C^r $(r \geq N+1)$ functions $\theta_i : D \to \mathbb{S}^1$ for $i = 1, \dots, M$, satisfying $0 < \theta_1 < \theta_2 < \cdots < \theta_M < T$ for all $x \in D$. For completeness, we take $\theta_0 \equiv 0$ and $\theta_{M+1} \equiv T$. We consider the following piecewise smooth system:

$$\dot{x} = \sum_{i=1}^{M} \varepsilon^i F_i(t,x) + \varepsilon^{M+1} R(t,x,\varepsilon). \tag{2-1}$$

with

$$F_i(t,x) = \begin{cases} F_i^0(t,x), & \text{if} \quad 0(x) < t < \theta_1, \\ F_i^1(t,x), & \text{if} \quad \theta_1(x) < t < \theta_2(x), \\ \quad \vdots & \\ F_i^M(t,x), & \text{if} \quad \theta_M(x) < t < T. \end{cases}$$

and

$$R(t,x,\varepsilon) = \begin{cases} R^0(t,x,\varepsilon), & \text{if} \quad 0 < t < \theta_1 \\ R^1(t,x,\varepsilon), & \text{if} \quad \theta_1 < t < \theta_2 \\ \quad \vdots & \\ R^M(t,x,\varepsilon), & \text{if} \quad \theta_M < t < T, \end{cases}$$

where $F_j^i : \mathbb{S}^1 \times D \to \mathbb{R}^d$, $R^j : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^d$ are C^r functions, $r \geq N+1$, for $i = 1, \dots, N$ and $j = 1, \dots, M$, and T-periodic in the variable t. Notice that the switching curve is given by $\Sigma = \{(\theta_i(x), x) : x \in D, \;\; i = 1, \dots, M+1\}$, see Figure 2.1.

Remark 11. *Throughout this work we shall denote* $F^j(t,x,\varepsilon) = \sum_{i=1}^{N} \varepsilon^i F_i^j(t,x) + \varepsilon^{N+1} R^j(t,x,\varepsilon)$.

Remark 12. *The form* (2-1) *is called standard form. To apply the Melnikov Theory, it is necessary to have a system in the standard form.*

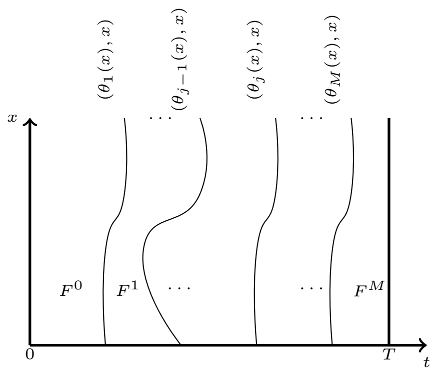


Figure 2.1: *Illustration fields in the Remark* (11).

There exists another theory which allows us to find limit cycles of systems of type (2-1), called Averaging Theory. The Melnikov Theory and the Averaging Theory are classical tools to investigate the existence of periodic solutions of pertubative system of type (2-1). In short, both theories provide a sequence of functions Δ_i, $i = 1, 2, ..., k$, which control the existence of isolated periodic solutions of (2-1). These functions Δ_i are known as Melnikov Functions and Averaging Functions, in each case. It is worth saying that for smooth systems the Melnikov functions are, in some sense, equivalent to Averaging functions at any order. Nevertheless, for piecewise smooth systems of type (2-1), the Melnikov function of order 2 has an extra term, which depends on the jump of discontinuity and on the geometry of the switching curve.

Consider the functions

$$f_1(x) = \int_0^T F_1(s,x)ds, \tag{2-2}$$

and

$$f_2(x) = \int_0^T \left[D_x F_1(s,x) \int_0^s F_1(t,x)dt + F_2(s,x) \right] ds, \tag{2-3}$$

and define the functions

$$\Delta_1(x) = f_1(x) \quad \text{and} \quad \Delta_2(x) = f_2(x) + f_2^*(x), \tag{2-4}$$

where f_1, f_2 are given by (2-2) and (2-3) respectively and

$$f_2^*(x) = \sum_{i=1}^{M} (F_1^{j-1}(\theta_j(x), x) - F_1^j(\theta_j(x), x)) D_x\theta_j(x) \int_0^{\theta_j(x)} F_1(s, x)ds. \tag{2-5}$$

In this section, we prove that the Melnikov functions of order 1 and 2 of (2-1) are given by Δ_1 and Δ_2 in (2-4), respectively, and we also show that they can be used to find limit cycles of system (2-1) for $\varepsilon > 0$ small.

Remark 13. *The functions given in* (2-2) *and* (2-3) *are called the Averaging functions of order* 1 *and* 2 *of the system* (2-1)*, respectively.*

Remark 14. *Note that if the vector field F_1 is continuous then $F_1^{j-1}(\theta_j(x), x) = F_1^j(\theta_j(x), x)$ $j = \overline{1, M}$, thus $f_2^* \equiv 0$. If $\theta_j(x)$ is vertical for all $j = \overline{1, M}$, then $D_x(\theta_j(x)) \equiv 0$ and so $f_2^* \equiv 0$. In both cases, the Melnikov functions coincide with Averaging functions up to order* 2*, see [19].*

2.0.1 Main Theorem

Now we establish and prove the main theorem of [6].

Theorem 2.0.1. *Consider system* (2-1) *and the functions Δ_1 and Δ_2 given by* (2-4)*. Then the following statements hold.*

- **i.** **(*First order*)** *Assume that $a^* \in D$ satisfies $\Delta_1(a^*) = 0$ and $det(J\Delta_1(a^*)) \neq 0$, then for $|\varepsilon| \neq 0$ sufficiently small, there is a unique T-periodic solution $x(t, \varepsilon)$ of system (2-1) such that $x(0, \varepsilon) \to a^*$ as $\varepsilon \to 0$.*
- **ii.** **(*Second order*)** *Assume that $\Delta_1(x) \equiv 0$ and $a^* \in D$ satisfies $\Delta_2(a^*) = 0$ and $det(J\Delta_2(a^*)) \neq 0$, then for $|\varepsilon| \neq 0$ sufficiently small, there exists a unique T-periodic solution $x(t, \varepsilon)$ of system (2-1) such that $x(0, \varepsilon) \to a^*$ as $\varepsilon \to 0$.*

Here $J\Delta_i(x)$ denotes the Jacobian matrix of Δ_i evaluated at x.

Proof. Let x_0 be a fixed element in D and consider the function

$$\begin{aligned} G_0 : \mathbb{S}^1 \times D \times (-\varepsilon_0, \varepsilon_0) &\to \mathbb{R} \\ (t, x, \varepsilon) &\to t - \theta_1(\varphi_0(t, x, \varepsilon)) \end{aligned}$$

where $\varphi_0(t, x, \varepsilon)$ is solution of

$$\begin{cases} \dot{x} = F^0(t, x, \varepsilon) \\ x(0) = x_0. \end{cases}$$

Note that G_0 is a smooth function and from (2-10) and $\varphi_0(x,t,0) = x$ for all $t \in \mathbb{S}^1$ and $x \in D$. Thus

$$\begin{aligned} G_0(\theta_0(x_0), x_0, 0) &= \theta_1(x_0) - \theta_1(\varphi_0(\theta_1(x_0), x_0, 0) \\ &= \theta_1(x_0) - \theta_1(x_0) \\ &= 0. \end{aligned}$$

Also, $\frac{\partial G_0}{\partial t}(\theta_1(x_0), x_0, 0) = 1 \neq 0$. From the Implicit Function Theorem, there exists a smooth function

$$\alpha_1 : (x_0 - \delta, x_0 + \delta) \times (-\varepsilon_0', \varepsilon_0') \to \mathbb{R}$$

for some $\delta > 0$ and $\varepsilon_0' \in (0, \varepsilon_0)$ such that $\alpha_1(x_0, 0) = \theta_1(x_0)$, and

$$\alpha_1(x,\varepsilon) = \theta_1(\varphi_0(\alpha_1(x,\varepsilon), x, \varepsilon), \quad \text{for each } \ x \in (x_0 - \delta, x_0 + \delta) \ \text{, and } \ \varepsilon \in (-\varepsilon_0, \varepsilon_0).$$

Hence, $\alpha_1(x_0, \varepsilon)$ denotes the time which is necessary to the trajectory $\varphi_0(\cdot, x, \varepsilon)$, starting at $\varphi_0(\alpha_0(x,\varepsilon), x, \varepsilon) \in D$ intersects the switching curve $\Sigma_1 = \{(\theta_1(x), x) : x \in D\}$. We refer $\alpha_1(x_0, \varepsilon)$ as the flying time of $\varphi_0(t, x, \varepsilon)$. Analogously, we construct the fiying time $\alpha_j(x_0, \varepsilon)$, $j = 1, \cdots, M$, which denotes the time sent by the trajectory $\varphi_{j-1}(\cdot, x, \varepsilon)$ starting at $\varphi_{j-1}(\alpha_{j-1}(x,\varepsilon), x, \varepsilon)$ to reach the curve $\Sigma_j = \{(\theta_j(x), x) : x \in D\}$ (see Figure 2.2). In this case, we have

$$\alpha_j(x,\varepsilon) = \begin{cases} 0, & \text{if } \ j = 0, \\ \theta_j(\varphi_{j-1}(\alpha_j(x,\varepsilon), x, \varepsilon)), & \text{if } \ 1 \le j \le M, \\ T, & \text{if } \ j = M+1. \end{cases} \tag{2-6}$$

Now, we write the solution of (2-1) as

$$\varphi(t,x,\varepsilon) = \begin{cases} \varphi_0(t,x,\varepsilon), & \text{if } \ 0 \le t \le \alpha_1(x,\varepsilon), \\ \varphi_1(t,x,\varepsilon), & \text{if } \ \alpha_1(x,\varepsilon) \le t \le \alpha_2(x,\varepsilon), \\ \quad \vdots & \\ \varphi_M(t,x,\varepsilon), & \text{if } \ \alpha_M(x,\varepsilon) \le t \le T. \end{cases} \tag{2-7}$$

Moreover

$$\frac{\partial \varphi_j}{\partial t}(t,x,\varepsilon) = F^j(t, \varphi_j(t,x,\varepsilon), \varepsilon), \quad for \quad j = \overline{1, M}, \tag{2-8}$$

where

$$\begin{cases} \varphi_0(0,x,\varepsilon) = x, \\ \varphi_j(\alpha_j(x,\varepsilon), x, \varepsilon) = \varphi_j(\alpha_j(x,\varepsilon), x, \varepsilon), \quad \text{for} \quad j = \overline{1, M} \end{cases} \tag{2-9}$$

See Figure 11.

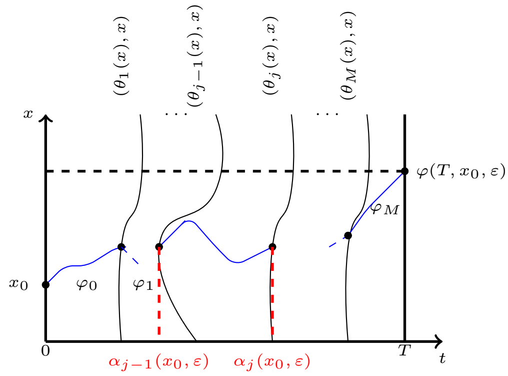


Figure 2.2: *Solution of a piecewise smooth system* (2-1).

Remark 15. *$\varphi_j(t, x, \varepsilon)$ and $\alpha_j(x, \varepsilon)$ are C^r functions for $j = 0, \ldots, M$. In fact, from Theorems 1.1.2 and 1.1.3, for $j = 0$ and $(x, \varepsilon) \in D \times (-\varepsilon_0, \varepsilon_0)$ there exist $\delta > 0$ and $a > 0$ such that the solution $\varphi_0(t, x, \varepsilon)$ of (2-1) is C^r in $[-a, a] \times B_\delta(x) \times B_\delta(\varepsilon)$ $(r \geq M + 1)$, and from (2-6) we have that α_0 is C^r. In a similar way we can prove the assertion for every j.*

We notice that the recurrence (2-8) describes a set of initial value problems. Therefore, we can write them in the integral form

$$\begin{aligned}
\varphi_0(t, x, \varepsilon) &= x + \int_0^t F^0(s, \varphi_0(s, x, \varepsilon), \epsilon) ds \\
\varphi_1(t, x, \varepsilon) &= x + \int_0^{\alpha_1(x,\varepsilon)} F^0(s, \varphi_0(s, x, \varepsilon), \varepsilon) ds + \int_{\alpha_1(x,\varepsilon)}^t F^1(s, \varphi_0(s, x, \varepsilon), \varepsilon) ds \\
&\ \ \vdots \\
\varphi_j(t, x, \varepsilon) &= \varphi_{j-1}(\alpha_j(t, \varepsilon), x, \varepsilon) + \int_{\alpha_j(x,\epsilon)}^t F^j(s, \varphi_j(s, x, \varepsilon), \varepsilon) ds,
\end{aligned} \tag{2-10}$$

for $j = \overline{1, M}$. So, we may compute

$$\begin{aligned}
\varphi_M(t, x, \varepsilon) =& x + \sum_{j=1}^{M} \int_{\alpha_{j-1}(x,\varepsilon)}^{\alpha_j(x,\varepsilon)} F^{j-1}(s, \varphi_{j-1}(s, x, \varepsilon), \varepsilon) ds \\
&+ \int_{\alpha_M(x,\varepsilon)}^t F^M(s, \varphi_M(s, x, \varepsilon), \varepsilon) ds.
\end{aligned} \tag{2-11}$$

We construct the displacement function

$$\Delta(x, \varepsilon) = \varphi(T, x, \varepsilon) - x = \varphi_M(T, x_0, \varepsilon) - x. \tag{2-12}$$

From Remark (15) the displacement function (2-12) is a C^r function. So, expanding Δ in

Taylor series, around $\varepsilon = 0$, we get

$$\Delta(x, \varepsilon) = \Delta_0(x) + \varepsilon\Delta_1(x) + \varepsilon^2\Delta_2(x) + O(\varepsilon^3). \tag{2-13}$$

Note that due to (2-11) and Remark 11, we have $\Delta_0(x) = 0$ since $\varphi_M(T, x, 0) = x$. The functions Δ_1 and Δ_2 are called the Melnikov functions of order 1 and 2 of (2-1), respectively. Due to (2-11), the displacement function (2-12) is given by

$$\Delta(x, \varepsilon) = \varphi_M(T, x, \varepsilon) - x = \sum_{j=1}^{M+1} \int_{\alpha_{j-1}(x,\varepsilon)}^{\alpha_j(x,\varepsilon)} F^{j-1}(s, \varphi_{j-1}(s, x, \varepsilon), \varepsilon) ds \tag{2-14}$$

Now we are interested in finding the Taylor expansion, around $\varepsilon = 0$, of the right-hand side of (2-14) and then match the terms with (2-13). In this sense we compute the Taylor expansion up to order 2 of the right hand side of 2-14. We shall prove that, using Remark (11), we have

$$\begin{aligned}\int_0^{\alpha_1(x,\varepsilon)} F^0(s, \varphi_0(s, x, \varepsilon), \varepsilon) ds = &\int_0^{\alpha_1(x,\varepsilon)} \left[\varepsilon F_1^0(s, \varphi_0(s, x, \varepsilon)) + \varepsilon^2 F_2^0(s, \varphi_0(s, x, \varepsilon))\right. \\ &\left. + \, O(\varepsilon^3)\right] ds \\ =&\varepsilon \left(\int_0^{\theta_1(x)} F_1^0(s, x) ds\right) + \varepsilon^2 \left(F_1^0(\theta_1(x), x)\frac{\partial \alpha_1}{\partial \varepsilon}(x, 0)\right. \\ &\left. + \int_0^{\theta_1(x)} \left[D_x F_1^0(s, x)\frac{\partial \varphi_0}{\partial \varepsilon}(s, x, 0) + F_2^0(s, x)\right] ds\right) \\ &+ O(\varepsilon^3).\end{aligned} \tag{2-15}$$

In fact, to prove the equality (2-15), consider

$$g(\varepsilon) = \int_0^{\alpha_1(x,\varepsilon)} \left[\varepsilon F_1^0(s, \varphi_0(s, x, \varepsilon)) + \varepsilon^2 F_2^0(s, \varphi_0(s, x, \varepsilon)) + O(\varepsilon^3)\right] ds \tag{2-16}$$

and

$$G(z, \varepsilon) = \int_0^{z} \left[\varepsilon F_1^0(s, \varphi_0(s, x, \varepsilon)) + \varepsilon^2 F_2^0(s, \varphi_0(s, x, \varepsilon)) + O(\varepsilon^3)\right] ds. \tag{2-17}$$

Thus

$$g(\varepsilon) = G(\alpha_1(x, \varepsilon), \varepsilon). \tag{2-18}$$

The idea is to find the Taylor expansion, around $\varepsilon = 0$, up to order 2 of $g(\varepsilon)$, and write

$$g(\varepsilon) = g(0) + \varepsilon g'(0) + \frac{1}{2}\varepsilon^2 g''(0) + \mathcal{O}(\varepsilon^3). \tag{2-19}$$

Due the definition of g we get that $g(0) = 0$. Moreover we get

$$g'(\varepsilon)\big|_{\varepsilon=0} = \left[\partial_z G(\alpha_1(x, \varepsilon), \varepsilon)\partial_\varepsilon \alpha_1(x, \varepsilon) + \partial_\varepsilon G(\alpha_1(x, \varepsilon), \varepsilon)\right]\big|_{\varepsilon=0} \tag{2-20}$$

and, from Fundamental Theorem of Calculus,

$$\partial_z G(z,\varepsilon) \quad = \varepsilon F_1^0(z,\varphi_0(s,x,\varepsilon)) + \varepsilon^2 F_2^0(z,\varphi_0(s,x,\varepsilon)) + \mathcal{O}(\varepsilon^2). \tag{2-21}$$

Thus

$$\partial_z G(\alpha_1(x,\varepsilon),\varepsilon)\partial_\varepsilon \alpha_1(x,\varepsilon)\Big|_{\varepsilon=0} = 0. \tag{2-22}$$

On the other hand we have

$$\begin{aligned}\partial_\varepsilon G(z,\varepsilon) = \int_0^z \Big[& F_1^0(s,\varphi_0(s,x,\varepsilon)) + \varepsilon D_x F_1^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial \varphi_0}{\partial \varepsilon}(s,x,\varepsilon) \\ & + 2\varepsilon F_2^0(s,\varphi_0(s,x,\varepsilon)) + \varepsilon^2 D_x F_2^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial \varphi_0}{\partial \varepsilon}(s,x,0) + \mathcal{O}(\varepsilon^2)\Big] ds.\end{aligned} \tag{2-23}$$

So,

$$\partial_\varepsilon G(\alpha_1(x,\varepsilon),\varepsilon)\Big|_{\varepsilon=0} = \int_0^{\alpha_1(x,0)} F_1^0(s,\varphi_0(s,x,0))ds = \int_0^{\theta_1(x)} F_1^0(s,x)ds. \tag{2-24}$$

Therefore,

$$g'(0) = \int_0^{\theta_1(x)} F_1^0(s,x)ds. \tag{2-25}$$

In order to conclude the calculation, we are interested in computing $\frac{1}{2}\varepsilon^2 g''(0)$. For this purpose we use (2-20) to compute

$$g''(\varepsilon)\Big|_{\varepsilon=0} = \partial_\varepsilon \left(\partial_z G(\alpha_1(x,\varepsilon),\varepsilon)\partial_\varepsilon \alpha(x,\varepsilon) + \partial_\varepsilon G(\alpha_1(x,\varepsilon),\varepsilon)\right)\Big|_{\varepsilon=0}. \tag{2-26}$$

From (2-21), we have

$$\begin{aligned}\partial_\varepsilon \left[\partial_z G(\alpha_1(x,\varepsilon),\varepsilon)\partial_\varepsilon \alpha(x,\varepsilon)\right] = & \left[\varepsilon D F_1^0(\alpha_1(x,\varepsilon),\varphi_0(s,x,\varepsilon))\nabla(\alpha_1(x,\varepsilon),\varphi_0(s,x,\varepsilon))\right. \\ & + F_1^0(\alpha_1(x,\varepsilon),\varphi_0(s,x,\varepsilon)) 2\varepsilon F_2^0(z,\varphi_0(s,x,\varepsilon)) \\ & + \varepsilon^2 D F_2^0(z,\varphi_0(s,x,\varepsilon))\nabla(\alpha_1(x,\varepsilon),\varphi_0(s,x,\varepsilon)) \\ & \left. + \mathcal{O}(\varepsilon)\right]\partial_\varepsilon \alpha_1(x,\varepsilon) + \left[\varepsilon F_1^0(\alpha_1(x,\varepsilon),\varphi_0(s,x,\varepsilon))\right. \\ & \left. + \varepsilon^2 F_2^0(\alpha_1(x,\varepsilon),\varphi_0(s,x,\varepsilon)) + \mathcal{O}(\varepsilon^2)\right]\partial_\varepsilon^2 \alpha_1(x,\varepsilon).\end{aligned} \tag{2-27}$$

Thus,

$$\begin{aligned}\partial_\varepsilon \left[\partial_z G(\alpha_1(x,\varepsilon),\varepsilon)\partial_\varepsilon \alpha(x,\varepsilon)\right]\Big|_{\varepsilon=0} & = F_1^0(\alpha_1(x,0),\varphi_0(x,s,0))\partial_\varepsilon \alpha_1(x,0) \\ & = F_1^0(\theta_1(x),x)\partial_\varepsilon \alpha_1(x,0).\end{aligned} \tag{2-28}$$

From (2-23) we have

$$\begin{aligned}\partial_\varepsilon^2 G(\alpha_1(x,\varepsilon),\varepsilon) =& \partial_\varepsilon\left(\int_0^{\alpha_1(x,\varepsilon)}\Big[F_1^0(s,\varphi_0(s,x,\varepsilon))+\varepsilon DF_1^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,\varepsilon)\right.\\ &+\ 2\varepsilon F_2^0(s,\varphi_0(s,x,\varepsilon))+\varepsilon^2 DF_2^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,\varepsilon)\\ &+\ \left.\mathcal{O}(\varepsilon^2)\Big]\,ds\right).\end{aligned} \tag{2-29}$$

So, consider

$$\begin{aligned}g_1(\varepsilon) =& \int_0^{\alpha_1(x,\varepsilon)}\Big[F_1^0(s,\varphi_0(s,x,\varepsilon))+\varepsilon DF_1^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,\varepsilon)\\ &+ 2\varepsilon F_2^0(s,\varphi_0(s,x,\varepsilon))+\varepsilon^2 DF_2^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,\varepsilon)+\mathcal{O}(\varepsilon^2)\Big]\,ds\end{aligned} \tag{2-30}$$

and

$$\begin{aligned}G_1(z,\varepsilon) =& \int_0^{z}\Big[F_1^0(s,\varphi_0(s,x,\varepsilon))+\varepsilon DF_1^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,\varepsilon)\\ &+ 2\varepsilon F_2^0(s,\varphi_0(s,x,\varepsilon))+\varepsilon^2 DF_2^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,\varepsilon)+\mathcal{O}(\varepsilon^2)\Big]\,ds.\end{aligned} \tag{2-31}$$

Notice that

$$g_1(\varepsilon) = G_1(\alpha_1(x,\varepsilon),\varepsilon), \tag{2-32}$$

thus

$$\partial_\varepsilon^2 G(\alpha_1(x,\varepsilon),\varepsilon) = \partial_\varepsilon g_1(\varepsilon) = \partial_z G_1(\alpha_1(x,\varepsilon),\varepsilon)\partial_\varepsilon\alpha_1(x,\varepsilon)+\partial_\varepsilon G_1(\alpha_1(x,\varepsilon),\varepsilon) \tag{2-33}$$

and from Fundamental of Theorem Calculus and (2-31), we get

$$\begin{aligned}\partial_z G_1(z,\varepsilon) =& F_1^0(z,\varphi_0(z,x,\varepsilon))+\varepsilon DF_1^0(z,\varphi_0(z,x,\varepsilon))\frac{\partial\varphi_0}{\partial\varepsilon}(z,x,\varepsilon)\\ &+ 2\varepsilon F_2^0(z,\varphi_0(z,x,\varepsilon))+\varepsilon^2 DF_2^0(z,\varphi_0(z,x,\varepsilon))\frac{\partial\varphi_0}{\partial\varepsilon}(z,x,\varepsilon)+\mathcal{O}(\varepsilon^2).\end{aligned} \tag{2-34}$$

Thus

$$\begin{aligned}\partial_z G_1(\alpha_1(x,\varepsilon),\varepsilon)\partial_\varepsilon\alpha_1(x,\varepsilon) = &\Big[F_1^0(\alpha_1(x,\varepsilon),\varphi_0(\alpha_1(x,\varepsilon),x,\varepsilon))\\ &+\ \varepsilon DF_1^0(\alpha_1(x,\varepsilon),\varphi_0(\alpha_1(x,\varepsilon),x,\varepsilon))\frac{\partial\varphi_0}{\partial\varepsilon}(\alpha_1(x,\varepsilon),x,\varepsilon)\\ &+\ 2\varepsilon F_2^0(\alpha_1(x,\varepsilon),\varphi_0(\alpha_1(x,\varepsilon),x,\varepsilon))\\ &+\ \varepsilon^2 DF_2^0(\alpha_1(x,\varepsilon),\varphi_0(\alpha_1(\alpha_1(x,\varepsilon),\varepsilon),x,\varepsilon))\\ &\cdot\ \frac{\partial\varphi_0}{\partial\varepsilon}(\alpha_1(x,\varepsilon),x,\varepsilon)+\mathcal{O}(\varepsilon^2)\Big]\,\partial_\varepsilon\alpha_1(x,\varepsilon)\end{aligned} \tag{2-35}$$

From (2-33) we need to compute

$$\partial_\varepsilon G_1(\alpha_1(x,\varepsilon),\varepsilon). \tag{2-36}$$

Due to (2-31), we get

$$\begin{aligned}\partial_\varepsilon G_1(z,\varepsilon) = \int_0^z \Big[& D_x F_1^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,\varepsilon) + D_x F_1^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,\varepsilon)\\ & + \varepsilon D_x^2 F_1^0(s,\varphi_0(s,x,\varepsilon))\left(\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,\varepsilon)\right)^2 + \varepsilon D F_1^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial^2\varphi_0}{\partial\varepsilon^2}(s,x,\varepsilon)\\ & + 2F_2^0(s,\varphi_0(s,x,\varepsilon)) + 2\varepsilon D_x F_2^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,\varepsilon)\\ & + 2\varepsilon D F_2^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,\varepsilon) + \varepsilon^2 D_x F_2^0(s,\varphi_0(s,x,\varepsilon))\left(\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,\varepsilon)\right)^2\\ & + \varepsilon^2 D F_2^0(s,\varphi_0(s,x,\varepsilon))\frac{\partial^2\varphi_0}{\partial\varepsilon^2}(s,x,\varepsilon) + \mathcal{O}(\varepsilon)\Big]\, ds.\end{aligned} \tag{2-37}$$

Now, from (2-35) and (2-37), we get

$$\begin{aligned}\partial_\varepsilon^2 G(\alpha_1(x,\varepsilon),\varepsilon)\Big|_{\varepsilon=0} =& F_1^0(\theta_1(x),x)\partial_\varepsilon\alpha_1(x,0) + \int_0^{\theta_1(x)} \Big[2D_x F_1^0(s,x)\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,0)\\ & + 2F_2^0(s,x)\Big]\, ds\end{aligned} \tag{2-38}$$

From (2-26), (2-28) and (2-38), it follows that

$$g''(0) = 2\left(\int_0^{\theta_1(x)} \left[D_x F_1^0(s,x)\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,0) + F_2^0(s,x)\right] ds + F_1^0(\theta_1(x),x)\partial_\varepsilon\alpha_1(x,0)\right). \tag{2-39}$$

From (2-16), (2-25) and (2-39) we obtain

$$\begin{aligned}g(\varepsilon) =& \varepsilon\int_0^{\theta_1(x)} F_1^0(s,x)ds + \varepsilon^2\Big(\int_0^{\theta_1(x)} \left[D_x F_1^0(s,x)\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,0) + F_2^0(s,x)\right] ds\\ & + F_1^0(\theta_1(x),x)\partial_\varepsilon\alpha_1(x,0)\Big) + \mathcal{O}(\varepsilon^3),\end{aligned} \tag{2-40}$$

which proves (2-15). In order to finish the computation of the right-hand side of (2-15) we have to compute the terms $\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,0)$ and $\partial_\varepsilon\alpha_1(x,0)$. Expanding the equation

$$\varphi_0(t,x,\varepsilon) = x + \int_0^t F^0(s,\varphi_0(s,x,\varepsilon),\varepsilon)ds,$$

around $\varepsilon = 0$, and using Remark 11, we have that

$$x + \varepsilon\frac{\partial\varphi_0}{\partial\varepsilon}(s,x,0) + O(\varepsilon^2) = x + \varepsilon\int_0^t F_1^0(s,x)ds + O(\varepsilon^2).$$

So we get

$$\frac{\partial \varphi_0}{\partial \varepsilon}(s,x,0) = \int_0^t F_1^0(s,x)ds. \tag{2-41}$$

In the same way, expanding the equation $\alpha_1(x,\varepsilon) = \theta_1(\varphi_0(\alpha_1(x,\varepsilon),x,\varepsilon))$, around $\varepsilon = 0$, and since $\varphi_0(t,x,0) = x$, we obtain

$$\begin{aligned}\alpha_1(x,0) + \varepsilon\frac{\partial \alpha_1}{\partial \varepsilon}(x,0) + O(\varepsilon^2) = \theta_1(x) + \varepsilon D_x\theta_1(x)\Bigg(&\frac{\partial \varphi_0}{\partial t}(\theta_1(x),x,0)\frac{\partial \alpha_1}{\partial \varepsilon}(x,0)\\ &+\frac{\partial \varphi_0}{\partial \varepsilon}(\theta_1(x),x,0)\Bigg) + O(\varepsilon^2)\end{aligned} \tag{2-42}$$

Now, from the definition of $\alpha_j(x,\varepsilon)$ $j = \overline{1,M}$, see (2-6), we have that $\alpha_1(x,0) = \theta_1(x)$. Due to (2-8) and Remark 11 we have $\frac{\partial\varphi_0}{\partial t}(\theta_1(x),x,0) = 0$ and, using (2-41), we get

$$\frac{\partial \alpha_1}{\partial \varepsilon}(x,0) = D_x\theta_1(x)\int_0^{\theta_1(x)} F_1^0(s,x)ds. \tag{2-43}$$

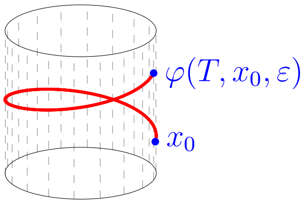
Replacing (2-41) and (2-43) in (2-15), the first term in the sum (2-14) is given by

$$\begin{aligned}\int_0^{\alpha_1(t,\varepsilon)} F^0(s,\varphi_0(s,x,\varepsilon),\varepsilon)ds =&\varepsilon\left(\int_0^{\theta_1(x)} F_1^0(s,x)ds\right)\\ &+\varepsilon^2\Bigg(\int_0^{\theta_1(x)}\left[D_xF_1^0(s,x)\int_0^s F_1^0(s,x)ds + F_2^0(s,x)\right]ds\\ &+F_1^0(\theta_1(x),x)D_x\theta_1(x)\int_0^{\theta_1(x)} F_1^0(s,x)ds\Bigg) + O(\varepsilon^3)\end{aligned} \tag{2-44}$$

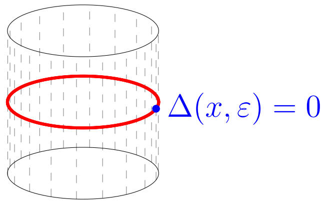
Similarly, we can compute other terms of the sum in (2-14). More specifically we obtain

$$\begin{aligned}\int_{\alpha_{j-1}(x,\varepsilon)}^{\alpha_j(t,\varepsilon)} F^{j-1}(s,\varphi_{j-1}(s,x,\varepsilon),\varepsilon)ds =&\varepsilon\left(\int_{\theta_{j-1}(x)}^{\theta_j(x)} F_{j-1}^0(s,x)ds\right)\\ &+\varepsilon^2\Bigg(F_1^{j-1}(\theta_j(x),x)D_x\theta_j(x)\int_0^{\theta_j(x)} F_1^0(s,x)ds\\ &+\int_{\theta_{j-1}}^{\theta_j(x)}\Bigg[DF_1^{j-1}(s,x)\int_0^s F_1^0(s,x)ds\\ &+F_2^{j-1}(s,x)\Bigg]ds - F_1^{j-1}(\theta_{j-1}(x),x)D_x\theta_{j-1}(x)\\ &\cdot\int_0^{\theta_{j-1}(x)} F_1^0(s,x)ds\Bigg) + O(\varepsilon^3)\end{aligned} \tag{2-45}$$

From (2-44) and (2-45) we have that the Melnikov functions of first and second order are given by equations (2-4) and (2-5), respectively. Now, from the definition of the displacement function (2-12) we have that the T-periodic solutions $x(t,\varepsilon)$ of system (2-1) satisfying $x(0,\varepsilon) = x$ are in one-to-one correspondence with the zeros of the equation $\Delta(x,\varepsilon) = 0$ (see Figure 2.3).



(a) $\Delta(x, \varepsilon) \neq 0$.



(b) $\Delta(x, \varepsilon) = 0$.

Figure 2.3: *Identifiying the points* $(0, x)$ *and* $(0, T)$ *in the Figure 2.2 we get that the zeros of* $\Delta(x, \varepsilon)$ *are in correspondence with* T*-periodic solutions* $x(t, \varepsilon)$ *of the system* (2-1).

From (2-13), if we define $\tilde{\Delta}(x, \varepsilon) = \Delta(x, \varepsilon)/\varepsilon = \Delta_1(x) + \mathcal{O}(\varepsilon)$ and from hypothesis of item 1 in Theorem 2.0.1, it follows that

$$\tilde{\Delta}(a^*, 0) = \Delta_1(a^*, 0) = 0 \quad \text{and} \quad det\left(\frac{\partial \tilde{\Delta}}{\partial x}(a^*)\right) = det\left(J\Delta_1(a^*)\right) \neq 0.$$

Therefore, from the Implicit Function Theorem, we get the existence of a unique C^r function $a(\varepsilon) \in D$ such that $a(0) = a^*$ and $\Delta(a(\varepsilon), \varepsilon) = \varepsilon\tilde{\Delta}(a(\varepsilon), \varepsilon) = 0$ for every $|\varepsilon| \neq 0$ sufficiently small. So, it is enough to define $x(0, \varepsilon) = a(\varepsilon)$ to get the conclusion of item 1 of Theorem 2.0.1.

Finally, assuming that $\Delta_1(x) \equiv 0$, define $\tilde{\Delta}(x, \varepsilon) = \Delta(x, \varepsilon)/\varepsilon^2 = \Delta_2(x) + \mathcal{O}(\varepsilon)$. The proof of statement 2 follows analogously to the proof of statement 1. □

Remark 16. *It is worth to say that Theorem 2.0.1 is a sufficient condition for finding limit cycles. In fact, it is possible that there exist zeros of the Poincaré map* Δ *which are not zeros of the Melnikov functions* Δ_i $i = 1, 2,$*. Hence, we can not say that this methodology detects all the limit cycles of* (2-1).

CHAPTER 3

Finding bifurcating limits cycles in planar system

3.1 Statement of the problem

Piecewise smooth systems appear in the field of nature, control systems [26], mechanics [8], nonlinear oscillations [4]. Thus many researches began to study the dynamical behavior of such systems in recent years. There is an interest in studying the existence of limit cycles and the maximum number of limit cycles if, it exists. Note that this problem is, in some sense, an extension of 16^{th} Hilbet's problem to piecewise smooth vectors fields. So there exists an increasing interest on establishing a uniform upper bound for the maximum number of limit cycles of piecewise polynomial differential systems.

There exists an extensive literature in the existence of limit cycles in piecewise smooth vector fields separated by an straight line, see for example [15, 16, 24]. It is worth saying that it does not exist a piecewise linear system separated by a straight line with more than 3 limit cycles, but if we change the straight line by a general curve, the statement is not true anymore. The existence and characterization of upper bounds to piecewise polynomial system in an open problem. In fact, in [7] and [29] the authors conjectured and proved, respectively, the existence of systems having an arbitrary number of limit cycles, which depends on the shape of the discontinuity set.

Recently, the study of piecewise linear systems with two zones separated by $y = x^n$ was explored in [6] and [3]. For this reason we are interested in finding limits cycles bifurcating from a linear center under a piecewise smooth polynomial perturbation of the form

$$Z(x,y) = \begin{cases} \begin{pmatrix} y + \sum_{i=1}^{k} \varepsilon^i P_i^+(x,y) \\ -x + \sum_{i=1}^{k} \varepsilon^i Q_i^+(x,y) \end{pmatrix}, & \text{if} \quad y \geq x^n \\ \\ \begin{pmatrix} y + \sum_{i=1}^{k} \varepsilon^i P_i^-(x,y) \\ -x + \sum_{i=1}^{k} \varepsilon^i Q_i^-(x,y) \end{pmatrix}, & \text{if} \quad y \leq x^n \end{cases} \tag{3-1}$$

In [6] the authors consider system (3-1) with $P_i^{\pm}$ and $Q_i^{\pm}$ being polynomials of degree 1, $k = 1, 2$ and $n = 3$, and using Melnikov functions they found limit cycles of system (3-1). In [3] the authors study (3-1) for a general $n \in \mathbb{Z}$, $k = 1, 2 \ldots, 6$, and polynomials $P_i^{\pm}$ and $Q_i^{\pm}$ are of degree 1, and using the Melnikov functions they found limit cycles of the system (3-1). In [39] the authors consider the same system (3-1) with $P_i^{\pm}$ and $Q_i^{\pm}$ being polynomials of any degree, $k = 1$ and $n \in \mathbb{Z}$ and they find lower and upper bounds for the maximum number of isolated zeroes of the Melnikov function of order 1.

Based on that, in this work, we study (3-1) and instead of considering that $P_i^{\pm}$ and $Q_i^{\pm}$ are polynomials of degree 1, we consider that $P_i^{\pm}$ and $Q_i^{\pm}$ are polynomials of degree 2, $i = 1, 2, \cdots, k$, given by

$$\begin{aligned} P_i^+(x,y) &= a_{0i} + a_{1i}x + a_{2i}y + a_{3i}xy + a_{4i}x^2 + a_{5i}y^2, \\ P_i^-(x,y) &= \alpha_{0i} + \alpha_{1i}x + \alpha_{2i}y + \alpha_{3i}xy + \alpha_{4i}x^2 + \alpha_{5i}y^2, \\ Q_i^+(x,y) &= b_{0i} + b_{1i}x + b_{2i}y + b_{3i}xy + b_{4i}x^2 + b_{5i}y^2, \\ Q_i^-(x,y) &= \beta_{0i} + \beta_{1i}x + \beta_{2i}y + \beta_{3i}xy + \beta_{4i}x^2 + \beta_{5i}y^2, \end{aligned} \tag{3-2}$$

and we use Melnikov Theory, developed in Chapter 2 to detect bifurcating limit cycles of (3-1) for $\varepsilon > 0$ small and we obtain lower bounds for the maximum number of limit cycles of system (3-1). We sumarize it in the following theorem, which is the main result of this work.

We denote by $m_\ell(n)$ the maximum number of simple zeros that the first non-vanishing Melnikov function Δ_ℓ can have for any choice of parameters $\alpha_{ji}, a_{ji}, \beta_{ji}, b_{ji} \in \mathbb{R}$, $i = 1, 2$, $j = 0, 1, 2$ and $\ell = 1, 2$.

Theorem 3.1.1. *Consider the planar piecewise polynomial differential system (3-1) where $P_i^{\pm}(x,y)$ and $Q_i^{\pm}(x,y)$ are given by (3-2). For $n \in \mathbb{Z}^+$ and $\ell = 1, 2$, $m_\ell(n)$ assume the following values:*

i. *$m_1(1) = 2$,$m_1(2) = 4$,$m_1(3) = 5$, $m_1(n) = 7$ for $n \geq 5$ odd, and $6 \leq m_1(n) \leq 7$ for $n \geq 4$ even;*

ii. *$m_2(1) = 3$.*

Moreover all the lower bounds above can be achieved. Consequently, $H(1) \geq 3$, $H(2) \geq 4$, $H(3) \geq 5$, $H(n) \geq 7$ for $n \geq 5$ odd, and $H(n) \geq 6$ for $n \geq 4$ even, where $H(n)$ is the maximum number of limit cycles of (3-1) with $P_i^{\pm}$ and $Q_i^{\pm}$, $i = 1, 2$ given by, (3-2).

This Theorem is proved in the remaining chapter. The results of item (**i**) of Theorem 3.1.1 is summarized in the following Table

n	$m_1(n)$
1	2
2	4
3	5
$n > 2$, n even	$6 \leq m_1(n) \leq 7$
$n > 3$, n odd	7

Table 3.1: *Values of $m_1(n)$.*

To conclude, we compare our results with the results obtained in references [6, 3, 37] in the following table. We can see that considering polynomial perturbations of degree 2 instead of polynomial perturbations of degree 1, we can detect the existence of more limit cycles through first order Melnikov Theory.

n	$m_1(n)$ in [6]	$m_1(n)$ in [3, 37]
1		1
2		3
3	3	3
$n > 2$, n even		4
$n > 3$, n odd		3

Table 3.2: *Lower bounds of $m_1(n)$ obtained in different works.*

Remark 17. *In [2], the authors prove that $m_1(n) = 6$ for $n \geq 4$ even, while we found $6 \leq m_1(n) \leq 7$ for $n \geq 4$ even.*

3.1.1 General procedure for finding Melnikov Functions

Note that Melnikov Theory applies to non-autonomous system and (3-1) is an autonomous system, so we must bring (3-1) into a non-autonomous context. In general we should do the following steps to compute the Melnikov functions of (3-1) and to study their zeroes.

1. Apply a change of variables at $Z(x, y)$ to polar coordinates (r, θ), i.e. consider $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
2. Take θ in the polar coordinates as an independent variable in order to obtain a system of the form $\frac{dr}{d\theta} = F(r, \theta, \varepsilon)$.
3. Compute the Taylor expansion of $F(r, \theta, \varepsilon)$ around $\varepsilon = 0$ up to the order k, writing $F(r, \theta, \varepsilon) = \displaystyle\sum_{i=1}^{k} \varepsilon^i F_i(r, \theta) + \mathcal{O}(\varepsilon^{k+1})$. Note that $\dot{r} = F(r, \theta, \varepsilon)$ is in the standard form (2-1).
4. Compute the Melnikov functions with $T = 2\pi$. So far the procedure is the same in all cases, but there is a change in the expressions depending on the discontinuity set.
5. Identify if $n \in \mathbb{Z}^+$ is odd or even in order to get the angle θ_1 between $y = x^n$ and the x-axis.
6. Obtain the Melnikov functions Δ_i $i = 1, 2, \ldots, n$ in term of $\cos(\theta_1)$, $\sin(\theta_1)$ and r and powers of them.
7. Use the relations $\cos(\theta_1) = \frac{x}{r}$, $\sin(\theta_1) = \frac{y}{r} = \frac{x^n}{r}$ and $r = \sqrt{x^2 + y^2} = \sqrt{x^2 + x^{2n}}$ and write $\Delta_i(x) = \displaystyle\sum_{j=1}^{r_i} \mu_j u_j(x)$, $\{\mu_j\}_{j=1}^{r_i} \subset \mathbb{R}$.
8. Check if this family $\{u_j\}_{j=1}^{r_i}$ is an ECT-system (see Definition 1.5.2) and use Theorem 1.5.2 or Theorem 1.5.3 to study the zeroes of Δ_i.

After following these steps we are able to detect limit cycles of (3-1) by meanof the proposed method.

3.2 Finding first order Melnikov functions

Now we compute the Melnikov function of (3-1) when $k = 1$, i.e.,

$$Z(x, y) = \begin{cases} \begin{pmatrix} y + \varepsilon P_1^+ + \varepsilon^2 P_2^+ \\ -x + \varepsilon Q_1^+ + \varepsilon^2 Q_2^+ \end{pmatrix}, & \text{if} \quad y \geq x^n, \\ \\ \begin{pmatrix} y + \varepsilon P_1^- + \varepsilon^2 P_2^- \\ -x + \varepsilon Q_1^- + \varepsilon^2 Q_2^- \end{pmatrix}, & \text{if} \quad y \leq x^n \end{cases} \tag{3-3}$$

where $P_i^{\pm}(x,y)$ and $Q_i^{\pm}(x,y)$ are given by (3-2) i.e.

$$\begin{aligned}
P_i^+(x,y) &= a_{0i} + a_{1i}x + a_{2i}y + a_{3i}xy + a_{4i}x^2 + a_{5i}y^2 \\
P_i^-(x,y) &= \alpha_{0i} + \alpha_{1i}x + \alpha_{2i}y + \alpha_{3i}xy + \alpha_{4i}x^2 + \alpha_{5i}y^2 \\
Q_i^+(x,y) &= b_{0i} + b_{1i}x + b_{2i}y + b_{3i}xy + b_{4i}x^2 + b_{5i}y^2 \\
Q_i^-(x,y) &= \beta_{0i} + \beta_{1i}x + \beta_{2i}y + \beta_{3i}xy + \beta_{4i}x^2 + \beta_{5i}y^2.
\end{aligned}$$

We denote $Z(x,y)$ restricted to the region $y \geq x^n$ by

$$\begin{pmatrix} F_{a1}(x,y) \\ F_{a2}(x,y) \end{pmatrix} = \begin{pmatrix} y + \varepsilon P_1^+ + \varepsilon^2 P_2^+, \\ -x + \varepsilon Q_1^+ + \varepsilon^2 Q_2^+ \end{pmatrix}, \tag{3-4}$$

and the vector field $Z(x,y)$ restricted to the region $y \leq x^n$ by

$$\begin{pmatrix} F_{d1}(x,y) \\ F_{d2}(x,y) \end{pmatrix} = \begin{pmatrix} y + \varepsilon P_1^- + \varepsilon^2 P_2^-, \\ -x + \varepsilon Q_1^- + \varepsilon^2 Q_2^- \end{pmatrix}. \tag{3-5}$$

According to last section, we must apply a change of variables to polar coordinates. Considering $x = r\cos(\theta)$ and $y = r\sin(\theta)$, (3-4) can be written as

$$\begin{aligned}
\dot{r} =& \varepsilon \big[\cos(\theta)(a_{01} + a_{02}\varepsilon + r\sin(\theta)(a_{21} + \varepsilon(a_{22} + b_{12}) + r^2\cos^3(\theta)(a_{41} + a_{42}\varepsilon) \\
&+ r\cos^2(\theta)(a_{11} + a_{12}\varepsilon + r\sin(\theta)(a_{31} + \varepsilon(a_{32} + b_{24}) + b_{41})) \\
&+ \sin(\theta)(b_{01} + b_{02}\varepsilon + r\sin(\theta)(b_{21} + b_{22}\varepsilon + r\sin(\theta)(b_{51} + b_{52}\varepsilon))) \\
&+ r\sin(\theta)(a_{51} + \varepsilon(a_{52} + b_{32}) + b_{31}) + b_{11}))\big] \\
\dot{\theta} =& \frac{1}{r}\,[\sin(\theta)(-\varepsilon(a_{01} + a_{02}\varepsilon) - r\sin(\theta)(\varepsilon(a_{21} + a_{22}\varepsilon) + r\varepsilon\sin(\theta)(a_{51} + a_{52}\epsilon) + 1)) \\
&+ \varepsilon\cos(\theta)(r\sin(\theta)(-a_{11} - a_{12}\varepsilon + r\sin(\theta)(-a_{31} - a_{32}\varepsilon + b_{51} + b_{52}\varepsilon) + b_{21} + b_{22}\varepsilon) \\
&+ b_{01} + b_{02}\varepsilon) + r\cos^2(\theta)(r\varepsilon\sin(\theta)(-a_{41} - a_{42}\varepsilon + b_{31} + b_{32}\varepsilon) + \varepsilon(b_{11} + b_{12}\varepsilon) - 1) \\
&+ r^2\varepsilon\cos^3(\theta)(b_{24}\varepsilon + b_{41})\big]
\end{aligned} \tag{3-6}$$

and (3-5) is written as

$$\begin{aligned}
\dot{r} =&\varepsilon\left[\cos(\theta)(\alpha_{01}+\alpha_{02}\varepsilon+r\sin(\theta)(\alpha_{21}+\varepsilon(\alpha_{22}+b_{12})+r^2\cos^3(\theta)(\alpha_{41}+\alpha_{42}\varepsilon)\right.\\
&+r\cos^2(\theta)(\alpha_{11}+\alpha_{12}\varepsilon+r\sin(\theta)(\alpha_{31}+\varepsilon(\alpha_{32}+\beta_{24})+\beta_{41}))\\
&+\sin(\theta)(\beta_{01}+\beta_{02}\varepsilon+r\sin(\theta)(\beta_{21}+\beta_{22}\varepsilon+r\sin(\theta)(\beta_{51}+\beta_{52}\epsilon)))\\
&+r\sin(\theta)(\alpha_{51}+\varepsilon(\alpha_{52}+b_{32})+\beta_{31})+\beta_{11}))]\\
\dot{\theta} =&\frac{1}{r}\left[\sin(\theta)(-\varepsilon(\alpha_{01}+\alpha_{02}\varepsilon)-r\sin(\theta)(\varepsilon(\alpha_{21}+\alpha_{22}\varepsilon)+r\varepsilon\sin(\theta)(\alpha_{51}+\alpha_{52}\varepsilon)+1))\right.\\
&+\varepsilon\cos(\theta)(r\sin(\theta)(-\alpha_{11}-\alpha_{12}\varepsilon+r\sin(\theta)(-\alpha_{31}-\alpha_{32}\varepsilon+\beta_{51}+\beta_{52}\varepsilon)+\beta_{21}+\beta_{22}\varepsilon)\\
&+\beta_{01}+\beta_{02}\varepsilon)+r\cos^2(\theta)(r\varepsilon\sin(\theta)(-\alpha_{41}-\alpha_{42}\varepsilon+\beta_{31}+\beta_{32}\varepsilon)+\varepsilon(\beta_{11}+\beta_{12}\varepsilon)-1)\\
&\left.+r^2\varepsilon\cos^3(\theta)(\beta_{24}\varepsilon+\beta_{41})\right]
\end{aligned} \tag{3-7}$$

Note that, in polar coordinates, the discontinuity set is given by

$$\sin(\theta)=r^{n-1}\cos^n(\theta). \tag{3-8}$$

Now we must consider θ as an independent variable in the systems (3-6) and (3-7). Doing $r'=\frac{dr}{d\theta}$, we obtain θ as independent variable in (3-6) and (3-7) respectively, and denote how as $r'=F_a(r,\theta,\varepsilon)$ and $r'=F_d(r,\theta,\varepsilon)$, here F_a represent 3-4 after this procedure and F_d represent 3-5 after this procedure. Computing the Taylor expansion of F_a and F_d up to order 2 around $\varepsilon=0$ we get

$$r'=\begin{cases}\varepsilon F_1^+(r,\theta)+\varepsilon^2F_2^+(r,\theta)+\mathcal{O}(\varepsilon^3), & \text{if } \sin(\theta)\geq r^{n-1}\cos(\theta),\\ \varepsilon F_1^-(r,\theta)+\varepsilon^2F_2^-(r,\theta)+\mathcal{O}(\varepsilon^3), & \text{if } \sin(\theta)\leq r^{n-1}\cos(\theta),\end{cases} \tag{3-9}$$

In this case, we have

$$\begin{aligned}
F_1^+(r,\theta)=&-a_{41}r^2\cos^3(\theta)-\cos(\theta)a_{01}+r\sin(\theta)(a_{21}+\beta_{11}+r(a_{51}+\beta_{31})\sin(\theta)))\\
&-r\cos^2(\theta)(a_{11}+r(a_{31}+b_{41})\sin(\theta))+\sin(\theta)(b_{01}+r\sin(\theta)(b_{21}+\beta_{51}r\sin(\theta))),
\end{aligned} \tag{3-10}$$

$$\begin{aligned}
F_2^+(r,\theta)=&-\frac{1}{r}\left[\left[a_{41}r^2\cos^3(\theta)+\cos(\theta)(a_{01}+r\sin(\theta)(a_{21}+b_{11}+r(a_{51}+b_{31})\sin(\theta)))\right.\right.\\
&+r\cos^2(\theta)(a_{11}+r(a_{31}+b_{41})\sin(\theta))+\sin(\theta)(b_{01}+r\sin(\theta)(b_{21}\\
&+\beta_{51}r\sin(\theta)))]\cdot\left[-\sin(\theta)\left(a_{01}+a_{51}r^2\sin^2(\theta)+a_{21}r\sin(\theta)\right)+r\cos^2(\theta)\right.\\
&\cdot(b_{11}+r(b_{31}-a_{41})\sin(\theta))+b_{41}r^2\cos^3(\theta)+\cos(\theta)(b_{01}+r\sin(\theta)(-a_{11}\\
&+b_{21}+r(b_{51}-a_{31})\sin(\theta)))]+r\left(a_{42}r^2\cos^3(\theta)+\cos(\theta)(a_{02}+r\sin(\theta)(a_{22}\right.\\
&+b_{12}+r(a_{52}+b_{32})\sin(\theta)))+r\cos^2(\theta)(a_{12}+r(a_{32}+b_{42})\sin(\theta))+\sin(\theta)(b_{02}\\
&\left.+r\sin(\theta)(b_{22}+\beta_{52}r\sin(\theta))))\right],
\end{aligned} \tag{3-11}$$

$$\begin{aligned}F_1^-(r,\theta) =& -\alpha_{41}r^2\cos^3(\theta) - \cos(\theta)(\alpha_{01} + r\sin(\theta)(\alpha_{21} + \beta_{11} + r(\alpha_{51} + \beta_{31})\sin(\theta)))\\ &- r\cos^2(\theta)(\alpha_{11} + r(\alpha_{31} + \beta_{41})\sin(\theta)) + \sin(\theta)(\beta_{01} + r\sin(\theta)(\beta_{21}\\ &+ \beta_{51} r\sin(\theta))),\end{aligned} \tag{3-12}$$

and

$$\begin{aligned}F_2^-(r,\theta) =& -\frac{1}{r}\left[\left[\alpha_{41}r^2\cos^3(\theta) + \cos(\theta)(\alpha_{01} + r\sin(\theta)(\alpha_{21} + \beta_{11} + r(\alpha_{51} + \beta_{31})\sin(\theta)))\right.\right.\\ &+\ r\cos^2(\theta)(\alpha_{11} + r(\alpha_{31} + \beta_{41})\sin(\theta)) + \sin(\theta)(\beta_{01} + r\sin(\theta)(\beta_{21}\\ &+\ \beta_{51}r\sin(\theta)))] \cdot \left[-\sin(\theta)\left(\alpha_{01} + \alpha_{51}r^2\sin^2(\theta) + \alpha_{21}r\sin(\theta)\right) + r\cos^2(\theta)\right.\\ &\cdot\ (\beta_{11} + r(\beta_{31} - \alpha_{41})\sin(\theta))\beta_{41}r^2\cos^3(\theta) + \cos(\theta)(\beta_{01} + r\sin(\theta)(-\alpha_{11} + \beta_{21}\\ &+\ r(\beta_{51} - \alpha_{31})\sin(\theta)))] + r\left(\alpha_{42}r^2\cos^3(\theta) + \cos(\theta)(\alpha_{02} + r\sin(\theta)(\alpha_{22} + \beta_{12}\right.\\ &+\ r(\alpha_{52} + \beta_{32})\sin(\theta))) + r\cos^2(\theta)(\alpha_{12} + r(\alpha_{32} + \beta_{42})\sin(\theta)) + \sin(\theta)(\beta_{02}\\ &+\ r\sin(\theta)(\beta_{22} + \beta_{52}r\sin(\theta))))].\end{aligned} \tag{3-13}$$

Thus, the original system (3-3) is brought to

$$r' = \varepsilon F_1(\theta,r) + \varepsilon^2 F_2(\theta,r) + \mathcal{O}(\varepsilon^3), \tag{3-14}$$

where for $i = 1,2$

$$F_i(r,\theta) = \begin{cases} F_i^-(\theta,r), & \text{if} \quad \sin(\theta) \leq r^{n-1}\cos^n(\theta)\\ F_i^+(r,\theta), & \text{if} \quad \sin(\theta) \geq r^{n-1}\cos^n(\theta)\end{cases} \tag{3-15}$$

So, system (3-3) is changed in the form (2-1), note that the equation $\sin(\theta) = r^{n-1}\cos^n(\theta)$ gives rise to two angles $\theta_1(r) \in [0, \frac{\pi}{2}]$ and $\theta_2(r) \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ such that the circle $x^2 + x^{2n} = r^2$ (recall that $y = x^n$) intersects at the points $(r\cos(\theta_1(r)), r\sin(\theta_1(r)))$ and $(r\sin(\theta_2(r)), r\cos(\theta_2(r)))$. See Figure 3.1 and 3.2. in this case (3-15) write as

$$F_i(r,\theta) = \begin{cases} F_i^-(r,\theta), & \text{if} \quad 0 \leq \theta \leq \theta_1(r),\\ F_i^+(r,\theta), & \text{if} \quad \theta_1(r) \leq \theta \leq \theta_2(r),\\ F_i^-(r,\theta), & \text{if} \quad \theta_2(r) \leq \theta \leq 2\pi.\end{cases} \tag{3-16}$$

Now we are able to compute the Melnikov function of first order given in (2-4) and obtain

$$\Delta_1(r) = \int_0^{\theta_1} F_1^-(\theta,r)d\theta + \int_{\theta_1}^{\theta_2} F_1^+(\theta,r)d\theta + \int_{\theta_2}^{2\pi} F_1^-(\theta,r)d\theta. \tag{3-17}$$

For simplicity, in the last expression we write θ_1 and θ_2 instead $\theta_1(r)$ and $\theta_2(r)$. Computing

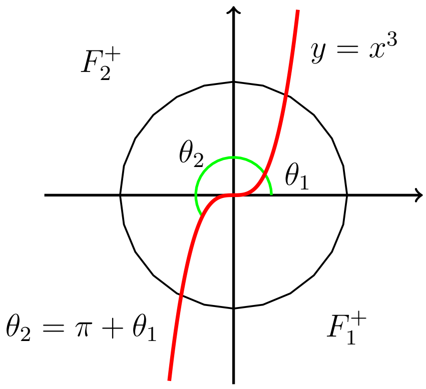


Figure 3.1: *Case* $n = 3$.

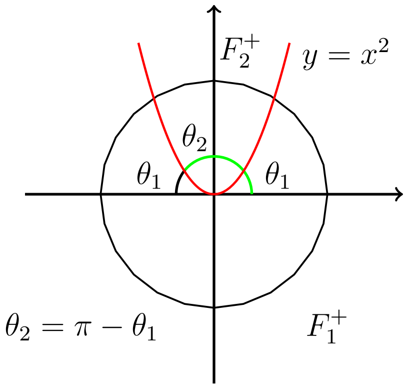


Figure 3.2: *Case* $n = 2$.

the integral (3-17) we get:

$$\begin{aligned}\Delta_1(r) =& -\frac{1}{12\pi}\left[3\pi r(a_{11}+\alpha_{11}+b_{21}+\beta_{21}) - 12\sin(\theta_1)\left(a_{01}-\alpha_{01}+a_{41}r^2-\alpha_{41}r^2\right)\right.\\ &+ 4r^2\cos^3(\theta_1)(a_{31}-\alpha_{31}+b_{41}-\beta_{41}-b_{51}+\beta_{51}) + 4r^2\sin^3(\theta_1)(a_{41}-\alpha_{41}\\ &\left.- a_{51}+\alpha_{51}-b_{31}+\beta_{31}) + 12\cos(\theta_1)\left(b_{01}-\beta_{01}+b_{51}r^2-\beta_{51}r^2\right)\right],\end{aligned} \tag{3-18}$$

when n is odd and when n is even

$$\begin{aligned}\Delta_1(r) =& \frac{1}{2\pi}\left[\cos(\theta_1)\left(r\sin(\theta_1)(a_{11}-\alpha_{11}-b_{21}+\beta_{21}) + 2\left(-b_{01}+\beta_{01}-b_{51}r^2+\beta_{51}r^2\right)\right)\right.\\ &- \frac{1}{2}r((2\theta_1+\pi)(\alpha_{11}+\beta_{21}) + a_{11}(\pi-2\theta_1) + b_{21}(\pi-2\theta_1)) - \frac{1}{3}2r^2\cos^3(\theta_1)(a_{31}\\ &\left.- \alpha_{31}+b_{41}-\beta_{41}-b_{51}+\beta_{51})\right].\end{aligned} \tag{3-19}$$

Now, it is sufficient to study the zeroes of Δ_1 given by (3-18) and (3-19).

3.2.1 Case $n = 1$

When $n = 1$, we have that

$$\cos(\theta_1) = \frac{x}{r}, \quad \sin(\theta_1) = \frac{x}{r}, \quad \text{and} \quad r^2 = 2x^2. \tag{3-20}$$

So, replacing (3-20) in (3-18) and since $r > 0$, we have that $\Delta_1(r) = 0$ if and only if $\overline{\Delta_1}(x) = 0$, where $\overline{\Delta_1}(x) = r\Delta_1(x)$. So, the maximum number of limit cycles obtained by the first order Melnikov method is the maximum number of positive zeros simples of the polynomial

$$\begin{aligned}\overline{\Delta_1}(x) =& \frac{x(-6a_{01} + 6\alpha_{01} + 6b_{01} - 6\beta_{01})}{6\pi} - \frac{1}{2}x^2(a_{11} + \alpha_{11} + b_{21} + \beta_{21}) \\ &- \frac{x^3(a_{31} - \alpha_{31} - 5a_{41} + 5\alpha_{41} - a_{51} + \alpha_{51} - b_{31} + \beta_{31} + b_{41} - \beta_{41} + 5b_{51} - 5\beta_{51})}{3\pi}.\end{aligned} \tag{3-21}$$

Considering

$$\begin{aligned} k_0 &= a_{31} - \alpha_{31} - 5a_{41} + 5\alpha_{41} - a_{51} + \alpha_{51} - b_{31} + \beta_{31} + b_{41} - \beta_{41} + 5b_{51} - 5\beta_{51}, \\ k_1 &= -6a_{01} + 6\alpha_{01} + 6b_{01} - 6\beta_{01}, \quad \text{and} \\ k_2 &= a_{11} + \alpha_{11} + b_{21} + \beta_{21} \end{aligned} \tag{3-22}$$

we get

$$\overline{\Delta_1}(x) = -\frac{k_0x^3}{3\pi} - \frac{k_1x}{6\pi} - \frac{k_2x^2}{2}. \tag{3-23}$$

It is worth mentioning that the change of parameters done in (3-22) is surjective, see Remark 18 below. Considering $F = [x^3, x, x^2]$, we obtain that

$$\begin{aligned} W_0(x) =& x^3, \\ W_1(x) =& -2x^3, \quad \text{and} \\ W_2(x) =& 2x^3. \end{aligned} \tag{3-24}$$

Since that $x \neq 0$ we get that the set F is an ECT-system in $(0, \infty)$ and so, from Theorem 1.5.2, we get that two is the maximum number of positive roots of polynomial (3-23). Therefore, when the switching curve is $y = x$, there is a choice of $[k_i]_{i=0}^2$, for which $\overline{\Delta_1}(x)$ has exactly 2 zeros.

Finally, using Theorem 2.0.1, it follows that the first order Melnikov function Δ_1 of (3-3) for $n = 1$ detects at most 2 limit cycles and there is a choice of polynomials $P_i^{\pm}$ and $Q_i^{\pm}$, $i = 1, 2$, for which (3-3) has at least 2 limit cycles.

Remark 18. *Let P be the parameter space given by*

$$\begin{aligned} P = \{(a, b, \alpha, \beta) : \ & \textit{where} \ \ a = (a_{01}, a_{11}, a_{21}, a_{31}, a_{41}, a_{51}), \ \ b = (b_{01}, b_{11}, b_{21}, b_{31}, b_{41}, b_{51}), \\ & \alpha = (\alpha_{01}, \alpha_{11}, \alpha_{21}, \alpha_{31}, \alpha_{41}, \alpha_{51}), \ \ \beta = (a_{01}, \beta_{11}, \beta_{21}, \beta_{31}, \beta_{41}, \beta_{51}), \ \ a_{i1}, b_{i2}, \alpha_{i1}, \\ & \beta_{i1} \in \mathbb{R} \ \ \forall \ \ i = 1, 2, \dots, 5\} \equiv \mathbb{R}^{24} \end{aligned}$$

and consider the linear change of parameters $T : P \to \mathbb{R}^3$ *given by* $T(a, b, \alpha, \beta) = (k_0, k_1, k_2)$*, where* $[k_i]_{i=0}^{2}$ *are given by* (3-22)*. Notice that* T *is a full rank linear transformation and therefore* T *is surjective. In fact, Consider the subspace* $S \subset P$ *given by* $S = span\{e_4, e_1, e_2\}$ *where* e_j *is the* j*-th element of the canonical basis of* $\mathbb{R}^{24}$*, then the map* $T|_S$ *is an isomorphism with matrix*

$$[T|_S] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that, the change of parameters done in (3-22) *is surjective.*

3.2.2 Case $n = 2$

When $n = 2$, we have that

$$\cos(\theta_1) = \frac{x}{r}, \quad r^2 = x^2 + x^4, \quad \sin(\theta_1) = \frac{x^2}{r}, \quad \text{and } \theta_1 = \tan^{-1}(x). \tag{3-25}$$

Replacing (3-25) in (3-19) and since $r > 0$, we have that $\Delta_1(r) = 0$ if, and only if, $\overline{\Delta_1}(x) = 0$, $\overline{\Delta_1}(x) = r\Delta_1(x)$. Therefore, the maximum number of limit cycles obtained by the first order Melnikov method is the maximum number of positive zeros of

$$\begin{aligned} \overline{\Delta_1}(x) = &\frac{1}{2\pi}\Bigg[x^3\Bigg(a_{11} - \alpha_{11} - \frac{2}{3}(a_{31} - \alpha_{31} + b_{41} - \beta_{41} - b_{51} + \beta_{51}) \\ &- b_{21} + \beta_{21} - 2(b_{51} - \beta_{51})\Bigg) + x(2\beta_{01} - 2b_{01}) - 2x^5(b_{51} - \beta_{51}) \\ &+ x^4\left(\tan^{-1}(x)(a_{11} - \alpha_{11} + b_{21} - \beta_{21}) - \frac{1}{2}\pi(a_{11} + \alpha_{11} + b_{21} + \beta_{21})\right) \\ &+x^2\left(\tan^{-1}(x)(a_{11} - \alpha_{11} + b_{21} - \beta_{21}) - \frac{1}{2}\pi(a_{11} + \alpha_{11} + b_{21} + \beta_{21})\right)\Bigg] \end{aligned} \tag{3-26}$$

Considering

$$\begin{aligned} k_0 &= a_{11} + b_{21} + \alpha_{11} + \beta_{21}, \\ k_1 &= k0 - 2\alpha_{11} - 2\beta_{21}, \\ k_2 &= 4a_{31} + 12b_{21} + 4b_{41} + 8b_{51} - 6k_0 + 12\alpha_{11} - 4\alpha_{31} - 4\beta_{41} - 8\beta_{51}, \\ k_3 &= 12b_{01} - 12\beta_{01}, \\ k_4 &= b_{51} - \beta_{51}, \end{aligned} \tag{3-27}$$

we get,

$$\overline{\Delta_1}(x) = -\frac{k_0 x^2}{4}\left(x^2+1\right) + \frac{k_1}{2\pi}x^2\tan^{-1}(x)\left(x^2+1\right) - \frac{k_2 x^3}{12\pi} - \frac{k_3 x}{12\pi} - \frac{k_4}{\pi}x^5. \tag{3-28}$$

It is worth mentioning that, following the same steps in Remark 18, we can show that the change of parameters done in (3-27) is surjective. Defining $F = [u_0 = -\frac{x^2}{4}\left(x^2+1\right), u_1 = x^2\tan^{-1}(x)\left(x^2+1\right), u_2 = x^3, u_3 = x, u_4 = x^5]$, then we obtain that

$$\begin{aligned} W_0(x) &= -\frac{x^2}{2}\left(x^2+1\right), \\ W_1(x) &= -\frac{1}{2}\pi x^4\left(x^2+1\right), \\ W_2(x) &= \frac{2\pi x^7}{x^2+1}, \\ W_3(x) &= -\frac{16\pi x^5}{\left(x^2+1\right)^2}, \quad \text{and} \\ W_4(x) &= -\frac{768\pi x^8}{\left(x^2+1\right)^3}. \end{aligned} \tag{3-29}$$

For $x \neq 0$ we get that the family $F = [u_0 = -\frac{x^2}{4}\left(x^2+1\right), u_1 = x^2\tan^{-1}(x)\left(x^2+1\right), u_2 = x^3, u_3 = x, u_4 = x^5]$ is an ECT-system in $(0,\infty)$ and, from Theorem 1.5.2, we get that four is the maximum number of positive roots of (3-26) when the switching curve is determined by $y = x^2$ and there is a choice for $[k_i]_{i=0}^4$, such that $\overline{\Delta_1}(x)$ has exactly 4 simple zeros. Finally, using Theorem 2.0.1, it follows that the first order Melnikov function Δ_1 of (3-3), for $n = 2$, detects at most 4 limit cycles and there is a choice of polynomials $P_i^{\pm}$ and $Q_i^{\pm}$, $i = 1, 2$, for which such that (3-3) has at least 4 limit cycles.

3.2.3 Case $n = 3$

When $n = 3$, we have that

$$\cos(\theta_1) = \frac{x}{r}, \quad r^2 = x^2 + x^4, \quad \text{and} \quad \sin(\theta_1) = \frac{x^2}{r}. \tag{3-30}$$

Replacing (3-30) in (3-18) and since $r > 0$, we have that $\Delta_1(r) = 0$ if, and only if, $\overline{\Delta_1}(x) = 0$, where $\overline{\Delta_1}(x) = r\Delta_1(x)$. So, the maximum number of limit cycles obtained by the first order Melnikov method is the maximum number of positive zeros of the polynomial

$$\begin{aligned} \overline{\Delta_1}(x) =& \frac{1}{2\pi}\left[-x^3(-12a_{01} + 12\alpha_{01} + 4a_{31} - 4\alpha_{31} + 4b_{41} - 4\beta_{41} + 8b_{51} - 8\beta_{51})\right. \\ &- x^7(12b_{51} - 12\beta_{51}) - x^6(3\pi a_{11} + 3\pi\alpha_{11} + 3\pi\beta_{21} + 3\pi b_{21}) - x^2\left(3\pi a_{11} + 3\pi\alpha_{11}\right. \\ &+ \left.3\pi\beta_{21} + 3\pi b_{21}\right) - x^9(-8a_{41} + 8\alpha_{41} - 4a_{51} + 4\alpha_{51} - 4b_{31} + 4\beta_{31}) - x^5\left(12\alpha_{41}\right. \\ &- \left.\left.12a_{41}\right) - x(12b_{01} - 12\beta_{01})\right] \end{aligned} \tag{3-31}$$

Considering

$$
\begin{aligned}
k_0 &= 4\beta_{31} - 8a_{41} - 4a_{51} - 4b_{31} + 8\alpha_{41} + 4\alpha_{51}, \\
k_1 &= 12\alpha_{41} - 12a_{41} \\
k_2 &= 3\pi a_{11} + 3\pi\alpha_{11} + 3\pi\beta_{21} + 3\pi b_{21}, \\
k_3 &= 12b_{01} - 12\beta_{01}, \\
k_4 &= -12a_{01} + 4a_{31} + 4b_{41} + 8b_{51} + 12\alpha_{01} - 4\alpha_{31} - 4\beta_{41} - 8\beta_{51}, \\
k_5 &= 12b_{51} - 12\beta_{51},
\end{aligned}
\tag{3-32}
$$

we get

$$
\overline{\Delta_1}(x) = -\frac{k_0 x^9}{12\pi} - \frac{k_1 x^5}{12\pi} - \frac{k_2\left(x^5+1\right)x^2}{12\pi} - \frac{k_3 x}{12\pi} - \frac{k_4 x^3}{12\pi} - \frac{k_5 x^7}{12\pi}. \tag{3-33}
$$

It is worth mentioning that, following the same steps in Remark 18, we can show that the change of parameters done in (3-32) is surjective. If, we define $F = [u_0 = -x^9, u_1 = x^5, u_2 = x, u_3 = x^3, u_4 = x^7, u_5 = x^2\left(1+x^5\right)]$, then we get

$$
\begin{aligned}
W_0(x) =& - x^9, \\
W_1(x) =& - 4x^{13}, \\
W_2(x) =& - 128x^{12}, \\
W_3(x) =& - 3072x^{12}, \\
W_4(x) =& 294912x^{15}, \quad \text{and} \\
W_5(x) =& 30965760x^{12},
\end{aligned}
\tag{3-34}
$$

From theorem 1.5.2 and $x \neq 0$ we get that the family $F = [u_0 = -x^9, u_1 = x^5, u_2 = x, u_3 = x^3, u_4 = x^7, u_5 = x^2\left(1+x^5\right)]$ is an ECT-system in $(0,\infty)$. From the Theorem 1.5.2 we get that, for the system (3-3) five is the maximum number of positive roots of polynomial (3-33) when the switching curve is given by $y = x^5$ and there is a choice for $[k_i]_{i=0}^5$, for which $\overline{\Delta_1}(x)$ has exactly 5 zeros. Finally, using Theorem 2.0.1, it follows that the first order Melnikov function Δ_1 of (3-3), for $n = 3$, detects at most 5 limit cycles and there is a choice of polynomials $P_i^{\pm}$ and $Q_i^{\pm}$ $i = 1, 2$ such that (3-3) has at least 5 limit cycles.

3.2.4 Case $y = x^{2k+1}$, for $k > 1$

When $n = 2k+1$, $k > 1$, we have that

$$
\cos(\theta_1) = \frac{x}{r}, \quad r^2 = x^2 + x^{2k+1}, \quad \text{and} \quad \sin(\theta_1) = \frac{x^{2k+1}}{r}. \tag{3-35}
$$

Replacing (3-35) in (3-18) and since $r > 0$, we have that $\Delta_1(r) = 0$ if, and only if, $\overline{\Delta_1}(x) = 0$, where $\overline{\Delta_1}(x) = r\Delta_1(x)$. So, the maximum number of limit cycles obtained by the first order Melnikov method is the maximum number of positive zeros of the polynomial

$$\begin{aligned}\overline{\Delta_1}(x) =& \frac{1}{2\pi}\Big[x\Big((12a_{01}-12\alpha_{01})x^{2k}-12\beta_{01}-12b_{01}\Big)+x^2\left(-3\pi a_{11}-3\pi\alpha_{11}-3\pi\beta_{21}\right.\\ &- 3\pi b_{21}+x^{4k}(-3\pi a_{11}-3\pi\alpha_{11}-3\pi\beta_{21}-3\pi b_{21})\Big)+x^3\left(-4a_{31}+4\alpha_{31}-4b_{41}\right.\\ &- 4\beta_{41}-8b_{51}+8\beta_{51}+x^{6k}(8a_{41}-8\alpha_{41}+4a_{51}-4\alpha_{51}+4_{b31}-4\beta_{31})\\ &+ (12a_{41}-12\alpha_{41})x^{2k}+(12\beta_{51}-12b_{51})x^{4k}\Big)\Big],\end{aligned} \tag{3-36}$$

Considering

$$\begin{aligned}k_0 &= 8a_{41}+4a_{51}+4b_{31}-8\alpha_{41}-4\alpha_{51}-4\beta_{31},\\ k_1 &= -12a41+12\alpha_{41},\\ k_2 &= 12b51-12\beta_{51},\\ k_3 &= 9a11\pi+9b21\pi+9\pi\alpha_{11}+9\pi\beta_{21},\\ k_4 &= -36a_{01}+36\alpha_{01},\\ k_5 &= 12a_{31}+12b_{41}+2k_2-12\alpha_{31}-12\beta_{41}, \quad \text{and}\\ k_6 &= 36b_{01}-36\beta_{01},\end{aligned} \tag{3-37}$$

we get,

$$\overline{\Delta_1}(x) = \frac{k_0x^{6k+3}}{12\pi}-\frac{k_1x^{2k+3}}{12\pi}-\frac{k_2x^{4k+3}}{12\pi}-\frac{k_3x^2\left(x^{4k}+1\right)}{36\pi}-\frac{k_4x^{2k+1}}{36\pi}-\frac{k_5x^3}{36\pi}-\frac{k_6x}{36\pi} \tag{3-38}$$

It is worth mentioning that, following the same steps in Remark 18 we can show that the change of parameters done in (3-37) is surjective.

Consider the ordered set of functions $F = [u_0 = x, u_1 = x^3, u_2 = x^{1+2k}, u_3 = x^{3+2k}, u_4 = x^{3+4k}, u_5 = x^{3+6k}, u_6 = x^2\left(1+x^{4k}\right)]$. Then, we get

$$\begin{aligned}W_0(x) =& -x,\\ W_1(x) =& 2x^3,\\ W_2(x) =& 8(k-1)kx^{2k+2},\\ W_3(x) =& 64k^2\left(k^2-1\right)x^{4k+2},\\ W_4(x) =& 2048(k-1)k^4(k+1)^2(2k+1)x^{8k+1},\\ W_5(x) =& 393216(k-1)k^7(k+1)^2(2k+1)^2(3k+1)x^{14k-1}, \quad \text{and}\\ W_6(x) =& 393216(k-1)k^7(k+1)^2(2k-1)(2k+1)^3(3k+1)(4k+1)x^{14k-5}\\ &\cdot\Big(\left(8k^2+2k-1\right)x^{4k}-6k-1\Big)\end{aligned} \tag{3-39}$$

Notice that since $x \neq 0$ the Wronskians $W_i(x) \neq 0$, for $i = 1, 2, \cdots, 5$ do not vanish in $(0, \infty)$, $k \in \mathbb{Z}^+ \setminus \{1\}$ and $W_6(x)$ has one positive zero for each $k \in \mathbb{Z}^+ \setminus \{1\}$, which is simple, see Remark 19 below. Thus from Theorem 1.5.3, we get that when the switching curve is given by $y = x^{2k+1}$, seven is the maximum number of positive roots of polynomial (3-36) and there is a choice for $[k_i]_{i=0}^6$, such that $\overline{\Delta_1}(x)$ has exactly seven zeros. Finally, using

Theorem 2.0.1, it follows that the first order Melnikov function Δ_1, of (3-3), for $n = 2k+1$, $k > 1$, detects at most 7 limit cycles and there is a choice of polynomials $P_i^{\pm}$ and $Q_i^{\pm}$, $i = 1, 2$. for which (3-3) has at least 7 limit cycles.

Remark 19. *Notice that equation $W_6(x) = 0$ has a positive solution*

$$x^* = \sqrt[4k]{\frac{6k+1}{8k^2+2k-1}}. \tag{3-40}$$

and computing the derivative of W_6, we get

$$\begin{aligned} W_6'(x) =& 393216(k-1)k^7(k+1)^2(2k-1)(2k+1)^3(3k+1)(4k+1)(14k-5)x^{14k-4} \\ &\cdot \left(\left(8k^2+2k-1\right)x^{4k} - 6k - 1\right) + 393216(k-1)k^7(k+1)^2(2k-1)(2k+1)^3 \\ &\cdot (3k+1)(4k+1)(14k-5)\left(8k^2+2k-1\right)(4k)x^{18k-5} \end{aligned}$$

and thus $W_6'(x^*) \neq 0$, for $k \neq -\frac{1}{6}$. Since $k \in \mathbb{Z}^+ \setminus \{1\}$, it follows that x^* is a simple zero of W_6.

3.2.5 Case $y = x^{2k}$, for $k > 1$

When $n = 2k$, $k > 1$, we have that

$$\cos(\theta_1) = \frac{x}{r}, \quad r^2 = x^2 + x^{2k}, \quad \text{and} \quad \sin(\theta_1) = \frac{x^{2k}}{r}. \tag{3-41}$$

Replacing (3-35) in (3-19) and since $r > 0$, we have that $\Delta_1(r) = 0$ if and only if $\overline{\Delta_1}(x) = 0$, where $\overline{\Delta_1}(x) = r\Delta_1(x)$. So, the maximum number of limit cycles obtained by the first order Melnikov method is the maximum number of positive zeros of

$$\begin{aligned} \overline{\Delta_1}(x) =& 6x\left(x^{2k}(a_{11} - \alpha_{11} - b_{21} + \beta_{21}) - 2b_{01} + 2\beta_{01} - 2(b_{51} - \beta_{51})\left(x^{4k} + x^2\right)\right) \\ &- 3\left(x^{4k} + x^2\right)\left[\pi(a_{11} + \alpha_{11} + b_{21} + \beta_{21}) + 2\tan^{-1}\left(x^{2k-1}\right)(-a_{11} + \alpha_{11}\right. \\ &\left. - b_{21} + \beta_{21})\right] + 4x^3(-a_{31} + \alpha_{31} - b_{41} + \beta_{41} + b_{51} - \beta_{51}). \end{aligned} \tag{3-42}$$

Considering

$$\begin{aligned} k_0 &= -a_{31} - b_{41} + b_{51} + \alpha_{31} + \beta_{41} - \beta_{51}, \\ k_1 &= a_{11} - b_{21} - \alpha_{11} + \beta_{21}, \\ k_2 &= -2b_{01} + 2\beta_{01}, \\ k_3 &= a_{31} + b_{41} + k_0 - \alpha_{31} - \beta_{41}, \\ k_4 &= 2b_{21} + k_1 + 2\alpha_{11}, \quad \text{and} \\ k_5 &= 2b_{21} + k_1 - 2\beta_{21}. \end{aligned} \tag{3-43}$$

we get,

$$\begin{aligned}\overline{\Delta_1}(x) =& 4k_0x^3 + 6k_1x^{2k+1} + 6k_2x + k_3\left(-12x^{4k+1} - 12x^3\right) + k_4\left(-3\pi x^{4k} - 3\pi x^2\right)\\ &+ 6k_5\left(x^{4k} + x^2\right)\tan^{-1}\left(x^{2k-1}\right).\end{aligned} \tag{3-44}$$

It is worth mentioning that, following the same steps in Remark 18 we can show that the change of parameters done in (3-43) is surjective.

Consider the ordered set of functions $F = [u_0 = x^3, u_1 = x^{2k+1}, u_2 = x, u_3 = -12x^{4k+1} - 12x^3, u_4 = -3\pi x^{4k} - 3\pi x^2, u_5 = \left(x^{4k} + x^2\right)\tan^{-1}\left(x^{2k-1}\right)]$. In this case we have that

$$\begin{aligned}
&W_0(x) = x^3,\\
&W_1(x) = 2(k-1)x^{2k+3},\\
&W_2(x) = 8(k-1)kx^{2k+2},\\
&W_3(x) = -1536(-1+k)k^3(-1+2k)x^{6k},\\
&W_4(x) = -4608\pi(1-2k)^2(k-1)k^3(4k-1)x^{6k-4}\left((4k-3)x^{4k} + x^2\right), \quad \text{and}\\
&W_5(x) = -\frac{147456}{\left(x^{4k}+x^2\right)^4}\pi(1-2k)^4(k-1)k^3(4k-1)x^{12k}P_k(x^{4k-2}),
\end{aligned} \tag{3-45}$$

where

$$\begin{aligned}P_k(x) =& (k-1)k(3k-1)(4k-3)x^3 - (3k-1)(k(3k(8k-15)+31)-8)x^2\\ &+ (3k-2)(k(k(4k-13)+6)-1)x + (k-1)(3k-2)(3k-1)\end{aligned} \tag{3-46}$$

We have that $W_i(x) \neq 0$ for $i = 1, 2, 3, 4$, in $(0, +\infty)$. Below we prove that $W_5(x)$ has two simple roots in $(0, +\infty)$. Notice that

$$\lim_{x\to\pm\infty} P_k(x) = \pm\infty. \tag{3-47}$$

Since the coefficient $(k-1)k(3k-1)(4k-3) > 0$, for $k > 1$, and

$$\begin{aligned}&P_k(0) = (k-1)(3k-2)(3k-1) > 0, \quad \text{and}\\ &P_k(2) = -3(7k-5)(k(k(8k-11)+9)-2) < 0\end{aligned} \tag{3-48}$$

From Bolzano's Theorem there exists a root of (3-46) in the interval $(0, 2)$. From (3-47) and (3-48) it follows from Bolzano's Theorem that there is a root in the interval $(2, +\infty)$ (see Figure 3.3). From $P_k(0) > 0$ and (3-47), we can apply Bolzano's Theorem again to conclude that there is a negative root of $P_k(x)$. Thus there exist two positive roots of the polynomial (3-46) which are simple. Applying Theorem 1.5.4 to the family F and using the expression of $W_5(x)$ in (3-45) we have that

$$v_0 = v_1 = v_2 = v_3 = v_4 = 0, \quad v_5 = 2, \text{ and } \mu_3 = \mu_4 = 0. \tag{3-49}$$

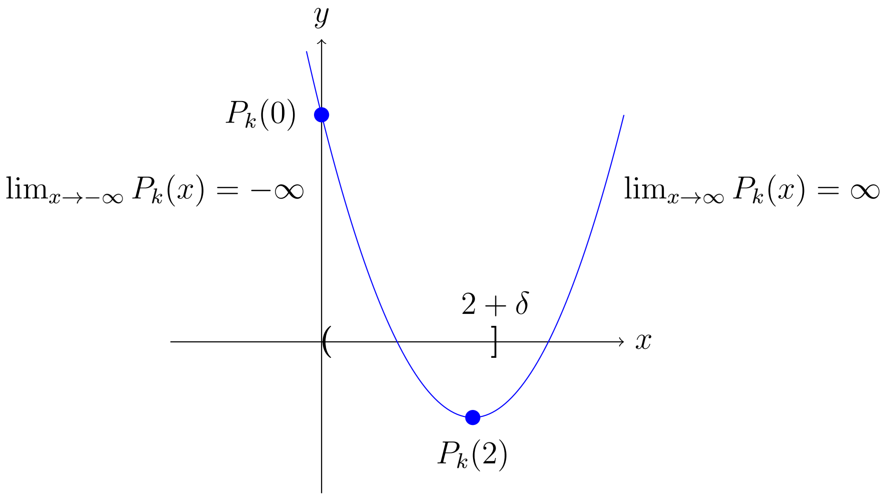


Figure 3.3: *Illustration of an polynomial of degree 3 with 2 positives real roots and the procedure with the polynomial* (3-46).

So, the number of isolated zeros of every element of $Span(F)$ does not exceed 7, i.e., $Z(F) \leq 7$. On the other hand, if we call the positive roots of polynomial (3-46) by $x_1 < x_2$ we can choose the interval $(0, x_1 + \frac{\delta}{3}]$, with $\delta = |x_2 - x_1|$, where the family F is an EC-system with accuracy 1 in the interval $[\overline{a}, x_1 + \frac{\delta}{3}]$ with $\overline{a} > 0$ sufficiently small. Using Theorem 1.5.3, we get that there exists an element of $Span(F)$ which has 6 isolated zeros, thus $6 \leq Z(F) \leq 7$ (see Figure 3.3). Since we can provide a choice of parameter to reach any $\{k_i\}_{i=0}^{5}$, we have that there is a choice of polynomials $P_i^{\pm}$ and $Q_i^{\pm}$ for which (3-3) has exactly 6 limit cycles. Thus $6 \leq m_1 \leq 7$.

3.3 Finding second order Melnikov functions

In this section we compute the second order Melnikov function of (3-3) which has the form

$$\begin{aligned} \Delta_2(r) = &\int_0^T \left[D_x F_1(s,x) \int_0^s F_1(t,x)dt + F_2(s,x) \right] ds \\ &+ \sum_{i=1}^{M} (F_1^{j-1}(\theta_j(x),x) - F_1^j(\theta_j(x),x)) D_x\theta_j(x) \int_0^{\theta_j(x)} F_1(s,x)ds. \end{aligned} \tag{3-50}$$

Now, from Section 3.2 we have two angles to describe the discontinuity set, namely $\theta_1(r) \in [0, \frac{\pi}{2}]$ and $\theta_2(r) \in [\frac{\pi}{2}, \frac{3\pi}{2}]$, where θ_2 depends on n and θ_1, thus we have that $M = 2$ in (3-50). Using (3-14) and

$$\Delta_2(r) = f_2(r) + f_2^*(r), \tag{3-51}$$

we can compute (3-50), where

$$\begin{aligned} f_2(r) =&\frac{1}{2\pi}\int_0^{\theta_1}\left[D_rF_1^-(\theta,r)\int_0^{\theta}F_1^-(s,r)ds+F_2^-(\theta,r)\right]d\theta \\ &+\frac{1}{2\pi}\int_{\theta_1}^{\theta_2}\left[D_rF_1^+(\theta,r)\left[\int_0^{\theta_1}F_1^-(s,r)ds+\int_{\theta_1}^{\theta}F_1^+(s,r)ds\right]+F_2^+(\theta,r)\right]d\theta \\ &+\frac{1}{2\pi}\int_{\theta_1}^{\theta_2}\left[D_rF_1^-(\theta,r)\left[\int_0^{\theta_1}F_1^-(s,r)ds+\int_{\theta_1}^{\theta_2}F_1^+(s,r)ds+\int_{\theta_2}^{\theta}F_1^-(s,r)ds\right]\right. \\ &\left.+F_2^-(\theta,r)\right]d\theta \end{aligned} \tag{3-52}$$

and

$$\begin{aligned} f_2^*(r) =& \left(F_1^-(\theta_1,r)-F_1^+(\theta_1,r)\right)\theta_1'\int_0^{\theta_1}F_1^-(s,r)ds \\ &+\left(F_1^+(\theta_2,r)-F_1^-(\theta_2,r)\right)\theta_2'\left[\int_0^{\theta_1}F_1^-(s,r)ds+\int_{\theta_1}^{\theta_2}F_1^+(s,r)ds\right] \end{aligned} \tag{3-53}$$

For simplicity, we write θ_i $i=1,2$ instead of $\theta_i(r)$. Using (3-10) and (3-12) we can find $D_rF_1^{\pm}(\theta,r)$ which are given by

$$\begin{aligned} D_rF_1^- =& \left(\frac{\alpha_{11}}{2}-\frac{\beta_{21}}{2}\right)\sin^2(\theta)+\left(\frac{\beta_{21}}{2}-\frac{\alpha_{11}}{2}\right)\cos^2(\theta)-\frac{\alpha_{11}}{2}+(-\alpha_{21}-\beta_{11})\sin(\theta)\cos(\theta) \\ &-\frac{\beta_{21}}{2}+\sin^3(\theta)\left(\frac{\alpha_{31}r}{2}+\frac{\beta_{41}r}{2}-\frac{\beta_{51}r}{2}\right)+\sin(\theta)\left(-\frac{\alpha_{31}r}{2}-\frac{\beta_{41}r}{2}-\frac{3\beta_{51}r}{2}\right) \\ &+\sin(\theta)\cos^2(\theta)\left(-\frac{1}{2}3\alpha_{31}r-\frac{3\beta_{41}r}{2}+\frac{3\beta_{51}r}{2}\right)+\cos^3(\theta)\left(-\frac{\alpha_{41}r}{2}+\frac{\alpha_{51}r}{2}+\frac{\beta_{31}r}{2}\right) \\ &+\cos(\theta)\left(-\frac{1}{2}3\alpha_{41}r-\frac{\alpha_{51}r}{2}-\frac{\beta_{31}r}{2}\right)+\sin^2(\theta)\cos(\theta)\left(\frac{3\alpha_{41}r}{2}-\frac{3\alpha_{51}r}{2}-\frac{3\beta_{31}r}{2}\right) \end{aligned} \tag{3-54}$$

and

$$\begin{aligned} D_rF_1^+ =& \left(\frac{a_{11}}{2}-\frac{b_{21}}{2}\right)\sin^2(\theta)+\left(\frac{b_{21}}{2}-\frac{a_{11}}{2}\right)\cos^2(\theta)-\frac{a_{11}}{2}+(-a_{21}-b_{11})\sin(\theta)\cos(\theta) \\ &-\frac{b_{21}}{2}+\sin^3(\theta)\left(\frac{a_{31}r}{2}+\frac{b_{41}r}{2}-\frac{b_{51}r}{2}\right)+\sin(\theta)\left(-\frac{a_{31}r}{2}-\frac{b_{41}r}{2}-\frac{3b_{51}r}{2}\right) \\ &+\sin(\theta)\cos^2(\theta)\left(-\frac{1}{2}3a_{31}r-\frac{3b_{41}r}{2}+\frac{3b_{51}r}{2}\right)+\cos^3(\theta)\left(-\frac{a_{41}r}{2}+\frac{a_{51}r}{2}+\frac{b_{31}r}{2}\right) \\ &+\cos(\theta)\left(-\frac{1}{2}3a_{41}r-\frac{a_{51}r}{2}-\frac{b_{31}r}{2}\right)+\sin^2(\theta)\cos(\theta)\left(\frac{3a_{41}r}{2}-\frac{3a_{51}r}{2}-\frac{3b_{31}r}{2}\right) \end{aligned} \tag{3-55}$$

Among the terms which appear in 3-51 we have θ_1' and θ_2' but their expressions depend on $y=x^n$ and it should be computed for each n.

3.3.1 Switching set $y = x$

Since $n = 1$ is an odd number, we have that

$$\theta_2 = \theta_1 + \pi, \tag{3-56}$$

See Figure 3.1. In this case $\theta_1 = \frac{\pi}{4}$, thus $\theta_1' = \theta_2' = 0$ and $f_2^*(r) = 0$ in (3-51). So we only need to compute the term $f_2(r)$. Since we are interested in using Theorem 2.0.1 we must impose that $\Delta_1 \equiv 0$, considering the formula for Δ_1 obtained in (3-21), it is sufficient to assume that

$$\begin{aligned} 0 &= a_{31} - \alpha_{31} - 5a_{41} + 5\alpha_{41} - a_{51} + \alpha_{51} - b_{31} + \beta_{31} + b_{41} - \beta_{41} + 5b_{51} - 5\beta_{51}, \\ 0 &= -6a_{01} + 6\alpha_{01} + 6b_{01} - 6\beta_{01}, \quad \text{and} \\ 0 &= a_{11} + \alpha_{11} + b_{21} + \beta_{21}. \end{aligned} \tag{3-57}$$

Computing $f_2(r)$ and replacing (3-57) in (3-51), we get

$$\Delta_2(x) = \sqrt{2}k_0 + \sqrt{2}k_1 x + k_2 x^2 + \sqrt{2}k_3 x^3, \tag{3-58}$$

where

$$\begin{aligned}
k_0 =& - a_{01}\alpha_{11} + \frac{a_{01}\alpha_{21}}{2} - \frac{a_{01}a_{21}}{2} + \frac{a_{01}b_{11}}{2} - \frac{a_{01}\beta_{11}}{2} + a_{01}b_{21} - 2a_{01}\beta_{21} - 2\alpha_{11}\beta_{01} + \beta_{02} \\
&+ a_{02} - \alpha_{02} + \frac{a_{21}b_{01}}{2} - \alpha_{11}b_{01} - \frac{\alpha_{21}b_{01}}{2} + \frac{b_{01}\beta_{11}}{2} - \frac{b_{01}b_{11}}{2} - b_{01}b_{21} - 2\beta_{01}\beta_{21} - b_{02}, \\
k_1 =& \pi a_{01}\alpha_{31} - \frac{5\pi a_{01}\alpha_{41}}{2} + \frac{5\pi a_{01}a_{41}}{2} - \frac{\pi a_{01}\alpha_{51}}{2} + \frac{\pi a_{01}a_{51}}{2} - \frac{\pi a_{01}\beta_{31}}{2} + \frac{\pi a_{01}b_{31}}{2} - \frac{\pi a_{12}}{2} \\
&- \frac{\pi a_{01}b_{41}}{2} + + \frac{\pi a_{01}\beta_{41}}{2} + \frac{7\pi a_{01}\beta_{51}}{2} - \frac{3\pi a_{01}b_{51}}{2} + \frac{\pi\alpha_{11}\alpha_{21}}{4} - \frac{\pi\alpha_{11}\beta_{11}}{4} - \frac{17\pi\alpha_{11}a_{21}}{64} \\
&+ \frac{\alpha_{11}a_{21}}{32} - \frac{\pi\alpha_{12}}{2} + \frac{a_{21}\alpha_{21}}{32} + \frac{a_{21}\beta_{11}}{32} + \frac{\pi\alpha_{21}\beta_{21}}{4} - \frac{\pi a_{21}b_{21}}{16} + \frac{\pi\alpha_{31}\beta_{01}}{2} - \frac{17\pi a_{21}\beta_{21}}{64} \\
&- \pi\alpha_{41}\beta_{01} - \pi a_{41}b_{01} - \frac{a_{21}\beta_{21}}{32} - \frac{\pi\alpha_{31}b_{01}}{2} - \frac{\pi\beta_{01}\beta_{31}}{2} - \frac{\pi b_{01}b_{31}}{2} + \pi\beta_{01}\beta_{51} - \pi b_{01}\beta_{51} \\
&+ \frac{\alpha_{11}b_{11}}{32} + \frac{15\pi\alpha_{11}b_{11}}{64} + \frac{\alpha_{21}b_{11}}{32} + \frac{b_{11}\beta_{11}}{32} + \frac{15\pi b_{11}\beta_{21}}{64} - \frac{\pi\beta_{11}\beta_{21}}{4} - \frac{\pi b_{11}b_{21}}{16} - \frac{\pi\beta_{22}}{2} \\
&- \frac{b_{11}\beta_{21}}{32} - \frac{\pi b_{22}}{2}, \\
k_2 =& \frac{17}{9}\sqrt{2}\alpha_{11}\alpha_{31} - \frac{14}{3}\sqrt{2}\alpha_{11}\alpha_{41} - \frac{4}{3}\sqrt{2}\alpha_{11}\alpha_{51} - \sqrt{2}\alpha_{11}\beta_{31} + \frac{2}{9}\sqrt{2}\alpha_{11}\beta_{41} + \frac{52}{9}\sqrt{2}\alpha_{11}\beta_{51} \\
&- \frac{\alpha_{21}\alpha_{31}}{\sqrt{2}} + \frac{1}{120}a_{21}\Big(51\sqrt{2}a_{31} + 20\alpha_{31} - 255\sqrt{2}\alpha_{41} + 224\sqrt{2}a_{41} - 51\sqrt{2}\alpha_{51} + 20\beta_{41} \\
&+ 32\sqrt{2}a_{51} - 51\sqrt{2}\beta_{31} + 72\sqrt{2}b_{31} - 40\sqrt{2}b_{41} + 51\sqrt{2}\beta_{41} - 136\sqrt{2}b_{51} + 255\sqrt{2}\beta_{51} \\
&+ 40\beta_{51}\Big) + \frac{\alpha_{21}\alpha_{41}}{3\sqrt{2}} + \frac{\alpha_{21}\alpha_{51}}{3\sqrt{2}} - \frac{\alpha_{21}\beta_{31}}{3\sqrt{2}} - \frac{\alpha_{21}\beta_{41}}{3\sqrt{2}} - \frac{3\alpha_{21}\beta_{51}}{\sqrt{2}} - \frac{\alpha_{31}\beta_{11}}{3\sqrt{2}} + \frac{8}{9}\sqrt{2}\alpha_{31}\beta_{21} \\
&+ \frac{\sqrt{2}\alpha_{32}}{3} - \frac{\sqrt{2}a_{32}}{3} + \frac{43}{9}\sqrt{2}\alpha_{11}a_{41} + \frac{28}{15}\sqrt{2}a_{41}b_{11} - \frac{3\alpha_{41}\beta_{11}}{\sqrt{2}} - \frac{11}{3}\sqrt{2}\alpha_{41}\beta_{21} - \frac{\alpha_{51}\beta_{11}}{3\sqrt{2}} \\
&+ 4\sqrt{2}a_{41}b_{21} + \frac{43}{9}\sqrt{2}a_{41}\beta_{21} + \frac{5\sqrt{2}a_{42}}{3} + \frac{5}{9}\sqrt{2}\alpha_{11}a_{51} + \frac{4}{15}\sqrt{2}a_{51}b_{11} - \frac{5}{3}\sqrt{2}\alpha_{51}\beta_{21}
\end{aligned} \tag{3-59}$$

$$
\begin{aligned}
&+\frac{4}{3}\sqrt{2}a_{51}b_{21}-\frac{5\sqrt{2}\alpha_{42}}{3}+\frac{5}{9}\sqrt{2}a_{51}\beta_{21}+\frac{\alpha_{31}b_{11}}{6}+\frac{11\alpha_{31}b_{11}}{60\sqrt{2}}-\frac{11\alpha_{41}b_{11}}{12\sqrt{2}}-\frac{11\alpha_{51}b_{11}}{60\sqrt{2}}\\
&+\frac{3}{5}\sqrt{2}b_{11}b_{31}-\frac{\beta_{11}\beta_{31}}{\sqrt{2}}-\frac{11b_{11}\beta_{31}}{60\sqrt{2}}-\frac{1}{3}\sqrt{2}b_{11}b_{41}+\frac{b_{11}\beta_{41}}{6}+\frac{\beta_{11}\beta_{41}}{3\sqrt{2}}-\frac{17}{15}\sqrt{2}b_{11}b_{51}\\
&+\frac{11b_{11}\beta_{41}}{60\sqrt{2}}+\frac{b_{11}\beta_{51}}{3}+\frac{\beta_{11}\beta_{51}}{3\sqrt{2}}+\frac{11b_{11}\beta_{51}}{12\sqrt{2}}+\sqrt{2}\alpha_{31}b_{21}-5\sqrt{2}\alpha_{41}b_{21}-\sqrt{2}\alpha_{51}b_{21}\\
&-\sqrt{2}b_{21}\beta_{31}-\frac{4}{3}\sqrt{2}b_{21}b_{41}+\sqrt{2}b_{21}\beta_{41}+\frac{5}{9}\sqrt{2}\beta_{21}\beta_{41}-\frac{\sqrt{2}\alpha_{52}}{3}-4\sqrt{2}b_{21}b_{51}+\frac{\sqrt{2}a_{52}}{3}\\
&+5\sqrt{2}b_{21}\beta_{51}+\frac{43}{9}\sqrt{2}\beta_{21}\beta_{51}-\frac{\sqrt{2}b_{24}}{3}+\frac{8}{9}\sqrt{2}\alpha_{11}b_{31}+\frac{8}{9}\sqrt{2}\beta_{21}b_{31}+\frac{\sqrt{2}b_{32}}{3}-\frac{\sqrt{2}\beta_{32}}{3}\\
&-\frac{5}{3}\sqrt{2}\alpha_{11}b_{41}-\frac{5}{3}\sqrt{2}\beta_{21}b_{41}-\frac{11}{3}\sqrt{2}\alpha_{11}b_{51}-\frac{11}{3}\sqrt{2}\beta_{21}b_{51}+\frac{5\sqrt{2}\beta_{52}}{3}-\frac{5\sqrt{2}b_{52}}{3}\\
&+\frac{\sqrt{2}\beta_{42}}{3},\quad \text{and}\\
k_3=&\frac{\pi\alpha_{31}a_{41}}{4}+\frac{\pi\beta_{41}a_{41}}{4}+\frac{5\pi\beta_{51}a_{41}}{4}-\frac{5\pi\alpha_{41}a_{41}}{4}-\frac{\pi\alpha_{51}+a_{41}}{4}+\frac{a_{51}b_{31}\pi}{4}-\frac{\pi\beta_{31}a_{41}}{4}\\
&-\frac{3b_{41}\pi a_{41}}{4}-\frac{5b_{51}\pi a_{41}}{4}+\frac{3a_{51}\pi a_{41}}{2}+\frac{b_{31}\pi a_{41}}{4}+\frac{a_{51}\pi\alpha_{31}}{4}+\frac{\pi\alpha_{31}\alpha_{41}}{4}+\frac{\pi\alpha_{31}\alpha_{51}}{4}\\
&+\frac{a_{51}\pi\beta_{41}}{4}+\frac{5\pi a_{41}^2}{4}+\frac{\pi\alpha_{51}\beta_{51}}{2}+\frac{5a_{51}\pi\beta_{51}}{4}-\frac{\pi\alpha_{41}\beta_{41}}{2}-\frac{5a_{51}\pi\alpha_{41}}{4}-\frac{a_{51}\pi\alpha_{51}}{4}\\
&-\frac{a_{51}\pi\beta_{31}}{4}-\frac{\pi\beta_{31}\beta_{41}}{4}-\frac{\pi\beta_{31}\beta_{51}}{4}-\frac{a_{51}b_{41}\pi}{4}-\frac{b_{31}b_{41}\pi}{4}-\frac{3a_{51}b_{51}\pi}{4}-\frac{b_{31}b_{51}\pi}{4}\\
&+\frac{a_{51}^2\pi}{4}.
\end{aligned}
\tag{3-60}
$$

If, we define $F=[u_0=\sqrt{2},u_1=\sqrt{2}x,u_2=x^2,u_3=\sqrt{2}x^3]$

$$
\begin{aligned}
W_0(x)&=\sqrt{2}\\
W_1(x)&=2\\
W_2(x)&=4\\
W_3(x)&=24\sqrt{2}\pi.
\end{aligned}
\tag{3-61}
$$

From Theorem 1.5.2 we get that the set $F=[u_0=\sqrt{2},u_1=\sqrt{2}x,u_2=x^2,u_3=\sqrt{2}x^3]$ is an ECT-system in $(0,\infty)$ and we get that three is the maximum number of positive roots of polynomial (3-58) when the switching curve is given by $y=x$ and there is a choice of $[k_i]_{i=0}^3$, such that $\Delta_2(x)$ has exactly 3 zeros. Finally, using Theorem 2.0.1, it follows that the second order Melnikov function Δ_2 of (3-3), for $n=1$, detects at most 3 limit cycles and there is a choice of polynomials $P_i^{\pm}$ and $Q_i^{\pm}$, $i=1,2$, for which (3-3) has at least 3 limit cycles.

3.4 Comparision with previous results

Finally, we compare the results obtained in Theorem 3.1.1 with the results obtained in [39]. For this purpose consider the following system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} y + \varepsilon p^{+}(x,y), \\ -x + \varepsilon q^{+}(x,y), \end{pmatrix}, & y \geq x^{m}, \\ \\ \begin{pmatrix} y + \varepsilon p^{-}(x,y), \\ -x + \varepsilon q^{-}(x,y), \end{pmatrix}, & y < x^{m}, \end{cases} \tag{3-62}$$

where $p^{\pm}(x,y) = \sum_{i+j=0}^{n} a_{i,j}^{\pm} x^i y^j$, $q^{\pm}(x,y) = \sum_{i+j=0}^{n} b_{i,j}^{\pm} x^i y^j$ are any polynomial of degree n and $m \geq 1$.

In [39] the authors find upper and lower bounds for the maximum number of limit cycles of (3-62) through the analysis of Melnikov functions. Below we enunciate the results obtained by the authors in [39] in order to compare with the Theorem 3.1.1. In the this section, $Z(m,n)$ denotes the upper bound of the number of limit cycles of system (3-62) obtained through first order Melnikov function.

Theorem 3.4.1. (***part (i) of Theorem 1.1 in [39]***) *Suppose that $m = 2k+1 (k \in \mathbb{N})$ By using the first order Melnikov function in ε, we have that for $(m,n) \in D_1 = \{(m,n) : 0 \leq n < m-1\}$*

$$\begin{aligned} Z(m,n) \leq& \frac{1}{2}\left[\frac{n+3}{2}\right]\left[\frac{n+5}{2}\right] + \left[\frac{n+2}{2}\right]\left[\frac{n+4}{2}\right] - 2, \\ Z(m,n) \geq& \left[\frac{n}{2}\right]\left[\frac{n+6}{2}\right] + \left[\frac{n-1}{2}\right] + 2 \end{aligned} \tag{3-63}$$

and the lower bound can be reached by some system of the form (3-62).

Theorem 3.4.2. (***part (i) of Theorem 1.2 in [39]***) *Let $\delta_n = 0$ if n is odd, and $\delta_n = -1$ if n is even. Suppose that $m = 2k (k \in \mathbb{N})$ By using the first order Melnikov function in ε, we have that for $(m,n) \in D_4 = \{(m,n) : \ 0 \leq n < m-1\}$*

$$\begin{aligned} Z(m,n) \leq& 4\left[\frac{n}{2}\right]^2 + (6\delta_n + 11)\left[\frac{n}{2}\right] + \frac{1}{2}\delta_n (5\delta_n + 17) + 4, \\ Z(m,n) \geq& \frac{1}{2}\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right] + 3\right) + \frac{1}{2}\left[\frac{n-1}{2}\right]\left(\left[\frac{n-1}{2}\right] + 7\right) + 3 \end{aligned} \tag{3-64}$$

and the lower bound can be reached by some system of the form (3-62).

Corollary 3.4.3. (***part (i,ii,iv) of the Corollary 1.2 in [39]***) *We have the following results:*

(i) $Z(1,n) = n,\ n \geq 1.$

(ii) $5 \leq Z(3,2) \leq 6$.

(iv) $Z(2,2) = 4$.

The cases that are of our interest occur when (m,n) takes values $(1,2)$, $(3,2)$, $(m,2)$ when $m \geq 2$ is even. The first three cases appear in Corollary 3.4.3. From Theorem 3.4.1, using (3-63), we have $6 \leq Z(m,2) \leq 7$ when $m > 3$ is odd and from Theorem 3.4.2 using (3-64), we have $5 \leq Z(m,2) \leq 7$ when $m > 2$ is even. Thus we have the following Table 3.3.

m	$Z(m,2)$ in this work	$Z(m,2)$ in [39].
1	2	2
2	4	4
3	5	$5 \leq Z(3,2) \leq 6$
$m > 2$, m even	$6 \leq Z(m,2) \leq 7$	$5 \leq Z(m,2) \leq 7$
$m > 3$, m odd	7	$6 \leq Z(m,2) \leq 7$

Table 3.3: *Comparison between $Z(m,2)$ obtained in this work and in [39].*

Recall that here $Z(m,2)$ is the maximum number of isolated zeros detected by Δ_1. Notice that our bounds agree with the bounds obtained in [39] when $n = 2$. Moreover we have improved the bounds for $m \geq 4$. For $m = 3$ the authors of [39] could obtain an improved bound, since they can reach 6 limit cycles, nevertheless they did not prove that this upper bound is reached.

CHAPTER 4

Final comments and next steps

As we seen in this work, Melnikov Theory is a useful tool to find lower bounds for the maximum number of limit cycles, as long as the system satisfies the hypothesis of Theorem 2.0.1. In this section we first discuss about some difficulties we had with such methodology during this work and then we present some further directions for our results.

1. In this work, we deal with systems of the form (3-1) and we apply Melnikov Theory up to order 2 to get Theorem 3.1.1, which provides lower bounds for the maximum number of limit cycles of system (3-1). Nevertheless, in order to apply Melnikov Theory, the system must be in the standard form (2-1) and it is not trivial to put the system in this form for a general class of systems in higher dimensions.
2. Following the ideas of this work, in order to succeed on finding the maximum number of zeros of a Melnikov function Δ_i, we should employ a surjective change of parameters to write Δ_i as a linear combination of the elements of an ECT-system and then we can obtain bounds for the number of zeros of Δ_i through Chebyshev Theory. In the problem approached in this work, for each switching curve $y = x^n$, $n \in \mathbb{N}$, we need to chose the parameters $[k_i]_{i=0}^{j_n}$ to identify the ECT-system, and there is not a unique way or method to employ such change of parameters, so finding this change of parameters is not always easy. In this case, we have obtained the expression of the second order Melnikov function associated to the curve $y = x^2$, nevertheless we were not able to provide a change of parameters to conclude the argument.
3. In general, the size of the Melnikov functions of (3-1) increases as long as we increase the order k or the degree of the switching curve n, so we need to compute them using a software (e.g. Mathematica). In many cases, the computer do not have enough power to compute them and in such cases, it is not possible to apply Melnikov Theory to detect limit cycles of the system.

With respect to the next steps and further directions of this work, we do the following considerations.

1. First, we wish to study the second order Melnikov functions associated to the switching curves $y = x^n$, $n \geq 2$. In some cases, we were able to compute the Melnikov functions but we could not apply Chebyshev Theory to study them yet. This could allows us to improve the lower bounds of $H(n)$.

2. In this work, we consider piecewise polynomial perturbations of degree 2 of the planar linear center and we have shown that we get improved lower bounds for the maximum number of limit cycles of the system in comparison with piecewise polynomial perturbations of degree 1. So, a natural extension of this work is to apply this methodology to piecewise polynomial perturbations of any degree of the planar linear center, in order to detect even more limit cycles.
3. In general, Theorem 2.0.1 can not detect all limit cycles of a system, but under certain conditions, it is possible to use a Melnikov function to detect all the limit cycles of a system. So, a possible direction of this work is to look for a new class of systems where all limit cycles can be detected by some Melnikov function. In this case, we could apply the same methodology to obtain an upper bound of the maximum number of limit cycles of such class of systems.
4. Finally, in this work we only consider piecewise polynomial perturbations of the planar linear center with switching curve given by $y = x^n$, so the problem remains open for another types of algebraic curves.

Bibliography

[1] Almeida, O. B. **Teoria do grau e aplicações**. Master's thesis, Programa de Pósgraduação em Matemática, Universidade Federal de Campina Grande, Centro de ciencias y Tecnólogia., 2006.

[2] Alves, A.; Ramirez, O. **Bifurcation of limit cycles by perturbing piecewise non-hamiltonian systems with nonlinear switching manifold**. *Nonlinear Analysis: Real World Applications*, 57:103188, 2021.

[3] Andrade, K. d. S.; Cespedes, O. A. R.; Cruz, D. R.; Novaes, D. D. **Higher order melnikov analysis for planar piecewise linear vector fields with nonlinear switching curve**. 6 2020.

[4] Andronov, A. A.; Vitt, A. A.; Khaikin, S. E. **Theory of Oscillators: Adiwes International Series in Physics**, volume 4. Elsevier, 2013.

[5] Arrowsmith, D.; Place, C.; Place, C. **An Introduction to Dynamical Systems**. An Introduction to Dynamical Systems. Cambridge University Press, 1990.

[6] Bastos, J. L.; Buzzi, C. A.; Llibre, J.; Novaes, D. D. **Melnikov analysis in nonsmooth differential systems with nonlinear switching manifold**. *Journal of Differential Equations*, 267:3748–3767, 9 2019.

[7] Braga, D. D. C.; Mello, L. F. **More than three limit cycles in discontinuous piecewise linear differential systems with two zones in the plane**. *International Journal of Bifurcation and Chaos*, 24(04):229–244, 2014.

[8] Brogliato, B.; Brogliato, B. **Nonsmooth mechanics**, volume 3. Springer, 1999.

[9] Buică, A.; Llibre, J. **Averaging methods for finding periodic orbits via brouwer degree**. *Bulletin des sciences mathematiques*, 128(1):7–22, 2004.

[10] Christopher, C.; Li, C. **Limit cycles of differential equations**. Springer Science & Business Media, 2007.

[11] Coddington, A.; Levinson, N. **Theory of Ordinary Differential Equations**. International series in pure and applied mathematics. McGraw-Hill Companies, 1955.

[12] Cortes, J. **Discontinuous dynamical systems**. *IEEE Control Systems Magazine*, 28(3):36–73, 2008.

[13] Croom, F. H. **A. F. Filippov, Differential equations with discontinuous righthand side, Mathematics and Its Applications**. Kluwer Academic Publisher, Dordrecht, 1988.

[14] G. Botelho, D. P.; Teixeira, E. **Fundamentos de Análise Funcional segunda edição**. Sociedade Braslileira de Matemática, Rio de Janeiro, second edition, 2015.

[15] Han, M.; Zhang, W. **On hopf bifurcation in non-smooth planar systems**. *Journal of Differential Equations*, 248(9):2399–2416, 2010.

[16] Huan, S.-M.; Yang, X.-S. **On the number of limit cycles in general planar piecewise linear systems of node-node types**. *Journal of Mathematical Analysis and Applications*, 411(1):340–353, 2014.

[17] Itikawa, J.; Llibre, J.; Novaes, D. D. **A new result on averaging theory for a class of discontinuous planar differential systems with applications**. *Revista matemática iberoamericana*, 33(4):1247–1265, 2017.

[18] Lima, E. **Curso de análise volumen 2, quinta edição**. Sociedad Brasileira de matemática, 1999.

[19] Llibre, J.; Mereu, A. C.; Novaes, D. D. **Averaging theory for discontinuous piecewise differential systems**. *Journal of Differential Equations*, 258(11):4007–4032, 2015.

[20] Llibre, J.; Mereu, A. C.; Novaes, D. D. **Averaging theory for discontinuous piecewise differential systems**. *Journal of Differential Equations*, 258(11):4007–4032, 2015.

[21] Llibre, J.; Novaes, D. D.; Rodrigues, C. A. **Averaging theory at any order for computing limit cycles of discontinuous piecewise differential systems with many zones**. *Physica D: Nonlinear Phenomena*, 353:1–10, 2017.

[22] Llibre, J.; Novaes, D. D.; Teixeira, M. A. **Higher order averaging theory for finding periodic solutions via brouwer degree**. *Nonlinearity*, 27(3):563, 2014.

[23] Llibre, J.; Novaes, D. D.; Teixeira, M. A. **On the birth of limit cycles for non-smooth dynamical systems**. *Bulletin des Sciences Mathématiques*, 139(3):229–244, 2015.

[24] Llibre, J.; Ponce, E. **Three nested limit cycles in discontinuous piecewise linear differential systems with two zones**. *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications and Algorithms*, 19(3):325–335, May 2012.

[25] LLIBRE, J.; ŚWIRSZCZ, G. **On the limit cycles of polynomial vector field**. *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, 18(2):203–214, 2011.

[26] MERKIN, D. R. **Introduction to the Theory of Stability**, volume 24. Springer Science & Business Media, 2012.

[27] NOVAES, D. **Orbitas periÓdicas de certas equaçóes diferenciais acopladas**. Master's thesis, Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas., 2012.

[28] NOVAES, D. **Regularization and minimal sets for nonsmooth dynamical systems**. PhD thesis, Instituto de Matemática, Estatística e Computação Científica, Universidade Estadual de Campinas., 2015.

[29] NOVAES, D. D.; PONCE, E. **A simple solution to the braga-mello conjecture**. *Int. J. Bifurc. Chaos*, 25:1550009:1–1550009:7, 2015.

[30] NOVAES, D. D.; TORREGROSA, J. **On extended chebyshev systems with positive accuracy**. *Journal of Mathematical Analysis and Applications*, 448(1):171–186, 2017.

[31] PADEN, B. E.; SASTRY, S. S. **A calculus for computing filippov's differential inclusion with application to the variable structure control of robot manipulators**. *IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS*, 34:73, 1987.

[32] PERKO, L. **Texts in Applied Mathematics 7 Differential Equations and Dynamical Systems Springer**. Springer, New york, 2001.

[33] RODRIGUES, H. M. **Equações diferenciais ordinárias**. Universidade de São Paulo, Sâo Paulo, 2012.

[34] ROUSSARIE, R. **Bifurcation of planar vector fields and Hilbert's sixteenth problem**, volume 164. Springer, 1998.

[35] S. KARLIN, W. S. **Tchebycheff Systems: With Applications in Analysis and Statistics, Pure and Applied Mathematics**. Interscience Publishers John Wiley and Sons, 1966.

[36] SANDERS, J. A.; VERHULST, F.; MURDOCK, J. **Averaging methods in nonlinear dynamical systems**, volume 59. Springer, 2007.

[37] SANTOS, C. C.; CESPEDES, O. A. R. **Melnikov analysis for planar piecewise linear vector fields with algebraic switching curve** $y^n - x^m = 0$. 2022.

[38] TAO, T. **An Introduction to Measure Theory**. Graduate studies in mathematics. American Mathematical Society, 2013.

[39] WANG, J.; ZHOU, J.; ZHAO, L. **On the number of limit cycles bifurcating from the linear center with an algebraic switching curve**. *arXiv:2012.03566*, 2020.

[40] YAKOVENKO, S. **Quantitative theory of ordinary differential equations and the tangencial Hilbert 16th problem, in On Finite-ness in Differential Equations and Diophantine Geometry**, volume 24. Amer. Math. Soc, 2005.