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Cohomology and Partial Differential Equations

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UNIVERSIDADE FEDERAL DE GOIÁS
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA

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ARTUR JORGE MARINHO

Cohomology and Partial Differential Equations

Cohomologia e Equações Diferenciais Parciais

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ATA DE DEFESA DE DISSERTAÇÃO

Ata nº **6** da sessão de Defesa de Dissertação de **Artur Jorge Marinho**, que confere o título de Mestre(a) em **Matemática**, na área de concentração em **Análise.**

Aos **20/06/2024**, a partir das **10:30**, via **Web Videoconferência**, realizou-se a sessão pública de Defesa de Dissertação intitulada **“Cohomology and Partial Differential Equations”**. Os trabalhos foram instalados pelo Orientador, Professor Doutor **Kaye Oliveira da Silva (IME/UFG)** com a participação dos demais membros da Banca Examinadora: Professor Doutor **Gaetano Siciliano (UNIBA/Italia)**, membro titular externo; Professor Doutor **Abiel Costa Macedo (IME/UFG)**, membro titular interno. Durante a arguição os membros da banca **não fizeram** sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Dissertação, tendo sido o candidato **aprovado** pelos seus membros. Proclamados os resultados pelo Professor Doutor **Kaye Oliveira da Silva**, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos Membros da Banca Examinadora, aos **décimo sexto dia do mês de julho do ano de dois mil e vinte e quatro.**

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Resumo

Marinho, A. J.. **Cohomology and Partial Differential Equations**. Goiânia, 2024. 145p. Dissertação de Mestrado. Programa de Pós Graduação em Matemática, Instituto de Matemática e Estatística (IME), Universidade Federal de Goiás (UFG).

Este texto trata de problemas de equações diferenciais parciais elípticas do ponto de vista da teoria de Morse. Como a teoria de Morse tem uma conexão com alguns conceitos da topologia algébrica, a teoria da cohomologia é empregada para mostrar um resultado de existência para uma classe de equações. O conceito de índice cohomológico será útil para mostrar resultados de múltiplas soluções para o problema de Brezis-Nirenberg para o p-Laplaciano.

Palavras–chave

Teoria de Morse, Índice Cohomológico, p-Laplaciano, problema de Brezis-Nirenberg, Equações Diferenciais Parciais Elípticas, Ligação, Pontos Críticos, Métodos Variacionais.

Abstract

Marinho, A. J.. **Cohomology and Partial Differential Equations**. Goiânia, 2024. 145p. MSc. Dissertation. Programa de Pós Graduação em Matemática, Instituto de Matemática e Estatística (IME), Universidade Federal de Goiás (UFG).

This text deals with elliptic partial differential equation problems by the Morse theoric point of view. Since Morse theory has a connection with some concepts from algebraic topology, cohomology theory is employed to show existence result for a class of equations. The concept of cohomological index will be useful to show multiple solution results for the Brezis-Nirenberg problem for the p-Laplacian.

Keywords

Morse Theory, Cohomological Index, p-Laplacian, Brezis-Niremberg problem, Elliptic Partial Differential Equations, Linking, Critical Points, Variational Methods.

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Introduction

Variational problems have always arised in science. Here are some examples; the Brachistochrone Problem, which asks for the shape of the best “ramp" connecting two points in such a way that a ball of mass m, restricted only to the gravitational pull, released from the rest at the higher point, would take the least time to arrive at its end point. The Principle of the Least Action or Hamilton’s Principle: the nature seems to always work in such a way that it do so with as minimal effort as possible, therefore some mathematical laws of nature can be formulated as variational principles. More precisely, given a Lagrangian system, a particle which goes from point x to point y will do so through a path that is critical for a certain functional.

Due to the importance of these problems, a significant amount of theory concerning the solvability of variational problems has naturally developed. Morse Theory has been a cornerstone in such studies, focusing on the topological behavior of the level sets of a function, along with a local study of isolated critical points. One good example of studying the existence of critical points through the study of the topology of level sets is the Mountain Pass Theorem, which detects a loss in path-connectedness when crossing certain levels. Another example in the same vein are the Saddle Point and Linking Theorems from Rabinowitz, both of which generalize the Mountain Pass Theorem. These theorems detect more general changes in the topology of level sets, thereby identifying critical points (see [27]). A common way of identifying these topological changes is by demonstrating that certain minimax values are finite. For instance, the Mountain Pass Theorem deals with values of the form

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)).$$

If c is finite, say $\infty > c > a$, then the set $\{u : f(u) \leq a\}$ is not path connected (here Γ is a family of continuous curves connecting two fixed points u_1 and u_2 with $f(u_i) \leq a$).

It is reasonable to think that not every critical point can be detected using the theorems cited above. Some problems require a different type of topological study. For example, the topological behavior of certain functionals is better understood through the Ljusternik-Schnirelmann category, while others are better analyzed using the Krasnoselskii genus, and so on (see [4], [26]).

This connection between finding critical points and topological invariance naturally leads to the use of tools from algebraic topology in Morse theory.

In 1953, a paper by Chung-Tao Yang titled "*On Theorems of Bursuk-Ulam, Kakutani-Yamabe-Yujobo, and Dyson, I*" was published (see [28]). This paper aimed to generalize the conclusions of these theorems to more general cases. To achieve this, Yang introduced a homological index for the family of compact Hausdorff spaces that possess a continuous involution without fixed points. This index was later called the Yang index. It was discovered that the Yang index is very useful for studying the topological behavior of certain problems with symmetries. Some applications of this index to Morse theory were made by Fadell and Rabinowitz [11] and K. Perera [22]. In 1978, Fadell and Rabinowitz published a paper generalizing the Yang index to more general types of symmetries, which has implications for Hamiltonian systems [12]. After this generalization, the index became known simply as the cohomological index. In 2007, using this cohomological index, M. Degiovanni and S. Lancelotti provided a generalization of Rabinowitz's Linking Theorem that uses cones instead of linear subspaces [7]. This generalization is very useful in problems involving the p-Laplace operator, as its eigenspaces are generally not linear.

This text will introduce the reader to the cohomological index and demonstrate its utility in handling variational problems, particularly those with $\mathbb{Z}_2$ symmetries. The focus will be on problems involving p-Laplacian type operators. Some abstract methods wll be developed for these kind of problems. As an application of these methods, we will be able to establish the existence of multiple solutions to the Brezis-Nirenberg problem.

$$-\Delta_p u = \lambda |u|^{p-2}u + |u|^{p^*-2}u, \quad u \in W_0^{1,p}(\Omega),$$

for any $\lambda > 0$. The Brezis-Nirenberg problem for $p = 2$ has the advantage that the eigenspaces of $-\Delta$ are linear, so the topological behavior of this problem can be easily understood by the most common methods such as the category or genus (see [5]). However, when $p \neq 2$ this property does not hold any longer, and we need to employ more sofisticated techniques. One way of doing this is by employing the cohomological index. This is just an example of the usefulness of the theory that will be developed in this text.

Since we will widely use the cohomological index through this text, Chapter 1 will be used to remember some basic concepts regarding homology and cohomology theories. We will define the Alexander-Spanier cohomology theory in Section 1.3, which will be the cohomology used to define the index.

Chapter 2 has the intetion to discuss, in a non-rigorous way, some intuitive ideas concerning Morse theory, showing why undertanding change in topology of sublevels is a good strategy for finding critical points. Section 2.2 will introduce an index which is very useful when one is trying to seek for critical values of an $\mathbb{Z}_2$-invariant functional. Section 2.3 will show that some problems requires more tools, other than that of the genus, to be dealt with, motivating the employment of the $\mathbb{Z}_2$-cohomological index, which is the index theory that this text is based on.

Chapter 3 will deal with the construction of the $\mathbb{Z}_2$-cohomological index. Some important properties of such an index will be stated.

Chapter 4 has the goal of remembering some basics about Morse theoretic aspects of critical point theory.

Using the theory discussed in Chapter 4, Chapter 5 will deal with general p-eigenvalue problems, culminating on the existence of an unbounded sequence of eigenvalues of the p-Laplacian operator using the cohomological index. The existence of this sequence will be used to find saddle points to some related problems.

Using the previous ideas, Chapter 6 and 7 will be used to develop and apply some abstract approaches that are useful in problems involving p-Laplacian type operators.

Such approaches will be able to study problems of the types

$$A_p u = \lambda B_p u + f(u), \quad A_p u = \lambda B_p u - \mu f(u) - g(u).$$

Chapter 8 will be used as the appendix, which contains a few useful results.

CHAPTER 1

A review on some topics on algebraic topology

1.1 Singular Homology

An introduction on algebraic topology can be found on [14] and [24]. For an interesting discussion about some problems that motivated the development of singular homology theory, the reader is referred to [15]. The reader interested in applications of algebraic topology other than those shown in this text might find [19] and [25] a very insteresting reading.

Let X be a topological space, and let

$$\Delta^q = \left\{ (t_0, \cdots, t_q) \in \mathbb{R}^{q+1} : \sum_{i=0}^{q} t_i = 1, \text{ and } t_i \geq 0 \text{ for all } i \right\}$$

be the standard q-simplex. A *singular q-simplex* in the space X is by definition a continuous map $\sigma : \Delta^q \to X$. Given an abelian group G, we define the formal linear combinations: $\sum g_i \sigma_i$, where $g_i \in G$, and σ_i are singular q-simplexes. These sums are called singular q-chains, or simply q-chains. The set of all singular q-chains is denoted by $C_q(X,G)$. Once the group G is known, we will simply denote $C_q(X)$ for the group $C_q(X,G)$.

Let us denote $\Delta^q_{(i)} = \left\{ (t_0, \ldots, t_q) \in \Delta^q : t_i = 0 \right\}$. There is a canonical homeomorphism $\Delta^{q-1} \to \Delta^q_{(i)}$. Given σ a q-singular simplex, let us denote by $\sigma^{(i)} : \Delta^{q-1} \to X$ the $(q-1)$-singular simplex defined as being the composition of the restriction $\sigma|_{\Delta^q_{(i)}}$ with the canonical homeomorphism $\Delta^{q-1} \to \Delta^q_{(i)}$.

Note that any homomorphism $C_q(X) \to C_p(X)$ is totally determined by its value on the singular q-simplexes. Given any singular q-simplex σ, we define the boundary operator $\partial_q : C_q(X) \to C_{q-1}(X)$ as

$$\partial_q \sigma = \sum_{j=0}^{q} (-1)^j \sigma^{(j)},$$

therefore given any formal sum $\sum_i g_i\sigma_i$, the boundary operator ∂_q is such that

$$\partial_q \sum_j g_j\sigma_j = \sum_j g_j \partial_q \sigma_j.$$

Often we write the boundary operator ∂_q simply as ∂ when this does not lead to ambiguities. It is not hard to verify that $\partial_q\partial_{q+1} := \partial_q \circ \partial_{q+1} = 0$, or more concisely $\partial^2 = 0$. Therefore we have the following sequence of homomorphisms of abelian groups

$$\cdots \longrightarrow C_{q+1}(X) \xrightarrow{\partial} C_q(X) \longrightarrow \cdots \longrightarrow C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\partial} 0 \tag{1.1}$$

with $\partial^2 = 0$ and the last ∂ on the right is defined to be identically equal zero. Such a sequence is called a *chain complex*. Let us define $Z_q(X,G) = \ker(\partial_q)$ and $B_q(X,G) = \mathrm{Im}\,(\partial_{q+1})$, and if there is no ambiguities we will denote the same sets simply as $Z_q(X)$ and $B_q(X)$ respectively. The equation $\partial_q\partial_{q+1} = 0$ is equivalent to the inclusion $B_q(X) \subset Z_q(X)$, so we can define the q^{th} homology group of the chain complex to be the quotient group

$$H_q(X,G) = Z_q(X,G)/B_q(X,G).$$

As always, we sometimes denote the q^{th} homology group just as $H_q(X)$, when G is already understood.

Elements of $Z_q(X)$ are called *cycles* and elements of $B_q(X)$ are called *boundaries*. Elements of $H_q(X)$ are cosets of $B_q(X)$, called homology classes. Two cycles representing the same homology class are said to be *homologous*. This means their difference is a boundary.

It is not difficult to prove the following simple results concerning singular homology groups of a topological space X (see [14], pg 109-110).

Proposition 1.1.1. *Corresponding to the decomposition of a space X into its path-components X_α there is an isomorphism of $H_q(X)$ with the direct sum $\bigoplus_\alpha H_q(X_\alpha)$.*

Proposition 1.1.2. *If X is nonempty and path-connected, then $H_0(X) \cong G$. Hence for any space X, $H_0(X)$ is a direct sum of G's, one for each path-component of X.*

Proposition 1.1.3. *If X is a point, then*

$$H_q(X) \cong \begin{cases} 0 & \text{if} \quad q > 0, \\ G & \text{if} \quad q = 0. \end{cases}$$

It is often very convenient to have a slightly modified version of homology for which a point has trivial homology groups in all dimensions, incuding zero. This is done by defining the *reduced* *homology groups* $\widetilde{H}_q(X)$ to be the homology groups of the augmented chain complex

$$\cdots \longrightarrow C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} G \longrightarrow 0$$

where $\varepsilon(\sum_i g_i\sigma_i) = \sum_i g_i$. Then we have

$$H_q(X) \cong \begin{cases} \widetilde{H}_q(X) \oplus G & \text{if} \quad q = 0, \\ \widetilde{H}_q(X) & \text{if} \quad q > 0. \end{cases}$$

Now let us take two topological spaces X and Y. For a continuous map $f : X \to Y$, an induced homomorphism $f_\sharp : C_q(X) \to C_q(Y)$ is defined by composing each singular q-simplex $\sigma : \Delta^q \to X$ with f to get a singular q-simplex $f_\sharp(\sigma) = f\sigma : \Delta^q \to Y$, then extending $f_\sharp$ linearly via $f_\sharp(\sum_i g_i\sigma_i) = \sum_i g_i f_\sharp(\sigma_i)$. It is easily seen that $f_\sharp\partial = \partial f_\sharp$, i.e. $f_\sharp$ defines a *chain map* from the singular chain complex of X to that of Y. Therefore we have a well defined induced homomorphism $f_* : H_q(X) \to H_q(Y)$, defined as $f_*[\gamma] = [f_\sharp\gamma]$.

Two maps $f, g : X \to Y$ are called homotopic if exists a continuous map $F : X \times [0,1] \to Y$ such that

$$F(\cdot, 0) = f \quad F(\cdot, 1) = g.$$

One of the most important properties of singular homology theory is the following.

Theorem 1.1.1. *If two maps $f, g : X \to Y$ are homotopic, then they induce the same homomorphism $f_* = g_* : H_q(X) \to H_q(Y)$ for all q.*

Definition 1.1.1. *Two topological spaces X and Y are said to have the same homotopy type if there are continuous maps $f : X \to Y$ and $g : Y \to X$, such that $f \circ g$ is homotopic with the identity on Y and $g \circ f$ is homotopic to the identity on X. If such maps exist, we say that f is a homotopy equivalence, and that g is the homotopy inverse of f. A space X is said to be contractible if there exists a homotopy equivalence $f : X \to P$, where P is a single point.*

Then we have the following corollary.

Corollary 1.1.1. *The maps $f_* : H_q(X) \to H_q(Y)$ induced by a homotopy equivalence $f : X \to Y$ are isomorphisms for all q.*

Example 1.1.1. *For example, if X is contractible, then by Proposition 1.1.3 we have $\widetilde{H}_q(X) = 0$ for all q. Or equivalently, $H_0(X) = G$ and $H_q(X) = 0$ for $q > 0$. An example of contractible space is any star-shaped subspace of a vector space. A star-shaped space $W \subset E$ with respect to a point $x_0 \in W$, where E is a vector space, is a subset such that for every point $y \in W$, the line $(1-t)y + tx_0$ belongs to W for all $t \in [0,1]$. The homotopy equivalence is the constant map $f : W \to \{x_0\}$, with homotopy inverse being the inclusion $i : \{x_0\} \to W$. For example every vector space is a contractible space.*

A topological pair (X,Y) is a topological space X together with a subspace $Y \subset X$. Given two topological pairs (X,Y) and (X',Y'), we say that a map $f:(X,Y) \to (X',Y')$ is continuous if $f: X \to X'$ is continuous with $f(Y) \subset Y'$. Naturally, we say that two maps $f,g:(X,Y) \to (X',Y')$ are homotopic if exists a continuous map

$$F:(X \times [0,1], Y \times [0,1]) \to (X',Y'),$$

such that

$$F(\cdot,0) = f, \quad F(\cdot,1) = g.$$

Let (X,Y) be a topological pair, since

$$\partial : C_q(X) \to C_{q-1}(X)$$

we have

$$\partial : C_q(Y) \to C_{q-1}(Y).$$

Therefore, the boundary operator induces a homomorphism $\overline{\partial}$ which makes the diagram

$$\begin{tikzcd}
C_q(X) \arrow[r] \arrow[d, "\partial"] & C_q(X)/C_q(Y) \arrow[d, "\overline{\partial}"] \\
C_{q-1}(X) \arrow[r] & C_{q-1}(X)/C_{q-1}(Y)
\end{tikzcd}$$

commutative. Then we have clearly that $\overline{\partial}\overline{\partial} = 0$. We call

$$C_q(X,Y) = C_q(X)/C_q(Y)$$

the singular q-relative chain group. We define $Z_q(X,Y)$, $B_q(X,Y)$ and $H_q(X,Y)$ in the same manner as we did for the case $Y = \emptyset$. We call $H_q(X,Y)$ the singular q-relative homology group.

As in the case $Y = \emptyset$, any continuous map $f:(X,Y) \to (X',Y')$ induces a homomorphism

$$f_* : H_q(X,Y) \to H_q(X',Y') \quad \text{for all} \quad q,$$

such that $\overline{\partial} f_* = f_* \overline{\partial}$.

We still have the homotopy invariance: if $f,g:(X,Y) \to (X',Y')$ are homotopic, then $f_* = g_*$. Also, we have the following very important results in relative homology grops:

Theorem 1.1.2 (Excision). *If $U \subset X$ satisfies $\overline{U} \subset int(Y)$, then the inclusion $i : (X \backslash U, Y \backslash U) \to (X,Y)$ induces isomorphisms*

$$i_* : H_q(X \backslash U, Y \backslash U) \cong H_q(X,Y), \quad \textit{for all} \quad q \geq 0.$$

A sequence of homomorphisms

$$\cdots \longrightarrow A_{q+1} \xrightarrow{\alpha_{q+1}} A_q \xrightarrow{\alpha_q} A_{q-1} \longrightarrow \cdots$$

is said to be *exact* if $\ker \alpha_q = \operatorname{Im} \alpha_{q+1}$ for each q.

Theorem 1.1.3 (Exactness). *If $Z \subset Y \subset X$ are three topological spaces, and define the injections $i : (Y,Z) \to (X,Z)$, and $j : (X,Z) \to (X,Y)$, then for each q there exists a homomorphism $\partial : H_q(X,Y) \to H_{q-1}(Y,Z)$ such that the sequence*

$$\cdots \longrightarrow H_q(Y,Z) \xrightarrow{i_*} H_q(X,Z) \xrightarrow{j_*} H_q(X,Y) \xrightarrow{\partial} H_{q-1}(Y,Z) \longrightarrow \cdots$$

is exact.

The following homology groups are often used, where S^n is the unit sphere on $\mathbb{R}^{n+1}$ and B^n the n-ball on $\mathbb{R}^n$, and P^n the real n-projective space.

$$H_q(S^n, G) \cong \begin{cases} 0 & q \neq n, \quad \text{when} \quad q,n \geq 1, \\ G & q = n \geq 1, \quad \text{and} \quad q = 0, n \geq 1, \\ G^2 & q = n = 0. \end{cases}$$

$$H_q(B^n, S^{n-1}, G) \cong \begin{cases} 0 & q \neq n, \\ G & q = n. \end{cases}$$

$$H_q(P^n, \mathbb{Z}_2) \cong \begin{cases} 0 & q > n, \\ \mathbb{Z}_2 & q \leq n. \end{cases}$$

1.2 Singular Cohomology

A cochain complex C over G is a sequence

$$\cdots \longrightarrow C^{q-1} \xrightarrow{\delta^{q-1}} C^q \xrightarrow{\delta^q} C^{q+1} \longrightarrow \cdots$$

of G-modules and homomorphisms δ^q, called coboundary operators, such that

$$\delta^q \delta^{q-1} = 0, \quad \forall q.$$

If $C'^q \subset C^q$ is a sequence of submodules such that $\delta^q(C'^q) \subset C'^{q+1}$, then $C' = \{C'^q, \delta^q\}$ is itself a cochain complex, called a subcomplex of C. Passing to quotients, δ^q induces a homomorphism

$$\overline{\delta}^q : C^q/C'^q \to C^{q+1}/C'^{q+1}$$

such that $\overline{\delta}^q \overline{\delta}^{q+1} = 0$, so $C/C' = \left\{C^q/C'^q, \overline{\delta}^q\right\}$ is also a cochain complex, called the quotient complex of C by C'.

The elements of C^q are called q-cochains and those of the subspaces $Z^q(C) = \ker \delta^q$, $B^q(C) = \operatorname{Im} \delta^{q-1}$ are called q-cocycles and q-coboundaries, respectively. We clearly have $B^q(C) \subset Z^q(C)$, and the quotient space

$$H^q(C) = Z^q(C)/B^q(C)$$

is called the q-th cohomology group of C. A cochain map $f : C \to C'$ of complexes $C = \{C^q, \delta^q\}$ and $C' = \{C'^q, \delta'^q\}$ consists of a sequence of homomorphisms $f^q : C^q \to C'^q$ such that $f^{q+1}\delta^q = \delta'^q f^q$ for all q. Then $f^q(Z^q(C)) \subset Z^q(C')$ and $f^q(B^q(C)) \subset B^q(C')$, so f induces homomorphisms

$$f^* : H^q(C) \to H^q(C'), \quad \forall q.$$

Given a topological space X and its singular chain complex $\left\{C_q(X,G), \partial_q\right\}_q$ we define the set of all singular q-cochains $C^q(X,G)$ as the dual of $C_q(X)$, i.e., $C^q(X,G) := \operatorname{Hom}(C_q(X,G), G)$. $C^q(X,G)$ is of course a G-module. The dual operator of ∂_q is denoted by δ^q, and it the homomorphism $\delta^q : C^{q-1}(X,G) \to C^q(X,G)$ defined as

$$(\delta^q f)(\sigma) = f(\partial_q \sigma), \quad \text{for all} \quad f \in C^{q-1}(X,G) \quad \text{and} \quad \sigma \in C_q(X,G).$$

Most of the time, when there are no ambiguities, we will just denote δ instead of δ^q. Since $\partial^2 = 0$ we have that $\delta^2 = 0$, i.e.

$$\delta^{q+1}\delta^q f = 0, \quad \forall\, f \in C^q(X,G).$$

Therefore $\{C^q(X;G), \delta^q\}$ is a cochain complex, called the singular cochain complex of X with coefficients in G.

More generally we can construc the singular cohomology as follows: For a topological pair

(X,Y), let

$$C^q(X,Y,G) = \text{Hom}\left(C_q(X,G)/C_q(Y,G),G\right)$$

and let

$$\overline{\delta}^q : C^{q-1}(X,Y,G) \to C^q(X,Y,G)$$

be the dual operator of the boundary operator $\overline{\partial} : C_q(X,Y,G) \to C_{q-1}(X,Y,G)$. Most of the time, when there are no ambiguities, we will denote $\overline{\delta}^q$ only by δ. It is easily seen that $C^q(X,Y,G)$ is isomorphic to the set

$$\left\{f \in C^q(X,G) : f(\sigma) = 0\, \forall \sigma \in C_q(Y,G)\right\}.$$

As usual, we define $Z^q(X,Y;Z) := \ker(\delta^q)$, $B^q(X,Y;G) = \text{Im}\,(\delta^{q-1})$ and

$$H^q(X,Y;G) = Z^q(X,Y;G)/B^q(X,Y;G).$$

In general, we have a canonical homomorphism

$$\alpha : H^q(X,Y;G) \to \text{Hom}\left(H_q(X,Y;G),G\right).$$

In the case where G is a field, α is surjective.

If $f : (X,Y) \to (X',Y')$ is continuous, then we can define a homomorphism

$$\overline{f}^* : C^q(X',Y';G) \to C^q(X,Y;G)$$

defined as

$$[\overline{f}^*(g)](\sigma) = g(f_\sharp\sigma)$$

for all $g \in C^q(X',Y';G)$ and $\sigma \in C^q(X,Y;G)$. Note that $\delta\overline{f}^* = \overline{f}^*\delta$, so it induces homomorphisms

$$f^* : H^q(X',Y';G) \to H^q(X,Y;G), \quad \forall q$$

given by

$$f^*[g] = [\overline{f}^* g].$$

We then have the following properties concerning singular cohomology

(1) If $f : (X,Y) \to (X',Y')$ is continuous, we have

   (a) If $f = 1_X$, then $f^* = 1_{H^*}$.
   (b) If $g : (X',Y') \to (X'',Y'')$ is continuous, then $(gf)^* = f^*g^*$.

(2) If $f,g : (X,Y) \to (X',Y')$ are homotopic, then $f^* = g^*$. If (X,Y) has the same homotopy type as (X',Y'), then $H^q(X,Y;G) \cong H^q(X',Y';G)$ for all q.

(3) (Excision) If $\overline{U} \subset \operatorname{int}(Y)$, then $H^*(X\backslash U, Y\backslash U;G) \cong H^*(X,Y;G)$.

(4) If $Z \subset Y \subset X$, then the sequence

$$\cdots \longrightarrow H^q(X,Y;G) \xrightarrow{j_*} H^q(X,Z;G) \xrightarrow{i_*} H^q(Y,Z;G) \xrightarrow{\delta} H^{q+1}(X,Y;G) \longrightarrow \cdots$$

is exact.

(5)

$$H^q(\{p\},G) \cong \begin{cases} G & q = 0 \\ 0 & q \neq 0. \end{cases}$$

Now suppose that G is also a ring, and let us consider the maps

$$\begin{aligned} \lambda_p &: \Delta^p \to \Delta^{p+q} \\ \beta_q &: \Delta^q \to \Delta^{p+q} \end{aligned}$$

to be

$$\lambda_p(t_0,\ldots,t_p) = (t_0,\ldots,t_p,0,\ldots,0), \quad \beta_q(t_0,\ldots,t_q) = (0,\ldots,0,t_0,\ldots,t_q).$$

For cochains $g \in C^p(X;G)$ and $h \in C^q(X;G)$, the *cup product* $g \smile h \in C^{p+q}(X;G)$ is the cochain whose value on a singular simplex $\sigma : \Delta^{p+q} \to X$ is given by the formula

$$(g \smile h)(\sigma) = g(\sigma \circ \lambda_p)h(\sigma \circ \beta_q).$$

It is easily seen that the cup product $\smile$ has the property

$$\delta(g \smile h) = \delta g \smile h + (-1)^p g \smile \delta h,$$

for $g \in C^p(X;G)$ and $h \in C^q(X;G)$. So it follows that there is an induced cup product

$$H^p(X;G) \times H^q(X;G) \xrightarrow{\smile} H^{p+q}(X;G).$$

It can also be shown that the cup product $\smile$ is not generally commutative. Namely, we have the following:

- The identity $g \smile h = (-1)^{pq} h \smile g$ holds for all $g \in H^p(X;G)$ and $h \in H^q(X;G)$, when G is a commutative ring.

Also, it is easily seen that for a continuous map $f : X \to Y$, the induced maps $f^* : H^q(Y;G) \to H^q(X;G)$ satisfy $f^*(g \smile h) = f^*g \smile f^*h$.

The cup product formula $(g \smile h)(\sigma) = g(\sigma \circ \lambda_p)h(\sigma \circ \beta_q)$ also gives relative cup products

$$\begin{aligned} H^p(X;G) \times H^q(X,A;G) &\xrightarrow{\smile} H^{p+q}(X,A;G) \\ H^p(X,A;G) \times H^q(X;G) &\xrightarrow{\smile} H^{p+q}(X,A;G) \\ H^p(X,A;G) \times H^q(X,A;G) &\xrightarrow{\smile} H^{p+q}(X,A;G), \end{aligned}$$

since if g or h vanishes on chains in A then so does $g \smile h$. There is a more general relative cup product

$$H^p(X,A;G) \times H^q(X,B;G) \xrightarrow{\smile} H^{p+q}(X,A \cup B;G)$$

when A and B are open subsets of X (see [14], pg 209).

The cup product is associative and distributive, so we can make it the multiplication in a ring structure on the cohomology groups of a space X. This is easy to do if we simply define $H^*(X;G)$ to be the direct sum of the groups $H^q(X;G)$, i.e. $H^*(X;G) = \bigoplus_{i=0}^{\infty} H^i(X;G)$. Elements of $H^*(X;G)$ are finite sums $\sum_i \alpha_i$, with $\alpha_i \in H^i(X;G)$, and the product of two such sums is defined to be $(\sum_i \alpha_i)(\sum_j \beta_j) = \sum_{i,j} \alpha_i \beta_j$, where of course we are denoting, by simplicity, $\alpha_i \beta_j = \alpha_i \smile \beta_j$.

Example 1.2.1. *If P^n is the n-dimensional real projective space, then it can be shown that $H^*(P^n;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]/(\alpha^{n+1})$ where α is the generator of the group $H^1(P^n;\mathbb{Z}_2)$. For example, if $n = 2$ then $H^*(P^2;\mathbb{Z}_2)$ consistis of the polynomials $a_0 + a_1\alpha + a_2\alpha^2$, with $a_i \in \mathbb{Z}_2$ and $\alpha^3 = 0$. if $n = \infty$, then we have $H^*(P^\infty;\mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha]$, where α is the generator of $H^1(P^\infty;\mathbb{Z}_2)$. The same is true for the complex projective spaces $\mathbb{C}P^n$ with $n \in \mathbb{N} \cup \{\infty\}$, but this time α is the generator of $H^2(\mathbb{C}P^n;\mathbb{Z}_2)$. (For a proof of those computations the reader is referred to [14],[24] or [15]).*

1.3 Alexander-Spanier Cohomology Theory

We now recall the construction and properties of the Alexsander-Spanier cohomology theory with coefficients in the field $\mathbb{Z}_2$. A basic reference for this theory can be found in [24].

Let (X,A) be a pair of spaces, i.e. a topological space X and a subspace $A \subset X$. For $q \geq 0$, let $C^q(X)$ be the vector space over $\mathbb{Z}_2$ of all functions $\varphi : X^{q+1} \to \mathbb{Z}_2$, where X^{q+1} is the product of $q+1$ copies of X, with adding and scallar multiplication defined pointwise, i.e.,

$$(\varphi + \varphi')(x) = \varphi(x) + \varphi'(x), \quad (c\varphi)(x) = c(\varphi(x))$$

for $\varphi, \varphi' \in C^q(X)$, $c \in \mathbb{Z}_2$, and $x = (x_0, \dots, x_q) \in X^{q+1}$. Define the coboundary operator $\delta^q : C^q(X) \to C^{q+1}(X)$ by

$$(\delta^q \varphi)(x_0, \dots, x_{q+1}) = \sum_{i=0}^{q+1} \varphi(x_0, \dots, \widehat{x_i}, \dots, x_{q+1})$$

where the symbol $\widehat{x_i}$ indicates that x_i is omitted. Then $\delta^q \delta^{q-1} = 0$, so $C(X) = \{C^q(X), \delta^q\}$ is a cochain complex.

We say that $\varphi \in C^q(X)$ is a locally zero if there is an open covering $\mathcal{U}$ of X such that φ vanishes on any $(q+1)$-tuple of X that lies in some $U \in \mathcal{U}$, i.e., φ vanishes on $\mathcal{U}^{q+1} := \bigcup_{U \in \mathcal{U}} U^{q+1}$. Let $C_0^q(X)$ be the subspace of $C^q(X)$ consisting of locally zero funcitons. If φ vanishes on $\mathcal{U}^{q+1}$, then $\delta^q \varphi$ vanishes on $\mathcal{U}^{q+2}$, so $C_0(X) = \left\{C_0^q(X), \delta^q\right\}$ is a subcomplex of $C(X)$. Let $\overline{C}(X) = C(X)/C_0(X) = \left\{C^q(X)/C_0^q(X), \overline{\delta}^q\right\}$. A continuous map $f : X \to Y$ of spaces induces a cochain map $f^\sharp : C(Y) \to C(X)$ by

$$(f^{\sharp q} \varphi)(x_0, \dots, x_q) = \varphi(f(x_0), \dots, f(x_q)), \quad \varphi \in C^q(Y).$$

If $\psi \in C_0^q(Y)$ and $\mathcal{V}$ is an open covering of Y such that ψ vanishes on $\mathcal{V}^{q+1}$, then $\mathcal{U} = f^{-1}\mathcal{V}$ is an open covering of X and $f^{\sharp q}\psi$ vanishes on $\mathcal{U}^{q+1}$. So $f^\sharp$ maps $C_0(Y)$ into $C_0(X)$ and hence induces a cochain map $\overline{f}^\sharp : \overline{C}(Y) \to \overline{C}(X)$.

If $i : A \subset X$, then $\overline{C}(X,A) = \ker \overline{i}^\sharp = \left\{\ker \overline{i}^{\sharp q}, \overline{\delta}^q\right\}$ is a subcomplex of $\overline{C}(X)$. Let $C(X,A)$ be the subcomplex of $C(X)$ consisting of functions that are locally zero on A. Then we see that $C_0(X) \subset C(X,A)$ and $\overline{C}(X,A) = C(X,A)/C_0(X)$. Define the Alexander-Spanier cohomology of (X,A) by

$$H^*(X,A) = H^*(\overline{C}(X,A)).$$

We write $H^*(X, \emptyset) = H^*(X)$. If $f : (X,A) \to (Y,B)$ is a continuous map of pairs, then $\overline{f}^\sharp : \overline{C}(Y,B) \to \overline{C}(X,A)$ is a cochain map and hence induces homomorphisms

$$f^* : H^*(Y,B) \to H^*(X,A)$$

such that $(gf)^* = f^* g^*$ for $(X,A) \xrightarrow{f} (Y,B) \xrightarrow{g} (Z,C)$ and $(1_{(X,A)})^* = 1_{H^*(X,A)}$.

The Alexander-Spanier cohomology satisfies the following axioms (see [24] for the proofs):

(i) Homotopy axiom: If $f_0 \simeq f_1 : (X,A) \to (Y,B)$, then

$$f_0^* = f_1^* : H^*(Y,B) \to H^*(X,A).$$

(ii) Exactness axiom: Each triple (X,A,B) with $B \subset A \subset X$ has an exact sequence

$$\cdots \longrightarrow H^q(X,A) \xrightarrow{j^*} H^q(X,B) \xrightarrow{i^*} H^q(A,B) \xrightarrow{\delta} H^{q+1}(X,A) \longrightarrow \cdots$$

where $i : (A,B) \subset (X,B)$, $j : (X,B) \subset (X,A)$, and δ is called the connecting map.

(iii) Excision axiom: If U is an open subset of X such that $\overline{U} \subset \mathrm{int}(A)$, then the inclusion $j : (X \backslash U, A \backslash U) \subset (X,A)$ induces an isomorphism

$$j^* : H^*(X,A) \cong H^*(X \backslash U, A \backslash U).$$

(iv) Dimension axiom: If X is a one-point space, then

$$H^q(X) \cong \begin{cases} \mathbb{Z}_2 & \text{if} \quad q = 0, \\ 0 & \text{if} \quad q \neq 0. \end{cases}$$

When $X \neq \emptyset$, in some occasions it is convenient to work with the reduced groups

$$\widetilde{H}^q(X) \cong \begin{cases} H^q(X)/\mathbb{Z}_2, & q = 0 \\ H^q(X), & q \geq 1. \end{cases}$$

They also fit into an exact sequence

$$\cdots \longrightarrow \widetilde{H}^{q-1}(X) \longrightarrow \widetilde{H}^{q-1}(A) \longrightarrow H^q(X,A) \longrightarrow \widetilde{H}^q(X) \longrightarrow \cdots$$

for a pair (X,A) with $A \neq \emptyset$.

There is a product $\smile: H^p(X) \times H^q(X) \to H^{p+q}(X)$, called the cup product, giving $H^*(X)$ the structure of a graded ring. To define this product, first define a product

$$\smile: C^p(X) \times C^q(X) \to C^{p+q}(X)$$

by

$$(\varphi \smile \psi)(x_0, \ldots, x_{p+q}) = \varphi(x_0, \ldots, x_p)\psi(x_p, \ldots, x_{p+q}).$$

If φ or ψ is locally zero, then so is $\varphi \smile \psi$, and hence $\smile$ induces a product

$$\smile: \overline{C}^p(X) \times \overline{C}^q(X) \to \overline{C}^{p+q}(X)$$

.

It is easy to verify that

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + \varphi \smile \delta\psi,$$

so if φ and ψ are cocycles, then so is $\varphi \smile \psi$, and if one of them is a coboundary in addition, then so is $\varphi \smile \psi$. Hence $\smile$ induces a product on cohomology classes.

This cohomology theory satisfies the following property. For the definition of *directed systems* see §2.2. on [21], or [24]. A neighborhood of a pair (A,B) in a topological space X is a pair (U,V) in X such that U is a neighborhood of A and V is a neighborhood of B. The set Λ of all neighborhoods of (A,B) is partially ordered downward (see [21]) by inclusion: $(U_\lambda, V_\lambda) \preceq (U_\mu, V_\mu)$ if $i_{\lambda\mu} : (U_\mu, V_\mu) \subset (U_\lambda, V_\lambda)$. Λ is directed since $(U_\lambda \cap U_\mu, V_\lambda \cap V_\mu) \subset (U_\lambda, V_\lambda), (U_\mu, V_\mu)$. Then it is easily seen that the collection $\{H^q(U_\lambda, V_\lambda)\}_{\lambda \in \Lambda}$ is a directed system. The maps $j_\lambda^* : H^q(U_\lambda, V_\lambda) \to H^q(A,B)$ induced by the inclusion $j_\lambda : (A,B) \subset (U_\lambda, V_\lambda)$ satisfy $j_\mu^* i_{\lambda\mu}^* = (i_{\lambda\mu} j_\mu)^* = j_\lambda^*$, so their limit

$$j^* = \varinjlim_{\Lambda} j_\lambda^* : \varinjlim_{\Lambda} H^q(U_\lambda, V_\lambda) \to H^q(A,B)$$

is well defined. Then we have the following two facts (see [24]).

Proposition 1.3.1. *If X is paracompact and A and B are closed, then j^* is an isomorphism for all q.*

Proposition 1.3.2. *Let us denote H_s^* the singular cohomology groups with $\mathbb{Z}_2$-coefficents. If X is paracompact and locally contractible, then there is an isomorphism $\mu : H^q(X) \to H_s^q(X)$.*

CHAPTER 2

Motivation

2.1 Some ideas on critical point theory

As mentioned in the introduction, many significant problems in science can be translated to finding critical points of a suitable functional defined on an appropriate space. Consequently, there is a substantial body of literature devoted to techniques for solving critical point problems. The problems of finding minima of a functional are the most natural ones, and such kind of problem do not even ask for regularity in the sense of differentiability. Therefore the problem of finding minima can deal with a broader class of situations. However, when a problem has regularity, we can go further and seek for another kind of critical points, the ones which are neither of minimum or maximum type, namely the saddle points.

This section will concern with some intuitive ideas that motivate this dissertation. The proper definitions regarding critical point theory will be given in the next section.

Given $f:\mathcal{M}\to\mathbb{R}$ a differentiable function, defined on a differentiable manifold $\mathcal{M}$, we will use the notations

$$f_a=\left\{u\in\mathcal{M}:f(u)\geq a\right\},\quad f^b=\left\{u\in\mathcal{M}:f(u)\leq b\right\},\quad f_a^b=f_a\cap f^b,$$

$$K=\left\{u\in\mathcal{M}:f'(u)=0\right\},\quad K_a^b=K\cap f_a^b,\quad K^c=K_c^c.$$

The best prototype of a saddle point is the unique critical point of the function $f:\mathbb{R}^2\to\mathbb{R}$ given by $f(x,y)=x^2-y^2$.

But how can we detect critical points of this kind? Well, we can first try to understand the case of the saddle x^2-y^2. The origin (the saddle point of f) can be characterized in the following way; let us take two points $x_1,x_2\in\mathbb{R}^2$ such that if $x_i=(a_i,b_i)$ then $b_1<0$ and $b_2>0$ (for example $(0,-1)$ and $(0,1)$). Then it is obvious, by the mean value theorem, that any continuous path $\gamma\colon[0,1]\to\mathbb{R}^2$ connecting x_1 and x_2 must cross the line $y=0$. But $f\geq0$ on the line $y=0$. We then conclude that for

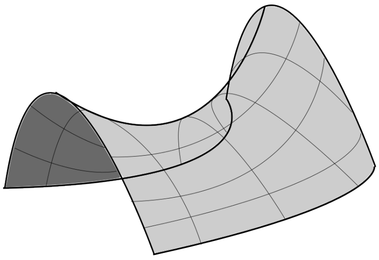


Figure 2.1:
Graph of $x^2 - y^2$ near the origin. A saddle.

any such a path, we have

$$\max_{t\in[0,1]} f(\gamma(t)) \geq 0.$$

Therefore, if we denote by Γ to be the set of continuous paths $\gamma : [0,1] \to \mathbb{R}^2$ connecting x_1 to x_2 ($\gamma(0) = x_1$ and $\gamma(1) = x_2$), we have

$$\inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)) \geq 0.$$

But it is easily seen that in fact the equality holds:

$$\inf_{\gamma\in\Gamma} \max_{t\in[0,1]} f(\gamma(t)) = 0. \tag{1.1}$$

Equation (1.1), as innocent as it might seem to be, it tells us a very important information; Consider $\varepsilon > 0$ such that $\max\{f(x_1), f(x_2)\} < -\varepsilon$. Then the set $f^{-\varepsilon}$ is not path connected (the points $x_1, x_2 \in f^{-\varepsilon}$ cannot be connected by a continuous path that lies entirely on $f^{-\varepsilon}$). Note, however, that for any $\delta > 0$, f^{δ} is always path connected, and f^0 is the “critical sublevel" in the sense that it is the last sublevel that is path connected, i.e. $0 = \inf\left\{\delta > 0 : f^{\delta} \text{ is path connected}\right\}$.

Just to be sure that such topological behaviour is not a coincidence, let us do something similar with a more general saddle point. Namely let us take the best prototype of a non-degenerate critical point with Morse index $\lambda > 0$ (see Morse Lemma 4.2.1). Let $f : \mathbb{R}^n \to \mathbb{R}$ be such a function. By the Morse Lemma, we can assume that f is given by

$$f(u_1, \ldots, u_n) = c - \sum_{i=1}^{\lambda} u_i^2 + \sum_{j=\lambda+1}^{n} u_j^2, \tag{1.2}$$

with $f(0) = c$.

Firt let us remember what a retract is; Let (X, A) be a pair, i.e., a topological space X and a subspace $A \subset X$. A is called a retract of X if there exists a continuous map $r : X \to A$ such that $ri = 1_A$, where 1_A is the identity map of A and $i : A \subset X$ is the inclusion map. This means that $r(x) = x$ for all

$x \in A$. Such a map r is called a retraction of X to A.

For example the map $r : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$, given by $x \mapsto x/|x|$ is a retraction of $\mathbb{R}^n \setminus \{0\}$ to S^{n-1}. And it can be shown that if B^n is a closed ball in $\mathbb{R}^n$, then there is no retraction $r : B^n \to \partial B^n$.

A is called a deformation retract of X if there is a retraction r of X to A such that if $i : A \subset X$, then $1_X \simeq ir$, where $\simeq$ means homotopy, i.e., the identity 1_X of X is homotopic to the map ir. So we have that $ri = 1_A$ and $ir \simeq 1_X$, which brings the idea of r being the *homotopy inverse* of i. When that happens, we say that the maps i and r are homotopy equivalences between X and A.

Let us consider the direct sum decomposition $\mathbb{R}^n = \mathbb{R}^\lambda \oplus \mathbb{R}^{n-\lambda}$, and on $\mathbb{R}^\lambda$ let us consider the ball centered at 0 with radius $\sqrt{\varepsilon}$

$$B^\lambda := B^\lambda_{\sqrt{\varepsilon}}(0) := \left\{ (u_1, \ldots, u_\lambda, 0, \ldots, 0) \in \mathbb{R}^n : \sum_{i=1}^{\lambda} u_i^2 \leq \varepsilon \right\} \subset \mathbb{R}^\lambda.$$

Then by employing the relative singular homology groups in dimension λ, it is easy to show that (see the figures below)

$$H_\lambda(f^c, \partial B^\lambda) \cong H_\lambda(\mathbb{R}^\lambda, \partial B^\lambda) \cong H_\lambda(B^\lambda, \partial B^\lambda) \cong H_\lambda(S^\lambda) \neq 0 \tag{1.3}$$

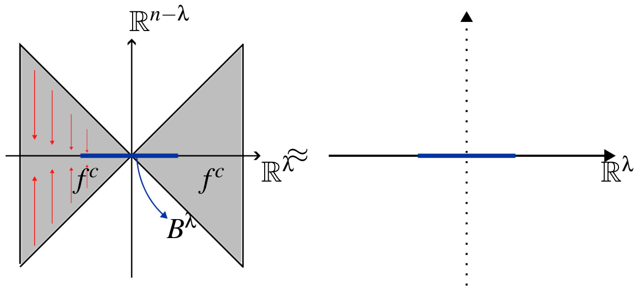


Figure 2.2:

while

$$H_\lambda(f^{c-\varepsilon}, \partial B^\lambda) \cong H_\lambda(\mathbb{R}^\lambda \setminus \{0\}, \partial B^\lambda) \cong H_\lambda(\partial B^\lambda, \partial B^\lambda) = 0 \tag{1.4}$$

Therefore, by (1.3) and (1.4) we see that the groups $H_\lambda(f^c, \partial B^\lambda)$ and $H_\lambda(f^{c-\varepsilon}, \partial B^\lambda)$ are not isomorphic. This implies that $f^{c-\varepsilon}$ is not a deformation retract of f^c, consequently $f^{c-\varepsilon}$ is not a deformation retract of $f^{c+\varepsilon}$. We might conclude that the change in topology between the sublevels $f^{c+\varepsilon}$ and $f^{c-\varepsilon}$ might not be a coincidence, and our guess is that it is related with the existence of a critical value between $c - \varepsilon$ and $c + \varepsilon$.

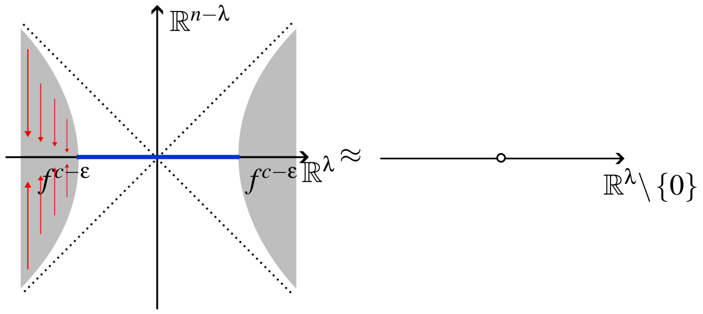


Figure 2.3:

Also, if our guess happens to be true, we just found a way of searching for critical values of a function f. Namely, by studying the topology of the sublevels f^a. However in some cases we can have a change in topology of the sublevels of a function without any critical values in between. For example the function $g(x) = e^{-x} - y^2$. It is easy to see that, for any $\varepsilon > 0$, g^{ε} is path connected but $g^{-\varepsilon}$ is not, and g has no critical points at all.

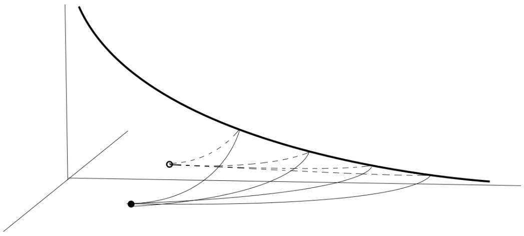


Figure 2.4:
Graph of $e^{-x} - y^2$.

A way of avoinding such situations is by asking the function to have some kind of compactness. The most common one is the *Palais-Smale* compactness condition.

Definition 2.1.1 (Palais-Smale compactness condition)**.** *A sequence $(x_n)_n$ is called a Palais-Smale sequence (PS-sequence) at c for f, if $f(x_n) \to c$ and $f'(x_n) \to 0$. We say that f satisfies the $(PS)_c$ condition if every PS-sequence at c possesses a subsequence that converges strongly to some x. We say that f satisfies the (PS) condition if it satisfies the $(PS)_c$ condition for all $c \in \mathbb{R}$.*

The result that connects the topology of sublevels with the existence of critical values of a function is the following (see [4], Theorem 3.2).

Theorem 2.1.1 (Second deformation lemma)**.** *Let M be a C^2 Finsler manifold (see Chapter 4). Suppose that $f \in C^1(M,\mathbb{R})$ satisfies the $(PS)_c$ condition for all $c \in [a,b]$ and that a is the only critical value of f in $[a,b)$. Assume that the connected components of K^a are only isolated points. Then f^a is a strong deformation retract of $f^b \setminus K^b$.*

It is easy to see that the function (1.2) satisfies the (PS) condition, and therefore the change in topology of its sublevels happens only because of the critical value $f(0) = c$.

Since the topological behaviour of a function near a non-degenerate critical point is well known, it is easy to see how to spot its critical value. For example, for the saddle $x^2 - y^2$, since we lose path conectdness when a critical value is crossed, we can spot the change in topology by studying paths, as the expression (1.1) shows us. And for the more general case of the function (1.2), since the change in topology occurs in dimension λ, we can spot its critical value using balls of dimension λ in the following way.

Consider

$$\Gamma := \left\{ \gamma \in C(B^\lambda, \mathbb{R}^n) : \gamma|_{\partial B^\lambda} = 1_{\partial B^\lambda} \right\}.$$

Since there is no continuous retraction from the ball B^λ to its boundary, we have that

$$\max_{u \in B^\lambda} f(\gamma(u)) \geq f(0), \quad \forall \gamma \in \Gamma.$$

Since $1_{B^\lambda} \in \Gamma$, it is easily seen that

$$\min_{\gamma \in \Gamma} \max_{u \in B^\lambda} f(\gamma(u)) = f(0). \tag{1.5}$$

So, at least for the simpler case of a non-degenerated critical point, we saw that by understanding the topological behaviour of the function f, we can try to find changes in the topology of the sublevels by an expression like (1.5).

However, in most problems we do not have the necessary hypothesis to talk about non-degeneracy of critical points, and therefore some generalizations of the previous ideas are necessary. For example, by mimicking the above ideas to a case of a function defined on a general Banach space, we have the following results.

Theorem 2.1.2 (Mountain pass theorem, Ambrosetti-Rabinowitz, 1973. See [27], Theorem 2.10)**.** *Let X be a Banach space, $\varphi \in C^1(X, \mathbb{R})$, $e \in X$ and $r > 0$ be such that $\|e\| > r$ and*

$$b := \inf_{\|u\|=r} \varphi(u) > \varphi(0) \geq \varphi(e).$$

If φ satisfies the $(PS)_c$ condition with

$$\begin{aligned} c: &= \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)), \\ \Gamma: &= \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e\}, \end{aligned}$$

then c is a critical value of φ.

Remark 2.1.1. *Note that the mountain pass theorem detects losses in path connectedness of the sublevels.*

Theorem 2.1.3 (Saddle-point theorem, Rabinowitz, 1978. See [27], Theorem 2.11)**.** *Let* $X = Y \oplus Z$ *be a Banach space with* $\dim Y < \infty$*. Define, for* $\rho > 0$*,*

$$M := \{u \in Y : \|u\| \leq \rho\}, \quad M_0 := \{u \in Y : \|u\| = \rho\}.$$

Let $\varphi \in C^1(X, \mathbb{R})$ *be such that*

$$b := \inf_Z \varphi > a := \max_{M_0} \varphi.$$

If φ *satisfies the* $(PS)_c$ *condition with*

$$\begin{aligned} c : &= \inf_{\gamma \in \Gamma} \max_{u \in M} \varphi(\gamma(u)), \\ \Gamma : &= \{\gamma \in C(M, X) : \gamma|_{M_0} = 1_{M_0}\}, \end{aligned}$$

then c *is a critical value of* φ*.*

Remark 2.1.2. *Note that the change in topology that the Saddle-point theorem is detecting is the same as in the function* (1.2)*.*

Theorem 2.1.4 (Linking Theorem, Rabinowitz, 1978. See [27])**.** *Let* $X = Y \oplus Z$ *be a Banach space with* $\dim Y < \infty$*. Let* $\rho > r > 0$ *and let* $z \in Z$ *be such that* $\|z\| = r$*. Define*

$$\begin{aligned} M : &= \{u = y + \lambda z : \|u\| \leq \rho, \lambda \geq 0, y \in Y\}, \\ M_0 : &= \{u = y + \lambda z : y \in Y, \|u\| = \rho \text{ and } \lambda \geq 0 \text{ or } \|u\| \geq \rho \text{ and } \lambda = 0\}, \\ N : &= \{u \in Z : \|u\| = r\}. \end{aligned}$$

Let $\varphi \in C^1(X, \mathbb{R})$ *be such that*

$$b := \inf_N \varphi > a := \max_{M_0} \varphi.$$

If φ *satisfies the* $(PS)_c$ *condition with*

$$c := \inf_{\gamma \in \Gamma} \max_{u \in M} \varphi(\gamma(u)), \quad \Gamma := \{\gamma \in C(M, X) : \gamma|_{M_0} = 1_{M_0}\},$$

then c *is a critical value of* φ*.*

More generally, one way of understanding the topological behaviour of sublevels of a functional which satisfies a compactness condition is as follows (see [26], Theorem 3.11).

Theorem 2.1.5 (Deformation Lemma)**.** *Suppose $\mathcal{M} \subset E$ is a C^1 complete submanifold of a Banach space E and $f \in C^1(\mathcal{M})$ satisfies the (PS) condition. Let $c \in \mathbb{R}$, $\bar{\varepsilon} > 0$ be given and let N be any neighborhood of $K^c := \{f(u) = c, f'(u) = 0\}$. Then there exist a number $0 < \varepsilon < \bar{\varepsilon}$ and a continuous 1-parameter family of homeomorphisms $\eta(\cdot,t)$ of $\mathcal{M}$, $0 \le t < \infty$, with the properties*

1. *$\eta(u,t) = u$, if $t = 0$, or $f'(u) = 0$, or $u \in \left(f^{-1}[c-\bar{\varepsilon}, c+\bar{\varepsilon}]\right)^c$*
2. *$f(\eta(u,t))$ is non-increasing in t for any $u \in \mathcal{M}$;*
3. *$\eta(f^{c+\varepsilon} \backslash N, 1) \subset f^{c-\varepsilon}$, and $\eta(f^{c+\varepsilon}) \subset f^{c-\varepsilon} \cup N$.*
4. *If $\mathcal{M}$ is symmetric and f is even, then $\eta(\cdot,t)$ can be taken to be odd for all t.*

It implies that if c is a regular value and f satisfies the $(PS)_c$ condition, then the family $\mathcal{D}_{c,\varepsilon}$ of maps $\eta \in C([0,1] \times \mathcal{M}, \mathcal{M})$ satisfying

1. $\eta(0,\cdot) = 1_{\mathcal{M}}$,
2. $\eta(t,\cdot)$ is a homeomorphism of $\mathcal{M}$ for all $t \in [0,1]$,
3. $\eta(t,\cdot)$ is the identity outside $f^{-1}[c-2\varepsilon, c+2\varepsilon]$ for all $t \in [0,1]$,
4. $f(\eta(\cdot,u))$ is non-increasing for all $u \in \mathcal{M}$,
5. $\eta(1, f^{c+\varepsilon}) \subset f^{c-\varepsilon}$

is nonempty for all small $\varepsilon > 0$

Definition 2.1.2. *We say that a family $\mathcal{F}$ of subsets $\mathcal{M}$ is deformation invariant under $\mathcal{D}_{c,\varepsilon}$, if*

$$M \in \mathcal{F}, \eta \in \mathcal{D}_{c,\varepsilon} \implies \eta(1,M) \in \mathcal{F}$$

Theorem 2.1.6 (Minimax Principle)**.** *Suppose $\mathcal{F}$ is a collection of subsets of $\mathcal{M}$ and*

$$c = \inf_{F \in \mathcal{F}} \sup_{u \in F} f(u)$$

is finite, and that $\mathcal{F}$ is deformation invariant under $\mathcal{D}_{c,\varepsilon}$ for all sufficently small $\varepsilon > 0$, and f satisfies $(PS)_c$ condition, then c is a critical value of f.

Remark 2.1.3. *The Minimax Principle tells us that we can seek for change in topology of sublevels through min-max numbers, using more general sets than just balls, as we did for the case of non-degenerated critical points.*

2.2 Kranoselskii Genus

More informations about the critical set of a functional can be found when the problem is endowed with some kind of symmetry. For example let us study the following problem.

Let X be a real Hilbert space with inner product $(\cdot,\cdot)$, norm $|\cdot| = \sqrt{(\cdot,\cdot)}$. Let $T : X \to X$ a continuous linear operator. Let us suppose, in addition, that T is compact, self-adjoint and $\varphi(x) = (Tx,x) \geq 0$ on X. Also let $S = \{x \in X : |x| = 1\}$ be the unit sphere of X.

Proposition 2.2.1. *Let X, S and T as above. Let $\mathcal{F}_n$ be the family of n-dimensional subspaces of X. Then the eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots$ of T are given by*

$$\lambda_n = \max_{\mathcal{F}_n} \min_{F\cap S} (Tx,x) = \min_{\mathcal{F}_{n-1}} \max_{F^\perp \cap S} (Tx,x).$$

Proof. Let us first prove the equality $\lambda_n = \min_{\mathcal{F}_{n-1}} \max_{F^\perp\cap S}(Tx,x)$, and the another one will follow similarly. Remember that $\lambda_n = \max_{S\cap X_{n-1}}(Tx,x)$, where $X_{n-1} = (\operatorname{span}\{e_1,\dots,e_{n-1}\})^\perp$. We claim that for every $F \in \mathcal{F}_{n-1}$ we have $F^\perp \cap \operatorname{span}\{e_1,\dots,e_n\} \neq \{0\}$, but this is clear since the codimension of $F^\perp$ is $n-1$. This then implies that $(F^\perp \cap S) \cap (\operatorname{span}\{e_1,\dots,e_n\} \cap S) \neq \emptyset$, so

$$\max_{F^\perp\cap S} (Tx,x) \geq \min_{\operatorname{span}\{e_1,\dots,e_n\}\cap S} (Tx,x),$$

and since on $\operatorname{span}\{e_1,\dots,e_n\}$ T has the form $Tx = \sum_{i=1}^n \lambda_i (x,e_i)e_i$, it is obvious that

$$\min_{\operatorname{span}\{e_1,\dots,e_n\}\cap S} (Tx,x) = \lambda_n.$$

Therefore λ_n is a lower bound for values of the form $\max_{F^\perp\cap S}(Tx,x)$ with $F \in \mathcal{F}_{n-1}$, and since it is attained for $F = \operatorname{span}\{e_1,\dots,e_{n-1}\}$ we are done.

Now let us prove the equality $\lambda_n = \max_{\mathcal{F}_n} \min_{F\cap S}(Tx,x)$. Let $F \in \mathcal{F}_n$. We claim that $(F\cap S) \cap (X_{n-1} \cap S) \neq \emptyset$. Note that it suffices to show that $F \cap X_{n-1} \neq \{0\}$. Suppose that $F \cap X_{n-1} = \{0\}$, then it implies that the codimension of X_{n-1} is at least n, which is a contradition since the codimension of X_{n-1} is of course $n-1$. So in fact we have $(F\cap S)\cap(X_{n-1}\cap S) \neq \emptyset$, which implies that $\min_{F\cap S}(Tx,x) \leq \max_{X_{n-1}}(Tx,x) \leq \lambda_n$, therefore $\max_{\mathcal{F}_n}\min_{F\cap S}(Tx,x) \leq \lambda_n$, so the equality follows since we have that it is attained for $F = \operatorname{span}\{e_1,\dots,e_n\}$. ☐

Denoting $f(x) = -(Tx,x)$ we have

$$c_n := -\lambda_n = \min_{F\in\mathcal{F}_n} \max_{F\cap S} f(x).$$

The above theorem remembers us about the Minimax Principle. Note, however, that the sets of the form $F \cap S$ where F is a vector subspace are not good to detect changes in homotopy type of the sublevels $f^{c_n+\varepsilon}$ and $f^{c_n-\varepsilon}$ since they are not deformation invariant.

First, let us denote $\mathcal{A}_n = \{F \cap S : F \text{ subspace}, \dim F = n\}$, so it is clear that

$$c_n = \min_{\mathcal{A}_n} \max_{A} f.$$

In order to detect the values c_n by means of a minimax number which fits the Minimax Principle, we need to add more sets to $\mathcal{A}_n$

So if one still wants to use the above idea to show that the value c_n is detecting a change in homotopy type of subleves, and therefore a critical value, it is needed to find a bigger set $\mathcal{Z}_n$, with $\mathcal{A}_n \subset \mathcal{Z}_n$ in such a way that if $A \in \mathcal{Z}_n$ and η is a deformation, then $\eta(1,A) \in \mathcal{Z}_n$ and

$$c_n = \min_{\mathcal{Z}_n} \max_{Z} f.$$

Since all the conditions above need to be satisfied, we can not choose a too large $\mathcal{Z}_n$. We can get a hint on how to do that using the following deformation lemma:

Since in our problem f is an even function, and the family $\mathcal{A}_n$ has only symmetric sets, Theorem 2.1.5-4. tells us that the wider family $\mathcal{Z}_n$ can only have symmetric sets. Indeed, if η is an odd deformation, $\eta(1,A)$ is always a symmetric set when $A \in \mathcal{A}_n$. So to define the family $\mathcal{Z}_n$ we need to understand what property the symmetric sets A and $\eta(1,A)$ have in common when $A \in \mathcal{A}_n$;

By definition, every $A \in \mathcal{A}_n$ is an $(n-1)$-sphere and the map $g : A \to \eta(1,A)$, given by $g(x) = \eta(1,x)$ is an odd homeomorphism. Now let us remember the following result:

Theorem 2.2.1 (Borsuk-Ulam)**.** *Let us denote S^k to be the k-sphere. If $g : S^n \to S^m$ is an odd continuous map, then $n \leq m$.*

So thanks to the Borsuk-Ulam theorem, both A and $\eta(1,A)$ have the property if $h : \eta(1,A) \to S^{m-1}$ is an odd continuous map, then $n \leq m$, or equivalently, if $h : \eta(1,A) \to \mathbb{R}^m \setminus \{0\}$ is an odd continuous map, then $n \leq m$. In fact, since A is an $(n-1)$-sphere, the assertion is evident for A. And if $h : \eta(1,A) \to \mathbb{R}^m \setminus \{0\}$ is odd and continuous, then the composition $A \xrightarrow{g} \eta(1,A) \xrightarrow{h} \mathbb{R}^m \setminus \{0\}$ is an odd map from A to $\mathbb{R}^m \setminus \{0\}$, so by Borsuk-Ulam theorem, $n \leq m$.

This implies that if we define the numbers

$$\begin{aligned} \gamma_1 &= \min\left\{k \in \mathbb{N} : \text{there exists an odd } f \in C(\eta(1,A); \mathbb{R}^k \setminus \{0\})\right\} \\ \gamma_2 &= \min\left\{k \in \mathbb{N} : \text{there exists an odd } f \in C(A; \mathbb{R}^k \setminus \{0\})\right\}, \end{aligned}$$

then $\gamma_1, \gamma_2 \geq n$. This motivates the following definition:

Definition 2.2.1 (Krasnoselski genus)**.** *Denote by Σ the class of all closed symmetric subsets of S. The Krasnoselski genus $\gamma : \Sigma \to \mathbb{N} \cup \{0, \infty\}$ is defined as $\gamma(\emptyset) = 0$,*

$$\gamma(X) = \min\left\{k \in \mathbb{N} : \text{ there exists an odd } f \in C(X; \mathbb{R}^k \setminus \{0\})\right\},$$

and if no such a finite k exists, we set $\gamma(X) = \infty$.

All the above discussion implies that the best way of defining $\mathcal{Z}_n$ is

$$\mathcal{Z}_n = \{A \in \Sigma : \gamma(A) \geq n\}.$$

It is clear that $\mathcal{A}_n \subset \mathcal{Z}_n$, and for any deformation η given by Theorem 2.1.5, $\eta(1,A) \in \mathcal{Z}_n$ for all $A \in \mathcal{Z}_n$. So we just need to prove that

$$-\lambda_n = c_n = \min_{\mathcal{A}_n} \max_{A} f = \min_{\mathcal{Z}_n} \max_{Z} f. \tag{2.6}$$

It is clear that $\min_{\mathcal{A}_n} \max_A f \geq \min_{\mathcal{Z}_n} \max_Z f$, so we just need to show that $\min_{\mathcal{A}_n} \max_A f \leq \min_{\mathcal{Z}_n} \max_Z f$. For let us remember that $f(x) = -(Tx,x)$, so

$$-\min_{\mathcal{Z}_n} \max_{Z} f = \max_{\mathcal{Z}_n} \min_{Z} (Tx,x).$$

Let $Z \in \mathcal{Z}_n$, then $\gamma(Z) \geq n$. We claim that $Z \cap X_{n-1} \neq \emptyset$. Suppose by contradiction that $Z \cap X_{n-1} = \emptyset$, and consider $p : X \to \operatorname{span}(\{e_1, \ldots, e_{n-1}\})$ be the orthogonal projection $p(u+v) = u$ where $u \in \operatorname{span}(\{e_1, \ldots, e_{n-1}\})$ and $v \in X_{n-1}$. Since Z does not intersects X_{n-1}, we have that $p(Z) \subset \operatorname{span}(\{e_1, \ldots, e_{n-1}\}) \setminus \{0\}$. So the composition

$$Z \xrightarrow{p} \operatorname{span}(\{e_1, \ldots, e_{n-1}\}) \setminus \{0\} \xrightarrow{\pi} S^{n-2}$$

is a continuous odd map, where $\pi(v) = v/|v|$. So by the Borsuk-Ulam theorem, $\gamma(Z) \leq n-1$, which is a contradiction. Therefore $Z \cap X_{n-1} \neq \emptyset$. This then implies that

$$\min_{Z} (Tx,x) \leq \max_{X_{n-1}} (Tx,x) = \lambda_n, \quad \forall Z \in \mathcal{Z}_n,$$

so

$$\max_{\mathcal{Z}_n} \min_{Z} (Tx,x) \leq \lambda_n,$$

or in other words,

$$\min_{\mathcal{Z}_n} \max_{Z} f \geq c_n$$

and consequently equation (2.6) holds. Now, since $\mathcal{Z}_n$ is deformatin invariant when the deformations are understood to be odd, the Minimax Principle shows us that the values c_n are in fact critical values of f.

It happens that the Krasnoselski genus (2.2.1) has been extensively used in the literature to prove existence and multiplicity results to a huge amount of problems that have a $\mathbb{Z}_2$-symmetry. It is clear that the definition of genus in (2.2.1) can be generalized to any metric space X which has an $\mathbb{Z}_2$-action acting on it, i.e., we have the following general definition of genus:

Definition 2.2.2. *Let (X,d) be a metric space and $I : X \to X$ a continuous map such that $I(I(x)) = x$ and $I(x) \neq x$ for all $x \in X$. We will denote $-x := I(x)$ (I is the $\mathbb{Z}_2$ action). Let*

$$\Sigma(X) = \{A \subset X : A \textit{ is closed and } -A = A\}.$$

Then the Krasnoselski genus (with relation with I) $\gamma : \Sigma(X) \to \mathbb{N} \cup \{0,\infty\}$ is defined by $\gamma(\emptyset) = 0$,

$$\gamma(A) = \min\left\{k \in \mathbb{N} : \textit{ there exists a continuous map } f : A \to \mathbb{R}^k \setminus \{0\} \textit{ with } f(-x) = -f(x)\right\}.$$

and $\gamma(A) = \infty$ otherwise.

It is not difficult to show that the genus satisfies the following properties (see [13], pg. 140 or [9], pg. 355.)

Proposition 2.2.2. *Using the same notations as in Definition 2.2.2 let $A, B, K \in \Sigma(X)$. Then we have that*

(g.1) *$\gamma(A) = 0$ if and only if $A = \emptyset$,*
(g.2) *If there exists a continuous map $f : A \to B$ with $f(-x) = -f(x)$, then $\gamma(B) \geq \gamma(A)$,*
(g.3) *If K is compact, then $\gamma(K) < \infty$ and there exists $\delta > 0$ such that $N_\delta(K) \in \Sigma(X)$ and $\gamma(N_\delta(K)) = \gamma(K)$,*
(g.4) *$\gamma(A \cup B) \leq \gamma(A) + \gamma(B)$.*

Remark 2.2.1. *If $A \in \Sigma(X)$ is a finite set, then $\gamma(A) = 1$. Indeed, if $A = \bigcup_{i=1}^{k} \{x_i, -x_i\}$, then the map $h : A \to S^0 = \{-1,1\}$ given by $h(x_i) = 1$ and $h(-x_i) = -1$ is odd and continuous.*

The genus is an example of an *index theory*. (see [26], § II-5.) The notion of genus generalizes the notion of a linear space (see [26], § II-5. Proposition 5.1.):

Proposition 2.2.3. *For any bounded symmetric neighborhood Ω of the origin in $\mathbb{R}^m$ there holds: $\gamma(\partial\Omega) = m$.*

Given an even smooth function $\phi : \mathcal{M} \to \mathbb{R}$, defined on complete the symmetric C^1-submanifold $\mathcal{M}$ of $X \setminus \{0\}$, we can generalize the procedure we did in the case of finding eigenvalues for T, to the case of finding critical values for f. As usual, define $\Sigma(\mathcal{M})$ as the family of closed symmetric subsets of $\mathcal{M}$, and $\mathcal{Z}_n = \left\{Z \in \Sigma(\mathcal{M}) : \gamma(Z) \geq n\right\}$. We then can define, for any $k \leq \gamma(\mathcal{M})$,

$$c_k = \inf_{Z \in \mathcal{Z}_k} \sup_{Z} \phi.$$

If c_k is finite and ϕ satisfies the $(PS)_{c_k}$ condition, by Proposition 2.2.4 it is a critical value for ϕ.

Remember that in the problem of the eigenvalues of T we worked above, it is clear that if successive eigenvalues $\lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+l-1} = \lambda$ coincide, then T has an l-dimensional eigenspace of eigenvectors $u \in X$ satisfying $Tu = \lambda u$. So we could ask if there is a similar result in the general non-linear case. In fact there is.

First let us remember the following deformation lemma (see [26]).

Proposition 2.2.4. *Suppose* $\mathcal{M} \subset X \backslash \{0\}$ *a symmetric* C^1 *submanifold of* X *and* $f : \mathcal{M} \to \mathbb{R}$ *an even function that satisfies the* (PS) *condition, and let* $c \in \mathbb{R}$. *Let us denote* $K^c = \big\{u \in \mathcal{M} : f'(u) = 0, f(u) = c\big\}$. *Given* $\varepsilon_0, \delta > 0$, *then there are* $\varepsilon > 0$ *and* $\eta \in C([0,1] \times \mathcal{M}, \mathcal{M})$ *satisfying*

(i) $\eta(0,x) = x$ *for all* $x \in \mathcal{M}$,
(ii) $\eta(t,\cdot)$ *is an odd homeomorphism for all* $t \in [0,1]$,
(iii) $\eta(1, f^{c+\varepsilon} \backslash N_\delta(K^c)) \subset f^{c-\varepsilon}$. *In particular, if* c *is not a critical value, then* $\eta(1, f^{c+\varepsilon}) \subset f^{c-\varepsilon}$.
(iv) $d(\eta(t,u),u) \leq \delta t \quad \forall (t,u) \in [0,1] \times \mathcal{M}$.

Proposition 2.2.5. *Suppose that for some* k, l *there holds*

$$-\infty < c_k = c_{k+1} = \cdots = c_{k+l-1} = c < \infty.$$

If ϕ *satisfies* $(PS)_c$ *condition, then* $\gamma(K^c) \geq l$. *And in particular, if* $l > 1$, K^c *is infinite.*

Proof. By the $(PS)_c$ condition, K^c is a symmetric compact set. Hence $\gamma(K^c)$ is well defined and by Proposition 2.2.2, there exists $\delta > 0$ such that $\gamma(N_\delta(K^c)) = \gamma(K^c)$. Let $\varepsilon > 0$ and η as in Proposition 2.2.4 such that $\eta(1, \phi^{c+\varepsilon} \backslash N_{\delta/2}) \subset \phi^{c-\varepsilon}$. Choose $Z \in \mathcal{M}$ closed such that $\gamma(Z) \geq k+l-1$ and $\sup_Z \phi < c+\varepsilon$.

Let $A = \eta(1,Z) \in \Sigma(\mathcal{M})$. By Proposition 2.2.4,

$$A \subset \phi^{c-\varepsilon} \cup N_\delta(K^c).$$

Moreover, by definition of $c = c_k$ it follows that

$$\gamma(\phi^{c-\varepsilon}) < k.$$

Thus by Proposition 2.2.2

$$\begin{aligned} \gamma(N_\delta(K^c)) &\geq \gamma(\phi^{c-\varepsilon} \cup N_\delta(K^c)) - \gamma(\phi^{c-\varepsilon}) \\ &> \gamma(A) - k \geq \gamma(A) - k \\ &\geq k+l-1-k = l-1; \end{aligned}$$

that is, $\gamma(N_\delta(K^c)) = \gamma(K^c) \geq l$, as claimed.

$\square$

In consequence, we have

Proposition 2.2.6. *Suppose $\phi : \mathcal{M} \to \mathbb{R}$ is an even C^1-functional and suppose ϕ satisfies the (PS) condition and is bounded from below on $\mathcal{M}$. Let*

$$\hat{\gamma}(\mathcal{M}) = \sup\left\{\gamma(K) : K \subset \mathcal{M} \text{ compact and symmetric } \right\}.$$

Then the functional ϕ possesses at least $\hat{\gamma}(M) \leq \infty$ pairs of critical points.

Corollary 2.2.1. *If $\mathcal{M}$ is compact and $\phi \in C^1(\mathcal{M})$ is even, then ϕ has at least $\gamma(M)$ pairs of critical points.*

Corollary 2.2.2. *Every even C^1-functional $\phi : S^n \to \mathbb{R}$ has at least $n+1$ pairs of critical points.*

- **Application to Elliptic Problems**.

Let Ω be an open smooth bounded subset of $\mathbb{R}^n$ with $n \geq 3$. Let $p \in (2, 2n/(n-2))$ and consider the following problem

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u & \text{in } \Omega \\ \qquad\quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

where $\lambda \geq 0$ is a constant, and Δ is the laplace operator $\Delta f = \sum_i \partial_i^2 f$. By definition, the weak solutions of (2.7) are the critical points of the functional $E : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$E(u) = \frac{1}{2}\left(\int_\Omega |\nabla u|^2 dx + \lambda \int_\Omega |u|^2 dx\right) - \frac{1}{p}\int |u|^p dx.$$

However, it is evident, by using the Lagrange Multipliers, that it suffices to search for critical points of the functional

$$\varphi(u) = \frac{1}{2}\left(\int_\Omega |\nabla u|^2 dx + \lambda \int_\Omega |u|^2 dx\right), \quad u \in H_0^{1,2}(\Omega),$$

restricted to the C^1-submanifold $S = \left\{u \in H_0^1(\Omega) : g(u) := \int_\Omega |u|^p dx = 1\right\}$. Indeed, if $\varphi'(u) = \mu g'(u)$ where $\mu \geq 0$ (since $\lambda \geq 0$, the only possibility for the Langrange multipliers are the non-negative ones) is a Lagrange multiplier, then $v = \mu^{1/(p-2)}u$ is a critical value for E.

First of all, to motivate the usage of Proposition 2.2.6 on this problem, let us see that the functional φ has a similar behavior as (Tx, x) in the problem of finding the eigenvalues for the linear map T we worked on. For that, we will need the following well known result (see [2], §9.8, pg. 311.):

Theorem 2.2.2. *There exist a Hilbert bases* $(e_n)_{n\geq 1}$ *of* $L^2(\Omega)$ *and a sequence* $0<\lambda_1\leq\lambda_2\leq\cdots$ *of real numbers with* $\lambda_n\to\infty$ *such that*

$$\begin{aligned} e_n &\in H_0^1(\Omega)\cap C^\infty(\Omega),\\ -\Delta e_n &= \lambda_n e_n \quad in \quad \Omega. \end{aligned}$$

Corollary 2.2.3. *The sequence* $(e_n/\sqrt{\lambda_n})$ *is a Hilbert basis of* $H_0^1(\Omega)$ *equipped with the inner product* $\int_\Omega \nabla u\nabla v$.

For simplicity, let us denote $v_i := e_i/\sqrt{\lambda_i}$. Then $(v_i)_{i\geq 1}$ is a Hilbert basis for $H_0^1(\Omega)$. Let us define

$$Y_n := \operatorname{span}\{v_1,\cdots,v_n\}.$$

Let us take $v\in Y_n$. Then $v=\sum_{i=1}^n \alpha_i v_i$, where $\alpha_i=\int\nabla v\nabla v_i$. Suppose in addition that $v\in S$. Since Y_n is finite dimensional, the norms $\|\cdot\|$ and $|\cdot|_p$ are equivalent on it. So there exists a positive constant c_n that depend only on Y_n such that $\|u\|^2\geq c_n|u|_p^2$ on it. Then we have

$$\begin{aligned} \varphi(v)=\varphi(\sum\alpha_i v_i) &= \frac{1}{2}\sum_i^n \alpha_i^2\left(\int_\Omega|\nabla v_i|^2dx+\lambda\int_\Omega|v_i|^2dx\right)\\ &= \frac{1}{2}\sum_{i=1}^n\alpha_i^2\left(1+\frac{\lambda}{\lambda_i}\right)\geq\frac{c_n}{2}\left(1+\frac{\lambda}{\lambda_n}\right). \end{aligned}$$

Therefore we have that

$$\inf_{Y_n\cap S}\varphi(v)\geq\beta_n,$$

where $\beta_n=\dfrac{c_n}{2}\left(1+\dfrac{\lambda}{\lambda_n}\right)$.

Now, let us take F to be an $(n-1)$-dimensional subspace of $H_0^1(\Omega)$. Then, we clearly have $F^\perp\cap Y_n\neq\{0\}$, and since S is radially homeomorphic to the unit sphere of $H_0^1(\Omega)$, we have $(F^\perp\cap S)\cap(Y_n\cap S)\neq\emptyset$. Consequently, denoting $\mathcal{F}_{n-1}$ to be the family of $(n-1)$-dimensional subspaces of $H_0^1(\Omega)$, we have

$$\inf_{\mathcal{F}_{n-1}}\sup_{F^\perp\cap S}\varphi\geq\inf_{Y_n\cap S}\varphi\geq\beta_n.$$

This shows us that φ behaves in a similar way as in the problem of Proposition 2.2.1. So it is tempting to use the tools developed in the study of the eigenvalues of T. In fact, it can be easily shown that φ satisfies the $(PS)_c$ condition for all c, and it is bounded from below on S. Therefore, by Theorem 2.2.6, if $\mathcal{A}_n = \{A \in S : A \text{ is closed symmetric and } \gamma(A) \geq n\}$, defining, for each n,

$$s_n = \inf_{A \in \mathcal{A}_n} \sup_{u \in A} \varphi(u),$$

we have that s_n is a critical value for φ on S for all n. Therefore we have the following

Theorem 2.2.3. *Problem* (2.7) *admits infinitely many distinct pairs of solutions.*

The sets A and $\eta(1,A)$ also have another common topological property. They have the same $\mathbb{Z}_2$-cohomological index. This cohomological index functions similarly to the genus, but it possesses some useful properties that the genus does not have. The definition of this index will be given later, but a motivation to use such an index instead of the genus is the following problem.

2.3 Motivation for this work

Let us consider the following PDE problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + \mu f(x,u) + |u|^{q-2}u & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $n \geq 1$, $p > 1$, $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian of u, $p < q < p^* = Np/(N-p)$ if $p < N$ and $p < q < \infty$ if $p \geq N$, $\lambda, \mu > 0$ are parameters, and f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$f(x,t) = |t|^{\sigma-2}t + o(|t|^{\sigma-1}) \quad \text{as} \quad t \to 0, \quad \text{uniformly} \quad \text{a.e. in} \quad \Omega \tag{3.9}$$

for some $1 < \sigma < p$, the sign condition

$$F(x,t) = \int_0^t f(x,s)ds > 0 \quad \text{for a.a.} \quad x \in \Omega \quad \text{and all} \quad t \in \mathbb{R} \backslash \{0\}, \tag{3.10}$$

and the growth condition

$$|f(x,t)| \leq a(|t|^{r-1} + 1) \quad \text{for a.a.} \quad x \in \Omega \quad \text{and all} \quad t \in \mathbb{R} \tag{3.11}$$

for some $a > 0$ and $p < r < q$.

It is well known that

$$\lambda_1 := \inf_{u\neq 0, u\in W_0^{1,p}(\Omega)} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} > 0 \tag{3.12}$$

It is easy to see that for $\lambda < \lambda_1$ the energy functional associated to (3.8) has the mountain pass geometry (see Theorem 2.3.2 below). Therefore to show the existence of nontrivial solution in such setting will not be big deal. This will be the first part of this section, where we will employ the mountain pass theorem to solve (3.8). However, since for $\lambda \geq \lambda_1$ the problem has no longer the mountain pass geometry, we will need to find out another kind of linking that detects the geometry of the sub-levels of the energy functional. For that, we will employ an index theory that involves cohomology theory, namely the $\mathbb{Z}_2$-cohomological index of Fadell-Rabinowitz.

2.3.1 The case $\lambda < \lambda_1$

Let us consider the energy functional associated to (3.8):

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p dx - \mu \int_\Omega F(x,u)dx - \frac{1}{q}\int_\Omega |u|^q dx,\ u \in W_0^{1,p}(\Omega) \tag{3.13}$$

Then we have the following

Theorem 2.3.1. *The functional* (3.13) *satisfies* $(PS)_c$ *condition for all* $c \in \mathbb{R}$.

We will need the following lemma to prove the above theorem.

Lemma 2.3.1. *Let* $p > 1$, *let* Ω *be an bounded subset of* $\mathbb{R}^N$ *and let* $\{u_n\} \subset W_0^{1,p}(\Omega)$ *be such that* $u_n \rightharpoonup u$ *in* $W_0^{1,p}(\Omega)$ *and*

$$\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\cdot \nabla T(u_n - u)dx \to 0$$

where

$$T(s) = \begin{cases} s & \text{if} \quad |s| \leq 1 \\ s/|s| & \text{if} \quad |s| > 1 \end{cases}$$

Then

1. *there exists a subsequence* (u_{n_k})*such that*

$$\nabla u_{n_k} \to \nabla u \quad a.e.\ on \quad \Omega$$

2.

$$\lim_{n\to\infty}\left(\int_\Omega |\nabla u_n|^p dx - \int_\Omega |\nabla(u_n-u)|^p dx\right) = \int_\Omega |\nabla u|^p dx$$

3. for any $1 \le q < p, u_n \to u$ *in* $W^{1,q}(\Omega)$.

Proof. ([6] Theorem 1.1). □

Proof of Theorem 2.3.1. Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ such that $E(u_n) \to c$ and $E'(u_n) \to 0$ on $W^{-1,p}(\Omega)$. Taking n big such that $-E'(u_n)u_n \le p\|u_n\|$, and denoting $d := \sup_n E(u_n)$ we have that

$$\begin{aligned} d + \|u_n\| &\ge E(u_n) - \frac{1}{p}E'(u_n)u_n \\ &= \int_\Omega \frac{\mu}{p} f(x,u_n)u_n - F(x,u_n)dx + \left(\frac{1}{p}-\frac{1}{q}\right)\int_\Omega |u_n|^q dx. \end{aligned} \tag{3.14}$$

Therefore, by the growth condition on f and Sobolev inequalities, there exist constants $c_1, c_2 > 0$ such that

$$\left|\int_\Omega \frac{\mu}{p} f(x,u_n)u_n - F(x,u_n)dx\right| \le c_1|u_n|_r^r + c_2 \tag{3.15}$$

then by (3.14) and (3.15) we have

$$|u_n|_q^q \le c_3|u_n|_r^r + \|u_n\| + c_4$$

for some constants $c_3, c_4 > 0$. And since $|\Omega| < \infty$ and $q > r$ there is a constant $c_5 > 0$ such that

$$|u_n|_r^r \le c_5(|u_n|_q)^{r/q}.$$

Therefore

$$\begin{aligned} |u_n|_r^r &\le c_5\left(c_3|u_n|_r^r + \|u_n\| + c_4\right)^{r/q} \\ &\le a_1(|u_n|_r^r)^{r/q} + a_2\|u_n\|^{r/q} + a_3 \end{aligned}$$

for some constants $a_i > 0$. And by doing the same procedure repeatedly together with the Sobolev inequality we have

$$|u_n|_r^r \le C_1(\|u_n\|^r)^{(r/q)^s} + \sum_{i=1}^{s} b_i\|u_n\|^{(r/q)^s} \tag{3.16}$$

for some constants $C_1, b_i > 0$ and some $s \in \mathbb{N}$ such that $r(r/q)^s < p$.

Similarly we have constants $C_2, g_j > 0$ and $m \in \mathbb{N}$ such that $p(p/q)^m < p$ and

$$|u_n|_p^p \leq C_2(\|u_n\|^p)^{(p/q)^m} + \sum_{j=1}^{m} g_j \|u_n\|^{(p/q)^j}. \tag{3.17}$$

Now, taking n bigger if necessary we have

$$\begin{aligned} d + \|u_n\| &\geq E(u_n) - \frac{1}{q}E'(u_n)u_n \\ &= \left(\frac{1}{p} - \frac{1}{q}\right)\|u_n\|^p - \lambda\left(\frac{1}{p} - \frac{1}{q}\right)|u_n|_p^p + \int \frac{1}{q}f(x,u_n)u_n - F(x,u_n)dx \\ &\geq A_1\|u_n\|^p - A_2|u_n|_p^p - A_3|u_n|_r^r - A_4 \end{aligned}$$

By (3.16), (3.17) and the above inequality we can easily check that $\|u_n\|$ is a bounded sequence.

We can then suppose that there exists $v \in W_0^{1,p}(\Omega)$ such that for all $s \in [1, p^*)$

$$\begin{aligned} u_n &\rightharpoonup v \quad &\text{in} \quad &W_0^{1,p}(\Omega), \\ u_n &\to v \quad &\text{in} \quad &L^s(\Omega), \\ u_n(x) &\to v(x) \quad &\text{a.e. in} \quad &\Omega. \end{aligned}$$

Taking T as in Lemma 2.3.1, since T is bounded, by dominated convergence theorem we have $\int_\Omega |T(u_n - v)|^s dx \to 0$ for all $s \geq 1$ and $T(u_n - v) \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$ (see Apendix). Therefore

$$\begin{aligned} &\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla v|^{p-2}\nabla v\right) \cdot \nabla T(u_n - v)dx = \\ &= E'(u_n)T(u_n - v) + \lambda\int |u_n|^{p-2}u_n T(u_n - u) + \mu\int f(x,u_n)T(u_n - v) + \\ &+ \int |u_n|^{q-2}u_n T(u_n - v) - \int |\nabla v|^{p-2}\nabla v \nabla T(u_n - v) \to 0, \quad n \to \infty. \end{aligned}$$

Hence by Lemma 2.3.1 we can assume that $\nabla u_n \to \nabla v$ pointwise a.e. on Ω.

Now, since

$$E'(u_n)v = \int \left(|\nabla u_n|^{p-2}\nabla u_n \nabla v - \lambda|u_n|^{p-2}u_n v - \mu f(x,u_n)v - |u_n|^{q-2}u_n v\right) dx = o(\|v\|) \tag{3.18}$$

and

$$E'(u_n)u_n = \int (|\nabla u_n|^p - \lambda|u_n|^p - \mu f(x,u_n)u_n - |u_n|^q)\, dx = o(\|u_n\|) \tag{3.19}$$

we have

$$\|u_n\|^p = |u_n|_q^q + \int (\lambda |v|^p + \mu f(x,v)v)dx + o(1), \tag{3.20}$$

and passing to the limit on (3.18)

$$\|v\|^p = |v|_q^q + \int (\lambda |v|^p + \mu f(x,v)v)dx \tag{3.21}$$

and by Lemma 2.3.1 we have

$$\|u_n - v\|^p = \|u_n\|^p - \|v\|^p + o(1) \tag{3.22}$$

Hence, combining the last three equations we conclude that $\|u_n - v\| \to 0$.

□

Now, employing the mountain pass theorem we have the

Theorem 2.3.2. *If $\lambda < \lambda_1$ then problem (3.8) has a nontrivial solution.*

Proof. By Theorem 2.3.1 the energy functional E satisties the $(PS)_c$ condition for all $c \in \mathbb{R}$.

Note that since by definition of the first eigenvalue

$$\lambda_1 = \inf_{u \neq 0, u \in W_0^{1,p}} \frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} > 0$$

Therefore, for all $u \in W_0^{1,p}(\Omega)$ we have

$$\frac{1}{\lambda_1} \int_\Omega |\nabla u|^p \geq \int_\Omega |u|^p$$

hence

$$-\lambda \int_\Omega |u|^p \geq -\frac{\lambda}{\lambda_1} \int_\Omega |\nabla u|^p$$

and finally

$$\int_\Omega |\nabla u|^p - \lambda \int_\Omega |u|^p \geq \left(1 - \frac{\lambda}{\lambda_1}\right) \int_\Omega |\nabla u|^p$$

Therefore, using the last inequality and the Sobolev inequalities we have

$$E(u) \geq c_1\|u\|^p - \mu c_2\|u\|^r - c_3\|u\|^q - \mu c_4$$

for positive constants c_i. Maximizing the right hand side of the above inequality we can easily see that there exists $\mu_0 > 0$ such that if $\mu \leq \mu_0$ then there exist $R_\mu > 0$ which

$$\inf_{\|u\|=R_\mu} E(u) > \delta.$$

for some $\delta > 0$.

Taking $v_1 \in W_0^{1,p}(\Omega)$ nonzero, then it is easy to see that $E(tv_1) \to -\infty$ as $t \to \infty$. This shows that there exist v with $\|v\| > R_\mu$ such that $E(v) < 0$.

Therefore, defining

$$\Gamma := \left\{\gamma \in C([0,1]; W_0^{1,p}(\Omega)) : \gamma(0) = 0, \gamma(1) = v\right\}$$

and

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} E(\gamma(t))$$

we have that $c > 0$ and by the Mountain Pass Theorem, c is a critical value for E

□

When $\lambda \geq \lambda_1$, the energy functional does not have the mountain pass geometry anymore. Also, the fact that the eigenspaces of the p-Laplacian are not linear spaces when $p \neq 2$ makes the usage of the genus considerably harder. To detect the topological behaviour of the functional E when $\lambda \geq \lambda_1$ we will need the $\mathbb{Z}_2$-cohomological index.

CHAPTER 3

$\mathbb{Z}_2$-Cohomological Index Theory

3.1 Basic definitions

Definition 3.1.1. *Let X be a topological space. A free $\mathbb{Z}_2$-action on X is a continuous function $T : X \to X$ with $T(T(x)) = x$ such that $T(x) \neq x$ for all $x \in X$.*

Remark 3.1.1. *Note that T as above is in fact a homeomorphism.*

Example 3.1.1. *Let $S^n \subset \mathbb{R}^{n+1}$ the n-dimensional sphere. Then the map $T : S^n \to S^n$ given by $T(x) = -x$ is a free $\mathbb{Z}_2$-action on S^n.*

Example 3.1.2. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x) = -x$. Then T is not a free $\mathbb{Z}_2$-action on $\mathbb{R}^n$ because $T(0) = 0$.*

Definition 3.1.2. *A topological space X is said to be paracompact if it is Hausdorff and every open cover of X has a locally finite partition of unity subordinated to it.*

In this chapter, free action will always stand for a free $\mathbb{Z}_2$-action, unless otherwise stated. To define the index we are looking for, we will need to do some constructions. First let

$$\mathbb{R}^\infty = \{(a_n)_{n\in\mathbb{N}} : \quad \text{there exists} \quad M \quad \text{such that} \quad a_i = 0 \quad \text{for} \quad i > M\}$$

with the norm $\|\cdot\| : \mathbb{R}^\infty \to \mathbb{R}$ given by $\|(a_n)\| = \sqrt{\sum a_i^2}$. We then have the infinite dimensional sphere

$$S^\infty := \{x \in \mathbb{R}^\infty : \|x\| = 1\}$$

Of course, since $\mathbb{R}^\infty$ is a metric space with the metric induced by the norm $\|\cdot\|$, we have that $\mathbb{R}^\infty$ is a paracompact space and therefore so is S^∞. Moreover, the antipodal map $x \mapsto -x$ is a free action on S^∞.

3.2 Construcion of the index

Let us denote $\mathcal{F}$ to be the set of all pairs (X,T) where X is a topological paracompact space and $T:X\to X$ is a free $\mathbb{Z}_2$-action. A topological space $X\in\mathcal{F}$ will mean a pair (X,T) and the action T will always be denoted by $T(x)=-x$. Given $X,Y\in\mathcal{F}$, an equivariant map $f:X\to Y$ is a map which satisfies $f(-x)=-f(x)$ for all $x\in X$. We will say that $E\subset X$ is invariant if $-E=E$. We have a very interesting relation between elements of $\mathcal{F}$ and the infinite dimensional sphere S^∞, namely

Proposition 3.2.1. *Let $X\in\mathcal{F}$ be a compact metric space. Then there is an odd continuous map $f:X\to S^\infty$, i.e. $f(-x)=-f(x)$.*

Proof. For each $x\in X$ let $B(x,\delta_x)$, $\delta_x>0$ be an open ball centered at x such that $B(x,2\delta_x)\cap B(-x,2\delta_x)=\emptyset$ and $\xi_x:X\to[0,1]$ a continuous function such that $\xi_x(y)=1$ in $B(x,\delta_x)$ and $\xi_x(y)=0$ ouside $B(x,2\delta_x)$. Then of course $X=\cup_{x\in X}B(x,\delta_x)$. Since X is compact we can extract a finite open subcover $X=\cup_{i=1}^m B(x_i,\delta_{x_i})$. Denote $\gamma_i:=\xi_{x_i}$ and $B_i:=B(x_i,\delta_{x_i})$. Now define $\alpha_i(x)=\gamma_i(-x)$. Then by construction we have $\operatorname{supp}\gamma_i\cap\operatorname{supp}\alpha_i=\emptyset$. Now it is clear that $\sum_{i=1}^m\gamma_i^2(x)+\alpha_i^2(x)>0$ for all $x\in X$. Define then

$$g_i:=\frac{\gamma_i}{\sqrt{\sum_{i=1}^m\gamma_i^2+\alpha_i^2}}\qquad h_i:=\frac{\alpha_i}{\sqrt{\sum_{i=1}^m\gamma_i^2+\alpha_i^2}}$$

Then we have

$$\sum_{i=1}^m(g_i^2+h_i^2)=1$$

Therefore, the function

$$f(x)=(g_1(x)-h_1(x),\dots,g_m(x)-h_m(x),0,0,\dots)$$

is an odd continuous function $f:X\to S^\infty$. □

Remark 3.2.1. *In fact the function f in the above proposition can be thought as a function $X\to S^N$ for some natural N.*

We have in fact much more than the last proposition. Namely, we have the

Theorem 3.2.1. *For any $X\in\mathcal{F}$ there is an odd continuous map $f:X\to S^\infty$. Moreover, if $f_1,f_2:X\to S^\infty$ are two odd continuous maps, then there exists a homotopy $H:[0,1]\times X\to S^\infty$ between f_1 and f_2 such that for each fixed t the map $H_t(x):=H(t,x)$ is an odd map from X to S^∞.*

Proof. By Theorem 7.5 in [10] it suffices to show that S^∞ is contractible. First let us define $T : [0,1]\times\mathbb{R}^\infty \to \mathbb{R}^\infty$ given by

$$T(t,x_1,x_2,\dots) = (1-t)(x_1,x_2,\dots)+t(0,x_1,x_2,\dots).$$

It is clear that T is a continuous map. We claim that $T(t,x)\neq 0$ for all $(t,x)\in[0,1]\times\mathbb{R}^\infty$ such that $x\neq 0$. In fact suppose that $T(t,x_1,x_2,\dots)=0$ with $(x_1,x_2,\dots)\neq 0$. Let $k:=\min\{i : x_i\neq 0\}$ then by the definition of T we would have $(1-t)x_k = 0$ hence $t=1$. But since $T(t,x_1,x_2,\dots)=0$ we finally get $(0,x_1,\dots)=0$ which implies $x_k=0$. This is a contradiction. Therefore in fact $T(t,x)\neq 0$ for all $x\neq 0$.

The map $H_1 : [0,1]\times S^\infty \to S^\infty$ given by $H_1(t,x) = T(t,x)/\|T(t,x)\|$ is a homotopy from the identity $1_{S^\infty} : S^\infty \to S^\infty$ to the map $Q : S^\infty \to S^\infty$ given by $Q(x_1,x_2,\dots) = (0,x_1,x_2,\dots)$. Now the map $G : [0,1]\times\mathbb{R}^\infty \to \mathbb{R}^\infty$ given by

$$G(t,x_1,x_2,\dots) = (1-t)(0,x_1,x_2,\dots)+t(1,0,0,\dots)$$

is a continuous map such that $G(t,x)\neq 0$ if $x\neq 0$, and defining $H_2(t,x) := G(t,x)/\|G(t,x)\|$ on $[0,1]\times S^\infty$ we have that H_2 is a homotopy from the function Q to the constant map $P(x) = (1,0,0,\dots)$

By composing the two homotopies H_1 and H_2 it gives a homotopy between the identity 1_{S^∞} and the constant map that sends S^∞ to the point $(1,0,0,\dots)$. □

Let $X\in\mathcal{F}$. We now consider $\widetilde{X} := \{\{x,-x\} : x\in X\}$ the orbit space of X with the topology induced by the projection $p : X\to\widetilde{X}$ given by $p(x) = \{x,-x\}$. In particular, we denote $\mathbb{R}P^\infty{:=}\widetilde{S^\infty}$ the infinite dimensional real projective space.

Remark 3.2.2. *Given $f : X\to S^\infty$ an odd map, there exists a unique continuous map $\widetilde{f} : \widetilde{X}\to\mathbb{R}P^\infty$ which makes the diagram below commute*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p"] & S^\infty \arrow[d, "p"] \\
\widetilde{X} \arrow[r, "\widetilde{f}"] & \mathbb{R}P^\infty
\end{tikzcd} \tag{2.1}$$

Where p are the canonical projections.

Definition 3.2.1. *In the same notation of the above remark, the map $\widetilde{f}$ is called a classifying map of X.*

Remark 3.2.3. *By Theorem 3.2.1 $\widetilde{f}$ is unique up to homotopy. Indeed, if $g : X\to S^\infty$ is another odd continuous map, then by Theorem 3.2.1 there exists a homotopy $H : [0,1]\times X\to S^\infty$ between f and g*

that is odd for each fixed t. Then the map $\widetilde{H} : [0,1] \times \overline{X} \to \mathbb{R}P^{\infty}$ *given by* $\widetilde{H}(t,\{x,-x\}) = p(H(t,x))$ *is a homotopy between* $\widetilde{f}$ *and* $\widetilde{g}$.

Definition 3.2.2. *Let* $X \in \mathcal{F}$ *and* $f : \widetilde{X} \to \mathbb{R}P^{\infty}$ *be the classifying map of X and*

$$f^* : H^*(\mathbb{R}P^{\infty}) \to H^*(\widetilde{X})$$

the induced homomorphism of the cohomoloy rings where H^q *is the Alexander-Spanier Cohomology groups. Let* ω *be the generating class in* $H^1(\mathbb{R}P^{\infty})$. *The* $\mathbb{Z}_2$*-cohomological index of X is defined by*

$$i(X) = \begin{cases} \sup\left\{k \geq 1 : f^*(w^{k-1}) \neq 0\right\} & \text{if} \quad X \neq \emptyset \\ 0 & \text{if} \quad X = \emptyset \end{cases}$$

The cohomological index as defined above has the following properties (see [21], Proposition 2.12).

Proposition 3.2.2. *The index* $i : \mathcal{F} \to \mathbb{N} \cup \{0,\infty\}$ *has the following properties.*

(i_1) $i(X) = 0$ *if and only if* $X = \emptyset$.

(i_2) *if* $f : X \to Y$ *is an equivariant map, then*

$$i(X) \leq i(Y)$$

(i_3) *If X is a symmetric subset of a normed linear space W and does not contain the origin, then*

$$i(X) \leq \dim W.$$

(i_4) *If* $X \in \mathcal{F}$ *and A is a closed invariant subset of X, then there is a closed invariant neighborhood N of A in X such that*

$$i(N) = i(A).$$

When X is a metric space and A is compact, N may be chosen to be a δ*-neighborhood* $N_\delta(A)$.

(i_5) *If* $X \in \mathcal{F}$ *and A and B are closed invariant subsets of X such that* $X = A \cup B$, *then*

$$i(A \cup B) \leq i(A) + i(B).$$

(i_6) *If SX is the set* $X \times [0,1]$ *with the sets* $X \times \{0\}$ *and* $X \times \{1\}$ *identified as different points, then*

$$i(SX) = i(X) + 1.$$

(i_7) *If* $X, A \in \mathcal{F}$, X_0 *and* X_1 *are closed invariant subsets of X such that* $X = X_0 \cup X_1$, *and* $\phi : A \times [0,1] \to X$ *is a continuous mapping satisfying* $\phi(-x,t) = -\phi(x,t)$ *for all* (x,t), $\phi(A \times [0,1])$

is closed in X, $\phi(A\times\{0\})\subset X_0$, *and* $\phi(A\times\{1\})\subset X_1$, *then*

$$i(\phi(A\times[0,1])\cap X_0\cap X_1)\geq i(A).$$

(i_8) *If* U *is a bounded closed symmetric neighborhood of the origin in a normed linear space* W, *then*

$$i(\partial U)=\dim W.$$

Corollary 3.2.1. *If* U *is a bounded closed symmetric neighborhood of the origin in a normed linear space* W, A *is a bounded symmetric subset of* $\overline{U^c}$, *and* $\psi: IA\to W$ *is an odd continuous mapping, where* $IA=\{tx: x\in A, t\in[0,1]\}$, *such that* $\psi(IA)$ *is closed and* $\psi|_A=1_A$, *then*

$$i(\psi(IA)\cap\partial U)\geq i(A).$$

Proof. Note that $\psi(IA)\cap\partial U=\psi(\psi^{-1}(\partial U))$ so by Proposition 3.2.2

$$i(\psi(IA)\cap\partial U)\geq i(\psi^{-1}(\partial U))$$

since ψ is odd. Also $\psi(0)=0$ (since ψ is odd), so there is a $\delta>0$ such that

$$V_\delta:=\{tx: x\in A, t\in[0,\delta)\}\subset\psi^{-1}(\overset{\circ}{U})$$

by continuity. We apply Proposition 3.2.2 (i_7) to the map

$$\begin{array}{rcll}\phi: A\times[0,1] & \to & IA_\delta\backslash V_\delta=(\phi^{-1}(U)\backslash V_\delta)\cup\phi^{-1}(\overline{U^c}) & \text{given by}\\ (x,t) & \mapsto & ((1-t)\delta+t)x. & \end{array}$$

Since V_δ is contained in the closed set $\psi^{-1}(U)$, so is its relative boundary ∂V_δ, so $\phi(A\times\{0\})=\partial V_\delta\subset\psi^{-1}(U)\backslash V_\delta$. Since ψ is the identity on $A\subset\overline{U^c}$, $\phi(A\times\{1\})=A\subset\psi^{-1}(\overline{U^c})$. Since ϕ is onto,

$$\phi(A\times[0,1])\cap(\psi^{-1}(U)\backslash V_\delta)\cap\psi^{-1}(\overline{U^c})=\psi^{-1}(\partial U)\backslash V_\delta=\psi^{-1}(\partial U)$$

so we have

$$i(\psi^{-1}(\partial U))\geq i(A).$$

□

Beyond the above properties, the index also have some interesting features, namely

Proposition 3.2.3. *Let* $X\in\mathcal{F}$ *with index* $k\geq 1$.

- *(i) If X is the disjoint union of a pair of subsets U, $-U$, then $k = 1$. In particular, $k = 1$ when X is a finite set.*
- *(ii) If X is compact, then $k < \infty$.*
- *(iii) If X is locally contractible, then for each finite $j \leq k$, X has a compact invariant subset C with $i(C) \geq j$. In particular, there is a compact invariant subset C with index k when $k < \infty$.*
- *(iv) If $k < \infty$ and A is an invariant subset of X with index k, then the rank $i^* : H^{k-1}(X) \to H^{k-1}(A)$, induces by $i : A \subset X$, is at least $1 + \delta_{k1}$. In particular, $\widetilde{H}^{k-1}(X) \neq 0$.*

Proof. (This is the same proof as in [21]) To prove item (i), note that we can define the map $f : X \to S^0 = \{-1, 1\}$ by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \in U \\ -1 & \text{if} \quad x \in -U. \end{cases}$$

Then f is equivariant, and therefore by (i_2) of Proposition 3.2.2 we have that $i(X) \leq i(S^0)$. By (i_8) in 3.2.2 $i(S^0) = 1$ hence $i(X) = 1$ by (i_1) of the same proposition.

(ii) For each $x \in X$ let U_x be a closed neighborhood of x such that $U_x \cap -U_x = \emptyset$ (this is possible because the action of $\mathbb{Z}_2$ on X is free, and each two disjoint closed sets can be separated by open neighborhoods). Since X is compact we can extract a finite collection $\{U_{x_i}\}_{i=1,\ldots,m}$ such that $X = \cup_i U_{x_i}$. Hence, by (i_2) and (i_5) on Proposition 3.2.2 we have that, since $X = \cup_{i=1}^m (U_{x_i} \cup -U_{x_i})$

$$i(X) \leq \sum_{i=1}^{m} i(U_{x_i} \cup -U_{x_i}) = m < \infty.$$

(iii) Recall that if X is locally contractible then there exists a natural isomorphism $\mu : H^q(X) \to H_s^q(X)$ where H_s^q stands for the singular cohomology groups. Also, there is a natural isomorphism $h : H_s^q(X) \to \mathrm{Hom}(H_q(X), \mathbb{Z}_2)$, where H_q stands for the singular homology groups. Note that the isomorphism h does not depend on the fact that X is locally contractible, but on the fact that $\mathbb{Z}_2$ is a field (see [14]). Now if C is any invariant subset of X, then we have the following commutative diagram

$$\begin{tikzcd}
H^{j-1}(\mathbb{R}P^\infty) \arrow[r, "\mu"] \arrow[d, "f^*"] & H_s^{j-1}(\mathbb{R}P^\infty) \arrow[r, "h"] \arrow[d, "f_s^*"] & \mathrm{Hom}\left(H_{j-1}(\mathbb{R}P^\infty), \mathbb{Z}_2\right) \arrow[d, "(f_*)^*"] \\
H^{j-1}(\widetilde{X}) \arrow[r, "\mu"] \arrow[d, "\widetilde{i}^*"] & H_s^{j-1}(\widetilde{X}) \arrow[r, "h"] \arrow[d, "\widetilde{i}_s^*"] & \mathrm{Hom}\left(H_{j-1}(\widetilde{X}), \mathbb{Z}_2\right) \arrow[d, "(\widetilde{i}_*)^*"] \\
H^{j-1}(\widetilde{C}) \arrow[r, "\mu"] & H_s^{j-1}(\widetilde{C}) \arrow[r, "h"] & \mathrm{Hom}\left(H_{j-1}(\widetilde{C}), \mathbb{Z}_2\right)
\end{tikzcd}$$

where f is a classifying map of X and $\widetilde{i} : \widetilde{C} \subset \widetilde{X}$ is induced by the inclusion $i : C \subset X$, and with $\widetilde{fi}$ serving as a classifying map for C. Now, since $j \geq k$ we have that the top left-hend corner give us $f^*(\omega^{j-1}) \neq 0$, so the map $\alpha \in \mathrm{Hom}\left(H_{j-1}(\widetilde{X}), \mathbb{Z}_2\right)$ given by $\alpha := h\mu f^*(\omega^{j-1})$ is nontrivial since both h and μ are isomorphisms. Then there exists $z \in H_{j-1}(\widetilde{X})$ such that $\alpha(z) \neq 0$. Since singular homology is a theory with compact supports, X has a compact invariant subset C such that z is the image of $\widetilde{i}_* : H_{j-1}(\widetilde{C}) \to H_{j-1}(\widetilde{X})$, say, $z = \widetilde{i}_* c$ for some $c \in H_{j-1}(\widetilde{C})$. By commutativeness of the diagram we have

$$(h\mu\widetilde{i}^* f^*(\omega^{j-1}))(c) = ((\widetilde{i}_*)^*\alpha)(c) = \alpha(\widetilde{i}_* c) = \alpha(z) \neq 0,$$

therefore $(\widetilde{fi})^*(\omega^{j-1}) = \widetilde{i}^* f^*(\omega^{j-1}) \neq 0$, so $i(C) \geq j$.

(iv) We have the commutative diagram

$$\begin{tikzcd}
H^{k-1}(\mathbb{R}P^\infty) \arrow[r, "\pi^*"] \arrow[d, "f^*"] & H^{k-1}(S^\infty) \arrow[r, "r^*"] \arrow[d, "\widetilde{f}^*"] & H^{k-1}(\mathbb{R}P^\infty) \arrow[r, "\smile\omega"] \arrow[d, "f^*"] & H^k(\mathbb{R}P^\infty) \arrow[d, "f^*"] \\
H^{k-1}(\widetilde{X}) \arrow[r, "\pi^*"] \arrow[d, "\widetilde{i}^*"] & H^{k-1}(X) \arrow[r, "r^*"] \arrow[d, "i^*"] & H^{k-1}(\widetilde{X}) \arrow[r, "\smile f^*\omega"] \arrow[d, "\widetilde{i}^*"] & H^k(\widetilde{X}) \arrow[d, "\widetilde{i}^*"] \\
H^{k-1}(\widetilde{A}) \arrow[r, "\pi^*"] & H^{k-1}(A) \arrow[r, "r^*"] & H^{k-1}(\widetilde{A}) \arrow[r, "\smile\widetilde{i}^* f^*\omega"] & H^k(\widetilde{A})
\end{tikzcd}$$

where f is the classifying map of X, $\widetilde{i} : \widetilde{A} \subset \widetilde{X}$ is induced by $i : A \subset X$, with $\widetilde{fi}$ serving as a classifying map for A, and the horizontal rows come from the Thom-Gysin exact sequence (see [14] or [24]). Since $i(X) = k$,

$$0 = f^*(\omega^k) = f^*(\omega^{k-1} \smile \omega) = f^*(\omega^{k-1}) \smile f^*\omega$$

so $f^*(\omega^{k-1}) \in \ker(\smile f^*\omega) = r^*(H^{k-1}(X))$ by exactness, say $f^*(\omega^{k-1}) = r^*\alpha$. Then from the lower middle square

$$r^* i^* \alpha = \widetilde{i}^* r^* \alpha = \widetilde{i}^* f^*(\omega^{k-1}) = (\widetilde{fi})^*(\omega^{k-1}) \neq 0$$

since $i(A) = k$, so $i^*\alpha \neq 0$. When $k = 1$, $i^*\alpha \neq \pi^* 1 = 1$, since $i^*\alpha \notin \ker r^* = \pi^*(H^0(\widetilde{A}))$, and $i^* 1 = 1$. The second assertion follows by taking $A = X$.

□

CHAPTER 4

Critical Point Theory

In this chapter we will develop some basic tools regarding Critical Point Theory

4.1 Basic definitions

Throuough all this work every Banach space will stand by a Banach space over the real numbers $\mathbb{R}$.

Definition 4.1.1 (Differentiability)**.** *Let X and Y two Banach spaces and $\Omega \subset X$ be an open set. A map $F : \Omega \to Y$ is said to be (Fréchet-)*differentiable *at $x_0 \in \Omega$ if there is an $F'(x_0) \in \mathcal{L}(X,Y)$ such that*

$$F(x_0+h) = F(x_0) + F'(x_0)h + \omega(x_0,h) \quad \textit{and} \quad \omega(x_0,h) = o(|h|) \quad \textit{as} \quad h \to 0$$

In this case $F'(x_0)$ is the (Fréchet-)derivative of F at x_0. If F is differentiable at each $x \in \Omega$ and $F' : \Omega \to \mathcal{L}(X,Y)$ is continuous, then F is said to be continuously differentiable in Ω, $F \in C^1(\Omega)$ for short.

Example 4.1.1. *Let $X = C([a,b]) := \{x : [a,b] \to \mathbb{R} : x \quad \text{is continuous}\}$ with the norm $\|x\| = \sup_{t\in[a,b]} |x(t)|$ and consider the map $F : X \to X$ defined by*

$$(Fx)(t) = \int_a^b k(t,s)f(s,x(s))ds \quad \textit{for} \quad t \in [a,b],$$

where $k : [a,b] \times [a,b] \to \mathbb{R}$, f and $\partial f/\partial x : [a,b] \times \mathbb{R} \to \mathbb{R}$ are continuous. We claim that F is continuously differentiable with $F'(x)$ given by

$$(F'(x)h)(t) = \int_a^b k(t,s)\frac{\partial f}{\partial x}(s,x(s))h(s)ds \quad \textit{for} \quad h \in X.$$

Indeed, it follows from the fact that $f(s,x+h) - f(s,x) - \dfrac{\partial f}{\partial x}(s,x) \to 0$ uniformly on s as $h \to 0$.

Example 4.1.2. *Let X be a Banach space and $\|\cdot\| : X \to \mathbb{R}$ its norm. We claim that $\|\cdot\|$ is not differentiable at 0. Indeed if it was true, then there would exist $T \in \mathcal{L}(X,\mathbb{R})$ such that*

$$\begin{aligned} \|h\| &= Th + o_1(\|h\|) \quad \text{and} \\ \|h\| &= -Th + o_1(\|h\|), \end{aligned}$$

therefore

$$\|h\| = o(\|h\|),$$

which is a contradiction.

Higher derivatives are defined in the usual way by induction, i.e. if F is k-times differentiable in a neighborhood of x_0 and the kth derivative $F^{(k)}(\cdot)$ is differentiable at x_0 then F is said to be $k+1$-times differentiable at x_0.

4.2 Critical Point Theory on Finsler Manifolds

First let us remember what is a manifold modeled by a Banach space.

Definition 4.2.1. *Let X be a Banach space, and let $\mathcal{M}$ be a connected Hausdorff space. We say that $\mathcal{M}$ is a Banach C^r manifold, $r \geq 1$ (integer), modeled on X, if*

- *There exist an open covering $\{U_i\}_{i\in\Gamma}$ of $\mathcal{M}$,*
- *For each $i \in \Gamma$ there exist a homeomorphism $\Psi_i : U_i \to \Psi_i(U_i) \subset X$,*
- *$\Psi_i \circ \Psi_j^{-1} : \Psi_j(U_i \cap U_j) \to \Psi_i(U_i \cap U_j)$ is a C^r-diffeomorphism, for all $i, j \in \Gamma$ when $U_i \cap U_j \neq \emptyset$.*

Differentiable maps between two C^r Banach manifolds, tangent $T\mathcal{M}$ and cotangent bundle $T^*\mathcal{M}$ are defined in the same way as in the finite dimensional case.

Let $(E, \pi, \mathcal{M})$ be a vector bundle. $\xi : \mathcal{M} \to E$ is called a section if $\pi \circ \xi = 1_{\mathcal{M}}$. A section ξ is called C^r if it is a C^r map from M to E.

A section of the tangent bundle is called a vector field, and a section of the cotangent bundle is called a co-vector field.

Definition 4.2.2. *A Finsler manifold is a C^1 Banach manifold $\mathcal{M}$ together with a continuous function $\|\cdot\| : T\mathcal{M} \to [0,\infty)$, called a Finsler structure on $T\mathcal{M}$, such that*

- *for each $u \in \mathcal{M}$, the restriction $\|\cdot\|_u$ of $\|\cdot\|$ to the tangent space $T_u\mathcal{M}$ at u is a norm,*
- *for each $u \in \mathcal{M}$ and $C > 1$, there is a trivializing neighbourhood U of u such that*

$$\frac{1}{C}\|\cdot\|_v \leq \|\cdot\|_u \leq C\|\cdot\|_v \quad \forall v \in U$$

The Finsler structure also induces a metric on $\mathcal{M}$ that is consistent with the topology of $\mathcal{M}$;

$$d(u,v) = \inf_{\sigma\in\Gamma} \int_0^1 \|\sigma'(t)\|_{\sigma(t)} dt$$

where $\Gamma = \left\{\sigma \in C^1([0,1],\mathcal{M}) : \sigma(0) = u, \sigma(1) = v\right\}$.

We say that the manifold $\mathcal{M}$ is a Hilbert-Riemannian manifold if it is a Finsler manifold with Finsler structure $\|\cdot\| : T\mathcal{M} \to [0,\infty)$ such that $\|\cdot\|_v$ is induced by an inner product on the Hilbert space $T_v\mathcal{M}$ for each $v \in \mathcal{M}$.

Lemma 4.2.1 (Morse Lemma). *Assume ϕ is a C^2-functional on a complete C^2-Riemannian manifold X modeled on a Hilbert space E. If v_0 is a non-degenerate critical point of ϕ (i.e. if $d^2\phi(v_0)$ is invertible), then there exists a Lipschitz homeomorphism H from a neighborhood W of 0 in E onto a neighborhood M of v_0 with $H(0) = v_0$ in such a way that*

$$\phi(H(z)) = \phi(v_0) + \|z_+\|^2 - \|z_-\|^2$$

where $z \to (z_-, z_+)$ corresponds to the decomposition of E into the positive and negative spaces E_+ and E_- associated with the operator $d^2\phi(v_0)$.

Proof. For simplicity let us assume that X is a Hilbert space and $v_0 = 0$. It will then suffice to show that there exists a homeomorphism H from a neighborhood W of 0 into a neighborhood M of 0 satisfying $H(0) = 0$ and for every $z \in X$

$$\phi(H(z)) = \phi(0) + \frac{1}{2}(d^2\phi(0)z, z).$$

First define the function $\psi : X \to \mathbb{R}$ given by

$$\psi(z) = \phi(z) - \phi(0) - \frac{1}{2}(d^2\phi(0)z, z)$$

and note that $\psi(0) = \psi'(0) = \psi''(0) = 0$. Therefore, by Taylor series we have

$$\psi(z) = \int_0^1 (1-s)(\psi''(sz)z, z)ds \quad \text{and} \quad \psi'(z) = \int_0^1 \psi''(sz)z ds.$$

Thus, for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $\|z\| \le \delta$ we have

$$|\psi(z)| \le \varepsilon\|z\|^2 \quad \text{and} \quad \|\psi'(z)\| \le \varepsilon\|z\| \tag{2.1}$$

Since 0 is a non-degenerate critical point, there exists $K > 0$ such that for all $z \in X$

$$K^{-1}\|z\| \leq \|\phi''(0)z\| \leq K\|z\|. \tag{2.2}$$

Indeed, if not, then there exists a sequence $(y_n) \in X$ such that either $1/n > \|\phi''(0)y_n/\|y_n\|\|$ or $\|\phi''(0)y_n/\|y_n\|\| > n$ for all n. Both cases leads to a contradiction.

Now let $\varepsilon < \frac{1}{2K}$ and $\delta > 0$ that satisfies (2.2) and define on $[0,1] \times B(0,\delta)$ the function

$$F(t,z) = (1-t)\left(\phi(0) + \frac{1}{2}(\phi''(0)z,z)\right) + t\phi(z).$$

Using (2.1) and (2.2) we can define the vector field on $B(0,\delta)$

$$f(t,z) = \begin{cases} -F_t(t,z)\|F_z(t,z)\|^{-2}F_z(t,z) & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0 \end{cases}$$

Note that for $z \neq 0$ we have

$$f(t,z) = -\psi(z)\|\phi''(0)z + t\psi'(z)\|^{-2}(\psi''z + t\psi'(z)).$$

Using (2.1) and (2.2) it is easy to see that for $\|z\| < \delta$

$$|f(t,z)| \leq 2K(K+\varepsilon)\varepsilon\|z\| \tag{2.3}$$

showing that f is continuous since $f(t,0) = 0$.

Taking, if necessary $\delta > 0$ smaller, we can also suppose that on $B(0,\delta)$

$$\|\psi''(z)\| \leq 1. \tag{2.4}$$

Using (2.1), (2.2) and (2.4) it is easy to find $K_1 > 0$ such that $\|f_z(t,z)\| \leq K_1$ for $\|z\| \leq \delta$ and $z \neq 0$. Using this together with the mean value theorem we have a constant $C > 0$ such that

$$\|f(t,z_1) - f(t,z_2)\| \leq C\|z_1 - z_2\|$$

for all $\|z\| <\leq \delta$.

It follows that the Cauchy problem

$$\begin{cases} \eta' & = f(t,\eta) \\ \eta(0) & = z \end{cases}$$

has a continuous solution η for z in some open neighborhood W of 0. Note that

$$\frac{d}{dt}F(t,\eta(t)) = F_t(t,\eta(t)) + (F_z(t,\eta(t)),\eta'(t)) = 0$$

and in particular

$$\phi(0) + \frac{1}{2}(\phi''(0)z,z) = F(0,z) = F(1,\eta(1,z)) = \phi(\eta(1,z))$$

which means that the homeomorphism $H(z) = \eta(1,z)$ verifies the claim of the lemma.

□

Lemma 4.2.2 (First Deformation Lemma, [21]). *If $c \in \mathbb{R}$, $\varepsilon_0, \delta > 0$ and C a bounded set containing K_c, and f satisfies $(PS)_c$, then there are $\varepsilon > 0$ and $\eta \in C([0,1] \times \mathcal{M}, \mathcal{M})$ satisfying*

(i) $d(\eta(t,u),u) \le \delta t \quad \forall (t,u) \in [0,1] \times \mathcal{M}$,

(ii) $\eta(t,\cdot)$ *is the identity outside* $f_{c-\varepsilon_0}^{c+\varepsilon_0}$ *for all* $t \in [0,1]$,

(iii) $f(\eta(t,u)) \le f(u) \quad \forall (t,u) \in [0,1] \times \mathcal{M}$,

(iv) $\eta(1, f^{c+\varepsilon} \backslash N_\delta(C)) \subset f^{c-\varepsilon}$.

(v) If $\mathcal{M}$ is a free $\mathbb{Z}_2$-space and f is even then η can be taken such that the map $\eta(t,\cdot)$ is odd for each fixed t.

Lemma 4.2.3 (Second Deformation Lemma, [4]). *If $-\infty < a < b \le +\infty$ and f has no critical values in $[a,b]$ and satisfies $(PS)_c$ for all $c \in [a,b] \cap \mathbb{R}$, then f^a is a deformation retract of f^b.*

Proposition 4.2.1. *If $c := \inf f(\mathcal{M})$ is finite and f satisfies $(PS)_c$, then c is a critical value of f.*

Proof. If not, let $\varepsilon > 0$ and $\eta \in C([0,1] \times \mathcal{M}, \mathcal{M})$ by Lemma 4.2.2 and take $u \in \mathcal{M}$ with $f(u) \le c + \varepsilon$. then $f(\eta(1,u)) \le c - \varepsilon$, a contradiction. □

Remark 4.2.1. *Note that the $(PS)_c$ condition in the above proposition is essential. Indeed if we take $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = e^x$ then $\inf f(\mathbb{R}) = 0$ but f has no critical points.*

4.3 Critical Groups

In Morse Theory, one of the most important concepts when one is studying the critical set of a functional is that of *critical groups*. Such groups are topological objects that detects the behaviour of a functional near an isolated critical point.

Definition 4.3.1. *Let u_0 be an isolated critical point of f, and let $c = f(u_0)$. We call*

$$C^q(f,u_0) = H^q(f^c \cap U, (f^c \backslash \{u_0\}) \cap U)$$

the q^{th} critical group of f at u_0 where $q \in \mathbb{N} \cup \{0\}$. Where U is a neighborhood of u_0 such that has no other critical points but u_0, and $H^*(X,Y)$ stands for the Alexander-Spanier *relative cohomology groups with coefficient group $\mathbb{Z}_2$.*

According to the excision property of the Alexsander-Spanier cohomology theory, the critical groups are well-defined; i.e., they do not depend on a special choice of the neighborhood U.

Example 4.3.1. *Letting u_0 be an isolated local minimum of $f : M \to \mathbb{R}$, then*

$$C^q(f,u_0) = \begin{cases} \mathbb{Z}_2 & q = 0 \\ 0 & q \neq 0 \end{cases}$$

In fact since u_0 is an isolated local minimum, U can be chosen in such a way that $f^c \cap U = \{u_0\}$, where $c = f(u_0)$, therefore $C^q(f,u_0) = H^q(\{u_0\})$ for all $q \geq 0$.

Example 4.3.2. *Let $f : M^n \to \mathbb{R}$ where M^n is a n-dimensional Riemannian manifold with $n < \infty$. If u_0 is an isolated local maximum, of f then*

$$C^q(f,u_0) = \begin{cases} \mathbb{Z}_2 & q = n \\ 0 & q \neq n \end{cases}$$

Indeed it suffices to take $U = B(u_0,\delta)$ for $\delta > 0$ sufficeiently small in such a way that $f^c \cap U = B(u_0,\delta)$. Then $C^q(f,u_0) = H^q(B(u_0,\delta), \partial B(u_0,\delta)))$.

Example 4.3.3 (Monkey Saddle)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by $f(x,y) = x^3 - 3xy^2$. Then we have*

$$C^q(f,0) = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 1 \\ 0 & q \neq 1. \end{cases}$$

Indeed note that we can choose an open neighborhood U of 0 such that $f^0 \cap U$ is the figure below Denoting $A := f^0 \cap U$ and $B := A \setminus \{0\}$ we have $C^q(f,0) = H^q(A,B)$. Since A is locally contractible

we can assume that $H^*(A,B)$ are the relative singular cohomology groups. First let us prove that $H^0(A,B) = 0$. Indeed let $[\sigma] \in H^0(A,B)$ then σ is a homomorphism $\sigma : C_0(A,B) \to \mathbb{Z}_2$ such that $\delta \circ \sigma = \sigma \circ \partial = 0$. By definition σ vanishes in any 0-singular simplex c_i that lies on B. We claim that

σ *also vanishes on the singular* 0*-singular simplex represented by* 0*. Indeed note that the* 1*-singular simplex* $[c,0]$ *with* c *being a point in* B *is such that* $\delta\circ\sigma([c,0])=0$ *therefore* $\sigma([0])-\sigma([c])=0$ *but since* $\sigma([c])=0$ *we have* $\sigma([0])=0$*. Hence* $\sigma=0$ *thus* $[\sigma]=0$*.*

Now consider the following part of the long exact sequence associated with the pair (A,B)*;*

$$0\to H^0(A,B)\to H^0(A)\to H^0(B)\to H^1(A,B)\to H^1(A)\to\cdots$$

which translates into the exact sequence

$$0\to\mathbb{Z}_2\to\mathbb{Z}_2\oplus\mathbb{Z}_2\oplus\mathbb{Z}_2\to H^1(A,B)\to 0$$

By exactness we have that $H^1(A,B)=\mathbb{Z}_2\oplus\mathbb{Z}_2$*.*

In a similiar fashion we can show that $H^2(A,B)=0$ *and the fact that* $H^q(A,B)=0$ *for* $q>2$ *is because we are dealing with a two-dimensional configuration.*

Now let us show an interesting result conserning the local behavior of a functional near a nondegenerate critical point;

Theorem 4.3.1. *Suppose that* $f\in C^2(M,\mathbb{R})$*, where* M *is a Hilbert-Riemannian manifold and* p *is a nondegenerate critical point of* f *with Morse index* j*. then*

$$C^q(f,p)\cong\begin{cases}\mathbb{Z}_2 & q=j\\ 0 & q\neq j\end{cases}$$

Proof. Since p is nondegenerate, using the Lemma Morse we can change coordinates if necessary and suppose that f has the form $f(x)=\frac{1}{2}(Ax,x)$ where A is a bounded, invertible, self-adjoint operator. Let $P_\pm$ be the orthogonal projection onto the positive (negative) subspace $H_\pm$ with respect with the spectral deocomposition of A. Then by functional calculus we have that $AP_+=\left((AP_+)^{\frac{1}{2}}\right)^2$ and $-AP_-=\left((-AP_-)^{\frac{1}{2}}\right)^2$ where $(AP_+)^{1/2}$ and $(-AP_-)^{1/2}$ are both bounded, positive and self-adjoint operators since both AP_+ and $-AP_-$ are positive and self-adjoint.

Therefore we have

$$\begin{aligned}f(x)&=\frac{1}{2}(Ax,x)=\frac{1}{2}(A(P_+x+P_-x),x)=\\&=\frac{1}{2}((AP_+x,x)-(-AP_-x,x))=\\&=\frac{1}{2}\left(\|(AP_+)^{1/2}x\|^2-\|(-AP_-)^{1/2}x\|^2\right)\end{aligned}$$

Let us denote $y_\pm = (\pm AP_\pm)^{\frac{1}{2}}x$. Given $\varepsilon > 0$ we have that if $B_\varepsilon := B(0,\varepsilon)$ then

$$B_\varepsilon \cap f^0 = \{x \in H : \|x\| < \varepsilon, \quad \|y_+\| \leq \|y_-\|\}.$$

Now we define the deformation $\eta : [0,1] \times (B_\varepsilon \cap f^0) \to (B_\varepsilon \cap f^0)$ given by

$$\eta(t,x) = P_-x + (1-t)P_+x.$$

We need to show that this is well defined. Indeed since $x \in B_\varepsilon$ it is evident that $P_-x + (1-t)P_+x \in B_\varepsilon$ so we need to verify that $\eta(t,x) \in f^0$ for all $x \in B_\varepsilon \cap f^0$ and $t \in [0,1]$. First note that $(AP_+)^{1/2}P_- = (-AP_-)^{1/2}P_+ = 0$, because $0 = ((AP_+)P_-x, P_-x) = \|(AP_+)^{1/2}P_-x\|^2$, the same holding for $(-AP_-)^{1/2}P_+$. Then we have that $\|(AP_+)^{1/2}(P_-x + (1-t)P_+x)\|^2 = (1-t)^2\|(AP_+)^{1/2}P_+x\|^2$. Now we claim that $\|(AP_+)^{1/2}P_+x\|^2 = \|(AP_+)^{1/2}x\|^2$ for all x. For that, just note that $\|(AP_+)^{1/2}P_+x\|^2 = (AP_+P_+x, P_+x) = (AP_+x, P_+x) = (AP_+x, x) = \|(AP_+)^{1/2}x\|^2$. Putting all those together we have that $\|(AP_+)^{1/2}\eta(t,x)\| \leq \|(AP_+)^{1/2}x\|$.

By similar arguments we have that $\|(-AP_-)^{1/2}\eta(t,x)\| = \|(-AP_-)^{1/2}x\|$. Therefore we have that in fact $\eta(t,x) \in f^0$ for all t and x in its domain.

It shows that η is a strong deformation retract from $(B_\varepsilon \cap f^0, B_\varepsilon \cap (f^0 \setminus \{0\}))$ to $(H_- \cap B_\varepsilon, (H_- \setminus \{0\}) \cap B_\varepsilon)$. Thus

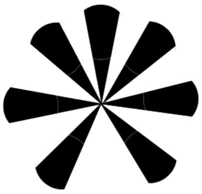
$$\begin{aligned} C^q(f,p) &\cong H^q\left(f^0 \cap B_\varepsilon, (f^0 \setminus \{0\}) \cap B_\varepsilon\right) \\ &\cong H^q\left(H_- \cap B_\varepsilon, (H_- \setminus \{0\}) \cap B_\varepsilon\right) \\ &\cong H^q\left(B^j, \partial B^j\right) \cong \begin{cases} \mathbb{Z}_2 & q = j \\ 0 & q \neq j \end{cases} \end{aligned}$$

if j is finite and B^j is the closed unit ball centered at the origin of H_-.

If $j = \infty$, since S^∞ is contractible we have $C^q(f,p) \cong 0$ for all q. □

Remark 4.3.1. *Note that, by the above theorem, the critical point of the example* (4.3.3) *is degenerate.*

Remark 4.3.2. *By the proof of the above theorem it is quite clear why the function in example* 4.3.3 *is degenerate at* 0*. It is because its sublevel f^0 can not be deformated in a linear subspace (as we deformated f^0 into H_- in the proof of the theorem). More generally, any function on $\mathbb{R}^2$ that has 0 as a critical point and a f^0 has the form of the figure below must be degenerate at* 0*;*



The critical groups have an important role in critical point theory because they represent the topological *"obstruction"* between two sublevels, more precisely we have

Proposition 4.3.1. *If $-\infty < a < b \leq +\infty$ and f has only a finite number of critical points at the level $c \in (a,b)$, has no other critical values in $[a,b]$, and satisfies $(PS)_c$ for all $c \in [a,b] \cap \mathbb{R}$, then*

$$H^q(f^b, f^a) \cong \bigoplus_{u \in K^c} C^q(f,u) \quad \forall q,$$

In particular,

$$\dim H^q(f^b . f^a) = \sum_{u \in K^c} \dim C^q(f,u) \quad \forall q.$$

Proof. By the Second Deformation Lemma 4.2.3 we have that f^b is a deformation retract of f^c and $f^c \backslash K^c$ is a deformation retract of f^a. Therefore

$$H^q(f^b, f^a) \cong H^q(f^c, f^a) \cong H^q(f^c, f^c \backslash K^c)$$

Therefore it is suffice to compute the group $H^q(f^c, f^c \backslash K^c)$. Since K^c is a finite set, let $\delta > 0$ small enough such that $K^c \subset \cup_{u \in K^c} B(u,\delta)$ and $B(u,\delta) \cap B(v,\delta) = \emptyset$ if $u \neq v$. Now define $Z := f^c \backslash \cup_{u \in K^c} B(u,\delta)$. Thus Z is a closed subset of f^c and it is contained in $f^c \backslash K^c$ which is an open subset of f^c. Therefore by the excision property of cohomology we have that $H^q(f^c, f^c \backslash K^c) \approx H^q(f^c \backslash Z, (f^c \backslash K^c) \backslash Z)$ but it is clear that

$$H^q(f^c \backslash Z, (f^c \backslash K^c) \backslash Z) \cong \bigoplus_{u \in K^c} H^q(f^c \cap B(u,\delta), f^c \cap B(u,\delta) \backslash \{u\}) \cong \bigoplus_{u \in K^c} C^q(f,u).$$

□

Note that the theorem is true under the hypothesis that c were the only critical value in $[a,b]$. We can use this to study the case where we have several critical points $[a,b]$ but still have only finite critical points associated to such critical values, namely

Proposition 4.3.2. *If $-\infty < a < b \leq +\infty$ are regular values and f has only a finite number of critical points in f_b^a and satisfies $(PS)_c$ for all $c \in [a,b] \cap \mathbb{R}$, then*

$$\dim H^q(f^b, f^a) \leq \sum_{u \in K_a^b} \dim C^q(f,u) \quad \forall q.$$

In particular, f has a critical point u with $a < f(u) < b$ and $C^q(f,u) \neq 0$ when $H^q(f^b, f^a) \neq 0$.

Proof. The idea here is to break down the interval $[a,b]$ in smaller ones in such a way that on each smaller interval we can apply Proposition 4.3.1. For that let us denote $c_1 < \cdots < c_k$ be the critical values of f on (a,b) and

$$a = a_1 < c_1 < a_2 < c_2 < \cdots c_{k-1} < a_k < c_k < a_{k+1} = b.$$

For sake of simplicity let us denote $f^{a_i} = X_i$ for $i = 1, \dots, k+1$. Then we have

$$X_1 \subset X_2 \subset \cdots \subset X_{k+1}.$$

We claim that

$$\dim H^q(X_{k+1}, X_1) \leq \sum_{i=1}^{k} H^q(X_{i+1}, X_i) \quad \forall q.$$

Indeed let us consider the long exact sequence associated with the triple (X_1, X_k, X_{k+1}) i.e. the long exact sequence induced by the inclusion of pairs

$$(X_k, X_1) \hookrightarrow (X_{k+1}, X_1) \hookrightarrow (X_{k+1}, X_k)$$

namely

$$\cdots \longrightarrow H^q(X_{k+1}, X_k) \xrightarrow{j^*} H^q(X_{k+1}, X_1) \xrightarrow{i^*} H^q(X_k, X_1) \longrightarrow \cdots$$

Then we have by exactness

$$\text{Im } i^* \cong H^q(X_{k+1}, X_1) / \ker i^* = H^q(X_{k+1}, X_1) / \text{Im } j^*$$

consequently

$$\dim H^q(X_{k+1}, X_1) = \text{rank } i^* + \text{rank } j^* \leq \dim H^q(X_k, X_1) + \dim H^q(X_{k+1}, X_k)$$

then by induction we have that in fact

$$\dim H^q(X_{k+1}, X_1) \leq \sum_{i=1}^{k} H^q(X_{i+1}, X_i) \quad \forall q.$$

therefore applying Proposition 4.3.1 to the intervals $[a_i, a_{i+1}]$ we have

$$\begin{aligned} \dim H^q(f^b, f^a) &\leq \sum_{i=1}^{k} \dim H^q(f^{a_{i+1}}, f^{a_i}) \\ &= \sum_{i=1}^{k} \sum_{u \in K^{c_i}} \dim C^q(f, u) \\ &= \sum_{u \in K_a^b} \dim C^q(f, u) \end{aligned}$$

□

Proposition 4.3.2 show us that under certain conditions, only by computing the cohomology groups of a space is enough to know wether a functional has critical points, for example

Example 4.3.4. *Let $\mathcal{M} = \mathbb{T}^2$ be the 2-dimensional torus. Let $f : \mathcal{M} \to \mathbb{R}$ be a C^1 function which has only a finite amount of critical points. Since $\mathcal{M}$ is compact we can choose $[a,b] \subset \mathbb{R}$ such that $f(\mathcal{M}) \subset (a,b)$. f satisfies $(PS)_c$ for all $c \in [a,b]$. Note that $H^q(f^b, f^a) \cong H^q(\mathbb{T}^2)$, so*

$$H^q(f^b, f^a) \cong \begin{cases} \mathbb{Z}_2 & q = 0,2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & q = 1 \\ 0 & q \geq 3. \end{cases}$$

and therefore by Proposition 4.3.2 we have

$$2 = \dim H^1(f^a, f^b) \leq \sum_{u \in K_a^b} \dim C^1(f,u).$$

Consequently f must have a critical point u with $C^1(f,u) \neq 0$ which is neither of maximum nor minimum by Examples 4.3.1 and 4.3.2.

Example 4.3.5. *More generally, any C^1 function f that has only a finite amount of critical points on a compact 2dimensional surface $\mathcal{M}$ of genus g must have a critical point u which is neither of maximum nor minimum since $H^1(\mathcal{M}) \cong \mathbb{Z}_2^{2g}$.*

Remark 4.3.3. *Note that the hypotesis of having only a finite amount of critical points on $[a,b]$ is crucial. For example if we take a constant function $f : \mathbb{T}^2 \to \mathbb{R}$ then all points are critical ones. Thus the critical groups of f are not well defined. Moreover all of its critical points are both of maximum and minimum nature.*

When the critical values of $f : W \to \mathbb{R}$ are bounded from below and f satisfies $(PS)_c$ for all $c \in \mathbb{R}$, the global behavior of f can be described by the critical groups at infinity

$$C^q(f, \infty) := H^q(W, f^a)$$

where a is less than all critical values. This is well defined by Lemma 4.2.3.

We have the following properties of the critical groups at infinity;

Proposition 4.3.3. *Assume that f satisfies $(PS)_c$ for all c.*

(i) If f is bounded from below, then

$$C^q(f, \infty) \cong \begin{cases} \mathbb{Z}_2 & q = 0 \\ 0 & q > 0 \end{cases}$$

(ii) If the critical values of f are bounded from below, then

$$C^q(f,\infty) \cong \widetilde{H}^{q-1}(f^a) \quad \forall q.$$

In particular $C^0(f,\infty) = 0$.

Proof. (i) It is suffice to choose a such that $f^a = \emptyset$, so $C^q(f,\infty) = H^q(W) = \delta_{q0}\mathbb{Z}_2$

(ii) It follows by the long exact sequence induced by the inclusions $f^a \hookrightarrow W \hookrightarrow (W, f^a)$ since W is contractible. □

Proposition 4.3.4. *If f has only a finite number of critical points and satisfies $(PS)_c$ condition for all c then*

$$\dim C^q(f,\infty) \leq \sum_{u\in K} \dim C^q(f,u) \quad \forall q.$$

Proof. Just take $b = \infty$ on Proposition 4.3.2. □

4.4 Linking

By Lemmas 4.2.2 and 4.2.3 we have that if a functional f does not have critical values in $[a,b]$ then f^a is homotopically equivalent to f^b i.e. their topology are quite similar. Therefore if it happens that the two sublevels f^a and f^b have very different topologies then it is likely to exists a critical value in $[a,b]$. One way to study the change in topology between two sublevels is by means of the relative cohomology groups, namely $H^q(f^b, f^a)$. Indeed under the hypothesis of Proposition 4.3.2 if the sublevels differ in topology in the sense of having $H^q(f^b, f^a) \neq 0$ then f must have a critical value in $[a,b]$.

Note that instead of using cohomology to study change in topology, we can also employ singular homology. All the previous results conserning the cohomology groups still hold when we use singular homology groups instead (see [4]).

Definition 4.4.1. *Let $\gamma \in C_q(A)$ be a q-dimensional singular simplex. Then $\gamma = \sum_{i=1}^k a_i\sigma_i$ where $\sigma_i : \Delta^q \to A$ is a continuous function for all i and $a_i \in \mathbb{Z}_2$. Then we call the set $|\gamma| := \cup_{i=1}^k \sigma_i(\Delta^q) \subset A$ the support of γ.*

Proposition 4.4.1. *Let $f : \mathcal{M} \to \mathbb{R}$ be a C^1 function defined on the C^1 Finsler manifold $\mathcal{M}$. Suppose that f satisfies $(PS)_c$ for all $c \in [a,b]$ and $H_q(f^b, f^a) \neq 0$. Then there exists a critical value $c \in [a,b]$, where H_* stands for singular homology groups.*

Proof. Let $0 \neq [\sigma] \in H_q(f^b, f^a)$ and define

$$c := \inf_{\gamma\in[\sigma]} \sup_{|\gamma|} f(x).$$

We claim that c is a critical value. Indeed, if it is not, then by Lemma 4.2.2 there exists a deformation $\eta : [0,1] \times \mathcal{M} \to \mathcal{M}$ and $\varepsilon > 0$ such that $\eta(0,\cdot) = 1_{\mathcal{M}}$ and $\eta(1, f^{c+\varepsilon}) \subset f^{c-\varepsilon}$.

Let $\gamma \in [\sigma]$ such that $\sup f(|\gamma|) \leq c+\varepsilon$. Since $1_{\mathcal{M}}$ is homotopic to the map $\eta(1,\cdot)$ we have $\eta(1,\cdot)_*\gamma \in [\sigma]$. But since $|\gamma| \subset f^{c+\varepsilon}$, $|\eta(1,\cdot)_*\gamma| \subset f^{c-\varepsilon}$ which leads to a contradiction by the definition of c. □

Propositions 4.4.1 and 4.3.2 tells us that computing the relative groups $H^q(f^b, f^a)$ is a good strategy to seek for critical values of a functional f. But how to guess which levels we should compute their relative groups? Well, we can choose those who are more likely to carry topological changes, namely those who are detected by the notion of linking. Here we will present the notion of cohomological linking.

Definition 4.4.2. *Let A and B be disjoint nonempty subsets of a Banach space W. We say that A cohomologically links B in dimension $q < \infty$ if the homomorphism*

$$i^* : \widetilde{H}^q(B^c) \to \widetilde{H}^q(A)$$

induced by $i : A \subset B^c$, is nontrivial.

At a first glance the above definition might seem a little bit strange. How can we detect changes in topology on the sublevels of a functional just by verifying the nontriviality of i^*? Well, to exemplify let us suppose that both A and B^c are locally contractible so we can assume, by means of the isomorphism μ, that H^q stands for the singular cohomology groups. Also, by means of the isomorphism h we have that $i : A \subset B^c$ is nontrivial in the singular homology groups $i_* : \widetilde{H}_q(A) \to \widetilde{H}_q(B^c)$. Let us consider now the long exact sequence induced by the inclusion $B^c \hookrightarrow W \hookrightarrow (W, B^c)$, i.e.

$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H}_{q+1}(W) \arrow[r] & H_{q+1}(W,B^c) \arrow[r, "\partial_*"] & \widetilde{H}_q(B^c) \arrow[r] & \widetilde{H}_q(W) \arrow[r] & \cdots . \end{tikzcd}$$

Where ∂_* is the connecting homomorphism wich associate a simplex σ in $C_{q+1}(W, B^c)$ with $\partial\sigma \in C_q(B^c)$ to its equivalence class $[\partial\sigma]$ on $\widetilde{H}_q(B^c)$. Since W is contractible, $\widetilde{H}_q(W) = 0$ for all q therefore by exactness ∂_* is an isomorphism

In the same fashion, but now analyzing the long exact sequence induced by the inclusions $A \hookrightarrow W \hookrightarrow (W, A)$

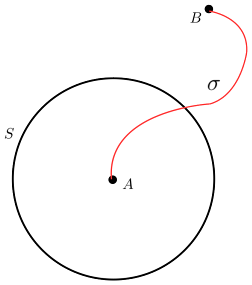
$$\begin{tikzcd} \cdots \arrow[r] & \widetilde{H}_{q+1}(W) \arrow[r] & H_{q+1}(W,A) \arrow[r, "\partial_*"] & \widetilde{H}_q(A) \arrow[r] & \widetilde{H}_q(W) \arrow[r] & \cdots \end{tikzcd}$$

we also have that ∂_* is an isomorphism. Therefore we have the commutative diagram

$$\begin{tikzcd}
H_{q+1}(W,A) \arrow[r, "\partial_*"] \arrow[d, "i_*"] & \widetilde{H}_q(A) \arrow[d, "i_*"] \\
H_{q+1}(W,B^c) \arrow[r, "\partial_*"] & \widetilde{H}_q(B^c)
\end{tikzcd}$$

hence the map $(\partial_*)^{-1}i_*\partial_* : H_{q+1}(W,A) \to H_{q+1}(W,B^c)$ is nontrivial, so there exists $[\sigma] \in H_q(W,A)$ such that $(\partial_*)^{-1}i_*[\sigma]\partial_* \neq 0$. By the construction of the connecting homomorphisms we have that any $(q+1)$-singular simplex γ such that $\partial\gamma = \partial\sigma$ happens to be in $(\partial_*)^{-1}i_*\partial_*[\sigma]$. But since $(\partial_*)^{-1}i_*\partial_*[\sigma] \neq 0$ then for any $\gamma \in (\partial_*)^{-1}i_*\partial_*[\sigma]$ we must have $|\gamma| \cap B \neq \emptyset$ (otherwise $\gamma \in C_{q+1}(B^c)$ and therefore $0 = [\gamma] = (\partial_*)^{-1}i_*\partial_*[\sigma]$, a contradiction). In short, we have that there exists a $q+1$-singular simplex σ which has its boundary on A (more precisely $|\partial\sigma| \subset A$) such that for any other $(q+1)$-singular simplex γ such that $\partial\gamma = \partial\sigma$ then $|\gamma| \cap B \neq \emptyset$.

This is the why the nontriviality of i^* is called a linking. For example when we have the mountain pass setting we have a linking in dimension zero. Namely, let $B = S$ be a sphere centered at the origin and A be two points $\{0, p\}$ where p has has lenght bigger than the radius of the sphere S. Then the set A links B cohomologically in dimension 0 (see the figure below).



By the figure it is clear that any other 1-singular simplex which has the same boundary as σ must have support which intersect S.

By detecting linkings by means of cohomology theory we have the algebra working together with the topology, which is a powerful combination which allows us to detect more complicated and useful linkings and consequently detect critical points as in the following result.

Proposition 4.4.2. *If A cohomologically links B in dimension q and*

$$f|_A \leq a < f|_B$$

then $H^0(W, f^a) = 0$ and $H^{q+1}(W, f^a) \neq 0$. If in addition, a is less than all critical values and f has only a finite number of critical points in $f^{-1}(a,\infty)$ and satisfies $(PS)_c$ for all c, then $C^0(f,\infty) = 0$, $C^{q+1}(f,\infty) \neq 0$, and f has a critical point u with $f(u) > a$ and $C^{q+1}(f,u) \neq 0$.

Proof. We have the commutative diagram

$$\begin{tikzcd}
\widetilde{H}^q(B^c) \arrow[d] & \widetilde{H}^q(f^a) \arrow[l] \arrow[dl, "i^*"] \\
\widetilde{H}^q(A) &
\end{tikzcd}$$

induced by the inclusions $A \hookrightarrow f^a \hookrightarrow B^c$. Since $i^* \neq 0$, by commutativity $\widetilde{H}^q(f^a) \neq 0$. By the long exact sequence induced by the inclusions $f^a \hookrightarrow W \hookrightarrow (W, f^a)$ and the fact that W is contractible, we have $H^{q+1}(W, f^a) \cong \widetilde{H}^q(f^a)$. Applying Proposition 4.3.4 we are done. ☐

Now let us use the concept of cohomological linking to construct examples of linkings. For that, let $\mathcal{M}$ be a bounded symmetric subset of $W\backslash\{0\}$ radially homeomorphic to the unit sphere $S = \{x \in W : \|x\| = 1\}$, i.e., the map $\pi : W\backslash\{0\} \to S$ given by $\pi(x) = x/\|x\|$ when restricted to $\mathcal{M}$ is a a homeomorphism onto S. Then the radial projection form $W\backslash\{0\}$ onto $\mathcal{M}$ is given by

$$\pi_{\mathcal{M}} = (\pi|_{\mathcal{M}})^{-1} \circ \pi.$$

Then we have, with the above notations :

Proposition 4.4.3. *Let $A_0 \neq \emptyset, B_0$ be disjoint closed symmetric subsets of $\mathcal{M}$ such that*

$$i(A_0) = i(\mathcal{M}\backslash B_0) < \infty.$$

Then $A = RA_0$ cohomologically links $B = \pi_{\mathcal{M}}^{-1}(B_0) \cup \{0\}$ in dimension $q = i(A_0) - 1$ for any $R > 0$, where $RA_0 = \{Rx : x \in A_0\}$.

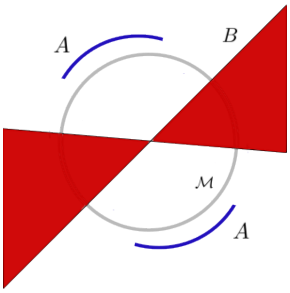


Figure 4.1:
geometric setting of A and B on $\mathbb{R}^2$

Proof. First let us show that $\mathcal{M}\backslash B_0$ and $W\backslash B$ have homotopic type. In fact let us define $H : [0,1] \times W\backslash B \to W\backslash B$ as $H(t,x) = (1-t)x + t(i_{\mathcal{M}}|_{\mathcal{M}\backslash B_0} \circ \pi_{\mathcal{M}}|_{W\backslash B})(x)$ where $i_{\mathcal{M}}$ is the inclusion $\mathcal{M} \hookrightarrow W\backslash\{0\}$. Therefore $(i_{\mathcal{M}}|_{\mathcal{M}\backslash B_0} \circ \pi_{\mathcal{M}}|_{W\backslash B}) \approx 1_{W\backslash B}$. Since $\pi_{\mathcal{M}}|_{W\backslash B} \circ i_{\mathcal{M}}|_{\mathcal{M}\backslash B_0} = 1_{\mathcal{M}\backslash B_0}$ our assertion follows.

This then implies that $\pi_{\mathcal{M}}|_{W\backslash B} : W\backslash B \to \mathcal{M}\backslash B_0$ induces isomorphisms on cohomology groups $(\pi_{\mathcal{M}}|_{W\backslash B})^* : \widetilde{H}^{q-1}(W\backslash B) \to \widetilde{H}^{q-1}(\mathcal{M}\backslash B_0)$ for all $q \geq 0$.

Also, we have that the restriction $\pi_{\mathcal{M}}|_{RA_0} : RA_0 \to A_0$ is a homeomorphism and therefore induces isomorphisms on cohomology groups $(\pi_{\mathcal{M}}|_{RA_0})^* : \widetilde{H}^{q-1}(A_0) \to \widetilde{H}^{q-1}(RA_0)$.

All that leads to the following commutative diagram

$$\begin{tikzcd}
\widetilde{H}^{q-1}(\mathcal{M}\backslash B_0) \arrow[r, "j^*"] \arrow[d, "\cong"] & \widetilde{H}^{q-1}(A_0) \arrow[d, "\cong"] \\
\widetilde{H}^{q-1}(W\backslash B) \arrow[r, "i^*"] & \widetilde{H}^{q-1}(A)
\end{tikzcd}$$

where the vertical isomorphisms are those of the above discussion, and j^* is the induced by the inclusion $A_0 \hookrightarrow \mathcal{M}\backslash B_0$.

By Proposition 3.2.3 (iv) we have that j^* is a nontrivial homomorphism, therefore by commutativity i^* is also nontrivial.

□

4.5 Critical Points of Even Functionals on Symmetric Finsler Manifolds

Suppose that $\mathcal{M}$ is a free $\mathbb{Z}_2$-space and also a C^1 Finsler manifold. Moreover, suppose that $f : \mathcal{M} \to \mathbb{R}$ is a C^1 even functional.

Let us denote by $\mathcal{F}$ the class of symmetric subsets of $\mathcal{M}$. For each $k \in \mathbb{N}$ let us define the family of sets

$$\mathcal{F}_k = \{M \in \mathcal{F} : i(M) \geq k\} \tag{5.5}$$

and the values

$$c_k = \inf_{M\in\mathcal{F}_k} \sup_{u\in M} f(u). \tag{5.6}$$

Lemma 4.5.1. *Suppose that f satisfies $(PS)_c$ condition, then there is an $\varepsilon > 0$ such that*

$$c - \varepsilon < c_k \leq \cdots \leq c_{k+m-1} < c + \varepsilon \implies i(K^c) \geq m.$$

Proof. By $(PS)_c$ condition K^c is compact, therefore by Proposition (3.2.2) there is $\delta > 0$ such that

$$i(K^c) = i(N_\delta(K^c)).$$

By Lemma (4.2.2) there exists an odd map $\eta \in C(\mathcal{M}, \mathcal{M})$ so that $\eta(f^{c+\varepsilon} \backslash N_\delta(K^c)) \subset f^{c-\varepsilon}$ for some small $\varepsilon > 0$.

Now note that $f^{c+\varepsilon} \subset \overline{f^{c+\varepsilon} \backslash N_\delta(K^c)} \cup N_\delta(K^c)$ and by Proposition (3.2.2) we have

$$i(f^{c+\varepsilon}) \leq i(K^c) + i(f^{c-\varepsilon}).$$

If we have $c - \varepsilon < c_k$ then $f^{c-\varepsilon} \notin \mathcal{F}_k$ so

$$i(f^{c-\varepsilon}) \leq k - 1.$$

Also, if $c_{k+m-1} < c + \varepsilon$ then there is a set $M \in \mathcal{F}_{k+m-1}$ such that $M \subset f^{c+\varepsilon}$, so

$$i(f^{c+\varepsilon}) \geq i(M) \geq k + m - 1.$$

Combining the last three inequalities we have the desired result. □

Proposition 4.5.1. *Let f and $\mathcal{M}$ be as in the beginning of this section.*

(i) If $-\infty < c_k = \cdots = c_{k+m-1} = c < +\infty$ and f satisfies $(PS)_c$, then $i(K^c) \geq m$. In particular, if $-\infty < c_k \leq \cdots \leq c_{k+m-1} < +\infty$ and f satisfies $(PS)_c$ for $c = c_k, \ldots, c_{k+m-1}$, then each c is a critical value and f has m distinct pairs of associated critical points.

(ii) if $-\infty < c_k < +\infty$ for all sufficiently large k and f satisfies (PS), then $c_k \nearrow +\infty$.

Proof. (i) Follows directly from Lemma 4.5.1.

(ii) If not, then since the sequence (c_k) is non-decreasing we have that there exists $b \in \mathbb{R}$ such that $c_k \to b$ as $k \to +\infty$, so for any $\varepsilon > 0$ we have that $c_k \in (b - \varepsilon, b + \varepsilon)$ for all k large enough. So, for any p, $b - \varepsilon < c_k \leq \cdots \leq c_{k+p-1} < b + \varepsilon$ hence by Lemma 4.5.1 $i(K^c) \geq p$ for any p, so $i(K^b) = +\infty$, but this is a contradiction since K^b is compact and by Proposition 3.2.3 $i(K^b) < +\infty$. □

We can easily compute the index of the sublevels of the functional $f : \mathcal{M} \to \mathbb{R}$ as follows.

Proposition 4.5.2. *Let $f : \mathcal{M} \to \mathbb{R}$ as in the introduction of this section.*

(i) If c_k is finite and f satisfies $(PS)_{c_k}$, then

$$i(\mathcal{M} \backslash f_{c_k}) < k \leq i(f^{c_k}). \tag{5.7}$$

(ii) If $c_k < c_{k+1}$ are finite and f satisfies $(PS)_c$ for $c = c_k, c_{k+1}$, then

$$i(f^{c_k}) = i(\mathcal{M} \backslash f_a) = i(f^a) = i(\mathcal{M} \backslash f_{c_{k+1}}) = k \quad \forall a \in (c_k, c_{k+1}).$$

Proof. (i) Suppose by contradiction that $i(\mathcal{M}\backslash f_{c_k}) \geq k$. Since $\mathcal{M}$ is locally contractible and f_{c_k} is closed, then $\mathcal{M}\backslash f_{c_k}$ is locally contractible, so by Proposition 3.2.3 there exists a compact symmetric subset C of $\mathcal{M}\backslash f_{c_k}$ such that $i(C) \geq k$. Then $C \in \mathcal{F}_k$ and since f is continuous $\sup_{u \in C} f(u) < c_k$ which is a contradiction by the definition of c_k. It gives the left inequality in (5.7).

Now let us prove the other inequality. By Proposition 3.2.2 there exists a closed neighborhood N of f^{c_k} such that

$$i(N) = i(f^{c_k}).$$

Since K^{c_k} is compact

$$\delta := \frac{1}{2}\mathrm{dist}(K^{c_k}, \mathcal{M}\backslash N) > 0.$$

By Lemma 4.2.2 there exist $\varepsilon > 0$ and an odd map $\eta \in C(\mathcal{M}, \mathcal{M})$ such that

$$\eta(f^{c_k+\varepsilon}\backslash N_\delta(K^{c_k})) \subset f^{c_k-\varepsilon}$$

and

$$d(\eta(u), u) \leq \delta \quad \forall u \in \mathcal{M}.$$

Therefore since $f^{c_k+\varepsilon} \subset f^{c_k+\varepsilon}\backslash N_\delta(K^{c_k}) \cup N_\delta(K^{c_k})$ and $N_{2\delta}(K^{c_k}) \subset N$ we have

$$\eta(f^{c_k+\varepsilon}) \subset f^{c_k-\varepsilon} \cup N \subset N$$

so since η is an odd map ,by Proposition 3.2.2 we have $i(f^{c_k+\varepsilon}) \leq i(N) = i(f^{c_k})$. By the definition of c_k there exists $M \in \mathcal{F}_k$ such that $M \subset f^{c_k+\varepsilon}$ so $i(f^{c_k+\varepsilon}) \geq k$ and consequently $i(f^{c_k}) \geq k$.

(ii) Employing Proposition 3.2.2 and (5.7) we have

$$k \leq i(f^{c_k}) \leq i(\mathcal{M}\backslash f_a) \leq i(f^a) \leq i(\mathcal{M}\backslash f_{c_{k+1}}) < k+1.$$

□

4.6 Pseudo-Index

Using the cohomological index already introduced, we are now able to develop a useful related tool. When one is seeking for positive critical points, it is convenient to employ the notion of pseudo-index introduced by Benci [1].

Definition 4.6.1. *Let $\mathcal{F}$ denote the class of symmetric subsets of $W \setminus \{0\}$ where W is a Banach space $\mathcal{M} \in \mathcal{F}$ be closed $0 \leq a < b \leq +\infty$, and denote by Γ the groups of odd homeomorphisms of W that* *are the identity outside* $f^{-1}(a,b)$*. Then the pseudo-index of* $M \in \mathcal{F}$ *related to i,* $\mathcal{M}$*, and* Γ *is defined by*

$$i^*(M) = \min_{\gamma \in \Gamma} i(\gamma(M) \cap \mathcal{M}).$$

We then have the following properties of the pseudo-index $i^* : \mathcal{F} \to \mathbb{N} \cup \{0, \infty\}$.

Proposition 4.6.1. *Let* $A, b \in \mathcal{F}$*.*

(i) If $A \subset B$*, then*

$$i^*(A) \leq i^*(B).$$

(ii) If $\eta \in \Gamma$*, then*

$$i^*(\eta(A)) = i^*(A).$$

(iii) If A and B are closed, then

$$i^*(A \cup B) \leq i^*(A) + i(B).$$

Proof. (i) Since $A \subset B$ then $\gamma(A) \subset \gamma(B)$ for any $\gamma \in \Gamma$, so by Proposition 3.2.2

$$i(\gamma(A) \cap \mathcal{M}) \leq i(\gamma(B) \cap \mathcal{M}),$$

so the assertion follows.

(ii) Note first that $\{\gamma\eta : \gamma \in \Gamma\} = \Gamma$. So we have that

$$i^*(\eta(A)) = \min_{\gamma \in \Gamma} i(\gamma\eta(A) \cap \mathcal{M}) = i^*(A).$$

(iii) For each $\gamma \in \Gamma$,

$$\begin{aligned} i(\gamma(A \cup B) \cap \mathcal{M}) &= i((\gamma(A) \cap \mathcal{M}) \cup (\gamma(B) \cap \mathcal{M})) \\ &\leq i(\gamma(A) \cap \mathcal{M}) + i(\gamma(B) \cap \mathcal{M}) \end{aligned}$$

Since γ is an odd homeomorphism,

$$i(\gamma(B) \cap \mathcal{M}) \leq i(\gamma(B)) = i(B).$$

□

Note that depending on the choice of $\mathcal{M}$ in the Definition 4.6.1 we have that the pseudo-index i^* is bounded. Indeed, we have by 4.6.1 $i^*(M) \leq i(\mathcal{M})$, so if we choose $\mathcal{M}$ so that $i(\mathcal{M}) < \infty$ then the pseudo-index relative to $\mathcal{M}$ is bounded.

For $k \leq i(\mathcal{M})$ in $\mathbb{N}$, let

$$\mathcal{F}_k^* = \{M \in \mathcal{F} : i^*(M) \geq k\}$$

and

$$c_k^* = \inf_{M \in \mathcal{F}_k^*} \sup_{u \in M} f(u).$$

Since $\mathcal{F}_{k+1}^* \subset \mathcal{F}_k^*$, we have $c_{k+1}^* \geq c_k^*$.

With the same notations of Definition 4.6.1 we have

Lemma 4.6.1. *If $c > 0$ and f is even and satisfies $(PS)_c$, then there is an $\varepsilon > 0$ such that*

$$c - \varepsilon < c_k^* \leq \cdots \leq c_{k+m-1}^* < c + \varepsilon \implies i(K^c) \geq m.$$

Proof. Let $\delta > 0$ such that

$$i(N_\delta(K^c)) = i(K^c).$$

By Lemma 4.2.2 there exist $\varepsilon > 0$ and an odd homeomorphism $\eta : W \to W$ such that

$$\eta(f^{c+\varepsilon} \backslash N_\delta(K^c)) \subset f^{c-\varepsilon}.$$

In Lemma 4.2.2 we can choose $\varepsilon > 0$ small enough such that η is the identity on $f^{-1}(a,b)$ so that $\eta \in \Gamma$.

By Proposition 4.6.1 we then have

$$i^*(f^{c+\varepsilon}) \leq i^*(\overline{f^{c+\varepsilon} \backslash N_\delta(K^c)}) + i(N_\delta(K^c)) \leq i^*(f^{c-\varepsilon}) + i(K^c).$$

Since $c + \varepsilon > c_{k+m-1}^*$ there exists, by the very definition of c_{k+m-1}^*, $M \in \mathcal{F}_k^*$ such that $M \subset f^{c+\varepsilon}$. Therefore by Proposition 4.6.1 $i^*(f^{c+\varepsilon}) \geq k + m - 1$.

Since $c - \varepsilon < c_k^*$, by the very definition of c_k^* we also have $i^*(f^{c-\varepsilon}) \leq k - 1$. So putting all those inequalities together we have the desired result. □

Proposition 4.6.2. *Assume that f is even,* *(i)* *If* $a < c_k^* = \cdots = c_{k+m-1}^* = c < b$ *and* f *satisfies* $(PS)_c$*, then* $i(K^c) \geq m$*. In particular, if* $a < c_k^* \leq \cdots \leq c_{k+m-1}^* < b$ *and* f *satisfies* $(PS)_c$ *for* $c = c_k^*, \dots, c_{k+m-1}^*$*, then each* c *is a critical value and* f *has* m *distincs pairs of associated critical points.*

(ii) *If* $a = 0$*,* $b = +\infty$*,* $0 < c_k^* < +\infty$ *for all sufficiently large* k*, and* f *satisfies* $(PS)_c$ *for all* $c > 0$*, then* $c_k^* \nearrow +\infty$*.*

Proof. (i) It is just a direct employment of the Lemma 4.6.1.

(ii) If that is not the case, then the points c_k^* accumulates in a point, so we can apply part (i) for any m and k big enough, concluding $i(K^c) = \infty$, which is a contradiction by Proposition 3.2.3 since K^c is compact. □

Proposition 4.6.3. *If* $\mathcal{M} \in \mathcal{F}$ *is bounded and radially homeomorphic to the unit sphere,* $U = \{tx : x \in \mathcal{M}, \quad t \in [0,1]\}$*,* $A, B \in \mathcal{F}$ *with* $A \subset U^c$ *compact and* $B \subset \mathcal{M}$*,*

$$i(A) \geq k+m-1, \quad i(\mathcal{M} \setminus B) \leq k-1$$

for some $k, m \geq 1$*, and*

$$\sup f(A) \leq a < \inf f(B) \leq \sup f(IA) < b$$

and f *is even and satisfies* $(PS)_c$ *for all* $c \in (a,b)$ *then* $a < c_k^* \leq \cdots \leq c_{k+m-1}^* < b$ *and hence there are* m *distinct pairs of critical points in* $f^{-1}(a,b)$*.*

Proof. We will aplly Proposition 4.6.2. Since $1_W \in \Gamma$ we have

$$i(M \cap \mathcal{M}) \geq i^*(M) > i(\mathcal{M} \setminus B)$$

and hance by Proposition 3.2.2 (i_2) M intersects B and since $\inf f(B) > a$ we have $c_k^* > a$. If $\gamma \in \Gamma$ we have $\gamma|_A = 1_A$ since $f \leq a$ on A. Applying Corollary 3.2.1 with $\psi = \gamma|_{IA}$ gives

$$i(\gamma(IA) \cap \mathcal{M}) \geq i(A) \geq k+m-1.$$

So $i^*(IA) \geq k+m-1$ and hence $IA \in \mathcal{F}_{k+m-1}^*$. Since $\sup f(IA) < b$, $c_{k+m-1}^* < b$. □

CHAPTER 5

p-Eigenvalues

One of the main interests of this work is to study the equation

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + \mu f(x,u) + |u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.1}$$

And because of the term $\lambda |u|^{p-2}u$, such equation depends on the values of a chosen λ. One way to study the dependence of the problem with the values of λ is to choose it based upon a study of the spectrum of $-\Delta_p$, i.e. based on the study of the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.2}$$

Because of that, let us put some effort in the study of problems like (0.2). No difficulties are added if we study a general case where (0.2) is just a special case.

5.1 General p-Laplacian Eigenvalue Problems

Let us consider the problem

$$A_p u = \lambda B_p u \tag{1.3}$$

in the dual $(W^*, \|\cdot\|^*)$ of a real reflexive Banach space $(W, \|\cdot\|)$, $\lambda \in \mathbb{R}$ and $A_p, B_p \in C(W, W^*)$ satisfies

- (A_1) A_p is $(p-1)$-homogeneous and odd for $p \in (1, \infty)$,
- (A_2) uniformly A_p is positive: $\exists c_0 > 0$ such that

$$(A_p u, u) \geq c_0 \|u\|^p \quad \forall u \in W,$$

(A_3) A_p is a potential operator: $\exists Q \in C^1(W,\mathbb{R})$ called a potential for A_p such that

$$Q'(u) = A_p u \quad \forall u \in W,$$

(A_4) A_p is of type (S): every sequence $(u_j) \subset W$ such that

$$u_j \rightharpoonup u, \quad (A_p u_j, u_j - u) \to 0$$

has a subsequence that converges strongly to u.

(B_1) B_p is $(p-1)$-homogeneous and odd,
(B_2) strictly positive:

$$(B_p u, u) > 0 \quad \forall u \neq 0,$$

(B_3) B_p is a compact potential operator.

We say that λ is an eigenvalue of (1.3) if there exists a $u \neq 0$ in W satisfying (1.3), called an eigenvalye associated with λ. The set $\sigma(A_p, B_p)$ of all eigenvalues is called the spectrum of the pair of operators (A_p, B_p).

For example, in the problem (0.2) we have

$$(A_p u, v) = -\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx$$

and

$$(B_p u, v) = \int_\Omega |u|^{p-2} u v dx$$

Proposition 5.1.1. *Suppose $A_p : W \to W^*$ satisfies (A_1) and (A_3). If I_p is its potential satisfying $I_p(0) = 0$, then*

$$I_p(u) = \frac{1}{p}(A_p u, u) \quad \forall u \in W.$$

Proof.

$$\begin{aligned} I_p(u) &= \int_0^1 (I_p'(tu), u) dt = \int_0^1 (A_p tu, u) dt \\ &= \int_0^1 t^{p-1} (A_p u, u) dt = (A_p u, u) \int_0^1 t^{p-1} dt \\ &= \frac{1}{p}(A_p u, u). \end{aligned}$$

□

Now note that if the equation $A_p u = \lambda B_p u$ is true for some $u \in W \setminus \{0\}$, then $(A_p u, u) = \lambda(B_p u, u)$, and since B_p satisfies (B_2) we can divide both sides by $(B_p u, u)$, getting

$$\frac{(A_p u, u)}{(B_p u, u)} = \lambda. \tag{1.4}$$

This shows that every eigenvalue of (0.2) is given by (1.4). So we are led to study the function $f : W \to \mathbb{R}$ given by

$$f(u) = \frac{(A_p u, u)}{(B_p u, u)}. \tag{1.5}$$

We claim that the critical levels of (1.5) are the eigenvalues of (0.2). Indeed, by Proposition 5.1.1 a potential functions for A_p and B_p are $I_p(u) = \frac{1}{p}(A_p u, u)$ and $J_p(u) = \frac{1}{p}(B_p u, u)$ respectively, so

$$\begin{aligned} (f'(u), v) &= \frac{J_p(u)(A_p u, v) - I_p(u)(B_p u, v)}{J_p(u)^2} \\ &= \left(\frac{1}{J_p(u)} A_p u - \frac{I_p(u)}{J_p(u)^2} B_p u, v \right) \end{aligned}$$

therefore if u is a critical point of f then

$$\frac{1}{J_p(u)} A_p u - \frac{I_p(u)}{J_p(u)^2} B_p u = 0$$

hence

$$A_p u = \frac{I_p(u)}{J_p(u)} B_p u = f(u) B_p u.$$

The converse is also true; if λ is an eigenvalue of (0.2) then λ is a critical value for f. Indeed let u be a eigenfunction associated with λ. Then $I_p(u)/J_p(u) = \lambda$ hence

$$\frac{1}{J_p(u)} A_p u - \frac{I_p(u)}{J_p(u)^2} B_p u = 0$$

so u is a critical point for f by the calculations we did above.

The above discussion shows us that in order to understand the eigenvalues of (0.2) it is suffice to study the set of critical points of the function (1.5). This is what we are going to do.

For that, first note that f is 0-homogeneous, i.e. $f(tu) = f(u)$ for all $t > 0$. This implies that we can study the values of f in any subset of W that is radially homeomorphic to the sphere, because we will not lose any important information. In particular we can study the function f when restricted to

$$\mathcal{M} := \left\{ u \in W : I_p(u) = 1 \right\}.$$

Since $(I'(u),u) = (A_p u,u) = pI_p(u)$, zero is the only critical value of I_p and hence by the implicit function theorem $\mathcal{M}$ is a C^1-Finsler manifold. Moreover, $\mathcal{M}$ is complete, symmetric, and radially homeomorphic to the unit sphere since I_p is continuous, even, and p-homogeneous. In fact, let us denote by S the unit sphere on W, i.e. $S := \{u \in W : \|u\| = 1\}$, and let us take $u \in S$ and consider the path $\gamma_u : \mathbb{R} \to W$ given by

$$\gamma_u(t) = tu.$$

We want to find the time t when $\gamma_u(t)$ hits $\mathcal{M}$. This is the time when $(A_p tu, tu) = p$, and since A_p satisfies (A_1) we have

$$t = \left(\frac{p}{(A_p u, u)} \right)^{\frac{1}{p}} = (I_p(u))^{-1/p}.$$

This shows that if we define the map $Q : S \to \mathcal{M}$ as

$$Q(u) = \frac{u}{(I_p(u))^{1/p}}.$$

We also have the restriction of the projection π on $\mathcal{M}$, $\pi|_{\mathcal{M}} : \mathcal{M} \to S$ given by

$$\pi|_{\mathcal{M}}(u) = \frac{u}{\|u\|}.$$

We claim that Q is a homeomorphism with inverse $\pi|_{\mathcal{M}}$. In fact, for all $u \in \mathcal{M}$

$$\begin{aligned}
(Q \circ \pi|_{\mathcal{M}})(u) &= Q(\pi|_{\mathcal{M}}(u)) = \frac{\pi|_{\mathcal{M}}(u)}{(I_p(\pi|_{\mathcal{M}}(u)))^{1/p}} \\
&= \frac{u/\|u\|}{(I_p(u/\|u\|))^{1/p}} = \frac{u/\|u\|}{\frac{1}{\|u\|}(I_p(u))^{1/p}} \\
&= \frac{u}{(I_p(u))^{1/p}} \\
&= u.
\end{aligned}$$

And for all $u \in S$

$$(\pi|_{\mathcal{M}} \circ Q)(u) = (\pi|_{\mathcal{M}}(Q(u))) = \frac{\dfrac{u}{(I_p(u))^{1/p}}}{\dfrac{1}{(I_p(u))^{1/p}}} = u.$$

Therefore $Q : S \to \mathcal{M}$ is in fact an homeomorphism with $\pi|_{\mathcal{M}}$ as its inverse.

Proposition 5.1.2. *Let $\mathcal{M}$ as in the discussion above. Then there are positive constants c_0, C_0 such that*

$$\left(\frac{p}{C_0}\right)^{1/p} \leq \inf_{u \in \mathcal{M}} \|u\| \leq \sup_{u \in \mathcal{M}} \|u\| \leq \left(\frac{p}{c_0}\right)^{1/p} \tag{1.6}$$

Proof. First let us prove that there exists a constant $C_0 > 0$ such that

$$\|A_p u\| \leq C_0 \|u\|^{p-1}$$

If not, there is a sequence $(u_j) \subset W \setminus \{0\}$ such that

$$\|A_p u_j\| > j^p \|u_j\|^{p-1} \quad \forall j.$$

Let $v_j = u_j/(j\|u_j\|)$. Then $\|v_j\| = 1/j \to 0$, but

$$\|A_p(v_j)\| = \frac{\|A_p u_j\|}{j^{p-1}\|u_j\|^{p-1}} > j \to \infty,$$

contradicting the continuity of A_p. Then we have

$$\begin{aligned} I_p(u) = \frac{1}{p}(A_p u, u) &\leq \frac{1}{p}\|A_p u\| \|u\| \\ &\leq \frac{C_0}{p}\|u\|^p \end{aligned}$$

and by (A_2) there exists a constant $c_0 > 0$ such that

$$\frac{c_0}{p}\|u\|^p \leq I_p(u).$$

Putting all those together we get

$$\frac{c_0}{p}\|u\|^p \leq I_p(u) \leq \frac{C_0}{p}\|u\|^p$$

consequently, if $u \in \mathcal{M}$ then $I_p(u) = 1$ leading to

$$\frac{c_0}{p}\|u\|^p \leq 1 \implies \sup_{u\in\mathcal{M}} \|u\| \leq \left(\frac{p}{c_0}\right)^{1/p}$$

and similarly

$$1 \leq \frac{C_0}{p}\|u\|^p \implies \left(\frac{p}{C_0}\right)^{1/p} \leq \inf_{u\in\mathcal{M}} \|u\|.$$

$\square$

Now we will show that not only the values of f are totally described when it is restricted to $\mathcal{M}$, but also the critical values. More precisely, we will show that the critical values of $f|_{\mathcal{M}}$ are exactly the eigenvalues of (0.2). But for that, we first need to understand what is the derivative of $f|\mathcal{M}$.

Since $\mathcal{M} = \{u \in W : I_p(u) = 1\}$ we have that the tangent space of $\mathcal{M}$ on the point u, $T_u\mathcal{M}$ is defined as

$$T_u\mathcal{M} = \{v \in W : (I_p'(u), v) = 0\} = \ker I_p'(u).$$

The dual space of $T_u\mathcal{M}$ is the space $T_u^*\mathcal{M} = \mathcal{L}(T_u\mathcal{M}, \mathbb{R})$ with the norm

$$\|B\|_u^* := \sup_{v\in T_u\mathcal{M}, \|v\|=1} (B, v).$$

Lemma 5.1.1. *If $L, M \in W^*$, then*

$$\|L|_{\ker M}\|^* = \min_{\mu\in\mathbb{R}} \|L - \mu M\|^*$$

Proof. For each $\mu \in \mathbb{R}$,

$$\|L|_{\ker M}\|^* = \sup_{v\in\ker M, \|v\|=1} (L, v) \leq \sup_{\|v\|=1} (L - \mu M, v) = \|L - \mu M\|^*.$$

By the Hahn-Banach theorem, there is $G \in W^*$ such that $G = L$ on $\ker M$ and

$$\|G\|^* = \|L|_{\ker M}\|^*.$$

Since $\ker M \subset \ker(L - G)$, $L - G = \mu M$ for some $\mu \in \mathbb{R}$ (because both $\ker M$ and $\ker(L - G)$ have codimension ≤ 1) for some $\mu \in \mathbb{R}$, so

$$\|L|_{\ker M}\|^* = \|L - \mu M\|^*.$$

□

Proposition 5.1.3. *If* g *is a* C^1*-functional defined in a neighborhood of* $\mathcal{M}$ *and* $\overline{g}$ *is its restriction to* $\mathcal{M}$*, then the norm of* $\overline{g}'(u) \in T_u^*\mathcal{M}$ *is given by*

$$\|\overline{g}'(u)\|_u^* = \min_{\mu\in\mathbb{R}} \|g'(u) - \mu I_p'(u)\|^*$$

Proof. By the above lemma we have

$$\begin{aligned}
\|\overline{g}'(u)\|_u^* &= \sup_{v\in\ker I_p'(u), \|v\|=1} (g'(u), v) \\
&= \|g'(u)|_{\ker I_p'(u)}\|^* \\
&= \min_{\mu\in\mathbb{R}} \|g'(u) - \mu I_p'(u)\|^*.
\end{aligned}$$

□

Now let us go back to the study of the function f given in defined as (1.5). As we saw, it is suffice to study the values of f when it is restricted to $\mathcal{M}$, but we want to study the set of critical points of f. What happens it is also suffice to study the critical points of $f|_{\mathcal{M}}$. Indeed we have the following

Lemma 5.1.2. *Let us denote* $\overline{f}$ *be the restriction* $f|_{\mathcal{M}}$ *of the function* f *given by* (1.5) *on the manifold* $\mathcal{M}$*. Then the eigenvalues of* (0.2) *coincide with the critical values of* $\overline{f}$*, i.e.,*λ *is an eigenvalue if and only if there is a* $u \in \mathcal{M}$ *such that* $\overline{f}'(u) = 0$ *and* $\overline{f}(u) = \lambda$*.*

Proof. By the definition of $\mathcal{M}$ we have that

$$\overline{f}(u) = \frac{p}{(B_p u, u)} = \frac{1}{J_p(u)}, \quad \forall u \in \mathcal{M}.$$

Note that if we define $g : W\setminus\{0\} \to \mathbb{R}$ by

$$g(u) = \frac{1}{J_p(u)}, \quad \forall u \in W\setminus\{0\},$$

then g is a function defined in a neighborhood of $\mathcal{M}$ such that $g|_{\mathcal{M}} = \overline{f}$. By Lemma 5.1.1 and Proposition 5.1.3,

$$\|\overline{f}'(u)\|_u^* = \|g'(u)|_{T_u\mathcal{M}}\|^* = \min_{\mu\in\mathbb{R}} \|g'(u) - \mu I'(u)\|^*.$$

Note that

$$(g'(u), h) = -\frac{(J_p'(u), h)}{(J_p(u))^2} = -\frac{(B_p u, h)}{(J_p(u))^2} = (-g(u)^2 B_p u, h)$$

therefore,

$$g'(u) = -\overline{f}(u)^2 B_p u \quad \forall u \in \mathcal{M}.$$

Since $I_p'(u) = A_p u$, we have then

$$\|\overline{f}\|_u^* = \min_{\mu\in\mathbb{R}} \|\overline{f}(u)^2 B_p u + \mu A_p u\|^* \quad \forall u \in \mathcal{M}. \tag{1.7}$$

Now we have $\overline{f}'(u) = 0$ with $u \in \mathcal{M}$, if and only if there exists some $\mu \in \mathbb{R}$ such that

$$\overline{f}(u)^2 B_p u + \mu A_p u = 0. \tag{1.8}$$

Suppose that $\overline{f}'(u) = 0$, then there exists $\mu \in \mathbb{R}$ that satisfies the above inequality, so applying u in both sides of (1.8) we have

$$\begin{aligned} &\mu(A_p u, u) + \overline{f}(u)^2 (B_p u, u) = 0 \\ \Longrightarrow \quad &\mu = -\overline{f}(u)^2 \frac{(B_p u, u)}{(A_p u, u)} = -\overline{f}(u)^2 \frac{1/\overline{f}(u)}{I_p(u)} = -\overline{f}(u) \end{aligned}$$

and substituting it in (1.8) we get

$$A_p u = \lambda B_p u \quad \text{with} \quad \lambda = \overline{f}(u).$$

Conversely, suppose that λ is an eigenvalue of (0.2), and let $u \in \mathcal{M}$ be an eigenfunction associated with λ. Then by (0.2) we have

$$\lambda = \frac{I_p(u)}{J_p(u)} = \frac{1}{J_p(u)} = \overline{f}(u)$$

so

$$\begin{aligned} &A_p u - \overline{f}(u) B_p = 0 \\ \Longrightarrow \quad &-\overline{f}(u) A_p u + \overline{f}(u)^2 B_p u = 0 \\ \Longrightarrow \quad &\mu A_p + \overline{f}(u)^2 B_p u = 0 \quad \text{with} \quad \mu = -\overline{f}(u) \end{aligned}$$

hence by (1.8) u is a critical value of $\overline{f}$ and $\overline{f}(u) = \lambda$.

□

Example 5.1.1. *Let us suppose that Ω is a bounded smooth open subset of $\mathbb{R}^N$, and $1 < p < \infty$ and* *let us consider the eigenvalue problem*

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.9}$$

Then we can apply all what we did to this particular case. And using the same notations we have, in this case

$$\mathcal{M} = \left\{ u \in W_0^{1,p}(\Omega) : \frac{1}{p}\int_\Omega |\nabla u|^p = 1 \right\},$$

$$J_p(u) = \frac{1}{p}\int_\Omega |u|^p,$$

$$\overline{f}(u) = \frac{p}{\int_\Omega |u|^p}.$$

Example 5.1.2. *In the same notations as in the last example, let*

$$\mathcal{V}(\Omega) := \begin{cases} L^{n/p}(\Omega) & \text{if} \quad 1 < p < n \\ \cup_{r>1} L^r(\Omega) & \text{if} \quad p = n \\ L^1(\Omega) & \text{if} \quad p > n. \end{cases}$$

and let $V \in \mathcal{V}(\Omega)$ *with* $V(x) > 0$ *almost everywhere, and consider the eigenvalue problem*

$$\begin{cases} -\Delta_p u = \lambda V(x) |u|^{p-2}u & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

In this case we have

$$\mathcal{M} = \left\{ u \in W_0^{1,p}(\Omega) : \frac{1}{p}\int_\Omega |\nabla u|^p \right\},$$

$$J_p(u) = \frac{1}{p}\int_\Omega V(x)|u|^p,$$

$$\overline{f}(u) = \frac{p}{\int_\Omega V(x)|u|^p}.$$

Lemma 5.1.3. *Let* $(L_j) \subset W^*$ *converges to* L *and* $(u_j) \subset W$ *converges weakly to* u, *then* $(L_j, u_j) \to$

(L,u), *in particular,* $(L_j, u_j - u) \to 0$.

Proof.

$$|(L_j,u_j)-(L,u)| \le \|L_j - L\|^* \|u_j\| + |(L,u_j)-(L,u)| \to 0$$

since $\|u_j\|$ is bounded. □

Lemma 5.1.4. *The function* $\overline{f} : \mathcal{M} \to \mathbb{R}$ *given by*

$$\overline{f}(u) = \frac{p}{(B_p u, u)} \quad \forall u \in \mathcal{M},$$

satisfies (PS).

Proof. ([21])

Let $(u_j) \subset \mathcal{M}$ such that

$$\overline{f}(u_j) \to c, \quad \|\overline{f}'(u)\|^*_{u_j} \to 0.$$

By Proposition 5.1.2, (u_j) is a bounded sequence, and since W is reflexive, we can assume, passing to a subsequence if necessary, that $u_j \rightharpoonup u$ for some $u \in W$. Since B_p satisfies (B_3), $B_p u_j$ converges strongly to some $L \in W^*$. By Lemma 5.1.3 we have

$$\overline{f}(u_j) = \frac{p}{(B_p u_j, u_j)} \to \frac{p}{(L,u)} \ne 0$$

so $c \ne 0$. By (1.7) there exists a sequence of real numbers μ_j such that

$$\mu_j A_p u_j + \overline{f}(u_j)^2 B_p u_j \to 0. \tag{1.11}$$

Applying (1.11) to u_j gives

$$\mu_j + \overline{f}(u_j) \to 0$$

so

$$\mu_j \to -c \ne 0.$$

Applying (1.11) to $u_j - u$ gives

$$(A_p u_j, u_j - u) \to 0$$

therefore by (A_4) $u_j \to u \in \mathcal{M}$ strongly. □

Example 5.1.3. *In the same configuration as in Example 5.1.1 we have that the map* $\overline{f} : \mathcal{M} \to \mathbb{R}$ *given by*

$$u \mapsto \frac{p}{\int_\Omega |u|^p}$$

satisfies (PS) *for all* $c \in \mathbb{R}$. *Such function appears quite frequently in applications.*

5.2 An unbounded sequence of eigenvalues

We can construct various types of unbounded sequence of eigenvalues of the problem (0.2). However, in order to search of useful linkigs it would be better to define an unbounded squence by means of $\mathbb{Z}_2$-cohomological index.

Let us denote

$$\overline{K}^c = \left\{ u \in \mathcal{M} : \overline{f}'(u) = 0, \overline{f}(u) = c \right\}.$$

By Lemma (5.1.4), $\overline{K}^c$ is a compact subset of $\mathcal{M}$.

As usual, let us denote $\mathcal{F}$ be the class of symmetric subsets of $\mathcal{M}$. Then, for each $k \in \mathbb{N}$ we can define

$$\mathcal{F}_k = \{A \in \mathcal{F} : i(M) \geq k\}$$

and

$$\lambda_k = \inf_{A \in \mathcal{F}_k} \sup_{u \in A} \overline{f}(u).$$

Of course we have that $\lambda_{k+1} \geq \lambda_k$ since $\mathcal{F}_{k+1} \subset \mathcal{F}_k$. We claim that $\lambda_k < \infty$ for each $k \in \mathbb{N}$ with $k \leq \dim W$. Indeed, if $\dim W = d \in \mathbb{R} \cup \{+\infty\}$ then for each $m \leq d$ let V_m be a subspace of W with $\dim V_m = m$. Then $i(V_m \cap \mathcal{M}) = m$ by Proposition 3.2.2 (i_8) and of course $-\infty < \sup \overline{f}(V_m \cap \mathcal{M}) < +\infty$.

Theorem 5.2.1. *Under assumptions* $(A_1) - (A_4)$ *and* $(B_1) - (B_3)$ *we have that the sequence* (λ_k) *satisfies*

(i) If $\lambda_k = \cdots = \lambda_{k+m-1} = \lambda$, *then* $i(\overline{K}^c) \geq m$. *In particular, every* λ_k *is a critical value for* $\overline{f}$.

(ii) The smallest eigenvalue, called the first eigenvalue, is

$$\lambda_1 = \min_{u \in \mathcal{M}} \overline{f}(u) = \min_{u \neq 0} \frac{I_p(u)}{J_p(u)} > 0$$

(iii) We have

$$i(\mathcal{M} \backslash \overline{f}_{\lambda_k}) < k \leq i(\overline{f}^{\lambda_k}).$$

And if $\lambda_k < \lambda < \lambda_{k+1}$*, then*

$$i(\overline{f}^{\lambda_k}) = i(\mathcal{M}\backslash \overline{f}_{\lambda}) = i(\overline{f}^{\lambda}) = i(\mathcal{M}\backslash \overline{f}_{\lambda_{k+1}}) = k.$$

(iv) If $\dim \mathcal{M} = \infty$*, then* $\lambda_k \nearrow +\infty$.

Proof. (i) and (iv) follows from Proposition 4.5.1.

(ii) follows from homogeneity and the fact that pairs of points are in $\mathcal{F}_1$ by Proposition 3.2.3.

(iii) Follows from Proposition 4.5.2. □

Let us now remember the notions of genus and cogenus of symmetric subsets. Two integer valued functions $\gamma^+, \gamma^- : \mathcal{F} \to \mathbb{N} \cup \{+\infty\}$ are defined as

$$\gamma^+(M) = \sup\left\{n \in \mathbb{N} : \exists\, \phi : S^{n-1} \to M \;\text{ odd and continuous}\right\}$$

and

$$\gamma^-(M) = \inf\left\{n \in \mathbb{N} : \exists\, \phi : M \to S^{n-1} \;\text{ odd and continuous}\right\}$$

and if no such ϕ exists we simply define $\gamma^{\pm}(M) = +\infty$. $\gamma^+(M)$ is called the genus of M and $\gamma^-(M)$ is called the cogenus of A.

Let us define

$$\mathcal{G}_k^+ := \left\{M \in \mathcal{F} : \gamma^+(M) \geq k\right\}$$

and

$$\mathcal{G}_k^- := \left\{M \in \mathcal{F} : \gamma^-(M) \geq k\right\}.$$

We can also define two more sequences of eigenvalues for (0.2), namely

$$\lambda_k^{\pm} = \inf_{M \in \mathcal{G}_k^{\pm}} \sup_{u \in M} \overline{f}(u). \tag{2.12}$$

The numbers $\lambda_k^{\pm}$ are the most standard ones treated in the literature when it comes to the study of critical points of a functional that possesses some kind of symmetry.

Proposition 5.2.1. *With the same notations as above*

(i) $\lambda_1^{\pm} = \lambda_1$,

(ii) $\lambda_k^- \le \lambda_k \le \lambda_k^+$.

Proof. (i) It suffices to note that antipodal points $\{u,-u\} \subset \mathcal{M}$ are such that

$$\gamma^{\pm}(\{u,-u\}) = 1,$$

Therefore

$$\lambda_1^{\pm} = \inf_{u \in \mathcal{M}} \overline{f}(u) = \lambda_1.$$

(ii) If $\phi_1 : S^{k-1} \to M$ and $\phi_2 : M \to S^{s-1}$ are odd continuous maps, then by Proposition 3.2.2 (i_2) we have

$$k \le i(M) \le s,$$

therefore

$$\gamma^+(M) \le i(M) \le \gamma^-(M). \tag{2.13}$$

Consequently

$$\mathcal{G}_k^+ \subset \mathcal{F}_k \subset \mathcal{G}_k^-,$$

which leads to the desired inequalities.

□

CHAPTER 6

An abstract method for a class of multiple solution problems.

We now have the necessary box of tools to deal with all cases of problem (3.8). In this section, we will develop an abstract method that can be applied to (3.8) as a particular case. Such method is due to (K. Perera, [23]).

6.1 Abstract configuration

As usual, let $(W,\|\cdot\|)$ be a uniformly convex Banach space with dual $(W^*,\|\cdot\|^*)$ and duality pairing $(\cdot,\cdot)$. We now consider the nonlinear operator equation

$$A_p u = \lambda B_p u + \mu f(u) + g(u) \tag{1.1}$$

in W^*, where $A_p, B_p, f, g \in C(W,W^*)$ are potential operators, and $\mu, \lambda > 0$ are parameters satisfying

(a_1) A_p is $(p-1)$-homogeneous and odd for some $p \in (1,+\infty)$: $A_p(tu) = |t|^{p-2}tA_p u$ for all $u \in W$ and $t \in \mathbb{R}$,

(a_2) $(A_p u, v) \leq \|u\|^{p-1}\|v\|$ for all $u, v \in W$, and equality holds if and only if $\alpha u = \beta v$ for some $\alpha, \beta \geq 0$, not both zero,

(b_1) B_p is $(p-1)$-homogeneous and odd: $B_p(tu) = |t|^{p-2}tB_p u$ for all $u \in W$ and $t \in \mathbb{R}$,

(b_2) $(B_p u, u) > 0$ for all $u \in W \setminus \{0\}$, and

$$(B_p u, v) \leq (B_p u, u)^{(p-1)/p}(B_p v, v)^{1/p} \quad \text{for all} \quad u, v \in W,$$

(b_3) B_p is a compact operator,

(f_1) the potential F of f with $F(0) = 0$ satisfies

$$\lim_{t\to 0} \frac{F(tu)}{|t|^p} = +\infty$$

uniformly on compact subsets of $W \setminus \{0\}$, (f$_2$) $F(u) > 0$ for all $u \in W \setminus \{0\}$,

(f$_3$) F is bounded on bounded subsets of W,

(g$_1$) the potential G of g with $G(0) = 0$ satisfies

$$G(u) = o(\|u\|^p) \quad \text{as} \quad u \to 0,$$

(g$_2$) $G(u) > 0$ for all $u \in W \setminus \{0\}$,

(g$_3$) G is bounded on bounded subsets of W,

(g$_4$)

$$\lim_{t \to +\infty} \frac{G(tu)}{t^p} = +\infty$$

uniformly on compact subsets of $W \setminus \{0\}$.

.

If I_p and J_p are the potentials of A_p and B_p respectively, with $I_p(0) = J_p(0) = 0$, then by Proposition 5.1.1 we have that

$$I_p(u) = \frac{1}{p}(A_p u, u) = \frac{1}{p}\|u\|^p, \quad J_p(u) = \frac{1}{p}(B_p u, u).$$

It is then clear that solutions of equation (1.1) coincinde with critical points of the C^1-functional

$$\mathcal{E}(u) = I_p(u) - \lambda J_p(u) - \mu F(u) - G(u), \quad u \in W. \tag{1.2}$$

The main idea of the method we will employ to solve (1.1) is to find a linking between two subsets of W that goes together with the behaviour of $\mathcal{E}$. More precisely, we will look for a kind of linking that detects changes in topology of the sublevels of $\mathcal{E}$. For that, we will employ the notion of cohomological linking given in Definition 4.4.2, and use the linking found in Proposition (4.4.3) to construct a more interesting one.

First let us define $\mathcal{M} = \left\{u \in W : I_p(u) = 1\right\}$. By the discussion after Proposition 5.1.1 we have that $\mathcal{M} \subset W \setminus \{0\}$ is a bounded complete symmetric C^1-Finsler manifold radially homeomorphic to the unit sphere S in W.

As we have set before Proposition 4.4.3, if $\pi : W \setminus \{0\} \to S$ is given by $u \mapsto u/\|u\|$, then we can define the radial projection of $W \setminus \{0\}$ onto $\mathcal{M}$ by

$$\pi_{\mathcal{M}} = (\pi|_{\mathcal{M}})^{-1} \circ \pi.$$

Using the linking given in Proposition 4.4.3 we can construct another useful linking in the following way (see [23]).

Theorem 6.1.1. *Let A_0 and B_0 be disjoint closed symmetric subsets of $\mathcal{M}$ such that*

$$i(A_0) = i(\mathcal{M}\backslash B_0) = k < +\infty.$$

Let $w_0 \in \mathcal{M}\backslash A_0$ and $0 \leq r < \rho < R$. Set

$$A_1 = \{\pi_{\mathcal{M}}((1-s)v + sw_0) : v \in A_0,\ 0 \leq s \leq 1\}$$

and

$$A = \{ru : u \in A_1\} \cup \{tv : v \in A_0,\ r \leq t \leq R\} \cup \{Ru : u \in A_1\}, \quad B = \{\rho w : w \in B_0\}.$$

Then A links B cohomologically in dimension k.

Proof. First we claim that A_1 is contractible. For, define the map $H_1 : A_1 \times [0,1] \to A_1$ given by

$$H_1(u,t) = \pi_{\mathcal{M}}((1-t)u + tw_0)$$

which is a contraction of A_1 to w_0. Now let us define the sets

$$A_2 = \{ru : u \in A_1\} \cup \{tv : v \in A_0, t \leq t \leq R\}, \quad A_3 = \{Rv : v \in A_0\},$$

and

$$B_1 = \{tw : w \in B_0,\ t \geq 0\}, \quad B_2 = \{tw : w \in B_0,\ t \geq \rho\}.$$

We claim that A_2 is also contractible. Indeed, let us define the map $H_2 : A_2 \times [0,1] \to A_2$ given by

$$H_2(u,t) = (1-t)u + tr\pi_{\mathcal{M}}(u)$$

which is a strong deformation retract of A_2 onto $\{ru : u \in A_1\}$, which is homeomorphic to A_1 and hence contractible.

Now, by considering the long exact sequences induced by the inclusions

$$W\backslash B_1 \hookrightarrow W\backslash B_2 \hookrightarrow (W\backslash B_2, W\backslash B_1)$$

and

$$A_3 \hookrightarrow A_2 \hookrightarrow (A_2, A_3)$$

we have the commutative diagram

$$\begin{tikzcd}
\widetilde{H}^{k-1}(W\backslash B_2) \arrow[r] \arrow[d] & \widetilde{H}^{k-1}(W\backslash B_1) \arrow[r] \arrow[d, "i_1^*"] & H^k(W\backslash B_2, W\backslash B_1) \arrow[r] \arrow[d, "i_2^*"] & \widetilde{H}^k(W\backslash B_2) \arrow[d] \\
\widetilde{H}^{k-1}(A_2) \arrow[r] & \widetilde{H}^{k-1}(A_3) \arrow[r, "\delta^*"] & H^k(A_2, A_3) \arrow[r] & \widetilde{H}^k(A_2)
\end{tikzcd}$$

where $i_1 : A_3 \hookrightarrow W\backslash B_1$ and $i_2 : (A_2, A_3) \hookrightarrow (W\backslash B_2, W\backslash B_1)$ are the inclusions. By Proposition 4.4.3 A_3 links B_1 cohomologically in dimension $k-1$ so i_1^* is nontrivial. Since A_2 is contractible $\widetilde{H}^*(A_2) = 0$ and hence δ^* is an isomorphism. By commutativity we have that $i_2^* \neq 0$.

Now let us define

$$A_4 = \{Ru : u \in A_1\}$$

and

$$B^* = \{tw : w \in B_0,\ 0 \leq t \leq \rho\},$$

and consider the commutative diagram

$$\begin{tikzcd}
H^k(W\backslash B, W\backslash B^*) \arrow[r, "\approx"] \arrow[d, "i_3^*"] & H^k(W\backslash B_2, W\backslash B_1) \arrow[d, "i_2^*"] \\
H^k(A, A_4) \arrow[r] & H^k(A_2, A_3)
\end{tikzcd}$$

induced by inclusions where $i_3 : (A, A_4) \hookrightarrow (W\backslash B, W\backslash B^*)$. The top arrow is an isomorphism by the excision property since $\{tw : w \in B_0,\ t > \rho\}$ is a closed subset of $W\backslash B$ contained in the open subset $W\backslash B^*$. Since $i_2^* \neq 0$, it follows by commutativity that $i_3^* \neq 0$.

In the same way we did before, we now consider the following commutative diagram

$$\begin{tikzcd}
\widetilde{H}^{k-1}(W\backslash B^*) \arrow[r] \arrow[d] & H^k(W\backslash B, W\backslash B^*) \arrow[r] \arrow[d, "i_3^*"] & \widetilde{H}^k(W\backslash B) \arrow[r] \arrow[d, "i^*"] & \widetilde{H}^k(W\backslash B^*) \arrow[d] \\
\widetilde{H}^{k-1}(A_4) \arrow[r] & H^k(A, A_4) \arrow[r, "j^*"] & \widetilde{H}^k(A) \arrow[r] & \widetilde{H}^k(A_4)
\end{tikzcd}$$

Note that A_4 is contractible since it is homeomorphic to A_1, hence $\widetilde{H}^*(A_4) = 0$. So j^* is an isomorphism by exactness. Since $i_3^* \neq 0$, by commutativity we have $i^* \neq 0$. □

Now that we have found a more interesting linking setting, we need to show that such linking detects changes in the topology of the sublevels of $\mathcal{E}$. For that porpuse, we need first show that under certain conditions, Theorem 6.1.1 is useful when searching for critical points of a functional. More precisely we have (see [23])

Theorem 6.1.2. *Let $\mathcal{E}$ be a C^1 functional on an infinite dimensional Banach space W and let A_0 and B_0 be disjoint closed symmetric subsets of $\mathcal{M}$ such that*

$$i(A_0) = i(\mathcal{M}\backslash B_0) = k < +\infty$$

Assume that there exist $w_0 \in \mathcal{M}\backslash A_0$, $0 \le r < \rho < R$, and $a < b$ such that, setting

$$\begin{aligned} A_1 &= \{\pi_{\mathcal{M}}((1-s)v + sw_0) : v \in A_0,\ 0 \le s \le 1\} \\ A^* &= \{tu : u \in A_1,\ r \le t \le R\} \\ B^* &= \{tw : w \in B_0,\ 0 \le t \le \rho\} \\ A &= \{ru : u \in A_1\} \cup \{tv : v \in A_0,\ r \le t \le R\} \cup \{Ru : u \in A_1\} \\ B &= \{\rho w : w \in B_0\}, \end{aligned}$$

we have

$$a < \inf_{B^*} \mathcal{E}, \quad \sup_{A} \mathcal{E} < \inf_{B} \mathcal{E}, \quad \sup_{A^*} \mathcal{E} < b,$$

If $\mathcal{E}$ satisfies $(PS)_c$ for all $c \in (a,b)$, then E has a pair of critical points u_1, u_2 with

$$\inf_{B^*} \mathcal{E} \le \mathcal{E}(u_1) \le \sup_{A} \mathcal{E}, \quad \inf_{B} \mathcal{E} \le \mathcal{E}(u_2) \le \sup_{A^*} \mathcal{E}.$$

If, in addition, $\mathcal{E}$ has only a finite number of critical points with the corresponding critical values in (a,b), then u_1 and u_2 can be chosen to satisfy

$$C^k(\mathcal{E}, u_1) \ne 0, \quad C^{k+1}(\mathcal{E}, u_2) \ne 0.$$

Proof. Since $B^* \cap A$ and $B \cap A^*$ are nonempty, $\inf \mathcal{E}(B^*) \le \sup \mathcal{E}(A)$ and $\inf \mathcal{E}(B) \le \sup \mathcal{E}(A^*)$. It will suffice, by Proposition 4.4.1 to show that for any numbers $\alpha < \beta < \gamma$ such that

$$a < \alpha < \inf_{B^*} \mathcal{E}, \quad \sup_{A} \mathcal{E} < \beta < \inf_{B} \mathcal{E}, \quad \sup_{A} \mathcal{E} < \gamma < b,$$

then

$$H^k(\mathcal{E}^\beta, \mathcal{E}^\alpha) \ne 0, \quad H^{k+1}(\mathcal{E}^\gamma, \mathcal{E}^\beta) \ne 0.$$

If that is true, then by Proposition 4.4.1 $\mathcal{E}$ has a pair of critical points u_1, u_2 with

$$\alpha \leq \mathcal{E}(u_1) \leq \beta, \quad \beta \leq \mathcal{E}(u_2) \leq \gamma$$

If $\inf \mathcal{E}(B^*)$ or $\sup \mathcal{E}(A)$ is a critical value of $\mathcal{E}$, we can take u_1 to be one of those levels. On the other hand, if neither of such numbers are critical levels, then since $\mathcal{E}$ satisfies $(PS)_c$ for all $c \in (a,b)$, then we can choose α and β in such a way that $\mathcal{E}$ has no critical values in $[\alpha, \inf \mathcal{E}(B^*)] \cup [\sup \mathcal{E}(A), \beta]$, therefore we would have

$$\inf_{B^*} \mathcal{E} < \mathcal{E}(u_1) < \sup_{A} \mathcal{E}$$

By a similar argument we can always guarantee that

$$\inf_{B} \mathcal{E} \leq \mathcal{E}(u_2) \leq \sup_{A^*} \mathcal{E}.$$

If E has only a finite number of critical points in (a,b), by Proposition 4.3.2 we can choose u_1 and u_2 such that

$$C^k(\mathcal{E}, u_1) \neq 0, \quad C^{k+1}(\mathcal{E}, u_2) \neq 0.$$

By Theorem 6.1.1 we have that the inclusion $i : A \hookrightarrow W \backslash B$ induces a nontrivial homomorphism in dimension k $i^* \widetilde{H}^k(W \backslash B) \to \widetilde{H}^k(A)$. Since $\mathcal{E} < \beta$ on A and $\mathcal{E} > \beta$ on B, we also have the inclusions $i_1 : A \hookrightarrow \mathcal{E}^\beta$ and $i_2 : \mathcal{E}^\beta \to W \backslash B$, leading to the commutative diagram

$$\begin{tikzcd}
\widetilde{H}^k(\mathcal{E}^\beta) \arrow[d, "i_1^*"] & \widetilde{H}^k(W \backslash B) \arrow[l, "i_2^*"'] \arrow[dl, "i^*"] \\
\widetilde{H}^k(A) &
\end{tikzcd}$$

Which implies that

$$i_1^* i_2^* = i^* \neq 0,$$

consequently both i_1^* and i_2^* are nontrivial homomorphisms.

Let us now show that $H^k(\mathcal{E}^\beta, \mathcal{E}^\alpha) \neq 0$. First note that since $\mathcal{E} > \alpha$ on B^* and $\alpha < \beta$, we have the inclusiongs

$$\mathcal{E}^\alpha \hookrightarrow W \backslash B^* \hookrightarrow W \backslash B, \quad \mathcal{E}^\alpha \hookrightarrow \mathcal{E}^\beta \hookrightarrow W \backslash B,$$

leading to the commutative diagram

$$\begin{tikzcd}
\widetilde{H}^k(W\backslash B) \arrow[r] \arrow[d, "i_2^*"] & \widetilde{H}^k(W\backslash B^*) \arrow[d] \\
\widetilde{H}^k(\mathcal{E}^\beta) \arrow[r, "j^*"] & \widetilde{H}^k(\mathcal{E}^\alpha)
\end{tikzcd}$$

where j is the inclusion $\mathcal{E}^\alpha \hookrightarrow \mathcal{E}^\beta$.

We claim that $W\backslash B^*$ is contractible. Indeed, let us take $\rho' > \rho$ and define the mapping $H_1 : (W\backslash B^*) \times [0,1] \to W\backslash B^*$ given by

$$H_1(u,t) = (1-t)u + t\rho'\pi_{\mathcal{M}}(u)$$

which is a strong deformation retract of $W\backslash B^*$ onto $\left\{\rho' u : u \in \mathcal{M}\right\}$, which is homeomorphic to the sphere S which is contractible since W is infinte dimensional. Then we have that $\widetilde{H}^*(W\backslash B^*) = 0$ so

$$i^* i_2^* = 0.$$

But as we have shown, $i_2^* \neq 0$, which means that i^* is not injective. Now consider the following part of the long exact sequence induced by the inclusions $\mathcal{E}^\alpha \hookrightarrow \mathcal{E}^\beta \hookrightarrow (\mathcal{E}^\beta, \mathcal{E}^\alpha)$

$$\cdots \xrightarrow{\delta^*} H^k(\mathcal{E}^\beta, \mathcal{E}^\alpha) \xrightarrow{j_1^*} \widetilde{H}^k(\mathcal{E}^\beta) \xrightarrow{i^*} \widetilde{H}^k(\mathcal{E}^\alpha) \xrightarrow{\delta^*} \cdots$$

Since i^* is not injective, by exactness we have

$$\operatorname{Im} j_1^* = \ker i^* \neq 0,$$

consequently

$$H^k(\mathcal{E}^\beta, \mathcal{E}^\alpha) \neq 0.$$

Now let us prove that $H^{k+1}(\mathcal{E}^\gamma, \mathcal{E}^\beta) \neq 0$. For that, first note that since $\mathcal{E} < \gamma$ on A^* and $\beta < \gamma$, we have the inclusions $A \hookrightarrow A^* \hookrightarrow \mathcal{E}^\gamma$ and $A \hookrightarrow \mathcal{E}^\beta \hookrightarrow E^\gamma$, which leads to the following commutative diagram

$$\begin{tikzcd}
\widetilde{H}^k(\mathcal{E}^\gamma) \arrow[r] \arrow[d, "i^*"] & \widetilde{H}^k(A^*) \arrow[d] \\
\widetilde{H}^k(\mathcal{E}^\beta) \arrow[r, "i_1^*"] & \widetilde{H}^k(A).
\end{tikzcd}$$

Note that A^* is contractible since we can define the strong deformation map $H_2 : A^* \times [0,1] \to A^*$

given by

$$H_2(u,t) = (1-t)u + tr\pi_{\mathcal{M}}(u)$$

that is a strong deformation of A^* onto $\{ru : u \in A_1\}$, which is contractible. Therfore $\widetilde{H}^*(A^*) = 0$ and therefore

$$i_1^* i^* = 0.$$

Since i_1^* is nontrivial, this implies that i^* is not surjective. Now consider the long exact sequence induced by the inclusions $\mathcal{E}^\beta \hookrightarrow \mathcal{E}^\gamma \hookrightarrow (\mathcal{E}^\gamma, \mathcal{E}^\beta)$:

$$\cdots \xrightarrow{j^*} \widetilde{H}^k(\mathcal{E}^\gamma) \xrightarrow{i^*} \widetilde{H}^k(\mathcal{E}^\beta) \xrightarrow{\delta^*} \widetilde{H}^{k+1}(\mathcal{E}^\gamma, \mathcal{E}^\beta) \longrightarrow \cdots.$$

Now, since i^* is not surjective, we have that, by exactness

$$\ker \delta^* = \mathrm{Im}\, i^* \neq \widetilde{H}^k(\mathcal{E}^\beta)$$

so $\delta^* \neq 0$ and consequently

$$H^{k+1}(\mathcal{E}^\gamma, \mathcal{E}^\beta) \neq 0.$$

□

Now, in order to show that the topological obstruction found in Theorem 6.1.2 also happens with the energy functional (1.2) we will need find suitable sets A_0 and B_0 which satisfies the hypothesis of Theorem 6.1.2.

For that, let us first prove the following.

Proposition 6.1.1. *If* (a_2) *holds, then*

(1) A_p *is strictly monotone:* $(A_pu - A_pv, u-v) > 0$ *for all* $u \neq v$ *in* W*,*

(2) A_p *satisfies* (S)*: every sequence* $(u_j) \subset W$ *such that* $u_j \rightharpoonup u$ *and* $(A_pu_j, u_j - u) \to 0$ *has a sequence that converges strongly to* u*.*

Proof. (1) Since (a_2) holds, we have

$$\begin{aligned} (A_pu - A_pv, u-v) &= (A_pu,u) - (A_pu,v) - (A_pv,u) + (A_pv,v) \\ &\geq \|u\|^p - \|u\|^{p-1}\|v\| + \|v\|^{p-1}\|u\| + \|v\|^p \\ &= (\|u\|^{p-1} - \|v\|^{p-1})(\|u\| - \|v\|) \geq 0 \end{aligned}$$

for all $u,v \in W$. If $(A_pu - A_pv, u-v) = 0$, then equality holds throughout and hence $(A_pu,v) = \|u\|^{p-1}\|v\|$ and $\|u\| = \|v\|$. Since (a_2) holds, the first equality implies that $\alpha u = \beta v$ for some $\alpha, \beta \geq 0$

not both zero. The second equality then implies that either $u = v = 0$, or $\alpha = \beta > 0$. In the latter case, $u = v$.

(2) Similarly we have

$$(A_p u_j - A_p u, u_j - u) \geq (\|u_j\|^{p-1} - \|u\|^{p-1})(\|u_j\| - \|u\|) \geq 0.$$

Since $(A_p u_j, u_j - u) \to 0$ and $u_j \rightharpoonup u$,

$$(A_p u_j - A_p u, u_j - u) \to 0$$

therefore $\|u_j\| \to \|u\|$. Then $u_j \to u$ strongly since W is uniformly convex.

$\square$

Now we are able to find useful sets A_0 and B_0 in order to apply Theorem 6.1.2 to our functional (1.2). We will find such sets by means of the study of the eigenvalues of the problem

$$A_p u = \lambda B_p u. \tag{1.3}$$

As we did in chapter 4 let us remember that eigenvalues of (1.3) coincide with the critical values of the C^1-functional $\Psi : \mathcal{M} \to \mathbb{R}$ defined as

$$\Psi(u) = \frac{1}{J_p(u)} \quad u \in \mathcal{M}.$$

Also, let us denote $\mathcal{F}$ the class of symmetric subsets of $\mathcal{M}$ and by $i(A)$ the cohomological index of $A \in \mathcal{F}$, and define

$$\mathcal{F} = \{A \in \mathcal{F} : i(A) \geq k\},$$

and set

$$\lambda_k = \inf_{A \in \mathcal{F}_k} \sup_{u \in A} \Psi(u), \quad k \in \mathbb{N}.$$

By Theorem 5.2.1 we have that

$$\lambda_1 = \inf_{u \in \mathcal{M}} \Psi(u) > 0$$

is the first eigenvalue and

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots$$

is an unbounded sequence of eigenvalues. Moreover, by Proposition 4.5.2, if $\lambda_k < \lambda_{k+1}$ then

$$i(\Psi^{\lambda_k}) = i(\mathcal{M}\setminus\Psi_{\lambda_{k+1}}) = k.$$

Now we are able to show the existence of a symmetric set A_0 that is suitable to our functional $\mathcal{E}$ in view of Theorem 6.1.2.

Theorem 6.1.3. *Assume that* $(a_1)-(b_3)$ *hold. If* $\lambda_k < \lambda_{k+1}$*, then the sublevel set* Ψ^{λ_k} *has a compact symmetric subset of index* k*.*

But first we will need the following lemma.

Lemma 6.1.1. *For each* $w \in W$*, the equation*

$$A_p u = B_p w \tag{1.4}$$

has a unique solution. Moreover, denoting

$$K : W \to W, \quad w \mapsto u$$

the solution map, then $K(tw) = tK(w)$ *for all* $t \in \mathbb{R}$ *and* K *is a compact operator.*

Proof. A solution of (1.4) can be obtained by minimizing the C^1-functional

$$\mathcal{Y}(u) = \frac{1}{p}(A_p u, u) - (B_p w, u).$$

Indeed, note that $\mathcal{Y}$ is coercive, since

$$\mathcal{Y}(u) \geq \frac{1}{p}\|u\|^p - \|B_p w\|^*\|u\|$$

and $p > 1$. So let u_k be a sequence in W such that

$$\mathcal{Y}(u_n) \to \inf_{u\in W} \mathcal{Y}(u),$$

so (u_k) is a bounded sequence and therefore converges weakly to some $u \in W$. Consequently

$$\frac{1}{p}\|u\| - (B_p w, u) \leq \liminf_{k\to\infty}\left(\frac{1}{p}\|u_k\| - (B_p w, u_k)\right) = \inf_{u\in W} \mathcal{Y}(u),$$

which implies that u is a solution (1.4). Suppose that u and v are two solutions of (1.4), then we have that

$$A_p u = B_p w = A_p v,$$

implying that $(A_p u - A_p v, u-v) = 0$, so by Proposition 6.1.1, $u = v$. Now suppose that $K(w) = u$, then $A_p u = B_p w$ and therefore, since (a_1) and (b_1) hold,

$$B_p(tw) = |t|^{p-2} t B_p w = |t|^{p-2} t A_p u = A_p(tu),$$

which means that $K(tw) = tu = tK(w)$.

Now, to show that K is compact, let (w_j) be a bounded sequence in W and let $u_j = Kw_j$. Since B_p is a compact operator,

$$A_p u_j = B_p w_j \to l$$

for a renamed subsequence and some $l \in W^*$. By (a_2),

$$\|u_j\|^p = (A_p u_j, u_j) = (B_p w_j, u_j) \leq \|B_p w_j\|^* \|u_j\|,$$

which implies that $\|u_j\|$ is bounded since $p > 1$ and $(B_p w_j)$ is bounded. Since W is reflexive, a further subsequence of (u_j) converges weakly to some $u \in W$. Then

$$(A_p u_j, u_j - u) = (A_p u_j - l, u_j - u) + (l, u_j - u) \to 0.$$

Since A_p is of type (S) (see Proposition 6.1.1), then (u_j) has a subsequence that converges strongly to u.

□

Proof of Theorem 6.1.3. We will use the same notations as in Lemma 6.1.1.

Let $w \in W \setminus \{0\}$ and led $u = Kw$. Then $u \neq 0$ since $(A_p u, w) = (B_p w, w) > 0$ by (b_2). It is easily seen that the radial projection of u on $\mathcal{M}$ is given by

$$\pi_{\mathcal{M}}(u) = \frac{u}{I_p(u)^{1/p}}.$$

Since J_p is p-homogeneous we have

$$\Psi(\pi_{\mathcal{M}}(u)) = \frac{1}{J_p(\pi_{\mathcal{M}}(u))} = \frac{I_p(u)}{J_p(u)}. \tag{1.5}$$

Also we have

$$\begin{aligned} I_p(u) = \frac{1}{p}(A_p u, u) = \frac{1}{p}(B_p w, u) &\leq \frac{1}{p}(B_p w, w)^{(p-1)/p}(B_p u, u)^{1/p} \\ &= J_p(w)^{(p-1)/p} J_p(u)^{1/p} \end{aligned}$$

by (b_2) and

$$J_p(w) = \frac{1}{p}(B_p w, w) = \frac{1}{p}(A_p u, w) \leq \frac{1}{p}\|u\|^{p-1}\|w\| = I_p(u)^{(p-1)/p} I_p(w)^{1/p} \tag{1.6}$$

by (a_2), so

$$\frac{I_p(u)}{J_p(u)} \leq \frac{I_p(w)}{J_p(w)}.$$

For $w \in \mathcal{M}$, $I_p(u) = 1$ and $1/J_p(w) = \Psi(w)$, so combining this with (1.5) gives

$$\Psi(\pi_{\mathcal{M}}(Kw)) \leq \Psi(w). \tag{1.7}$$

Let

$$C_0 = \overline{\pi_{\mathcal{M}}(K(\Psi^{\lambda_k}))}.$$

By (1.7) we have that

$$\pi_{\mathcal{M}}(K(\Psi^{\lambda_k})) \subset \Psi^{\lambda_k},$$

and since Ψ^{λ_k} is closed we have

$$\pi_{\mathcal{M}}(K(\Psi^{\lambda_k})) \subset C_0 \subset \Psi^{\lambda_k}.$$

By Lemma 6.1.1 $\pi_{\mathcal{M}} \circ K$ is an odd continuous map on Ψ^{λ_k}, then by the monotonicity of the index (Proposition 3.2.2) we have

$$i(\Psi^{\lambda_k}) \leq i(\pi_{\mathcal{M}}(K(\Psi^{\lambda_k}))) \leq i(C_0) \leq i(\Psi^{\lambda_k}).$$

Since $\lambda_k < \lambda_{k+1}$, by Proposition 4.5.2 we have $i(C_0) = k$. Since $I_p(w) = 1$ and $J_p(w) \geq 1/\lambda_k$ for $w \in \Psi^{\lambda_k}$, (1.6) implies that $\overline{K(\Psi^{\lambda_k})} \subset W \setminus \{0\}$. Since K is compact, Ψ^{λ_k} is bounded and $\pi_{\mathcal{M}}$ is continuous, C_0 is compact. □

Putting Theorem 6.1.3 and Proposition 4.5.2 together, we can apply Theorem 6.1.2 to our energy functional (1.2). Namely we have the following.

Theorem 6.1.4. *Assume that* (a_1)-(g_4) *hold and* $\mathcal{E}$ *satisfies* $(PS)_c$ *for all* $c \in \mathbb{R}$. *If* $\lambda > 0$, *then there exists* $\mu_0 > 0$ *such that equation* (1.1) *has two nontrivial solutions* u_1, u_2 *with*

$$\mathcal{E}(u_1) < 0 < \mathcal{E}(u_2)$$

for $0 < \mu < \mu_0$.

Proof. We will apply Theorem 6.1.2 to $\mathcal{E}$. The sets A_1, A, A^*, B, B^* in what follows are the same as in Theorem 6.1.2.

If $0 < \lambda < \lambda_1$, take $A_0 = \emptyset$ and $B_0 = \mathcal{M}$. Otherwise, if $\lambda \geq \lambda_1$, then since (λ_k) is an unbounded sequence, there exists $k \geq 1$ such that $\lambda_k \leq \lambda < \lambda_{k+1}$. By Theorem 6.1.3 there exists a compact symmetric subset A_0 of $\Psi^{\lambda_k} \subset \mathcal{M}$ such that $i(A_0) = k$. Now, let $B_0 = \Psi_{\lambda_{k+1}}$. By Proposition 4.5.2 we have $i(\mathcal{M} \setminus B_0) = k$. Then

$$i(A_0) = i(\mathcal{M} \setminus B_0) = k < +\infty.$$

Let us note first that, for $u \in \mathcal{M}$ and $t > 0$, since $\Psi(u) = 1/J_p(u)$,

$$\mathcal{E}(tu) = t^p \left(1 - \frac{\lambda}{\Psi(u)}\right) - \mu F(tu) - G(tu). \tag{1.8}$$

For $w \in B_0$, since (g_1) holds and $\Psi(w) \geq \lambda_{k+1}$, we get

$$\begin{aligned} \mathcal{E}(tw) &\geq t^p \left(1 - \frac{\lambda}{\lambda_{k+1}}\right) - \mu F(tw) - G(tw) \\ &\geq t^p \left(1 - \frac{\lambda}{\lambda_{k+1}} + o(1)\right) - \mu F(tw) \quad \text{as} \quad t \to 0. \end{aligned}$$

Therefore, since B_0 is bounded, $\lambda < \lambda_{k+1}$ and (f_3) holds, there are $\rho, \mu_0 > 0$ such that

$$\inf_B \mathcal{E} > 0,$$

for all $0 < \mu < \mu_0$, where B is defined as in Theorem 6.1.2.

Now let us fix $\mu \in (0, \mu_0)$ and let $w_0 \in B_0$, and let A_1 defined as in Theorem 6.1.2. Since A_0 is compact, so is A_1. For $u \in A_1$, (1.8) together with (g_2) gives

$$\mathcal{E}(tu) \leq t^p \left(1 - \mu \frac{F(tu)}{t^p}\right).$$

Since A_1 is compact, it follows form this and (f_1) that there exists $0 < r < \rho$ such that

$$\sup \{\mathcal{E}(ru) : u \in A_1\} < 0. \tag{1.9}$$

Similarly, (1.8) together with (f_2) gives

$$\mathcal{E}(tu) \leq t^p \left(1 - \frac{G(tu)}{t^p}\right)$$

for $u \in A_1$, and it follows from this and (g_4) that there exists $R > \rho$ such that

$$\sup\{\mathcal{E}(Ru) : u \in A_1\} < 0. \tag{1.10}$$

For $v \in A_0$,

$$\mathcal{E}(tv) < -t^p \left(\frac{\lambda}{\Psi(v)} - 1\right) \leq 0$$

since $\Psi(v) \leq \lambda_k \leq \lambda$. Since A_0 is compact, it follows from this that

$$\sup\{\mathcal{E}(tv) : v \in A_0,\ r \leq t \leq R\} < 0. \tag{1.11}$$

Combining (1.9)-(1.11) gives

$$\sup_A \mathcal{E} < 0.$$

Since $\mathcal{E}$ is bounded on bounded sets, we also have that there exist $a < b \in \mathbb{R}$ such that

$$a < \inf_{B^*} \mathcal{E}, \quad \sup_{A^*} \mathcal{E} < b.$$

Therefore, by Theorem 6.1.2, $\mathcal{E}$ has two nontrivial critical points u_1, u_2 with

$$\mathcal{E}(u_1) \leq \sup_A \mathcal{E} < 0 < \inf_B \mathcal{E} \leq \mathcal{E}(u_2).$$

□

Remark 6.1.1. *Theorem 6.1.4 gives us two nontrivial critical points. If $\mathcal{E}$ has not infinitely many critical points, one way to find a third one is to search for local minimum type critical points. Indeed, if $\lambda \geq \lambda_k$ with $k \geq 1$ then we have that*

$$C^k(\mathcal{E}, u_1) \neq 0, \quad C^{k+1}(\mathcal{E}, u_2) \neq 0.$$

And if we find an isolated local minimum u_3 for $\mathcal{E}$, then by Example 4.3.1 we would have

$$C^k(\mathcal{E}, u_3) = \begin{cases} \mathbb{Z}_2 & \text{if} \quad k = 0 \\ 0 & \text{if} \quad k \neq 0. \end{cases}$$

which implies that $u_3 \neq u_1$ and $u_3 \neq u_2$.

6.2 Some applications of Theorem 6.1.4

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The principal application of Theorem 6.1.4 is on our Problem (3.8). By Theorem 2.3.1 we just need to check that E defined in (3.13) satisfies (a_1)-(g_4). In this case we have that A_p, B_p, f and g in $C(W_0^{1,p}(\Omega), W^{-1,p'}(\Omega))$ are given by

$$(A_p u, v) = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v, \quad (B_p u, v) = \int_\Omega |u|^{p-2} uv$$

$$(f(u), v) = \int_\Omega f(x,u)v, \quad (g(u), v) = \int_\Omega |u|^{q-2} uv$$

(a_1). Let $u \in W_0^{1,p}(\Omega)$ and note that, for all $v \in W_0^{1,p}(\Omega)$ and $t \in \mathbb{R}$,

$$\begin{aligned} (A_p(tu), v) &= \int_\Omega |\nabla(tu)|^{p-2} \nabla(tu) \nabla v \\ &= |t|^{p-2} t \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v. \\ &= |t|^{p-2} t (A_p u, v) \\ &= (|t|^{p-2} t A_p u, v). \end{aligned}$$

Logo de fato $A_p(tu) = |t|^{p-2} t A_p u$

(a_2) Let $u, v \in W_0^{1,p}(\Omega)$, then

$$\begin{aligned} (A_p u, v) &= \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \leq \int_\Omega |\nabla u|^{p-1} |\nabla v| \\ &\leq \|u\|^{p-1} \|v\|, \end{aligned}$$

where the first inequality is due the Cauchy-Schwarz Inequality, and the second one by Holder Inequality.

Suppose now that $(A_p u, v) = \|u\|^{p-1} \|v\|$ with $u, v \neq 0$. Then

$$\begin{aligned} \|u\|^{p-1} \|v\| &= (A_p u, v) = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v \leq \int_\Omega |\nabla u|^{p-2} |\nabla u| |\nabla v| && (2.12) \\ &\leq \|u\|^{p-1} \|v\|, && (2.13) \end{aligned}$$

which implies that

$$\int_\Omega |\nabla u|^{p-1}|\nabla v| = \left(\int_\Omega |\nabla u|^p\right)^{(p-1)/p}\left(\int_\Omega |\nabla v|^p\right)^{1/p}$$
$$= \|u\|^{p-1}\|v\| \int_\Omega \left(\frac{|\nabla u|^p}{\frac{p}{p-1}\|u\|^p} + \frac{|\nabla v|^p}{p\|v\|^p}\right) \geq \int_\Omega |\nabla u|^{p-1}|\nabla v|$$

therefore

$$\frac{|\nabla u|^p}{\frac{p}{p-1}\|u\|^p} + \frac{|\nabla v|^p}{p\|v\|^p} = \frac{|\nabla u|^{p-1}|\nabla v|}{\|u\|^{p-1}\|v\|} \quad \text{a.e. on} \quad \Omega,$$

hence by Young's inequality it implies that

$$\frac{|\nabla u|^p}{\|u\|^p} = \frac{|\nabla v|^p}{\|v\|^p} \implies \frac{|\nabla u|}{\|u\|} = \frac{|\nabla v|}{\|v\|} \implies |\nabla u| = \frac{\|u\|}{\|v\|}|\nabla v| \quad \text{a.e. on} \quad \Omega. \tag{2.14}$$

Now, by inequality (2.12) we get

$$\nabla u \nabla v = |\nabla u||\nabla v|,$$

and by Cauchy-Schwarz Inequality there exists a function $f : \Omega \to \mathbb{R}$ such that

$$\nabla u(x) = f(x)\nabla v(x) \quad \text{a.e. on} \quad \Omega. \tag{2.15}$$

Putting together (2.14) and (2.15) we have that f is a constant nonzero function so there exists $0 \neq \alpha \in \mathbb{R}$ such that

$$\nabla u = \alpha \nabla v \quad \text{for almost everywhere} \quad x \in \Omega,$$

and consequently

$$u = \alpha v \quad \text{a.e. on} \quad \Omega.$$

(b_1) It is analogous as in (a_1).

(b_2) The fact that $(B_p u, u) > 0$ on $W_0^{1,p}(\Omega) \setminus \{0\}$ is evident. And the inequality

$$(B_p u, v) \leq (B_p u, u)^{(p-1)/p}(B_p v, v)^{1/p}$$

follows from Holder inequality.

(b_3) Let $\{u_k\}_{k\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ be a bounded sequence. Then there exists $u \in W_0^{1,p}(\Omega)$ such that, taking a subsequence if necessary, $u_k \rightharpoonup u$. By the imbedding theorems, we can assume, taking a further subsequence if necessary, that

$$\begin{aligned} u_k &\to u && \text{in} \quad L^p(\Omega), \\ u_k(x) &\to u(x) && \text{a.e. on} \quad \Omega. \end{aligned}$$

By Theorem 8.0.2, taking an even further subsequence, we can assume that there exists $v \in L^p(\Omega)$ such that

$$|u_k(x)| \leq v(x) \quad \text{a.e. on} \quad \Omega.$$

Now note that, by Holder inequality,

$$|(B_p u_k - B_p u, v)| \leq \left(\int_\Omega \left| |u_k|^{p-2} u_k - |u|^{p-2} u \right|^{p/(p-1)} \right)^{(p-1)/p} \|v\| \quad \forall v \in W_0^{1,p}(\Omega),$$

and consequently

$$\|B_p u_k - B_p u\|^* \leq \left(\int_\Omega \left| |u_k|^{p-2} u_k - |u|^{p-2} u \right|^{p/(p-1)} \right)^{(p-1)/p} \tag{2.16}$$

Since

$$\left| |u_k|^{p-2} u_k - |u|^{p-2} u \right|^{p/(p-1)} \leq C \left(|v|^p + |u|^p \right) \quad \text{a.e. on} \quad \Omega,$$

for some constant $C > 0$, and

$$\left| |u_k(x)|^{p-2} u_k(x) - |u(x)|^{p-2} u(x) \right|^{p/(p-1)} \to 0 \quad \text{a.e. on} \quad \Omega \quad \text{as} \quad k \to +\infty,$$

therefore, by Lebesgue Theorem 8.0.1 and inequality (2.16) we have

$$B_p u_k \to B_p u \quad \text{on} \quad W^{-1,p'}(\Omega).$$

(f_1) By the condition (3.9) we have

$$\int_0^t f(x,s) dx = c_1 |t|^\sigma + o(1) |t|^\sigma \quad \text{as} \quad t \to 0, \quad \text{uniformly} \quad \text{a.e. in} \quad \Omega \tag{2.17}$$

for some $1 < \sigma < p$ and a constant $c_1 > 0$. And rememeber that

$$F(u) = \int_\Omega \left(\int_0^{u(x)} f(x,s) ds \right) dx$$

is the potential for f with $F(0) = 0$.

Now let $K \subset W_0^{1,p}(\Omega)\setminus\{0\}$ a compact subset. So we have that $K \subset L^r(\Omega)\cap L^\sigma(\Omega)\setminus\{0\}$ is a bounded subset. It is worth noting that K is a set bounded from below in $L^p(\Omega)\cap L^r(\Omega)$, i.e. there exists a constant $M > 0$ such that $\int_\Omega |u|^r, \int_\Omega |u|^\sigma > M$ for all $u \in K$.

By (2.17) and (3.11) we have that

$$\int_0^t f(x,s)ds = c_1|t|^\sigma + h(t),$$

such that $|h(t)| \le c_2|t|^r$ for all $t \in \mathbb{R}$ and for some constants $c_1, c_2 > 0$ with $p < r < q$. Therefore we have, for all $u \in K$,

$$F(tu) = \int_\Omega \left(\int_0^{tu(x)} f(x,s)ds\right) dx \ge c_1|t|^\sigma \int_\Omega |u|^\sigma - c_2|t|^r \int_\Omega |u|^r$$

and since K is bounded on $L^r(\Omega)$

$$F(tu) \ge C_1\left(|t|^\sigma - |t|^r\right) \quad \forall u \in K,$$

for some positive constant C_1. Consequently, since $1 < \sigma < p < r$,

$$\lim_{t\to 0} \frac{F(tu)}{|t|^p} \to +\infty \quad \text{uniformly on} \quad K.$$

(f_2) It follows from assumption (3.10).

(f_3) It follows from assumptions (3.9) and (3.11) that there are positive constants a_1 and a_2 such that

$$\int_0^t f(x,s)ds \le a_1|t|^\sigma + a_2|t|^r \quad \text{for all} \quad t \in \mathbb{R}$$

and consequenlty, by Sobolev inequalities,

$$F(u) \le \widetilde{C_1}\|u\|^\sigma + \widetilde{C_2}\|u\|^r \quad \forall u \in W_0^{1,p}(\Omega),$$

for some positive constants $\widetilde{C_1}$ and $\widetilde{C_2}$.

(g_1) By Sobolev inequality there exists a constant $B > 0$ such that

$$\int_\Omega |u|^q \le B\|u\|^q,$$

and the assertion follows since $p < q$.

(g_2) It is evident since $G(u) = \frac{1}{q}\int_\Omega |u|^q = \frac{1}{q}|u|_q^q$.

(g_3) Since $q < p^*$, there exists, by the Sobolev inequality, a positive constant $C > 0$ such that

$$|u|_q \le C\|u\|, \quad \text{for all} \quad u \in W_0^{1,p}(\Omega),$$

which implies that G is bounded on bounded subsets of $W_0^{1,p}(\Omega)$.

(g_4) Let $K \subset W_0^{1,p}(\Omega) \setminus \{0\}$ be a compact subset. By the Sobolev inequality, K is a also a compact subset of $L^q(\Omega) \setminus \{0\}$. Therefore there exists a constant $\widetilde{C} > 0$ such that

$$|u|_q \geq \widetilde{C} \quad \forall u \in K,$$

and therefore, for $t > 0$,

$$\frac{G(tu)}{t^p} = \frac{1}{q}|t|^q |u|_q^q \geq c\frac{t^q}{t^p} = ct^{q-p}, \quad \forall u \in K,$$

for some positive constant c. And since $q - p > 0$ we have that

$$\lim_{t \to +\infty} \frac{G(tu)}{t^p} \to +\infty \quad \text{uniformly on} \quad K.$$

This shows that we can apply Theorem 6.1.4 to the problem (3.8), namely we have proved the following result.

Theorem 6.2.1. *Let $1 < p < q$, with $q < p^*$ if $p < N$ and $q < \infty$ if $p \geq N$, let $\lambda > 0$, and let f be a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying (3.9)-(3.11) for some $1 < \sigma < p < r < q$. Then $\exists\, \mu_0 > 0$ such that problem (3.8) has two nontrivial solutions u_1, u_2 with*

$$E(u_1) < 0 < E(u_2)$$

for $0 < \mu < \mu_0$.

Another application of Theorem 6.1.4 is the following; Consider the Kirchhoff type problem

$$\begin{cases} -\left(\int_\Omega |\nabla u|^2\right)\Delta u = \lambda u^3 + \mu f(x,u) + |u|^{q-2}u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.18}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N = 1, 2$ or 3, $4 < q < 6$ if $N = 3$ and $4 < q < \infty$ if $N = 1$ or 2, $\lambda, \mu > 0$ are parameters, and f is a Carathéodory funcion on $\Omega \times \mathbb{R}$ satisfying (3.9)-(3.11) for some $1 < \sigma < 4 < r < q$. Solutions of this problem coincide with critical points of the functional

$$\mathcal{E}(u) = \frac{1}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2 - \frac{\lambda}{4}\int_\Omega u^4 dx - \mu\int_\Omega F(x,u)dx - \frac{1}{q}\int_\Omega |u|^q dx, \quad u \in H_0^1(\Omega),$$

where $F(x,t) = \int_0^t f(x,s)ds$. As in Theorem 2.3.1, $\mathcal{E}$ satisfies $(PS)_c$ condition for all $c \in \mathbb{R}$. We can apply Theorem 6.1.4 with $W = H_0^1(\Omega)$ and the operators $A_p, B_p, f, g \in C(H_0^1(\Omega), H^{-1}(\Omega))$ given by

$$(A_p u, v) = \left(\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u \nabla v dx, \quad (B_p u, v) = \int_\Omega u^3 v dx,$$

$$(f(u),v) = \int_\Omega f(x,u)v dx, \quad (g(u),v) = \int_\Omega |u|^{q-2}uv dx,$$

for all $u, v \in H_0^1(\Omega)$.

By verfifying, in a similar way as we just did, that for problem (2.18), (a_1)-(g_4) hold, we have the following result.

Theorem 6.2.2. *Let* $q > 4$, *with* $q < 6$ *if* $N = 3$ *and* $q < \infty$ *if* $N = 1$ *or* 2, *let* $\lambda > 0$, *and let* f *be a Carathéodory function on* $\Omega \times \mathbb{R}$ *satisfying* (3.9)-(3.11) *for some* $1 < \sigma < 4 < r < q$. *Then* $\exists\, \mu_0 > 0$ *such that problem* (2.18) *has two nontrivial solutions* u_1, u_2 *with*

$$\mathcal{E}(u_1) < 0 < \mathcal{E}(u_2)$$

for $0 < \mu < \mu_0$.

6.3 The critical version of Problem 3.8

Now let us consider the case of Problem 3.8 when it is allowed to have $q = p^*$, namely consider the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + \mu f(x,u) + |u|^{p^*-2}u & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.19}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $1 < p < N$, $\lambda, \mu > 0$ are parameters, and f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying (3.9)-(3.11) for some $1 < \sigma < p < r < p^*$. Solutions of this problem coincide with critical points of the functional

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda}{p}\int_\Omega |u|^p dx - \mu \int_\Omega F(x,u)dx - \frac{1}{p^*}\int_\Omega |u|^{p^*} dx, \quad u \in W_0^{1,p}(\Omega). \tag{3.20}$$

Since $q = p^*$, the compact imbeddings are no longer valid, therefore $(PS)_c$ condition is no more guaranteed for all $c \in \mathbb{R}$. However, functional (3.20) satisfies a local (PS) condition. In that fashion, to apply the theory so far developed we will need to make some adjustments.

The main result of this section will be the following.

Theorem 6.3.1. *Let* $p > 1$, $N \geq p^2$ *and* $\lambda > 0$ *with* $\lambda \notin \sigma(-\Delta_p)$, *and let* f *be a Carathéodory function on* $\Omega \times \mathbb{R}$ *satisfying* (3.9)-(3.11) *for some* $1 < \sigma < p < r < p^*$. *Then* $\exists\, \mu_0$, $\omega > 0$ *such that problem* (3.19) *has two nontrivial solutions* u_1, u_2 *with*

$$E(u_1) < 0 < E(u_2) < \frac{1}{N}S^{N/p} - \omega\mu$$

for $0 < \mu < \mu_0$.

In order to prove the above theorem, we will need to adapt Theorem 6.1.4 in the following way.

Theorem 6.3.2. *Assume that* (a_1)-(g_4) *hold and* $\exists c_\mu > 0$ *such that* $\mathcal{E}$ *satisfies* $(PS)_c$ *condition for all* $c < c_\mu$. *If* $0 < \lambda < \lambda_1$, *assume that* $\exists w_0 \in \mathcal{M}$ *such that*

$$\sup_{t \geq 0} \mathcal{E}(tw_0) < c_\mu \tag{3.21}$$

for all sufficiently small $\mu > 0$. *If* $\lambda_k \leq \lambda < \lambda_{k+1}$, *assume that thee exist a compact symmetric subset* C *of* Ψ^λ *with* $i(C) = k$ *and* $w_0 \in \mathcal{M} \backslash C$ *such that*

$$\sup_{v \in C,\ s,t \geq 0} \mathcal{E}(sv + tw_0) < c_\mu \tag{3.22}$$

for all sufficiently small $\mu > 0$. *Then* $\exists \mu_0 > 0$ *such that equation* (1.1) *has two nontrivial solutions* u_1, u_2 *with*

$$\mathcal{E}(u_1) < 0 < \mathcal{E}(u_2) < c_\mu$$

for $0 < \mu < \mu_0$.

Proof. Here we will use the same notations as in Theorem 6.1.4.

If $0 < \lambda < \lambda_1$, as in Theorem 6.1.4, take $A_0 = \emptyset$ and $B_0 = \mathcal{M}$. Then

$$A^* \subset \{tw_0 : t \geq 0\}$$

and hence $\sup \mathcal{E}(A^*) < c_\mu$ for all sufficiently small $\mu > 0$ by (3.21). Therefore Theorem 6.1.4 can be apply for any $b \in (\sup \mathcal{E}(A^*), c_\mu]$ and $a < \inf \mathcal{E}(B^*)$.

If $\lambda_k \leq \lambda < \lambda_{k+1}$, let $A_0 = C$ and $B_0 = \Psi_{\lambda_{k+1}}$, then by Proposition 4.5.2,

$$i(A_0) = i(\mathcal{M} \backslash B_0) = k.$$

Also, we have

$$A^* \subset \{sv + tw_0 : v \in C,\ s,t \geq 0\},$$

and hence $\sup \mathcal{E}(A^*) < c_\mu$ for all small $\mu > 0$ by (3.22). Therefore we can apply Theorem 6.1.4 for $a < \inf \mathcal{E}(B^*)$ and $b = c_\mu$.

□

We will employ Theorem 3.19 in order to seek critical points of the functional E defined in (3.20). For that porpuse, let us first show that there exists $c_\mu > 0$ such that the functional 3.20 satisfies $(PS)_c$ for all $c < c_\mu$.

Let

$$S = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{(\int_\Omega |u|^{p^*} dx)^{p/p^*}} > 0 \tag{3.23}$$

be the best Sobolev constant.

We then have the following.

Lemma 6.3.1. *Let* $0 < \mu \leq 1$. *Then there exists* $\omega > 0$ *such that* E *satisfies the* $(PS)_c$ *condition for all*

$$c < \frac{1}{N} S^{N/p} - \omega\mu. \tag{3.24}$$

Proof. Suppose that

$$\begin{aligned} E(u_j) &\to c \quad \text{on} \quad W_0^{1,p}(\Omega),\\ E'(u_j) &\to 0 \quad \text{on} \quad W^{-1,p'}(\Omega). \end{aligned}$$

Then

$$E(u_j) = \int_\Omega \left(\frac{1}{p} |\nabla u_j|^p - \frac{\lambda}{p} |u_j|^p - \mu F(x,u_j) - \frac{1}{p^*} |u_j|^{p^*} \right) dx = c + o(1), \tag{3.25}$$

and

$$\begin{aligned} (E'(u_j),v) = \int_\Omega \left(|\nabla u_j|^{p-2} \nabla u_j \nabla v - \lambda |u_j|^{p-2} u_j v - \mu f(x,u_j) v - |u_j|^{p^*-2} u_j v \right) dx \\ = o(\|v\|) \quad \forall v \in W_0^{1,p}(\Omega). \end{aligned} \tag{3.26}$$

Taking $v = u_j$ in (3.26) gives

$$\int_\Omega \left(|\nabla u_j|^p - \lambda |u_j|^p - \mu f(x,u_j) u_j - |u_j|^{p^*} \right) dx = o(\|u_j\|). \tag{3.27}$$

As in Theorem 2.3.1, (u_j) is a bounded sequence, so a renamed subsequence converges weakly to some u in $W_0^{1,p}(\Omega)$ and strongly in $L^s(\Omega)$ for all $s \in [1, p^*)$, and almost everywhere in Ω. Let us write $v_j = u_j - u$.

Therefore, for (3.27) we have

$$\|u_j\|^p = |u_j|_{p^*}^{p^*} + \int_\Omega (\lambda |u|^p + \mu f(x,u) u) \, dx + o(1). \tag{3.28}$$

As in Theorem 2.3.1, $\nabla u_j(x) \to \nabla u(x)$ a.e. on Ω, and consequently, since $|\nabla u_j|^{p-2} \nabla u$ is bounded on $L^{p/(p-1)}(\Omega : \mathbb{R}^N)$, we have that $|\nabla u_j|^{p-2} \nabla u_j \rightharpoonup |\nabla u|^{p-2} \nabla u$ on $L^{p/(p-1)}(\Omega : \mathbb{R}^N)$. Therefore, taking

$v = u$ in (3.26) and passing to the limit gives

$$\|u\|^p = |u|_{p^*}^{p^*} + \int_\Omega (\lambda |u|^p + \mu f(x,u)u)\, dx. \tag{3.29}$$

Again as in Theorem 2.3.1 we have

$$\|v_j\|^p = \|u_j\|^p - \|u\|^p + o(1). \tag{3.30}$$

Also, by the Brézis-Lieb lemma,

$$|v_j|_{p^*}^{p^*} = |u_j|_{p^*}^{p^*} - |u|_{p^*}^{p^*} + o(1),$$

and combining (3.28) and (3.29) with the last two equalities we get, by the definition of S,

$$\|v_j\|^p = |v_j|_{p^*}^{p^*} + o(1) \le \frac{\|v_j\|^{p^*}}{S^{p^*/p}} + o(1),$$

so

$$\|v_j\|^p \left(S^{N/(N-p)} - \|v_j\|^{p^2/(N-p)} \right) \le o(1). \tag{3.31}$$

Now, from (3.25) we have

$$c = \frac{1}{p}\|u_j\|^p - \frac{1}{p^*}|u_j|_{p^*}^{p^*} - \int_\Omega \left(\frac{\lambda}{p}|u|^p + \mu F(x,u) \right) dx + o(1),$$

this together with (3.28)-(3.30) gives

$$c = \frac{1}{N}\|v_j\|^p + \int_\Omega H(x,u)dx + o(1), \tag{3.32}$$

where

$$H(x,t) = \frac{1}{N}|t|^{p^*} + \mu \left(\frac{1}{p} f(x,t)t - F(x,t) \right).$$

By condition (3.11) on f, we have

$$\frac{1}{p} f(x,t)t - F(x,t) \ge -a_1 \left(|t|^r + 1 \right) \quad \text{for a.e. } x \in \Omega \text{ and all } t \in \mathbb{R}$$

for some $a_1 > 0$. Since $r < p^*$, there exists a constant $a_2 > 0$ such that

$$H(x,t) \ge -a_2 \mu \quad \text{a.e. in } \Omega \text{ and all } t \in \mathbb{R}.$$

Then (3.32) gives

$$\|v_j\|^p \leq N(c+\omega\mu)+o(1)$$

for some $\omega > 0$.

Combining this with (3.31) shows that $v_j \to 0$ when (3.24) holds. □

Now we have found the constant c_μ of Theorem 3.19, namely

$$c_\mu = \frac{1}{N}S^{N/p} - \omega\mu.$$

Let us first consider the case where $\lambda_k < \lambda < \lambda_{k+1}$ with $k \geq 1$ on Theorem 6.3.1. The case $0 < \lambda < \lambda_1$ will be treated later. So in order to apply Theorem 6.3.2, we will need to show the existence of the compact symmetric subset $C \subset \Psi^\lambda$ with $i(C) = k$ satisfying (3.22).

Consider now the operator K first defined in Lemma 6.1.1 and used in in the proof of Theorem 6.1.3 and the set C_0 constructed in the same such a theorem. It can be proved that there exists some $m \geq 2$ such that $K^m(\Psi^{\lambda_k})$ is a bounded subset of $L^\infty(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\Omega)$ of index k (see [8], Theorem 2.3). Therefore we can assume that C_0 is a subset of Ψ^{λ_k} that is also bounded in $L^\infty(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\Omega)$.

By a translation, we can assume without loss of generality that $0 \in \Omega$. Let $\delta_0 = \text{dist}(0, \partial\Omega)$, and $\eta : [0,\infty) \to [0,1]$ be a smooth function such that $\eta(s) = 0$ for $s \leq 3/4$ and $\eta(s) = 1$ for $s \geq 1$, and set

$$u_\delta(x) = \eta\left(\frac{|x|}{\delta}\right)u(x), \quad 0 < \delta \leq \delta_0/2$$

for $u \in C_0$. Remember that $\mathcal{M} = \left\{u \in W^{1,p}_0(\Omega) : \|u\|^p = p\right\}$ and $\pi_{\mathcal{M}} : W^{1,p}_0(\Omega)\setminus\{0\} \to \mathcal{M}$ defined as $\pi_{\mathcal{M}}(u) = p^{1/p}u/\|u\|$ is the radial projection onto $\mathcal{M}$. Let us define

$$C_\delta = \{\pi_{\mathcal{M}}(u_\delta) : u \in C_0\}.$$

Then we have the following.

Lemma 6.3.2. *If $\delta > 0$ is sufficiently small, then C_δ is a compact symmetric subset of Ψ^λ with $i(C) = k$.*

Proof. By the above discussion, we have that functions in C_0 are bounded in $C^1(B_{\delta_0/2}(0))$ and belong to Ψ^{λ_k}. Note that

$$\nabla u_\delta(x) = \eta'\left(\frac{|x|}{\delta}\right)u(x)\frac{x}{\delta|x|} + \eta\left(\frac{|x|}{\delta}\right)\nabla u(x),$$

therefore

$$\begin{aligned}\int_\Omega |\nabla u_\delta|^p dx &= \int_{\Omega\setminus B_\delta(0)} |\nabla u_\delta|^p dx + \int_{B_\delta(0)} |\nabla u_\delta|^p dx \\ &\leq \int_{\Omega\setminus B_\delta(0)} |\nabla u|^p dx + 2^p \int_{B_\delta(0)} \left(\frac{1}{\delta^p}|\eta'|^p|u|^p + |\eta|^p|\nabla u|^p\right) dx,\end{aligned}$$

and since η is of class C^∞, $|\eta(y)| \leq 1$, and C_0 is bounded $\in C^1(B_{\delta_0/2}(0))$ and $2^p > 1$, we have

$$\begin{aligned}\int_\Omega |\nabla u_\delta|^p dx &\leq \int_\Omega |\nabla u|^p dx + (2^p-1)\int_{B_\delta(0)} |\eta|^p|\nabla u|^p dx + 2^p \int_{B_\delta(0)} \frac{1}{\delta^p}|\eta'|^p|u|^p dx \\ &\leq p + c_1\delta^{N-p} + c_2\delta^N\end{aligned}$$

for some positive constants c_1 and c_2. We also have that

$$\int_\Omega |u_\delta|^p dx \geq \int_{\Omega\setminus B_\delta(0)} |u|^p dx = \int_\Omega |u|^p dx - \int_{B_\delta(0)} |u|^p dx \geq \frac{p}{\lambda_k} - c_3\delta^N$$

for some $c_3 > 0$. So

$$\Psi(\pi_{\mathcal{M}}(u_\delta)) = \frac{\int_\Omega |\nabla u_\delta|^p dx}{\int_\Omega |u_\delta|^p dx} \leq \frac{p + c_1\delta^{N-p} + c_2\delta^N}{\dfrac{p}{\lambda_k} - c_3\delta^N}.$$

Denoting by

$$I(\delta) = \frac{p + c_1\delta^{N-p} + c_2\delta^N}{\dfrac{p}{\lambda_k} - c_3\delta^N},$$

then I is well defined and continuous for δ small enough and $I(0) = \lambda_k$. Therefore since $\lambda_k < \lambda$ we have that for all $\delta > 0$ small enough we have $I(\delta) < \lambda$. Therefore $v = \Psi(\pi_{\mathcal{M}}(u_\delta)) \in \Psi^k$ for all sufficiently small $\delta > 0$.

Since C_0 is a compact symmetric set and $u \mapsto v$ is an odd continuous map of C_0 onto C_δ, C_δ is also a compact symmetric set and

$$i(C_\delta) \geq i(C_0) = k$$

by the monotonicity of the index. On the other hand, since $C_\delta \subset \Psi^\lambda \subset \mathcal{M}\setminus\Psi_{\lambda_{k+1}}$ by Proposition 4.5.2,

$$i(C_\delta) \leq i(\mathcal{M}\setminus\Psi_{\lambda_{k+1}}) = k.$$

So $i(C_\delta) = k$. □

Now we are ready to prove Theorem 6.3.1.

Proof of Theorem 6.3.1 . Let us first prove when $\lambda \in (\lambda_k, \lambda_{k+1})$.

Let us consider the Aubin-Talenti functions

$$u^*_\varepsilon(x) = \frac{c_{N,p}\varepsilon^{(N-p)/p^2}}{\left(\varepsilon + |x|^{p/(p-1)}\right)^{(N-p)/p}}, \quad \varepsilon > 0.$$

It is well-known that such functions minimizes (3.23) when $\Omega = \mathbb{R}^N$. The constant $c_{N,p} > 0$ is chosen so that

$$\int_{\mathbb{R}^N} |\nabla u^*_\varepsilon|^p dx = \int_{\mathbb{R}^N} |u^*_\varepsilon|^p dx = S^{N/p}.$$

Fix $\delta > 0$ so small that C_δ is a compact symmetric subset of Ψ^λ with $i(C_\delta) = k$ (see Lemma 6.3.2). Let $\theta : [0,\infty) \to [0,1]$ be a smooth function such that $\theta(s) = 1$ for $s \leq 1/4$ and $\theta(s) = 0$ for $s \geq 1/2$, and set

$$u_\varepsilon(x) = \theta\left(\frac{|x|}{\delta}\right) u^*_\varepsilon(x), \quad \widetilde{u}_\varepsilon(x) = \frac{u_\varepsilon(x)}{\left(\int_{\mathbb{R}^N} |u_\varepsilon|^{p^*} dx\right)^{1/p^*}}, \quad \varepsilon > 0.$$

We have the well known estimates (see [16])

$$\int_{\mathbb{R}^N} |\nabla \widetilde{u}_\varepsilon|^p dx \leq S + a_1 \varepsilon^{(N-p)/p},$$

$$\int_{\mathbb{R}^N} |\widetilde{u}_\varepsilon|^{p^*} dx = 1,$$

$$\int_{\mathbb{R}^N} |\widetilde{u}_\varepsilon|^p dx \geq \begin{cases} a_2 \varepsilon^{p-1} & \text{if } N > p^2 \\ a_2 \varepsilon^{p-1} |\log \varepsilon| & \text{if } N = p^2. \end{cases}$$

for some $a_1, a_2 > 0$.

Let

$$w_0 = \pi_{\mathcal{M}}(\widetilde{u}_\varepsilon).$$

By the construction of C_δ in Lemma 6.3.2 it is clear that functions in C_δ have their support in $\Omega \backslash B_{3\delta/4}(0)$, while the support of w_0 is in $\overline{B_{\delta/2}(0)}$, therefore $w_0 \in \mathcal{M} \backslash C_\delta$. Let $v \in C_\delta$ and $s, t \geq 0$. Since v and w_0 have disjoint support,

$$E(sv + tw_0) = E(sv) + E(tw_0). \tag{3.33}$$

By (3.10),

$$E(sv) \leq \frac{s^p}{p}\left(\int_\Omega |\nabla v|^p dx - \lambda \int_\Omega |v|^p dx\right) = -s^p\left(\frac{\lambda}{\Psi(v)} - 1\right) \leq 0 \tag{3.34}$$

since $v \in \Psi^\lambda$. Moreover,

$$E(tw_0) \leq \frac{t^p}{p}\left(\int_\Omega |\nabla w_0|^p dx - \lambda \int_\Omega |w_0|^p dx\right) - \frac{t^{p^*}}{p^*}\int_\Omega |w_0|^{p^*} dx,$$

and maximizing the right-hand side over all $t \geq 0$, and using the above integral estimates, we have

$$\begin{aligned} E(tw_0) &\leq \frac{1}{N}\frac{(\int_\Omega |\nabla w_0|^p dx - \lambda \int_\Omega |w_0|^p dx)^{p^*/(p^*-p)}}{(\int_\Omega |w_0|^{p^*} dx)^{p/(p^*-p)}} \\ &= \frac{1}{N}\frac{(\int_\Omega |\nabla \widetilde{u}_\varepsilon|^p dx - \lambda \int_\Omega |\widetilde{u}_\varepsilon|^p dx)^{p^*/(p^*-p)}}{(\int_\Omega |\widetilde{u}_\varepsilon|^{p^*} dx)^{p/(p^*-p)}} \\ &\leq \frac{1}{N}\begin{cases} \left(S + a_1\varepsilon^{(N-p)/p} - \lambda a_2 \varepsilon^{p-1}\right)^{N/p} & \text{if } N > p^2 \\ \left(S + a_1\varepsilon^{p-1} - \lambda a_2 \varepsilon^{p-1}|\log\varepsilon|\right)^p & \text{if } N = p^2. \end{cases} \end{aligned} \tag{3.35}$$

It follows from (3.33)-(3.35) that

$$\sup_{v \in C_\delta,\ s,t \geq 1} E(sv + tw_0) < \frac{1}{N}S^{N/p}$$

if $\varepsilon > 0$ os sufficiently small. Then for $\mu > 0$ sufficiently small we also have that

$$\sup_{v \in C_\delta,\ s,t \geq 1} E(sv + tw_0) < \frac{1}{N}S^{N/p} - \omega\mu.$$

Note that this can be done since the above calculations does not depend on μ.

The case when $0 < \lambda < \lambda_1$ is handled the same way, but the only modification is that we does not need the set C_δ anymore.

□

CHAPTER 7

The Brezis-Nirenberg problem for the p-Laplacian

In this section we will apply the theory so far developed, with some adjustments, to the famous Brezis-Nirenberg problem. Such kind of problem was first studied by Brezis and Nirenberg in their seminal paper [3] for the case $p = 2$.

Consider the problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u + |u|^{p^*-2} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{0.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $1 < p < N$, $p^* = Np/(N-p)$ is the critical Sobolev exponent, and $\lambda > 0$ is a parameter.

In order to show that under certain conditions problem (0.1) has nontrivial solutions, we will develop an abstract approach to a class of equations that includes (0.1) as a special case.

7.1 Abstract approach

Assume that W is a uniformly convex Banach space and let $(W^*, \|\cdot\|^*)$ be its dual with duality pairing $(\cdot,\cdot)$. Consider the nonlinear operator equation

$$A_p u = \lambda B_p u + f(u) \tag{1.2}$$

in W^*, where $A_p, B_p, f \in C(W, W^*)$ are potential operators, with f odd, satisfying the following assumptions, and $\lambda \in \mathbb{R}$ is a parameter:

(A'_1) A_p is $(p-1)$-homogeneous and odd for some $p \in (1, +\infty)$: $A_p(tu) = |t|^{p-2}tA_p u$ for all $u \in W$ and $t \in \mathbb{R}$,

(A'_2) $(A_p u, v) \leq \|u\|^{p-1}\|v\|$ for all $u, v \in W$, and equality holds if and only if $\alpha u = \beta v$ for some $\alpha, \beta \geq 0$, not both zero,

(B'_1) B_p is $(p-1)$-homogeneous and odd: $B_p(tu) = |t|^{p-2}tB_p u$ for all $u \in W$ and $t \in \mathbb{R}$,

(B'_2) $(B_p u, u) > 0$ for all $u \in W \setminus \{0\}$, and

$$(B_p u, v) \leq (B_p u, u)^{(p-1)/p} (B_p v, v)^{1/p} \quad \text{for all} \quad u, v \in W,$$

(B'_3) B_p is a compact operator,

(F'_1) the potential F of f with $F(0) = 0$ satisfies $F(u) = o(\|u\|^p)$ as $u \to 0$,

(F'_2)

$$\lim_{t \to +\infty} \frac{F(tu)}{|t|^p} = +\infty$$

uniformly on compact subsets of $W \setminus \{0\}$.

By Lemma 5.1.1, solutions of (1.2) coincide with critical points of the C^1-functional

$$\mathcal{E}(u) = I_p(u) - \lambda J_p(u) - F(u), \quad u \in W,$$

where

$$I_p(u) = \frac{1}{p}(A_p u, u) = \frac{1}{p}\|u\|^p, \quad J_p(u) = \frac{1}{p}(B_p u, u)$$

are the potentials of A_p and B_p satisfying $I_p(0) = 0 = J_p(0)$, respectively.

Moreover, eigenvalues of the eigenvalue problem

$$A_p u = \lambda B_p u$$

coincide with critical values of the C^1-functional

$$\Psi(u) = \frac{1}{(B_p u, u)}, \quad u \in \mathcal{S},$$

where $\mathcal{S} = \big\{u \in W : (A_p u, u) = 1\big\}$ is the unit sphere in W. Denote by $\mathcal{F}$ the class of symmetric subsets of $\mathcal{S}$, let $\mathcal{F}_k = \{M \in \mathcal{F} : i(M) \geq k\}$, and set

$$\lambda_k = \inf_{M \in \mathcal{F}_k} \sup_{u \in M} \Psi(u), \quad k \geq 1.$$

Then $\lambda_1 = \inf \Psi(\mathcal{S}) > 0$ is the first eigenvalue and by Theorem 5.2.1 $\lambda_1 \leq \lambda_2 \leq \cdots$ is an unbounded sequence of eigenvalues.

Denote $\mathcal{A}^*$ the class of symmetric subsets of $W \setminus \{0\}$. Let $\mathcal{N}$ be a closed symmetric subset of $W \setminus \{0\}$ and let Γ denote the group of homeomorphisms of W that are the identity outside $\mathcal{E}^{-1}(0, b)$. As in Definition 4.6.1, let us define the pseudo-index of a set $A \in \mathcal{A}^*$ related to the cohomological index i (see Definition 3.2.2), $\mathcal{N}$, and Γ as

$$i^*(A) = \min_{\gamma \in \Gamma} i(\gamma(A) \cap \mathcal{N}). \tag{1.3}$$

For $j \geq 1$, let

$$\mathcal{A}_j^* = \{M \in \mathcal{A}^* : M \text{ is compact and } i^*(M) \geq j\}$$

and set

$$c_j^* = \inf_{M \in A_j^*} \max_{u \in M} \mathcal{E}(u).$$

Also, suppose that W is a closed linear subspace of a Banach space D. We assume that $\mathcal{E}$ has the following compactness properties

(C_1) there exists $c^* > 0$ such that $\mathcal{E}$ satisfies the $(PS)_c$ condition for all $c \in (0, c^*)$,

(C_2) there exist $b > c^*$ and for each $c \in [c^*, b)$ a set $M_c \subset D \setminus \{0\}$ such that

- (i) every $(PS)_c$ sequence (u_n) has either a subsequence that converges strongly to a point in K_c, or a renamed subsequence that converges weakly to a point in K_{c-c^*} and satisfies

  $$\operatorname{dist}(u_n, M_c) \to 0 \quad \text{or} \quad \operatorname{dist}(u_n, -M_c) \to 0,$$

- (ii) $N_\delta(M_c) \cap N_\delta(-M_c) = \emptyset$ for all sufficiently small $\delta > 0$.

We then have the following result.

Theorem 7.1.1. *Assume* (C_1) *and* (C_2). *If* $0 < c_{k+1}^* \leq \cdots \leq c_{k+l}^* < b$ *for some* $k \geq 0$ *and* $l \geq 3$, *then* $\mathcal{E}$ *has at least* $(l-1)/2$ *distinct pairs of critical points at levels in* $(0, b)$.

First we will need to prove the following two lemmas.

Lemma 7.1.1. *Assume* (C_2) *and let* $c \in [c^*, b)$, $B = K^c \cup M_c \cup -M_c$, *and* $\delta > 0$. *Then there exist* $\varepsilon_0 > 0$ *and for each* $\varepsilon \in (0, \varepsilon_0)$ *a map* $\eta \in C(W \times [0,1], W)$ *satisfying*

- (i) $\eta(\cdot, 0)$ *is the identity,*
- (ii) $\eta(\cdot, t)$ *is an odd homeomorphism of* W *for all* $t \in [0,1]$,
- (iii) $\eta(\cdot, t)$ *is the identity outside* $\mathcal{E}_{c-2\varepsilon}^{c+2\varepsilon} \setminus N_{\delta/3}(B)$ *for all* $t \in [0,1]$,
- (iv) $\|\eta(u,t) - u\| \leq \delta/4$ *for all* $(u,t) \in W \times [0,1]$,
- (v) $\mathcal{E}(\eta(u, \cdot))$ *is nonincreasing for all* $u \in W$,
- (vi) $\eta(\mathcal{E}^{c+\varepsilon} \setminus N_\delta(B), 1) \subset \mathcal{E}^{c-\varepsilon}$.

Proof. By (C_2) it is clear that there exists $\varepsilon_0 > 0$ such that for each $\varepsilon \in (0, \varepsilon_0)$,

$$\|\mathcal{E}'(u)\|^* \geq \frac{32\varepsilon}{\delta} \quad \forall u \in \mathcal{E}_{c-2\varepsilon}^{c+2\varepsilon} \backslash N_{\delta/3}(B). \tag{1.4}$$

Let V be an odd pseudo-gradient vector field for $\mathcal{E}$, i.e., a locally Lipschitz continuous mapping from $\{u \in W : \mathcal{E}'(u) \neq 0\}$ to W satisfying

$$\|V(u)\| \leq \|\mathcal{E}'(u)\|^*, \quad 2(\mathcal{E}'(u), V(u)) \geq \left(\|\mathcal{E}'(u)\|^*\right)^2, \quad V(-u) = -V(u), \tag{1.5}$$

this is possible because $\mathcal{E}$ is even.

Now take an even Locally Lipschitz continuous mapping $g : W \to [0,1]$ such that $g = 0$ outside $\mathcal{E}_{c-2\varepsilon}^{c+2\varepsilon} \backslash N_{\delta/3}(B)$ and $g = 1$ on $\mathcal{E}_{c-2\varepsilon}^{c+2\varepsilon} \backslash N_{2\delta/3}(B)$, for instance

$$g(x) = \frac{\operatorname{dist}(x, \left(\mathcal{E}_{c-2\varepsilon}^{c+2\varepsilon} \backslash N_{\delta/3}(B)\right)^c)}{\operatorname{dist}(x, \left(\mathcal{E}_{c-2\varepsilon}^{c+2\varepsilon} \backslash N_{\delta/3}(B)\right)^c) + \operatorname{dist}\left(x, \mathcal{E}_{c-2\varepsilon}^{c+2\varepsilon} \backslash N_{2\delta/3}(B)\right)}.$$

Let $\eta(u,t)$, $0 \leq t \leq T(u) \leq +\infty$ be the maximal solution of

$$\eta' = \begin{cases} -4\varepsilon g(\eta) \dfrac{V(\eta)}{\|V(\eta)\|^2} & \text{if} \quad u \in \mathcal{E}_{c-2\varepsilon}^{c+2\varepsilon} \backslash N_{\delta/3}(B), \\ 0 & \text{otherwise}, \end{cases} \qquad t > 0, \quad \eta(u,0) = u \in W.$$

By (1.4) and (1.5) it is easily seen that

$$\|\eta(u,t) - u\| \leq \frac{\delta t}{4}$$

so $\|\eta(u,\cdot)\|$ is bounded if $T(u) < +\infty$. Therefore, as in [20] §6, Theorem (3) we have $T(u) = +\infty$ and consequently (i)-(iv) easily follow.

Since

$$\frac{d}{dt}(\mathcal{E}(\eta(u,t))) \leq -2\varepsilon g(\eta) \leq 0 \tag{1.6}$$

by (1.5), (v) holds.

Now let $u \in \mathcal{E}^{c+\varepsilon} \backslash N_\delta(B)$ and suppose $\eta(u,1) \notin \mathcal{E}^{c-\varepsilon}$. Then by (v) we have that $\eta(u,t) \in \mathcal{E}_{c-\varepsilon}^{c+\varepsilon}$ for all $t \in [0,1]$ and $\eta(u,t) \notin N_{2\delta/3}(B)$ by (iv). So $\eta(u,t) \in \mathcal{E}_{c-\varepsilon}^{c+\varepsilon} \backslash N_{2\delta/3}(B)$ and hence $g(\eta(u,t)) = 1$ for all $t \in [0,1]$, so (1.6) gives

$$\mathcal{E}(\eta(u,t)) \leq \mathcal{E}(u) - 2\varepsilon \leq c - \varepsilon,$$

a contradiction, hence (vi) follows. $\square$

Lemma 7.1.2. *Assume* (C_1) *and* (C_2). *If* $0 < c_j^* = c_{j+1}^* = c < b$, *then* K^c *is an infinite set.*

Proof. If $c \in (0,c^*)$, then $\mathcal{E}$ satisfies $(PS)_c$ by (C_1) and as in Proposition 4.6.2 the desired conclusion follows. So suppose $c \in [c^*,b)$ and let $B = K^c \cup M_c \cup -M_c$. Suppose that K^c is finite. Then K^c consists of a finite number of pairs of antipodal points, then (C_2) implies that for sufficiently small $\delta > 0$, $N_\delta(B)$ is the disjoint union of $\pm N_\delta(M_c)$ and a finite number of pairs of closed balls centered at antipodal points. So there is an odd continuous map from $N_\delta(B)$ to S^0 and hence $i(N_\delta(B)) \leq 1$.

Let ε_0, $\varepsilon \in (0,\varepsilon_0)$ and η as in Lemma 7.1.1. Since $c^*_{j+1} = c$, there exists $M \in \mathcal{A}^*_{j+1}$ such that $M \subset \mathcal{E}^{c+\varepsilon}$ and hence by Proposition 4.6.1

$$j+1 \leq i^*(M) \leq i^*(\mathcal{E}^{c+\varepsilon}). \tag{1.7}$$

Take $\varepsilon < \min\{c/2,(b-c)/2\}$ and let $\gamma = \eta(\cdot,1)$ so $\gamma \in \Gamma$ because γ is an odd homeomorphism of W and is the identity outside $\mathcal{E}^{-1}(0,b)$. Since $\mathcal{E}^{c+\varepsilon} = N_\delta(B) \cup \mathcal{E}^{c+\varepsilon} \backslash N_\delta(B)$ we have

$$i^*(\mathcal{E}^{c+\varepsilon}) \leq i^*(\mathcal{E}^{c-\varepsilon}) + i(N_\delta(B)) \tag{1.8}$$

by Lemma 7.1.1 and Proposition 4.6.1.

Since $c = c^*_j$, we have

$$i^*(\mathcal{E}^{c-\varepsilon}) \leq j-1 \tag{1.9}$$

By (1.7)-(1.9) we have

$$i(N_\delta(B)) \geq 2,$$

which is a contradiction. □

Proof of Theorem 7.1.1. In view of Lemma 7.1.2 we can assume that

$$0 < c^*_{k+1} < \cdots < c^*_{k+l} < b.$$

We claim that if $\mathcal{E}$ satisfies $(PS)_{c^*_{k+i}}$ condition, then c^*_{k+i} is a critical value for it. Indeed let us denote $c = c^*_{k+i}$ and let $\delta, \varepsilon > 0$ such that $i(K^c) = i(N_\delta(K^c))$ and $\gamma = \eta(\cdot,1)$ given in Lemma 4.2.2 such that γ is an odd homeomorphism and $\gamma \in \Gamma$.

Since

$$\mathcal{E}^{c+\varepsilon} = N_\delta(K^c) \cup \mathcal{E}^{c+\varepsilon} \backslash N_\delta(K^c),$$

we have

$$\gamma(\mathcal{E}^{c+\varepsilon}) \subset \gamma(N_\delta(K^c)) \cup \gamma(\mathcal{E}^{c+\varepsilon} \backslash N_\delta(K^c)).$$

And since $\gamma \in \Gamma$, $\gamma(\mathcal{E}^{c+\varepsilon} \backslash N_\delta(K^c)) \subset \mathcal{E}^{c-\varepsilon}$ it implies that

$$i^*(\mathcal{E}^{c+\varepsilon}) \leq i^*(\mathcal{E}^{c-\varepsilon}) + i(N_\delta(K^c)) \tag{1.10}$$

by Proposition 4.6.1.

By definition of $c = c^*_{k+i}$ we have

$$k+i \leq i^*(\mathcal{E}^{c+\varepsilon}), \quad \text{and} \quad i^*(\mathcal{E}^{c-\varepsilon}) \leq k+i-1. \tag{1.11}$$

Then

$$i(N_\delta(K^c)) \geq 1$$

by (1.10) and (1.11). And since $i(K^c) = (N_\delta(K^c))$, $K^c \neq \emptyset$ by Proposition 3.2.2.

On the other hand, if $\mathcal{E}$ does not satisfies the $(PS)_{c^*_{k+i}}$ condition, then $c^*_{k+i} \in [c^*, b)$ by (C_1) and $\mathcal{E}$ has a $(PS)_{c^*_{k+i}}$ sequence with no convergent subsequence. Then $c^*_{k+i} - c^*$ is a critical level of $\mathcal{E}$ by (C_2). So c^*_{k+i} or $c^*_{k+i} - c^*$ is a critical level of $\mathcal{E}$ in $(0,b)$ for each i such that $c^*_{k+i} \neq c^*$, and it follows that $\mathcal{E}$ has at least $(l-1)/2$ distinct critical levels in $(0,b)$.

□

Now we prove a key result, which is essential in showing that equation (1.2) has nontrivial solutions.

Theorem 7.1.2. *Assume (C_1) and (C_2). Let A_0 and B_0 be symmetric subsets of the unit sphere $S = \{u \in W : \|u\| = 1\}$ such that A_0 is compact, B_0 is closed, and*

$$i(A_0) \geq k+l, \quad i(S \backslash B_0) \leq k \tag{1.12}$$

for some $k \geq 0$ and $l \geq 3$. Assume that there exist $R > r > 0$ such that, setting

$$A = \{Ru : u \in A_0\}, \quad B = \{ru : u \in B_0\}, \quad X = \{tu : u \in A, \ t \in [0,1]\},$$

we have

$$\sup_A \mathcal{E} \leq 0 < \inf_B \mathcal{E}, \quad \sup_X \mathcal{E} < b.$$

Then $\mathcal{E}$ has at least $(l-1)/2$ distinct pairs of critical points at levels in $(0,b)$.

Proof. In the definition of the pseudo-index (1.3), we take $\mathcal{N}$ to be the sphere

$$S_r = \{u \in W : \|u\| = r\},$$

and we will show that

$$0 < \inf_B \mathcal{E} \leq c^*_{k+1} \leq \cdots \leq c^*_{k+l} \leq \sup_X \mathcal{E} < b,$$

and apply Theorem 7.1.1. We note that A and $S_r \backslash B$ are radially homeomorphic to A_0 and $S \backslash B_0$, respectively, and hence

$$i(A) \geq k+l, \quad i(S_r \backslash B) \leq k \tag{1.13}$$

by Proposition 3.2.2 and (1.12).

If $M \in \mathcal{A}^*_{k+1}$, then (1.13) gives

$$i(S_r \backslash B) \leq k < k+1 \leq i^*(M) \leq i(M \cap S_r)$$

since the identity is in Γ, so M intersects B by Proposition 3.2.2-(i_2). Hence

$$c^*_{k+1} \geq \inf_B \mathcal{E}.$$

For $\gamma \in \Gamma$, consider the continuous map

$$\varphi : A \times [0,1] \to W, \quad \varphi(u,t) = \gamma(tu).$$

We have

$$\varphi(A \times [0,1]) = \gamma(X),$$

which is compact. Since γ is odd,

$$\varphi(-u,t) = -\varphi(u,t)$$

for all $(u,t) \in A \times [0,1]$ and $\varphi(A \times [0,1]) = \{\gamma(0)\} = \{0\}$. Since $\mathcal{E} \leq 0$ on A, $\gamma|_A$ is the identity and hence $\varphi(A \times \{1\}) = A$. Applying the piercing property (i_7) of Proposition 3.2.2 with $X_0 = \{u \in W : \|u\| \leq r\}$ and $X_1 = \{u \in W : \|u\| \geq r\}$ gives

$$i(\gamma(X) \cap S_r) = i(\varphi(A \times [0,1]) \cap X_0 \cap X_1) \geq i(A) \geq k+l$$

by (1.13). Hence $i^*(X) \geq k+l$. So $X \in \mathcal{A}^*_{k+l}$ and hence $c^*_{k+l} \leq \sup_X \mathcal{E}$. □

Using Theorem 7.1.2 we can prove the following.

Theorem 7.1.3. *Assume* (C_1) *and* (C_2)*. Let* B_0 *and* C_0 *be symmetric subsets of* S *such that* C_0 *is compact,* B_0 *is closed and*

$$i(C_0) \geq k-m, \quad i(S\backslash B_0) \leq k \tag{1.14}$$

for some $k \geq m \geq 0$*. Assume that there exist an odd continuous map* $\varphi : S^N \to S\backslash C_0$*,* $N \geq m+2$ *and* $R > r > 0$ *such that, setting*

$$A_0 = \begin{cases} \varphi(S^N) & \text{if } C_0 = \emptyset \\ \left\{\pi((1-t)v+tw) : v \in C_0,\ w \in \varphi(S^N),\ t \in [0,1]\right\} & \text{if } C_0 \neq \emptyset, \end{cases}$$

$$A = \{Ru : u \in A_0\}, \quad B = \{ru : u \in B_0\}, \quad X = \{tu : u \in A,\ t \in [0,1]\},$$

we have

$$\sup_A \mathcal{E} \leq 0 < \inf_B \mathcal{E}, \quad \sup_X \mathcal{E} < b.$$

Then $\mathcal{E}$ *has at least* $(N-m)/2$ *distinct pairs of critical points at levels in* $(0,b)$*.*

Proof. Let us first consider the case where $C_0 \neq \emptyset$.

Recall that ΣC_0 denotes the suspension of C_0, which is obtained as the quotient space of $C_0 \times [-1,1]$ with $C_0 \times \{1\}$ and $C_0 \times \{-1\}$ collapsed to different points. Let $\Sigma^{N+1}C_0$ be the $(N+1)$-fold suspension consisting of points $(v,t_1,\dots,t_{N+1})$, where $v \in C_0$ and $t_j \in [-1,1]$ for $j = 1,\dots,N+1$, with the appropriate identifications for $t_j = \pm 1$, for instance if $N = 1$ then we have, under such identifications

$$(v,t_1,\pm 1) = (w,\bar{t}_1,\pm 1), \quad \text{for all } v,w \in C_0,\ t_1,\bar{t}_1 \in [-1,1]$$

$$(v,\pm 1,t_2) = (w,\pm 1,t_2), \quad \text{for all } v,w \in C_0,\ t_2 \in [-1,1].$$

Now set

$$p_0 = \prod_{l=1}^{N+1}(1-|t_l|), \quad p_j = |t_j| \prod_{l=j+1}^{N+1}(1-|t_l|), \ \text{for } j = 1,\dots,N, \quad p_{N+1} = |t_{N+1}|.$$

It is worth noting that the maps

$$\Sigma^{N+1}C_0 \to \mathbb{R}, \quad (v,t_1,\dots,t_{N+1}) \mapsto p_j$$

are well defined and continuous for all $j = 0,1,\dots,N+1$, indeed the continuous property is evident once it is well defined, and it is in fact well defined since the p_j are constant on collapsed points. Let $\{e_1,\dots,e_{N+1}\}$ be the standard unit basis of $\mathbb{R}^{N+1}$, and let

$$\varpi : \mathbb{R}^{N+1}\setminus\{0\} \to S^N, \quad x \mapsto \frac{x}{|x|}$$

be the radial projection onto S^N. Let us define the map

$$\begin{aligned}\Sigma^{N+1}C_0 \to A_0, \quad (v,t_1,\dots,t_{N+1}) &\mapsto \pi\left(p_0 v + (1-p_0)\varphi\left(\frac{\sum_{j=1}^{N+1} p_j \operatorname{sgn} t_j e_j}{\sqrt{\sum_{j=1}^{N+1} p_j^2}}\right)\right)\\ &= \pi\left(p_0 v + (1-p_0)\varphi\left(\varpi\left(\sum_{j=1}^{N+1} p_j \operatorname{sgn} t_j e_j\right)\right)\right).\end{aligned}$$

Such map is well defined, odd and continuous since

$$\sum_{j=1}^{N+1} p_j \operatorname{sgn} t_j e_j \neq 0, \quad \text{for all } (v,t_1,\dots,t_{N+1}) \in \Sigma^{N+1}C_0,$$

and continuous because $p_j \operatorname{sgn} t_j = t_j \prod_{l=j+1}^{N+1}(1-|t_l|)$.

Therefore

$$i(A_0) \geq i(\Sigma^{N+1}C_0) = i(C_0) + N + 1 \geq k + N - m + 1$$

by Proposition 3.2.2- (i_2), (i_6) and (1.14). So the conclusion follows from Theorem 7.1.2.

If $C_0 = \emptyset$, then $k = m$ by (1.14) and Proposition 3.2.2- (i_1), and

$$i(A_0) = i(\varphi(S^N)) \geq i(S^N) = N + 1 = k + N - m + 1$$

by Proposition 3.2.2 (i_2) and (i_8), so the conclusion follows from Theorem 7.1.2.

□

Now we have the tools to prove the most important theorem of this abstract setting, which will allows us to prove the existence of nontrivial solutions of problem (0.1).

Theorem 7.1.4. *Suppose* (A_1')-(F_2') *hold and* $\mathcal{E}$ *satisfies* (C_1) *and* (C_2). *Assume that* $\lambda < \lambda_{k+1}$, *there exists a compact symmetric subset* C_0 *of* S *with* $i(C_0) \geq k - m$ *for some* $0 \leq m \leq k$, *and there exists an odd continuous map* $\varphi : S^N \to S\setminus C_0$, $N \geq m+2$ *such that*

$$\sup_{w\in\varphi(S^N),\, t\geq 0} \mathcal{E}(tw) < b \quad \text{if } C_0 = \emptyset \tag{1.15}$$

and

$$\sup_{v\in C_0,\ w\in\varphi(S^N),\ s,t\geq 0} \mathcal{E}(sv+tw)<b \quad \text{if } C_0\neq\emptyset. \tag{1.16}$$

Then equation (1.2) *has* $N-m$ *distinct pairs of nontrivial solutions satisfying*

$$0<\mathcal{E}(u)<b.$$

Proof. We apply Theorem 7.1.3 with $B_0=\Psi_{\lambda_{k+1}}$. By Proposition 4.5.2, $i(S\backslash\Psi_{\lambda_{k+1}})\leq k$. For $u\in S$ and $t>0$,

$$\mathcal{E}(tu)=\frac{t^p}{p}\left(1-\frac{\lambda}{\Psi(u)}\right)-F(tu).$$

Since $\Psi(u)>0$ by (B_2'), this gives

$$\frac{t^p}{p}\left(1-\frac{\lambda^+}{\Psi(u)}\right)-F(tu)\leq\mathcal{E}(tu)\leq\frac{t^p}{p}\left(1+\frac{\lambda^-}{\Psi(u)}\right)-F(tu),$$

where $\lambda^{\pm}=\max\{\pm\lambda,0\}$. Since $\lambda^+<\lambda_{k+1}$, the first inequality and (F_1') imply that $\inf_B\mathcal{E}>0$ if $r>0$ is sufficiently small. Since A_0 is a compact subset of $W\setminus\{0\}$, the second inequality and (F_2') imply that $\sup_A\mathcal{E}\leq 0$ if $R>r$ is sufficiently large. By (1.15) and (1.16), $\sup_X\mathcal{E}<b$. □

7.2 Compactness condition for problem (0.1)

In order to show the existence of nontrivial solutions of (0.1) we will apply Theorem 7.1.4 to the energy functional associated to (0.1)

$$E(u)=\frac{1}{p}\int_\Omega|\nabla u|^p dx-\frac{\lambda}{p}\int_\Omega|u|^p dx-\frac{1}{p^*}\int_\Omega|u|^{p^*}dx, \quad u\in W_0^{1,p}(\Omega). \tag{2.17}$$

Firstly we need to show that E satisfies the compactness conditions (C_1) and (C_2). Let us first set some notations. Let

$$S_{N,p}=\inf_{u\in D^{1,p}(\mathbb{R}^N)\setminus\{0\}}\frac{\int_{\mathbb{R}^N}|\nabla u|^p dx}{\left(\int_{\mathbb{R}^N}|u|^{p^*}dx\right)^{p/p^*}} \tag{2.18}$$

denote the best Sobolev constant. Recall that the infimum in (2.18) is attained on the functions

$$u_{\varepsilon,y}(x)=\frac{c_{N,p}\varepsilon^{(N-p)/p(p-1)}}{\left(\varepsilon^{p/(p-1)}+|x-y|^{p/(p-1)}\right)^{(N-p)/p}}, \quad \varepsilon>0, y\in\mathbb{R}^N, \tag{2.19}$$

where the constant $c_{N,p} > 0$ is chosen so that

$$\int_{\mathbb{R}^N} |\nabla u_{\varepsilon,y}|^p dx = \int_{\mathbb{R}^N} |u_{\varepsilon,y}|^{p^*} dx = S_{N,p}^{N/p}.$$

Solutions of the equation

$$-\Delta_p u = |u|^{p^*-2} u \tag{2.20}$$

in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ coincide with critical points of the functional

$$E_\infty(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx, \quad u \in \mathcal{D}^{1,p}(\mathbb{R}^N).$$

Denote by $\mathbb{R}^N_+ = \{x = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_N > 0\}$ the upper-half space in $\mathbb{R}^N$ and by $\mathcal{D}_0^{1,p}(\mathbb{R}^N_+)$ the closure of $C_0^\infty(\mathbb{R}^N_+)$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ after extending by zero on $\mathbb{R}^N \backslash \mathbb{R}^N_+$. Set

$$c^* = \frac{1}{N} S_{N,p}^{N/p}.$$

We will need the following nonexistence result.

Lemma 7.2.1. *Let* $1 < p < N$, $\mu \in \mathbb{R}_+$ *and let* $u \in \mathcal{D}_0^{1,p}(\mathbb{R}^N_+)$ *be a nonnegative solution of the equation*

$$-\Delta_p u = \mu u^{p^*-1} \quad \text{in} \quad \mathbb{R}^N_+.$$

Then $u \equiv 0$.

Proof. [17], Theorem 1.1. □

Lemma 7.2.2. *Let* u *be a nontrivial weak solution of the equation* (2.20) *in* $\mathcal{D}^{1,p}(\mathbb{R}^N)$ *or* $\mathcal{D}_0^{1,p}(\mathbb{R}^N_+)$. *Then*

$$E_\infty(u) \geq c^*. \tag{2.21}$$

If $u \in \mathcal{D}(\mathbb{R}^N_+)$, *then this inequality is strict. If* u *is sign-changing, then*

$$E_\infty(u) \geq 2c^*. \tag{2.22}$$

Proof. Let u be a solution of (2.20) in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ or $\mathcal{D}_0^{1,p}(\mathbb{R}^N_+)$. Since u satisfies the equality (2.20) (in the weak sense) we have that

$$\int |\nabla u|^p dx = \int |u|^{p^*} dx \tag{2.23}$$

and this together with the Sobolev inequality gives us

$$\int |\nabla u|^p \geq S_{N,p} \left(\int |u|^{p^*} \right)^{p/p^*} = S_{N,p} \left(\int |\nabla u|^p dx \right)^{p/p^*},$$

and consequently

$$\left(\int |\nabla u|^p \right) \geq S_{N,p}^{N/p}. \tag{2.24}$$

By (2.23) and (2.24) we have

$$\begin{aligned} E_\infty(u) &= \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{1}{p^*} \int_{\mathbb{R}^N} |u|^{p^*} dx \\ &= \left(\frac{1}{p} - \frac{1}{p^*} \right) \int_{\mathbb{R}^N} |\nabla u|^p dx \\ &\geq \frac{1}{N} S_{N,p}^{N/p}, \end{aligned}$$

which is (2.21). Now suppose that $u \in \mathcal{D}_0^{1,p}(\mathbb{R}_+^N)$ and $E_\infty(u) = c^*$. By what we just did above we get

$$\int_{\mathbb{R}^N} |\nabla u|^p dx = S_{N,p}^{N/p} = \int_{\mathbb{R}^N} |u|^{p^*} dx,$$

therefore

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^p}{(\int_{\mathbb{R}^N} |u|^{p^*} dx)^{p/p^*}} = S_{N,p},$$

and consequently u minimizes (2.18). We can suppose $u \geq 0$, otherwise take $|u|$ instead, which will clearly minimize (2.18) as well. So by Lagrange multipliers we have that u satisfies

$$-\Delta_p u = \mu u^{p^*-1} \quad \text{in} \quad \mathbb{R}_+^N$$

for some $\mu \in \mathbb{R}$, so by Lemma 7.2.1 we have $u \equiv 0$, which is a contradiction, because $E_\infty(u) > 0$, therefore the inequality (2.21) must be strict. Now suppose u is sign-changing. Since u satisfies (2.20), we have in particular

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla u^\pm dx = \int_{\mathbb{R}^N} |u|^{p^*-2} u u^\pm dx$$

which leads to

$$\int_{\mathbb{R}^N} |\nabla u^\pm|^p dx = \int_{\mathbb{R}^N} |u^\pm|^{p^*} dx,$$

and similarly as we did before,

$$E_\infty(u^+) \geq c^*, \quad E_\infty(u^-) \geq c^*,$$

and noting that $E_\infty(u) = E_\infty(u^+) + E_\infty(u^-)$ we have (2.22). □

Lemma 7.2.3. *Let (Ω_k) be a sequence of open subsets of Ω such that $\Omega_k \subset \Omega_{k+1}$ and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. Let $p > 1$, $\{v_n\} \subset W^{1,p}(\Omega)$ be such that $v_n \rightharpoonup v$ in $W^{1,p}(\Omega)$ and, for every k,*

$$\lim_{n\to\infty} \int_{\Omega_k} (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v)\nabla T(v_n - v)dx = 0,$$

where T is as in Lemma 2.3.1. Then

1. *$\nabla v_n \to \nabla v$ a.e. on Ω passing if necessary to a subsequence,*
2. *$\lim_{n\to\infty} (\int_\Omega |\nabla v_n|^p - |\nabla(v_n - v)|^p - |\nabla v|^p dx) = 0$,*
3. *$|\nabla v_n|^{p-2}\nabla v_n - |\nabla(v_n - v)|^{p-2}\nabla(v_n - v) \to |\nabla v|^{p-2}\nabla v$ in $L^{p/(p-1)}(\Omega : \mathbb{R}^N)$.*

Proof. [17] Theorem 3.3. □

Lemma 7.2.4. *Let $p > 1$ and define $A : \mathbb{R}^N \to \mathbb{R}^N$ by $A(y) = |y|^{p-2}y$. Let $(u_n) \subset L^p(\Omega;\mathbb{R}^N)$ be such that $u_n \to u$ a.e. on Ω and $\sup_n |u_n|_p < \infty$. Then*

$$\lim_{n\to\infty} \int_\Omega |A(u_n) - A(u_n - u) - A(u)|^{p/(p-1)} dx = 0.$$

Proof. [17] Lemma 3.2. □

Lemma 7.2.5. *Let $c \in \mathbb{R}$, $\{u_n\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega)$ and $u \in W_0^{1,p}(\Omega)$ such that*

$$\begin{aligned}
&u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,p}(\Omega),\\
&u_n \to u \quad \text{a.e. on} \quad \Omega,\\
&E(u_n) \to c,\\
&E'(u_n) \to 0 \quad \text{in} \quad W^{-1,p'}(\Omega).
\end{aligned}$$

Then, passing if necessary to a subsequence, $\nabla u_n \to \nabla u$ a.e. on Ω and $E'(u) = 0$. Moreover $v_n := u_n - u$ is such that

1. *$\lim_{n\to\infty} (\|u_n\|^p - \|v_n\|^p) = \|u\|^p$,*
2. *$E_\infty(v_n) \to c - E(u)$,*
3. *$E'_\infty(v_n) \to 0$ in $W^{-1,p'}(\Omega)$.*

Proof. Let us take $T : \mathbb{R} \to \mathbb{R}$ defined as $T(s) = s$ if $|s| \leq 1$ and $T(s) = s/|s|$ if $|s| > 1$. Since T is bounded, and $u_n \to u$ a.e. on Ω, by the Dominated Convergence Theorem, we have that

$\int_\Omega |T(u_n-u)|^q dx \to 0$ as $n\to\infty$ for all $q>1$. Moreover $T(u_n-u) \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$. Hence we have

$$\begin{aligned}
&\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u\right)\nabla T(u_n-u)dx \\
= \;& (E'(u_n),T(u_n-u)) - \int_\Omega |\nabla u|^{p-2}\nabla u \nabla T(u_n-u)dx \\
+ \;& \int_\Omega \left(\lambda |u_n|^{p-2}u_n + |u_n|^{p^*-2}u_n\right) T(u_n-u)dx \to 0.
\end{aligned}$$

Statement 1. and the fact that $\nabla u_n \to \nabla u$ a.e. on Ω follows from Lemma 7.2.3. This together with the compact embedding and Brezis-Lieb lemma we have statement 2.:

$$\begin{aligned}
E_\infty(v_n) &= E(v_n)+o(1) \\
&= E(u_n)-E(u)+o(1) \\
&= c-E(u)+o(1).
\end{aligned}$$

Since $\nabla u_n \to \nabla u$ a.e. on Ω, and $|\nabla u_n|^{p-2}\nabla u_n$ is bounded in $L^{p/(p-1)}(\Omega,\mathbb{R}^N)$, $|\nabla u_n|^{p-2}\nabla u_n \rightharpoonup h$ in $L^{p/(p-1)}(\Omega,\mathbb{R}^N)$, and by Proposition 8.0.1, $h=|\nabla u|^{p-2}\nabla u$. Similarly we have that $|u_n|^{p-1}u_n \rightharpoonup |u|^{p-2}u$ in $L^{p/(p-1)}(\Omega)$ and $|u_n|^{p^*-2}u_n \rightharpoonup |u|^{p^*-2}u$ in $L^{p^*/(p^*-1)}(\Omega)$. Since $E'(u_n)\to 0$ in $W^{-1,p'}(\Omega)$, we then have that

$$\begin{aligned}
(E'(u),v) &= \int_\Omega \left(|\nabla u|^{p-2}\nabla u\nabla v - \lambda|u|^{p-2}uv - |u|^{p^*-2}uv\right)dx \\
&= \lim_{n\to\infty}\int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n\nabla v - \lambda|u_n|^{p-2}u_n v - |u_n|^{p^*-2}u_n v\right)dx \\
&= \lim_{n\to\infty}(E'(u_n),v)=0,
\end{aligned}$$

for all $v\in W_0^{1,p}(\Omega)$, therefore $E'(u)=0$.

Finally, Lemma 7.2.4 yields statement 3. :

$$\begin{aligned}
E'_\infty(v_n) &= E'(v_n)+o(1) \\
&= E'(u_n)-E'(u)+o(1) \\
&= o(1).
\end{aligned}$$

□

Lemma 7.2.6. *Let* $\{y_n\}_{n\in\mathbb{N}}\subset\Omega$ *and* $\{\varepsilon_n\}_{n\in\mathbb{N}}\subset(0,\infty)$ *be such that*

$$\varepsilon_n^{-1} dist\,(y_n,\partial\Omega)\to\infty \tag{2.25}$$

Assume that the sequence $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ *and the sequence*

$$v_n(x) = \varepsilon_n^{(N-p)/p} u_n(\varepsilon_n x + y_n)$$

are such that

$$\begin{aligned} &v_n \rightharpoonup v \quad in \quad \mathcal{D}^{1,p}(\mathbb{R}^N),\\ &v_n \to v \quad a.e.\ on \quad \mathbb{R}^N,\\ &E_\infty(u_n) \to c,\\ &E'_\infty(u_n) \to 0 \quad in \quad W^{-1,p'}(\Omega). \end{aligned}$$

Then, passing to a subsequence, $\nabla v_n \to \nabla v$ *a.e. on* $\mathbb{R}^N$ *and* $E'_\infty(v) = 0$. *Moreover, the sequence*

$$w_n = u_n - (\varepsilon_n)^{(p-N)/p} v(\varepsilon_n^{-1}(\cdot - y_n))$$

satisfies

- *i)* $\lim_{n\to\infty}(\|u_n\|^p - \|w_n\|^p) = \|v\|^p$,
- *ii)* $E_\infty(w_n) \to c - E_\infty(v)$,
- *iii)* $E'_\infty(w_n) \to 0$ *in* $W^{-1,p'}(\Omega)$.

Proof. 1) We first prove the existence of a subsequence on ∇v_n converging a.e. on $\mathbb{R}^N$.

For $h \in C_0^\infty(\mathbb{R}^N)$, we define

$$h_n = \varepsilon_n^{(p-N)/p} h((\cdot - y_n)/\varepsilon_n).$$

Let us also define $B_k := B(0,k)$. For every n large enough, if $h \in C_0^\infty(B_k)$ then, by assumption (2.25), $h_n \in C_0^\infty(\Omega)$. Indeed, the only chance of $h_n(z) \neq 0$ is that $\varepsilon^{-1}(z - y_n) \in B_k$, so we must have necessarily that $z \in \varepsilon_n B_k + y_n$. Therefore the support of h_n is in $\varepsilon_n B_k + y_n$. Now, since $(\varepsilon_n)^{-1}\text{dist}\,(y_n, \partial\Omega) \to \infty$, there exists $N_k \in \mathbb{N}$ such that if $n > N_k$ we have

$$\text{dist}\,(y_n, \partial\Omega) > \varepsilon_n k,$$

but this means that $\varepsilon_n B_k + y_n = B(y_n, \varepsilon_n k) \subset \Omega$ for all $n > N_k$. Take n big such that $h_n \in C_0^\infty(\Omega)$, then we have, by changes of variables

$$|(E'_\infty(v_n), h)| = |(E'_\infty(u_n), h_n)| \leq \|E'_\infty(u_n)\|^* \|h_n\| = \|E'_\infty(u_n)\|^* \|h\|.$$

Hence, by apporoximation we have $E'_\infty(v_n) \to 0$ in $W^{-1,p'}(B_k)$. Let $\rho \in C_0^\infty(\mathbb{R}^N)$ be such that $0 \leq \rho \leq 1$

and

$$\rho(x) = \begin{cases} 1, & |x| \le k, \\ 0, & |x| \ge k+1. \end{cases}$$

Consider the vector field

$$f_n = |\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v.$$

We have

$$\left|\int_{B_k} f_n \cdot \nabla T(v_n - v) dx\right| \le \left|\int_{\mathbb{R}^N} f_n \cdot \nabla [\rho T(v_n - v)]\right| + \left|\int_{\mathbb{R}^N} T(v_n - v) f_n \cdot \nabla \rho\right|.$$

Since T is bounded, it is clear that

$$\int_{\mathbb{R}^N} T(v_n - v) f_n \cdot \nabla \rho dx \to 0, \quad \text{as} \quad n \to \infty.$$

Moreover

$$\begin{aligned} \int_{\mathbb{R}^N} f_n \cdot \nabla [\rho T(v_n - v)]\, dx &= (E'_\infty(v_n), \rho T(v_n - v)) \\ &+ \int_{\mathbb{R}^N} |v_n|^{p^*-2} v_n \rho T(v_n - v) dx \\ &+ \int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \cdot \nabla [\rho T(v_n - v)]\, dx \end{aligned}$$

so that

$$\int_{\mathbb{R}^N} f_n \cdot \nabla [\rho T(v_n - v)]\, dx \to 0, \quad n \to \infty.$$

Finally,

$$\int_{B_k} f_n \cdot \nabla T(v_n - v) dx \to 0, \quad n \to \infty,$$

and it suffices to employ Lemma 7.2.3.

2) Using Lemma 7.2.3, we obtain

$$\lim_{n \to \infty} (\|v_n\|^p - \|v_n - v\|^p) = \|v\|^p$$

or

$$\lim_{n \to \infty} (\|u_n\|^p - \|w_n\|^p) = \|v\|^p.$$

It follows then from the Brezis-Lieb Lemma that

$$\begin{aligned} E_\infty(w_n) &= E_\infty(v_n - v) \\ &= E_\infty(v_n) - E_\infty(v) + o(1) \\ &= E_\infty(u_n) - E_\infty(v) + o(1) \\ &= c - E_\infty(v) + o(1). \end{aligned}$$

3) Since, for every $h \in C_0^\infty(\mathbb{R}^N)$, we have, simlilarly as in the proof of Lemma 7.2.5, that

$$(E'_\infty(v_n), h) \to 0, \quad (E'_\infty(v_n), h) \to (E'_\infty(v), h),$$

so it is clear that $E'_\infty(v) = 0$.

4) For $g \in C_0^\infty(\Omega)$ we define

$$g_n(x) = \varepsilon_n^{(N-p)/p} g(\varepsilon_n x + y_n).$$

By combining Lemma 7.2.5 and Lemma 7.2.4, we obtain, uniformly on $g \in C_0^\infty(\Omega)$, $\|g\| = 1$,

$$\begin{aligned} (E'_\infty(w_n), g) &= (E'_\infty(v_n - v), g_n) \\ &= (E'_\infty(v_n), g_n) - (E'_\infty(v), g_n) + o(1) \\ &= (E'_\infty(u_n), g) + o(1) = o(1). \end{aligned}$$

□

Theorem 7.2.1. *Let $c \in \mathbb{R}$ and let $(u_n) \subset W_0^{1,p}(\Omega)$ be a $(PS)_c$ sequence for E. Then, passing to a subsequence if necessary, there exist a possibly nontrivial solution $u \in W_0^{1,p}(\Omega)$ of problem (0.1), $k \in \mathbb{N} \cup \{0\}$, nontrivial solutions v_i, $i = 1, \ldots, k$ of equation (2.20) in H_i, where H_i is $\mathbb{R}^N$ or (up to a translation and a rotation) $\mathbb{R}^N_+$, with $v_i \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ if $H_i = \mathbb{R}^N$ and $v_i \in \mathcal{D}_0^{1,p}(\mathbb{R}^N_+)$ if $H_i = \mathbb{R}^N_+$, and sequences $(y_n^i) \subset \overline{\Omega}$ and $(\varepsilon_n^i) \subset \mathbb{R}_+$ such that*

$$(\varepsilon_n^i)^{-1} dist\,(y_n^i, \partial\Omega) \to \infty \quad as \quad n \to \infty \quad if \quad H_i = \mathbb{R}^N,$$

$$(\varepsilon_n^i)^{-1} dist\,(y_n^i, \partial\Omega) \quad is\ bounded \quad if \quad H_i = \mathbb{R}^N_+,$$

$$\left\| u_n - u - \sum_{i=1}^k \left(\varepsilon_n^i\right)^{-(N-p)/p} v_i\left((\cdot - y_n^i)/\varepsilon_n^i\right) \right\| \to 0 \quad as \quad n \to \infty, \tag{2.26}$$

$$\|u_n\|^p \to \|u\|^p + \sum_{i=1}^k \|v_i\|^p \quad as \quad n \to \infty,$$

$$E(u) + \sum_{i=1}^k E_\infty(v_i) = c. \tag{2.27}$$

Proof. For sake of clarity we divide the proof in 6 steps.

(1) Suppose $(u_n) \subset W_0^{1,p}(\Omega)$ is such that

$$E(u_n) \to c, \quad E'(u_n) \to 0. \tag{2.28}$$

We claim that (u_n) is a bounded sequence. Indeed, since $E'(u_n) \to 0$, for all n big enough we have $|(E'(u_n), u_n)| \leq p\|u_n\|$, and denoting $d := \sup E(u_n)$, we have

$$d + \|u_n\| \geq E(u_n) - \frac{1}{p}(E'(u_n), u_n) = \frac{1}{N}\int_\Omega |u|^{p^*} dx. \tag{2.29}$$

Since Ω has finite measure and $p < p^*$, by the Holder inequality there is a positive constant c_1 such that

$$\int_\Omega |u|^p dx \leq c_1 \left(\int_\Omega |u|^{p^*} dx\right)^{p/p^*},$$

so by (2.29) we get

$$\int_\Omega |u_n|^p dx \leq c_2 (d + \|u_n\|)^{p/p^*} \leq c_3 + c_4 \|u_n\|^{p/p^*}$$

for some constants $c_2, c_3, c_4 > 0$. Since $p < p^*$, we also have

$$d + \|u_n\| \geq E(u_n) - \frac{1}{p^*}(E'(u_n), u_n) = \frac{1}{N}\left[\int_\Omega |\nabla u|^p dx - \lambda \int_\Omega |u_n|^p dx\right].$$

By the last two inequalities we have

$$D_1 + D_2 \|u_n\|^{p/p^*} + D_3 \|u_n\| \geq D_4 \|u_n\|^p$$

for some constants $D_1, D_2, D_3, D_4 > 0$, and since $p > 1$ we have that $\|u_n\|$ is bounded.

(2) Passing if necessary to a subsequence, we can assume $u_n \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$ and $u_n \to v_0$ a.e. on Ω for some $v_0 \in W_0^{1,p}(\Omega)$. By Lemma 7.2.5, it follows that $E'(v_0) = 0$ and $u_n^1 := u_n - v_0$ is such that

i) $\|u_n^1\|^p = \|u_n\|^p - \|v_0\|^p + o(1)$,
ii) $E_\infty(u_n^1) = c - E(v_0)$,
iii) $E'_\infty(u_n^1) \to 0$, in $W^{-1,p'}(\Omega)$.

(3) If $u_n^1 \to 0$ in $L^{p^*}(\Omega)$, since $E'_\infty(u_n^1) \to 0$, we have that $u_n^1 \to 0$ in $W_0^{1,p}(\Omega)$ and the proof is complete. Otherwise we can assume that

$$\int_\Omega |u_n^1|^{p^*} dx > \delta$$

for some $0<\delta<(S_{N,p}/2)^{N/p}$. Introducing the Levy concentration function

$$Q_n(r):=\sup_{y\in\overline{\Omega}}\int_{B(y,r)}|u_n^1|^{p^*}dx,$$

since $Q_n(0)=0$ and $Q_n(\infty)>\delta$, and Q_n is continuous, there exists a sequence $\{\varepsilon_n^1\}_{n\in\mathbb{N}}\subset(0,\infty)$ and a sequence $\{y_n^1\}_{n\in\mathbb{N}}\subset\overline{\Omega}$ such that

$$\delta=\sup_{y\in\overline{\Omega}}\int_{B(y,\varepsilon_n^1)}|u_n^1|^{p^*}dx=\int_{B(y_n^1,\varepsilon_n^1)}|u_n^1|^{p^*}dx.$$

We define on

$$\Omega_n:=\frac{1}{\varepsilon_n^1}\left(\Omega-y_n^1\right)$$

the sequence $v_n^1(x)=(\varepsilon_n^1)^{(N-p)/p}u_n^1(\varepsilon_n^1x+y_n^1)$. We can assume that $v_n^1\rightharpoonup v_1$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and $v_n^1\to v_1$ a.e. on $\mathbb{R}^N$. Observe also that

$$\delta=\sup_{y\in\mathbb{R}^N}\int_{B(y,1)}|v_n^1|^{p^*}dx=\int_{B(0,1)}|v_n^1|^{p^*}dx. \tag{2.30}$$

(4) We claim that $v_1\neq 0$. Indeed let $f_n=\left(f_n^1,\dots,f_n^N\right)\in\left(L^{p'}(\Omega)\right)^N$ defined by the representation

$$(E'_\infty(u_n^1),h)=\sum_{i=1}^N\int_\Omega f_n^i\frac{\partial h}{\partial x_i}dx,\quad\forall h\in W_0^{1,p}(\Omega).$$

Define $g_n:=(\varepsilon_n^1)^{(N-p)/p}f_n(\varepsilon_n^1x+y_n^1)$. It is clear that

$$(E'_\infty(v_n^1),h)=\sum_{i=1}^N\int_\Omega g_n^i\frac{\partial h}{\partial x_i}dx,\quad\forall h\in W_0^{1,p}(\Omega)$$

and, since $E'_\infty(u_n^1)\to 0$,

$$\sum_{i=1}^N\int_{\Omega_n}|g_n^i|^{p'}dx=\sum_{i=1}^N\int_\Omega|f_n^i|^{p'}dx=o(1).$$

Suppose, by contradiction, that $v_1=0$. Then we can assume that $v_n^1\to 0$ in $L^p_{\text{loc}}(\mathbb{R}^N)$. Take $h\in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}h\subset B(y,1)$ for some $y\in\mathbb{R}^N$. From the Holder inequality, it follows that

$$\int|h|^p|v_n^1|^{p^*}dx\le\left(\int_{\operatorname{supp}h}|v_n^1|^{p^*}dx\right)^{p/N}\left(\int|hv_n|^{p^*}dx\right)^{(N-p)/N}$$

and from the Sobolev inequality

$$\int |h|^p |v_n^1|^{p^*} dx \le S_{N,p}^{-1} \left(\int_{\operatorname{supp} h} |v_n^1|^{p^*} dx \right)^{p/N} \int |\nabla (h v_n^1)|^p dx.$$

Hence, since $v_n^1 \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$, we have

$$\begin{aligned}
\int_{\Omega_n} |\nabla (h v_n^1)|^p dx &= \int |h|^p |\nabla v_n^1|^p + o(1) \\
&= \int |\nabla v_n^1|^{p-2} \nabla v_n^1 \nabla (|h|^p v_n^1) dx + o(1) \\
&= \int |h|^p |v_n^1|^{p^*} dx + \sum_{i=1}^N \int_{\Omega_n} g_n^i \frac{\partial (|h|^p v_n^1)}{\partial x_i} dx + o(1) \\
&\le S_{N,p}^{-1} \delta^{p/N} \int |\nabla (h v_n^1)|^p dx + o(1) \\
&\le \frac{1}{2} \int |\nabla (h v_n^1)|^p dx + o(1).
\end{aligned}$$

As a consequence we have that $\nabla v_n^1 \to 0$ in $L^p_{\mathrm{loc}}(\mathbb{R}^N)$ (just use a partition of unity) and by the Sobolev inequality we have that $v_n^1 \to 0$ in $L^{p^*}_{\mathrm{loc}}(\mathbb{R}^N)$. Because of (2.30), this is a contradiction. Hence $v_1 \neq 0$.

(5) Since Ω is bounded we may assume $y_n^1 \to y_0^1 \in \overline{\Omega}$ and $\varepsilon_n^1 \to \varepsilon_0^1 \ge 0$. If $\varepsilon_0^1 > 0$ then, as a consequence of the fact that $u_n^1 \rightharpoonup 0$ in $W_0^{1,p}(\Omega)$, we have $v_n^1 \rightharpoonup 0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ and this is impossible. Then we can assume that $\varepsilon_n^1 \to 0$, and we have two possibilities, the first one is that

$$\sup_{n \in \mathbb{N}} \left((\varepsilon_n^1)^{-1} \operatorname{dist}(y_n^1, \partial\Omega) \right) < \infty.$$

In that case, we have $y_0^1 \in \partial\Omega$, and therefore the limit domain of v_1 is a halspace H (H consists of the halspace determined by the tangent space $T_{y_0^1}\partial\Omega$, and contains the inward normal vector at y_0^1). The second possibility is that

$$(\varepsilon_n^1)^{-1} \operatorname{dist}(y_n^1, \partial\Omega) \to \infty.$$

In that case we have that the limit domain of v_1 is $\mathbb{R}^N$. As in Lemma 7.2.6 it is easy to see that v_1 satisfies

$$\begin{aligned}
-\Delta_p u &= |u|^{p^*-2} u \quad \text{in} \quad \mathcal{H} \\
u &= 0 \quad \text{on} \quad \partial\mathcal{H},
\end{aligned}$$

i.e. $E'_\infty(v_1) = 0$, where $\mathcal{H}$ is either a halfspace H or $\mathbb{R}^N$, depending whether the expression $\left((\varepsilon_n^1)^{-1} \operatorname{dist}(y_n^1, \partial\Omega) \right)$ is bounded or not. The sequence

$$u_n^2(x) := u_n^1(x) - (\varepsilon_n^1)^{(p-N)/p} v_1((x - y_n^1)/\varepsilon_n^1)$$

satisfies

$$\begin{aligned}
&\|u_n^2\|^p = \|u_n\|^p - \|v_0\|^p - \|v_1\|^p + o(1)\\
&E_\infty(u_n^2) = c - E(v_0) - E_\infty(v_1) + o(1)\\
&E'_\infty(u_n^2) = o(1).
\end{aligned}$$

(6) We have that the above procedure iterates constructing sequences $\{v^i\}$, $\{\varepsilon_n^i\}$ and $\{v_n^i\}$ but only a finite number of iterations is allowed, since $E_\infty(v_i) \geq c^*$ by Lemma 7.2.2, and this concludes the proof.

□

Now let us set

$$M = \left\{u_{\varepsilon,y} : \varepsilon > 0, y \in \mathbb{R}^N\right\}$$

where $u_{\varepsilon,y}$ is given by (2.19). Denote $D = \mathcal{D}^{1,p}(\mathbb{R}^N)$. Then we have

Theorem 7.2.2. *The functional E satisfies the $(PS)_c$ condition for all $c < c^*$. If $c^* \leq c < 2c^*$ and (u_n) is a $(PS)_c$ sequence for E such that $u_n \rightharpoonup u$ but not strongly, then $u \in K_{c-c^*}$ and*

$$dist\,(u_n - u, M) \to 0 \quad or \quad dist\,(u_n - u, -M)$$

for a renamed subsequence. Moreover, setting

$$M_c = K_{c-c^*} + M = \{u + v : u \in K_{c-c^*}, v \in M\},$$

we have $M_c \subset D\setminus\{0\}$ where $D = \mathcal{D}^{1,p}(\mathbb{R}^N)$ and $N_\delta(M_c) \cap N_\delta(-M_c) = \emptyset$ for all sufficiently small $\delta > 0$.

Proof. Suppose (u_n) is a $(PS)_c$ sequence such that $u_n \rightharpoonup u$. Using Lemma 2.3.1 and Proposition 8.0.1 it is easy to see that $E'(u) = 0$. Therefore

$$E(u) = E(u) - \frac{1}{p}E'(u)u = \frac{1}{N}\int_\Omega |u|^{p^*}dx.$$

By (7.2.1) we have $E_\infty(v_i) \geq c^*$ for $i = 1, \dots, k$, and by (2.27) we also have

$$\frac{1}{N}\int_\Omega |u|^{p^*}dx + kc^* \leq c. \tag{2.31}$$

If $c < c^*$, this implies $k = 0$, so $u_n \to u$ by (2.26).

Suppose $c^* \leq c < 2c^*$. Then $k \leq 1$ by (2.31). If $k = 0$, then $u_n \to u$ as before, so suppose $k = 1$. Then

$$\left\| u_n - u - \left(\varepsilon_n^1\right)^{-(N-p)/p} v_1\left((\cdot - y_n^1)/\varepsilon_n^1\right) \right\| \to 0 \quad \text{as} \quad n \to \infty$$

by (2.26) and

$$E(u) + E_\infty(v_1) = c \tag{2.32}$$

by (2.27). If $c = c^*$, then $u = 0$ by 2.31 and hence $E_\infty(v_1) = c^*$ by (2.32). So by Lemma 7.2.2, v_1 does not change sign, so either $v_1 \in M$ or $v_1 \in -M$ by [**?**]. If $c^* < c < 2c^*$, then $E_\infty(v_1) < 2c^*$ by (2.32) and hence v_1 does not change sign by Lemma 7.2.2. So v_1 is a constant sign nontrivial solution of (2.20) in $\mathcal{D}^{1,p}(\mathbb{R}^N)$ by Lemma 7.2.1. Then either $v_1 \in M$ or $v_1 \in -M$ by [**?**]. In particular $E_\infty(v_1) = c^*$, so $E(u) = c - c^*$ by 2.32 and hence $u \in K_{c-c^*}$. Since $c - c^* < c^*$ and hence K_{c-c^*} is compact by the first part of the theorem, the rest follows as in [5], Lemma 9. □

7.3 Existence of nontrivial solutions for problem (0.1)

In this section we will apply Theorem 7.1.4 to our problem (0.1). We have already verified that the functional E given in (2.17) satisfies the compactness conditions (C_1) and (C_2) with

$$c^* = \frac{1}{N} S_{N,p}^{N/p}, \quad b = \frac{2}{N} S_{N,p}^{N/p}$$

by means of Theorem 7.2.2 in the last section. Now it remains to prove the existence of a suitable compact symmetric set C_0 and the odd map φ as in Theorem 7.1.4. Our setting is the following:

$D = \mathcal{D}^{1,p}(\mathbb{R}^N)$, $W = W_0^{1,p}(\Omega)$, and the operators $A_p, B_p, f \in C(W_0^{1,p}(\Omega), W^{-1,p'}(\Omega))$ are given by

$$(A_p u, v) = \int_\Omega |\nabla u|^{p-2} \nabla u \nabla v dx, \quad (B_p u, v) = \int_\Omega |u|^{p-2} uv dx, \quad (f(u), v) = \int_\Omega |u|^{p^*-2} uv dx$$

for $u, v \in W_0^{1,p}(\Omega)$.

In a similar manner as we did in Section 6.2 it is easily seen that $(A_1') - (F_2')$ hold. Now let us put some effort in the proof of the existence of the compact symmetric set C_0. After a translation we can assume without loss of generality that $0 \in \Omega$. Fix $0 < \delta_0 < \operatorname{dist}(0, \partial\Omega)$ and let $\pi : W_0^{1,p}(\Omega) \setminus \{0\} \to S$, $u \mapsto u/\|u\|$ be the radial projection onto S. We then have the following important fact.

Lemma 7.3.1. *For each $k \geq 1$ and sufficiently small $0 < \delta < \delta_0$, there exists a compact symmetric subset $C_{k,\delta}$ of S with $i(C_{k,\delta}) \geq k$ such that $v = 0$ on $B_{3\delta/4}(0)$ for all $v \in C_{k,\delta}$ and*

$$\sup_{v \in C_{k,\delta},\, s \geq 0} E(sv) \leq \begin{cases} 0 & \text{if} \quad \lambda > \lambda_k \\ a\delta^{(N-p)N/p} & \text{if} \quad \lambda = \lambda_k \end{cases} \tag{3.33}$$

for some constant $a > 0$.

Proof. We have $\lambda_k = \cdots = \lambda_l < \lambda_{l+1}$ for some $l \geq k$. By Degiovanni and Lancelotti ([?], Theorem 2.3), the sublevel Ψ^{λ_l} has a compact symmetric subset C_l of index l that is bounded in $L^\infty(\Omega) \cap C^{1,\alpha}_{\text{loc}}(\Omega)$. Let $\xi : [0,\infty) \to [0,1]$ be a smooth function such that $\xi(s) = 0$ for $s \leq 3/4$ and $\xi(s) = 1$ for $s \geq 1$. For $u \in C_l$, set

$$u_\delta(x) = \xi\left(\frac{|x|}{\delta}\right) u(x),$$

and let

$$C_{k,\delta} = \{\pi(u_\delta) : u \in C_l\}.$$

Since C_l is a compact symmetric set and $u \mapsto \pi(u_\delta)$ is an odd continuous map of C_l onto $C_{k,\delta}$, $C_{k,\delta}$ is also a compact symmetric set and

$$i(C_{k,\delta}) \geq i(C_l) = l \geq k$$

by Proposition 3.2.2.

Let $u \in C_l$, $v = \pi(u_\delta)$ and note that Note that

$$\nabla u_\delta(x) = \xi'\left(\frac{|x|}{\delta}\right) \frac{x}{\delta|x|} u(x) + \xi\left(\frac{|x|}{\delta}\right) \nabla u(x).$$

Since u is bounded in $C^1(B_{\delta_0}(0))$ and belongs to Ψ^{λ_k},

$$\begin{aligned} \int_\Omega |\nabla u_\delta|^p dx &\leq \int_{\Omega \setminus B_\delta(0)} |\nabla u|^p dx + \int_{B_\delta(0)} \left| |\xi'| \frac{1}{\delta} \frac{x}{|x|} u + \xi \nabla u \right|^p dx \\ &\leq \int_{\Omega \setminus B_\delta(0)} |\nabla u|^p dx + 2^p \int_{B_\delta(0)} |\xi'|^p \frac{|u|^p}{\delta^p} dx + 2^p \int_{B_\delta(0)} |\xi|^p |\nabla u|^p dx \\ &\leq 1 + (2^p - 1) \int_{B_\delta(0)} |\nabla u|^p dx + c_2 \delta^{N-p} \\ &\leq 1 + c_1 \delta^N + c_2 \delta^{N-p} \leq 1 + a_1 \delta^{N-p} \quad \text{if} \quad 0 < \delta < \delta_0 \quad \text{is small enough,} \end{aligned}$$

where $c_1, c_2, a_1 > 0$ are constants. We also have

$$\int_\Omega |u_\delta| dx \geq \int_{\Omega\setminus B_\delta(0)} |u|^p dx = \int_\Omega |u|^p dx - \int_{B_\delta(0)} |u|^p dx \geq \frac{1}{\lambda_k} - a_2\delta^N$$

for some constant $a_2 > 0$. So

$$\int_\Omega |v|^p = \frac{\int_\Omega |u_\delta|^p dx}{\int_\Omega |\nabla u_\delta|^p dx} \geq \frac{\dfrac{1}{\lambda_k} - a_2\delta^N}{1 + a_1\delta^{N-p}} \geq \frac{1}{\lambda_k} - a_3\delta^{N-p} \tag{3.34}$$

provided $0 < \delta < \delta_0$ is small enough, where $a_3 > 0$ is come constant. Cosequently, since Ω is bounded, the Holder inequality implies that

$$\int_\Omega |v|^{p^*} dx \geq a_4 \tag{3.35}$$

for some constant $a_4 > 0$.

Let $s \geq 0$, then we have

$$E(sv) = \frac{s^p}{p}\int_\Omega |\nabla v|^p dx - \frac{\lambda s^p}{p}\int_\Omega |v|^p dx - \frac{s^{p^*}}{p^*}\int_\Omega |v|^{p^*} dx,$$

and maximizing the right-hand side over $s \geq 0$ gives

$$E(sv) \leq \frac{1}{N} Q(v)^{N/p},$$

where

$$Q(v) = \frac{(\int_\Omega |\nabla v|^p dx - \lambda \int_\Omega |v|^p dx)^+}{(\int_\Omega |v|^{p^*} dx)^{p/p^*}}.$$

By (3.34) and (3.35),

$$Q(v) \leq b_1 \left(1 - \frac{\lambda}{\lambda_k} + b_2\delta^{N-p}\right)^+$$

for some constants $b_1, b_2 > 0$, so (3.33) follows for sufficiently small $\delta > 0$. □

Now let us construct the odd continuous map φ.

Lemma 7.3.2. *For each* $\varepsilon > 0$ *and* $0 < \delta < \delta_0$ *with* $\varepsilon \ll \delta$, *there exists an odd continuous map*

$\varphi_{\varepsilon,\delta} : S^N \to S$ such that $w = 0$ on $\Omega \backslash B_{3\delta/4}(0)$ for all $w \in \varphi_{\varepsilon,\delta}(S^N)$ and

$$\sup_{w\in\varphi_{\varepsilon,\delta}(S^N),\,t\geq 0} E(tw) \leq \begin{cases} \dfrac{2}{N} S_{N,p}^{N/p}\left[1 + a_1\left(\dfrac{\varepsilon}{\delta}\right)^{(N-p)/(p-1)} - a_2\varepsilon^p\right]^{N/p} & \text{if } N > p^2 \\ \dfrac{2}{p^2} S_{N,p}^{N/p}\left[1 + a_1\left(\dfrac{\varepsilon}{\delta}\right)^{p} - a_2\varepsilon^p \log\left(\dfrac{\delta}{\varepsilon}\right)\right]^{p} & \text{if } N = p^2 \end{cases} \tag{3.36}$$

for some constants $a_1, a_2 > 0$.

Proof. Let us consider the functions given in (2.19) when $y = 0$, namely

$$u_\varepsilon(x) = u_{\varepsilon,0}(x) = \frac{c_{N,p}\varepsilon^{(N-p)/p(p-1)}}{\left(\varepsilon^{p/(p-1)} + |x|^{p/(p-1)}\right)^{(N-p)/p}}, \quad \varepsilon > 0.$$

Let $\zeta : [0,\infty) \to [0,1]$ be a smooth function such that $\zeta(s) = 1$ for $s \leq 1/8$ and $\zeta(s) = 0$ for $s \geq 1/4$, and set

$$u_{\varepsilon,\delta}(x) = \zeta\left(\frac{|x|}{\delta}\right) u_\varepsilon(x), \quad \varepsilon > 0,\ 0 < \delta < \delta_0.$$

As in [8]-Lemma 3.1 we have the estimates

$$\int_\Omega |\nabla u_{\varepsilon,\delta}|^p dx \leq S_{N,p}^{N/p}\left[1 + a_3\left(\frac{\varepsilon}{\delta}\right)^{(N-p)/(p-1)}\right], \tag{3.37}$$

$$\left(\int_\Omega |u_{\varepsilon,\delta}|^{p^*} dx\right)^{p/p^*} \geq S_{N,p}^{N/p}\left[1 - a_4\left(\frac{\varepsilon}{\delta}\right)^{N/(p-1)}\right], \tag{3.38}$$

$$\int_\Omega |u_{\varepsilon,\delta}|^p \geq \begin{cases} a_5\varepsilon^p - a_6\varepsilon^p\left(\dfrac{\varepsilon}{\delta}\right)^{(N-p^2)/(p-1)} & \text{if } N > p^2 \\ a_5\varepsilon^p \log\left(\dfrac{\delta}{\varepsilon}\right) - a_6\varepsilon^p & \text{if } N = p^2 \end{cases} \tag{3.39}$$

for some constants $a_i > 0$, $i = 3, \ldots, 6$.

Let S^{N-1} be the unit sphere in $\mathbb{R}^N$, let

$$S_+^N = \left\{x = (x'\sqrt{1-s^2}, s) : x' \in S^{N-1}, s \in [0,1]\right\}$$

be the upper hemisphere in $\mathbb{R}^{N+1}$, and define a continuous map $\varphi_{\varepsilon,\delta} : S_+^N \to S$ by

$$\varphi_{\varepsilon,\delta}(x) = \pi(u_{\varepsilon,\delta}(\cdot - (1-s)x'/2) - (1-s)u_{\varepsilon,\delta}(\cdot + x'/2)).$$

Let $w = \varphi_{\varepsilon,\delta}(x)$ and note that $w = 0$ outside $B_{3\delta/4}(0)$. For $t \geq 0$,

$$E(tw) = \frac{t^p}{p}\int_\Omega |\nabla w|^p dx - \frac{\lambda t^p}{p}\int_\Omega |w|^p dx - \frac{t^{p^*}}{p^*}\int_\Omega |w|^{p^*} dx,$$

and maximizing the right-hand side over $t \geq 0$ gives

$$E(tw) \leq \frac{1}{N} Q(w)^{N/p},$$

where

$$Q(w) = \frac{\int_\Omega |\nabla w|^p dx - \lambda \int_\Omega |w|^p dx}{(\int_\Omega |w|^{p^*} dx)^{p/p^*}}$$

Noting that $Q(w) = \pi(u_{\varepsilon,\delta}(\cdot - (1-s)x'/2) - (1-s)u_{\varepsilon,\delta}(\cdot + x'/2))$ and $u_{\varepsilon,\delta}(\cdot - (1-s)x'/2)$ and $u_{\varepsilon,\delta}(\cdot + x'/2)$ have disjoint suppports gives

$$Q(w) = \frac{1+(1-s)^p}{[1+(1-s)^{p^*}]^{p/p^*}} Q(u_{\varepsilon,\delta}),$$

and maximizing the right-hand side over $s \in [0,1]$, gives

$$Q(w) \leq 2^{p/N} Q(u_{\varepsilon,\delta}).$$

So

$$E(tw) \leq \frac{2}{N} Q(u_{\varepsilon,\delta})^{N/p}. \tag{3.40}$$

Since $\varphi_{\varepsilon,\delta}$ is odd on S^{N-1} and E is an even functional, $\varphi_{\varepsilon,\delta}$ can be extended to an odd continuous map from S^N to S such that (3.40) holds for all $w \in \varphi_{\varepsilon,\delta}(S^N)$ and $t \geq 0$. Since $\varepsilon \ll \delta$,

$$Q(u_{\varepsilon,\delta}) \leq \begin{cases} S + a_3\left(\dfrac{\varepsilon}{\delta}\right)^{(N-p)/(p-1)} - a_4\varepsilon^p & \text{if } N > p^2 \\ S + a_3\left(\dfrac{\varepsilon}{\delta}\right)^p - a_4\varepsilon^p \log\left(\dfrac{\delta}{\varepsilon}\right) & \text{if } N = p^2 \end{cases}$$

for some constants $a_3, a_4 > 0$ by (3.37)-(3.39), so (3.36) follows. □

Now we have the following multiplicity result for problem (0.1).

Theorem 7.3.1. *Let* $N \geq p^2$.

(*i*) *If* $0 < \lambda < \lambda_1$ *or* $\lambda_k < \lambda < \lambda_{k+1}$ *for some* $k \geq 1$, *then problem* (0.1) *has* $N/2$ *distinct pairs of nontrivial solutions.*

(*ii*) *If* $\lambda = \lambda_1$ *and* $N \geq 3$, *then problem* (0.1) *has* $N-1$ *distinct pairs of nontrivial solutions.*

(*iii*) *If* $\lambda_{k-m} < \lambda = \lambda_{k-m+1} = \cdots = \lambda_k < \lambda_{k+1}$ *for some* $k > m \geq 1$ *and* $N \geq m+2$*, then problem* (0.1) *has* $N - m$ *distinct pairs of nontrivial solutions.*

These solutions satisfy

$$0 < E(u) < \frac{2}{N} S_{N,p}^{N/p} \tag{3.41}$$

Proof. Let $0 < \delta < \delta_0$. Lemma 7.3.2 gives an odd continuous map $\varphi_{\varepsilon,\delta} : S^N \to S$ satisfying

$$\sup_{w \in \varphi_{\varepsilon,\delta}(S^N),\, t \geq 0} E(tw) < \frac{2}{N} S_{N,p}^{N/p} \tag{3.42}$$

for all sufficiently small $\varepsilon > 0$.

(*i*) If $0 < \lambda < \lambda_1$, Theorem 7.1.4 with $k = m = 0$ and $C_0 = \emptyset$ gives N distinct pairs of nontrivial solutions satisfying (3.41). If $\lambda_k < \lambda < \lambda_{k+1}$, Lemma 7.3.1 gives a compact symmetric subset $C_{k,\delta}$ of S with $i(C_{k,\delta}) \geq k$ satisfying

$$\sup_{v \in C_{k,\delta},\, s \geq 0} E(sv) = 0 \tag{3.43}$$

when $\delta > 0$ is sufficiently small. Let $v \in C_{k,\delta}$, $w \in \varphi_{\varepsilon,\delta}(S^N)$, and $s, t \geq 0$. Since $v = 0$ on $B_{3\delta/4}(0)$ and $w = 0$ outside $B_{3\delta/4}(0)$, $\varphi_{\varepsilon,\delta}(S^N) \subset S \backslash C_{k,\delta}$ and

$$E(sv + tw) = E(sv) + E(tw). \tag{3.44}$$

It follows from (3.42)-(3.44) that

$$\sup_{v \in C_{k,\delta}, w \in \varphi_{\varepsilon,\delta}(S^N), s,t \geq 0} E(sv + tw) < \frac{2}{N} S_{N,p}^{N/p}.$$

So Theorem 7.1.4 with $m = 0$ again gives N distinct pairs of nontrivial solutions satisfying (3.41). □

We also have a multiplicity result for problem (0.1) which makes no assumption on the spectrum of the p-Laplacian.

Theorem 7.3.2. *If* $N^2/(N+1) > p^2$*, then problem* (0.1) *has* N *distinct pairs of nontrivial solutions satisfying*

$$0 < E(u) < \frac{2}{N} S_{N,p}^{N/p} \tag{3.45}$$

for all $\lambda > 0$.

Proof. Since $N^2/(N+1) > p^2$ implies $N > p^2$ we have that the case where $0 < \lambda < \lambda_1$ or $\lambda_k < \lambda < \lambda_{k+1}$ for some $k \geq 1$ is covered in Theorem 7.3.1 (i), so we assume that $\lambda = \lambda_k < \lambda_{k+1}$ for some $k \geq 1$. Lemma 7.3.1 gives a compact symmetric subset $C_{k,\delta}$ of S with $i(C_{k,\delta}) \geq k$ satisfying

$$\sup_{v \in C_{k,\delta},\, s \geq 0} E(sv) \leq a\delta^{(N-p)N/p} \tag{3.46}$$

for some constant $a > 0$ when $0 < \delta < \delta_0$ is sufficiently small. Lemma 7.3.2 gives an odd continuous map $\varphi_{\varepsilon,\delta} : S^N \to S$ satisfying

$$\sup_{w \in \varphi_{\varepsilon,\delta}(S^N),\, t \geq 0} E(tw) \leq \frac{2}{N} S_{N,p}^{N/p} \left[1 + a_1 \left(\frac{\varepsilon}{\delta}\right)^{(N-p)/(p-1)} - a_2 \varepsilon^p\right]^{N/p} \tag{3.47}$$

for some constants $a_1, a_2 > 0$ for all sufficiently small $\varepsilon > 0$. Let $v \in C_{k,\delta}$, $w \in \varphi_{\varepsilon,\delta}(S^N)$, and $s, t \geq 0$. Since $v = 0$ on $B_{3\delta/4}(0)$ and $w = 0$ outside $B_{3\delta/4}(0)$, $\varphi_{\varepsilon,\delta}(S^N) \subset S \backslash C_{k,\delta}$ and

$$E(sv + tw) = E(sv) + E(tw). \tag{3.48}$$

Take $\delta = \varepsilon^\alpha$ with $0 < \alpha < 1$. Then

$$\delta^{(N-p)N/p} = \varepsilon^{p + \left[(N-p)N\alpha - p^2\right]/p}, \quad \left(\frac{\varepsilon}{\delta}\right)^{(N-p)/(p-1)} = \varepsilon^{p + \left[N - p^2 - (N-p)\alpha\right]/(p-1)}. \tag{3.49}$$

Since $N^2/(N+1) > p^2$,

$$0 < \frac{p^2}{(N-p)N} < \frac{N-p^2}{N-p} < 1$$

so we can take $p^2/(N-p)N < \alpha < (N-p^2)/(N-p)$, combine (3.46)-(3.49), and take ε sufficiently small to get

$$\sup_{v \in C_{k,\delta},\, w \in \varphi_{\varepsilon,\delta}(S^N),\, s,t \geq 0} E(sv + tw) < \frac{2}{N} S_{N,p}^{N/p}.$$

So Theorem 7.1.4 with $m = 0$ gives N distinct pairs of nontrivial solutions satisfying (3.45). □

CHAPTER 8

Appendix

Theorem 8.0.1 (Dominated Convergence Theorem). *Let $\Omega \subset \mathbb{R}^N$ be open and let $\{u_k\}_{k\in\mathbb{N}} \subset L^1(\Omega)$ be a subsequence such that*

1. *$u_k(x) \to u(x)$ a.e. on Ω as $k \to +\infty$;*
2. *there exists $v \in L^1(\Omega)$ such that for all k, $|u_k(x)| \leq v(x)$ a.e. on Ω.*

Then $u \in L^1(\Omega)$ and $u_k \to u$ in the $L^1(\Omega)$ norm, namely $\int_\Omega |u_k - u| dx \to 0$.

Theorem 8.0.2. *Let $\Omega \subset \mathbb{R}^N$ be open and let $\{u_k\}_{k\in\mathbb{N}} \subset L^p(\Omega)$, $p \in [1, +\infty]$, be a sequence such that $u_k \to u$ in $L^p(\Omega)$ as $k \to +\infty$. Then there exist a subsequence $\left\{u_{k_j}\right\}_{j\in\mathbb{N}}$ and a function $v \in L^p(\Omega)$ such that*

1. *$u_{k_j}(x) \to u(x)$ a.e. on Ω as $j \to +\infty$;*
2. *for all j, $|u_{k_j}(x)| \leq v(x)$ a.e. on Ω.*

Theorem 8.0.3 (Egoroff's Theorem). *Suppose $\mu(X) < \infty$, and $f_1, f_2, \dots, f : X \to \mathbb{R}$ are measurable functions on X such that $f_n \to f$ almost everywhere. Then for every $\varepsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \varepsilon$ and $f_n \to f$ uniformly on $X \setminus E$.*

Proof. Without loss of generality we may assume that $f_n \to f$ everywhere on X. For $k, n \in \mathbb{N}$ let

$$E_n(k) = \bigcup_{m=n}^{\infty} \{x \in X : |f_m(x) - f(x)| \geq 1/k\}.$$

Then, for fixed k, $E_n(k)$ decreases as n increases, and $\cap_{n=1}^{\infty} E_n(k) = \emptyset$, so since $\mu(X) < \infty$, by the Dominated Convergence Theorem, $\mu(E_n(k)) \to 0$ as $n \to \infty$. Given $\varepsilon > 0$ and $k \in \mathbb{N}$, choose n_k so large that $\mu(E_{n_k}(k)) < \varepsilon/2^k$ and let $E = \cup_{k=1}^{\infty} E_{n_k}(k)$. Then $\mu(E) \leq \varepsilon$, and we have $|f_n(x) - f(x)| < 1/k$ for $n > n_k$ and $x \in X \setminus E$. Thus $f_n \to f$ uniformly on $X \setminus E$. □

Proposition 8.0.1. *Let $\Omega \subset \mathbb{R}^N$ be an open set with finite Lebesgue measure, i.e. $\mu(\Omega) < \infty$. Let $\{f_n\}_{n\in\mathbb{N}} \subset L^p(\Omega)$ with $p \in [1, \infty]$ such that $f_n \rightharpoonup f$ (f_n converges weakly to f) and $f_n(x) \to g(x)$ almost everywhere on Ω. Then $f = g$ almost everywhere on Ω.*

Proof. Let us denote $h_n = f_n - f$, then $h_n \rightharpoonup 0$ and $h_n(x) \to g(x) - f(x)$ a.e. Suppose by contradiction that $g \neq f$ on a set of positive measure $X \subset \Omega$. Then clearly exists $\varepsilon > 0$ and a subset E of positive measure of Ω such that either $g - f \geq \varepsilon$ or $f - g \geq \varepsilon$. So let $E \subset \Omega$ with $\mu(E) > 0$ such that $g - f \geq \varepsilon$ on E. By the Egoroff's Theorem (Theorem 8.0.3), we can replace E by a smaller subset of positive measure such that $h_n(x) \to g(x) - f(x)$ uniformly on E. But since $h_n \rightharpoonup 0$ we have, on one hand

$$\int_\Omega h_n \chi_E dx \to 0,$$

and by the uniformly convergence we have on the other hand

$$\int_\Omega h_n \chi_E dx \to \int_\Omega (g - f)\chi_E dx \geq \varepsilon\mu(E) > 0,$$

which is a contradition.

$\square$

Bibliography

[1] BENCI, V. **On critical point theory for indefinite functionals in the presence of symmetries**. *Transactions of the American Mathematical Society*, 274(2):533–572, 1982.

[2] BREZIS, H. **Functional Analysis, Sobolev Spaces and Partial Differential Equations**. Universitext. Springer New York, 2010.

[3] BREZIS, H.; NIRENBERG, L. **Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents**. 36(4):437–477, July 1983.

[4] CHANG, K.; ZHANG, G. **Infinite Dimensional Morse Theory and Multiple Solution Problems**. Progress in Nonlinear Differential Equat Series. Birkhäuser, 1993.

[5] CLAPP, M.; WETH, T. **Multiple solutions for the brezis-nirenberg problem**. *Advances in Differential Equations*, 10, 04 2005.

[6] DE VALERIOLA, S.; WILLEM, M. **On some quasilinear critical problems**. *Mathematics Subject Classification. 35J20*, 9, 01 2000.

[7] DEGIOVANNI, M.; LANCELOTTI, S. **Linking over cones and nontrivial solutions for p-Laplace equations with p-superlinear nonlinearity**. *Annales de l'Institut Henri Poincaré C, Analyse non linéaire*, 24(6):907–919, 2007.

[8] DEGIOVANNI, M.; LANCELOTTI, S. **Linking solutions for p-laplace equations with nonlinearity at critical growth**. *Journal of Functional Analysis*, 256(11):3643–3659, 2009.

[9] DEIMLING, K. **Nonlinear Functional Analysis**. Dover books on mathematics. Dover Publications, 2013.

[10] DOLD, A. **Partitions of unity in the theory of fibrations**. *Annals of Mathematics*, 78:223, 1963.

[11] FADELL, E. R.; RABINOWITZ, P. H. **Bifurcation for odd potential operators and an alternative topological index**. *Journal of Functional Analysis*, 26(1):48–67, 1977.

[12] FADELL, E.R., R. P. **Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for hamiltonian systems.** *Inventiones mathematicae*, 45:139–174, 1978.

[13] GHOUSSOUB, N. **Duality and Perturbation Methods in Critical Point Theory**. Cambridge Tracts in Mathematics. Cambridge University Press, 1993.

[14] HATCHER, A. **Algebraic Topology**. Algebraic Topology. Cambridge University Press, 2002.

[15] MASSEY, W. **Singular Homology Theory**. Graduate Texts in Mathematics. Springer New York, 2012.

[16] MERCURI, C.; PERERA, K. **New multiplicity results for critical p-Laplacian problems**. *Journal of Functional Analysis*, 283:109536, 05 2022.

[17] MERCURI, C.; WILLEM, M. **A global compactness result for the p-Laplacian involving critical nonlinearities**. *Discrete and Continuous Dynamical Systems*, 28:469–493, 2010.

[18] MILNOR, J.; SPIVAK, M.; WELLS, R. **Morse Theory. (AM-51), Volume 51**. Princeton University Press, 1969.

[19] MILNOR, J.; STASHEFF, J. **Characteristic Classes**. Annals of mathematics studies. Princeton University Press, 1974.

[20] PALAIS, R. S. **Morse Theory on Hilbert manifolds**. *Topology*, 2(4):299–340, 1963.

[21] PERERA, K.; AGARWAL, R.; O'REGAN, D. **Morse Theoretic Aspects of p-Laplacian Type Operators**. Mathematical surveys and monographs. American Mathematical Society, 2010.

[22] PERERA, K. **Nontrivial critical groups in p-Laplacian problems via the Yang index**. *Topological Methods in Nonlinear Analysis*, 21(2):301 – 309, 2003.

[23] PERERA, K. **An abstract critical point theorem with applications to elliptic problems with combined nonlinearities**. *Calculus of Variations and Partial Differential Equations*, 60, 10 2021.

[24] SPANIER, E. **Algebraic Topology**. McGraw-Hill series in higher mathematics. Springer, 1989.

[25] STEENROD, N. **The Topology of Fibre Bundles. (PMS-14)**. Princeton University Press, 1951.

[26] STRUWE, M. **Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems**. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics. Springer Berlin Heidelberg, 2013.

[27] WILLEM, M. **Minimax Theorems**. Progress in Nonlinear Differential Equations and Their Applications. Birkhäuser Boston, 1997.

[28] YANG, C.-T. **On Theorems of Bursuk-Ulam, Kakutani-Yamabe-Yujobo and Dyson, I**. *Annals of Mathematics*, 60(2):262–282, 1954.