Universidade Federal de Goiás (UFG)
Instituto de Matemática e Estatística (IME)
Programa de Pós-Graduação em Matemática

Arlam Cassio Pereira da Silva Junior

Classification of Real Clifford Algebras and of Polynomials' roots with algebras of $\mathbb{R}^4$ coefficients

Goiânia
2025

UNIVERSIDADE FEDERAL DE GOIÁS
INSTITUTO DE MATEMÁTICA E ESTATÍSTICA

TERMO DE CIÊNCIA E DE AUTORIZAÇÃO (TECA) PARA DISPONIBILIZAR VERSÕES ELETRÔNICAS DE TESES

E DISSERTAÇÕES NA BIBLIOTECA DIGITAL DA UFG

Na qualidade de titular dos direitos de autor, autorizo a Universidade Federal de Goiás (UFG) a disponibilizar, gratuitamente, por meio da Biblioteca Digital de Teses e Dissertações (BDTD/UFG), regulamentada pela Resolução CEPEC nº 832/2007, sem ressarcimento dos direitos autorais, de acordo com a Lei 9.610/98, o documento conforme permissões assinaladas abaixo, para fins de leitura, impressão e/ou download, a título de divulgação da produção científica brasileira, a partir desta data.

O conteúdo das Teses e Dissertações disponibilizado na BDTD/UFG é de responsabilidade exclusiva do autor. Ao encaminhar o produto final, o autor(a) e o(a) orientador(a) firmam o compromisso de que o trabalho não contém nenhuma violação de quaisquer direitos autorais ou outro direito de terceiros.

1. Identificação do material bibliográfico

[X] Dissertação   [ ] Tese   [ ] Outro*:_____________

*No caso de mestrado/doutorado profissional, indique o formato do Trabalho de Conclusão de Curso, permitido no documento de área, correspondente ao programa de pós-graduação, orientado pela legislação vigente da CAPES.

Exemplos: Estudo de caso ou Revisão sistemática ou outros formatos.

2. Nome completo do autor

Arlam Cassio Pereira da Silva Junior

3. Título do trabalho

Classification of real Clifford algebras and of polynomials’ roots with algebras of R^4 coefficients

4. Informações de acesso ao documento (este campo deve ser preenchido pelo orientador)

Concorda com a liberação total do documento [X] SIM   [ ] NÃO[1]

[1] Neste caso o documento será embargado por até um ano a partir da data de defesa. Após esse período, a possível disponibilização ocorrerá apenas mediante:
a) consulta ao(à) autor(a) e ao(à) orientador(a);
b) novo Termo de Ciência e de Autorização (TECA) assinado e inserido no arquivo da tese ou dissertação.
O documento não será disponibilizado durante o período de embargo.
Casos de embargo:
- Solicitação de registro de patente;
- Submissão de artigo em revista científica;
- Publicação como capítulo de livro;
- Publicação da dissertação/tese em livro.

Obs. Este termo deverá ser assinado no SEI pelo orientador e pelo autor.

Documento assinado eletronicamente por **Arlam Cassio Pereira Da Silva Junior**, **Discente**, em 01/07/2025, às 10:43, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

Documento assinado eletronicamente por **Jhone Caldeira Silva**, **Professor do Magistério Superior**, em 01/07/2025, às 14:32, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

A autenticidade deste documento pode ser conferida no site https://sei.ufg.br/sei/controlador_externo.php?acao=documento_conferir&id_orgao_acesso_externo=0, informando o código verificador **5467418** e o código CRC **21834109**.

Referência: Processo nº 23070.029074/2025-66 SEI nº 5467418

Arlam Cassio Pereira da Silva Junior

Classificação das Álgebras de Clifford Reais e de Raízes Polinômias com coeficientes de álgebras de $\mathbb{R}^4$

Classification of Real Clifford Algebras and of Polynomials' roots with algebras of $\mathbb{R}^4$ coefficients

Dissertação apresentada ao Programa de Pós–Graduação em Matemática, do Instituto de Matemática e Estatística (IME), da Universidade Federal de Goiás (UFG), como requisito para obtenção do título de Mestre em Matemática.

Área de concentração: Álgebra

Orientador: Prof. Dr. Jhone Caldeira Silva

Goiânia
2025

Ficha de identificação da obra elaborada pelo autor, através do
Programa de Geração Automática do Sistema de Bibliotecas da UFG.

Silva Junior, Arlam Cassio Pereira da
Classification of Real Clifford Algebras and of Polynomials’ roots with algebras of R^4 coefficients [manuscrito] / Arlam Cassio Pereira da Silva Junior. - 2025.
95 f.: il.

Orientador: Prof. Dr. Jhone Caldeira Silva.
Dissertação (Mestrado) - Universidade Federal de Goiás, Instituto de Matemática e Estatística (IME), Programa de Pós-Graduação em Matemática, Goiânia, 2025.
Bibliografia.
Inclui tabelas, lista de figuras, lista de tabelas.

1. Clifford Algebras. 2. Classification of Real Clifford Algebras. 3. Polynomials with algebras of R^4. 4. Classification of Polynomials’ roots. I. Silva, Jhone Caldeira, orient. II. Título.

CDU 512.5

UNIVERSIDADE FEDERAL DE GOIÁS

INSTITUTO DE MATEMÁTICA E ESTATÍSTICA

ATA DE DEFESA DE DISSERTAÇÃO

Ata nº **13** da sessão de Defesa de Dissertação de **Arlam Cassio Pereira da Silva Junior**, que confere o título de Mestre em **Matemática**, na área de concentração em **Álgebra**.

Ao **vigésimo sétimo dia do mês de junho do ano de dois mil e vinte e cinco**, a partir das **08h00**, via Web videoconferência, realizou-se a sessão pública de Defesa de Dissertação intitulada **"Classification of real Clifford algebras and of polynomials' roots with algebras of R^4 coefficients".** Os trabalhos foram instalados pelo Orientador, Professor Doutor **Jhone Caldeira Silva - IME/UFG** com a participação dos demais membros da Banca Examinadora: Professor Doutor **Ricardo Nunes de Oliveira - IME/UFG**, membro titular interno; Professor Doutor **Igor dos Santos Lima MAT/UnB**, membro titular externo. Durante a arguição os membros da banca **não fizeram** sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Dissertação, tendo sido o candidato **aprovado** pelos seus membros. Proclamados os resultados pelo Professor Doutor **Jhone Caldeira Silva**, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos Membros da Banca Examinadora, ao **vigésimo sétimo dia do mês de junho do ano de dois mil e vinte e cinco.**

TÍTULO SUGERIDO PELA BANCA

---

sei! assinatura eletrônica

Documento assinado eletronicamente por **Jhone Caldeira Silva**, **Professor do Magistério Superior**, em 27/06/2025, às 09:31, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

---

sei! assinatura eletrônica

Documento assinado eletronicamente por **Igor dos Santos Lima**, **Usuário Externo**, em 27/06/2025, às 09:35, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

---

sei! assinatura eletrônica

Documento assinado eletronicamente por **Ricardo Nunes De Oliveira**, **Professor do Magistério Superior**, em 27/06/2025, às 09:35, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

A autenticidade deste documento pode ser conferida no site https://sei.ufg.br/sei/controlador_externo.php?acao=documento_conferir&id_orgao_acesso_externo=0, informando o código verificador **5411633** e o código CRC **1536E01E**.

Referência: Processo nº 23070.029074/2025-66 SEI nº 5411633

Arlam Cassio Pereira da Silva Junior

Bacharel em Física pelo Instituto de Física na UFG - Universidade Federal de Goiás. Durante a graduação, foi monitor no Instituto de Física da UFG e pesquisador do CNPq em dois trabalhos de iniciação científica no Instituto de Física.

To God, my family and friends.

Acknowledgments

First of all, I want to thank God, who gave me the oportunity to continue my studies in Mathematics even in troubled times, and to my family and friends, who gave me support throughout my academic carrer so far. Now, special thanks are given to my advisor Jhone, who took me in to face an endevour that both of us were not so sure how it would turn out to be in the end; and to my friend Almério, who helped me with a lot of the technical difficulties I faced during my learning process with LaTeX (especially the beatiful diagrams that are, to some extent, better appreciated in its digital format), and listened to some of my expositions related to my work, which helped me clarify some concepts and ideas I had at the time, obviously everything under a good cup of coffee whenever possible.

Wisdom and folly both are like meats that are wholesome and unwholesome, and courtly or simple words are like town-made or rustic vessels - both kinds of food may be served in either kind of dish.

St. Augustine of Hippo, *Confessions - Book V.*

Abstract

Cassio Pereira da Silva Junior, Arlam. **Classification of Real Clifford Algebras and of Polynomials' roots with albegras of $\mathbb{R}^4$ coefficients**. Goiânia, 2024. 92 p. MSc. Dissertation. Programa de Pós-graduação em Matemática Strictu Sensu, Instituto de Matemática e Estatística, Universidade Federal de Goiás.

Clifford Algebras is an abstract construct related to bilinear vector spaces, whilst Polynomials with $\mathbb{R}^4$ coefficients is a generalization of the study of polynomials with complex coefficients. The study of tensor product and tensor algebras is first presented as a theoretical base to be build upon. The definition of a Clifford Algebra over the real numbers and its universal property are studied. The classification of Clifford Algebras related to real finite-dimensional vector spaces is presented with the use of the Periodicity Theorem. Then the presentation eight algebras defined in $\mathbb{R}^4$ is done. The classification of roots from one-sided polynomials with quartenionic coefficients is first presented, followed by the one-sided polynomials over noncommutative algebras of $\mathbb{R}^4$, and then by the two-sided quaternionic polynomial's case.

Keywords

Clifford Algebras, Classification of Real Clifford Algebras, Polynomials with algebras of $\mathbb{R}^4$, Classification of Polynomials' roots.

Contents

List of Figures

List of Tables

Introduction

Clifford Algebras, also known as *Geometric algebras*, were first introduced by British mathematician William Kingdon Clifford, who generalized the idea of Grassman Algebras [1], and they were later developed by other mathematicians, like Lipschitz, Cartan and Chevalley to name a few.

By its construction, Clifford Algebras are typically related to multilinear algebras, quartenions, the theory of spinors and spin groups (see [2], [3], [4], [5], [6]).

In physics, one of the earliest use of such algebras was done by notorious English mathematician and physicist Paul Dirac, where, by trying to write a relativistic wave equation to model the motion of the electron (the Dirac Equation), he ended up writing four matrices, hitorically noted as γ_i, $0 \leq i \leq 3$, called the *Dirac matrices* or the *gamma matrices*, that satisfy the relation

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\eta_{i+1j+1},$$

with η being the diagonal matrix with entries $\eta_{11} = 1$ and $\eta_{22} = \eta_{33} = \eta_{44} = -1$. This realation is the same done in Definition 2.1.1 (see Equation 2.1). For more information about the Dirac matrices and the Dirac Equation, see Reference [7].

In geometry, Clifford Algebras play a role in the Conformal Algebra, as an example in the *conformal algebra of twistors* (see References [8] and [9]). Likewise, we can also see some applications in neural networks, encryption, signal and image processing and descrete mathematics. The curious reader can see Reference [10] for more details.

Also related to quartenions is the study of quaternionic polynomials, *i.e.*, polynomials with quartenionic coefficients, the study of polynomials over other noncommutative algebras of $\mathbb{R}^4$, and the type of roots such polynomial may present. The set of Quartenions, in particular, has seen some applications in physics as a way to express the Lorentz transformation in the Theory of Relativity, as well as in Classica Mechanics, like in some scattering problems and in modeling the rotation of a body in 3D space. As such, Quartenions also are used in computer graphics to for the animation of 3D bodies (see References [11] and [12]).

This work here presented has two objectives, which can be divided into two parts: the first being the classification of Clifford Algebras for finite-dimensional real vector spaces and the classification of one and two-sided polynomials over noncommutative algebras of $\mathbb{R}^4$. To do so, the following "roadmap" will be used as the structure of this work:

- We begin by presenting the foundation necessary in Chapter 1 which stablishes the "building blocks" necessary to define the Clifford Algebras. Begining with modules, the definition of a tensor product between modules will be presented

(Definition 1.1.4) together with some of its most important properties, for example its universal nature (Theorem 1.1.2), all being encapsulated in Corollary 1.1.2. The definition of a graded tensor product (Definition 1.2.2) and of an algebra (Definition 1.2.3) are also present, followed by the theorem defining the graded tensor algebra (Theorem 1.2.2). This is all done whilst presenting some examples to illustrate as well as to antecipate some future results.

- Chapter 2 treats with Clifford Algebras proper, they being defined in Definition 2.1.1 and one example being constructed based on the graded tensor algebra (Theorem 2.1.1). The dimension of a Clifford Algebra for a finite-dimensional vector space and its relation with the dimension of the original vector space will also be studied (Theorem 2.1.4); likewise the isomorphisms between a series of Clifford Algebras shall be proved throughout Section 2.2, culminating in Corollary 2.2.2, the Periodicity Theorem (Theorem 2.2.4) and Corollary 2.2.5. This three results will be the pillars in order to classify the Clifford Algebras, which are summarized in Table 2.2 and Table 2.2.

- Chapter 3 starts the second part of this work. It involves the study and classification of polynomials' roots with coefficients in algebras of $\mathbb{R}^4$, more specifically the noncommutative algebras of Quartenions, Coquartenions, Nectarines and Conectarines (Table 3.4). Because of their noncommutative nature, as well as some of them having zero-divisors, three cases are studied: one-sided quaternionic polynomials, one-sided polynomials over the other three noncommutative algebras (denoted by $\mathcal{A}$), and two-sided quaternionic polynomials.The idea behind classifying zeros in such polynomials is related to how many there are in the equivalent class of a given zero, such class being defined from the equivalent relation of quasi-similarity (Definition 3.1.5). From this, a series of properties are presented, in special the Iteration Process (Theorem 3.2.1 for the quaternionic case, Theorem 3.3.1 for the other noncommutative algebras) and how to rewrite a polynomial with degree n as a linear polynomial (Corollary 3.2.2 for $\mathbb{H}$, Corollary 3.3.1 for $\mathcal{A}$, and Corollary 3.4.2 for the two-side quaternionic polynomial). From this results we shall classify the zeros of each case.

As a final consideration, while the bibliography related to the topics presented is extensive, not all results are easily proven or find to be found. So, in order to give a more complete look into the proofs, the calculations made follow our vision.

Chapter 1

The Tensor Product

This chapter is dedicated to some fundamental concepts that are noneless fundamental for the theory to be studied in Chapter 2. Although there're a lot of results in it, we're still focusing only on the more important parts related to the aim of this work, namely to study the Clifford Algebras. Thus, the reader who wants to delve into a more specific point here presented, or to see other applications of the structures presented in this chapter, can look into References [13], [14], [15] and [16].

1.1 The tensor product over modules

We begin this section by defining some basic algebraic constructions used throughout this essay.

Definition 1.1.1. *Let E be an additive group and R be a commutative ring with unity, i.e., $1 \in R$. Then E is a* ***module over R (or an R-module)*** *if there exists an binary operation, called scalar multiplication, with the following properties: for any $x, y \in E$, and any $\alpha, \beta \in R$,*

(i) $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$;

(ii) $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$;

(iii) $(\alpha\beta) \cdot x = \alpha(\beta \cdot x)$;

(iv) $1 \cdot x = x$.

In this context, the elements of R are called ***scalars*** *(hence the name).*

Example 1.1.1.

(i) Let V be a vetor space over the field F. Since, by definition, every field is a ring with unity, we can conclude every vector space is a module over its field.

(ii) Let R be a ring with unity, and let I be an ideal of R. If we consider the abelian group $(I, +)$ and the scalar multiplication $. : R \times I \to I$ as just the multiplication of elements in R, then I is an R-module, for if we take arbitraries $x \in I$ and $\alpha \in R$, then $\alpha . x \in I$.

(iii) Let $(G, +)$ be any abelian group. We can define the scalar multiplication $. : \mathbb{Z} \times G \to G$ as

$$(n, g) \mapsto \begin{cases} 0.g = 0, \\ n.g = \underbrace{g + ... + g}_{n\text{-times}}, \text{ for } n > 0, \\ n.g = -(\underbrace{g + ... + g}_{-n\text{-times}}), \text{ for } n < 0. \end{cases}$$

In such case, G is a module over $\mathbb{Z}$.

From now on we will omit the "dot" symbolizing the scalar product since, by context, it will always be clear to understand if it's the product between two scalars or the product between a scalar and an element of the abelian group E.

Definition 1.1.2. *Let $E_1, ..., E_n$ and F be modules over R. A function*

$$f : \prod_{i=1}^{n} E_i \to F$$

*is said to be a **multilinear map** (or an **R-multilinear map**) if*

$$f(x_1, ..., \alpha x_i + x_i', ..., x_n) = \alpha f(x_1, ..., x_i, ..., x_n) + f(x_1, ..., x_i', ..., x_n)$$

for all $\alpha \in R$ and any $1 \leq i \leq n$. The set of all multilinear maps from $\prod_{i=1}^{n} E_i$ into F will be denoted as $\mathcal{L}(\prod_{i=1}^{n} E_i; F)$. For the particular case of $n = 1$, the adjective "multilinear" is dropped in favor of the more usual (and intuitive) descriptive "linear".

Example 1.1.2.

(i) Let V be a vector space over the real field $\mathbb{R}$, and let $\langle ., . \rangle$ be an inner product in V. Then, by definition of an inner product, $\langle ., . \rangle$ is a multilinear mapping between $V \times V$ and $\mathbb{R}$.

(ii) The determinant of a matrix can also be seen as an multilinear mapping, as it can be seen in [13], [14] and [17].

In order to define the tensor product between two modules, it becomes necessary to introduce the idea of free-modules.

Definition 1.1.3. *Let E be an R-module and let S be a subset of E. Additionally, let $i : S \to E$ be the insertion of S in E. We say that E is a **free module** on S when, to every function $f : S \to F$, F being an R-module, there exists an unique linear map $t : E \to F$ with $t \circ i = f$. In this case, we also say that S is a **set of free generators** for E, or that S is a **basis** for E.*

The name *basis* may bring recolections from the concept of a basis in linear algebra. This is no coincidence since a vector space is nothing more than a module over a field. Moreover, the concept of a basis generating the vector space means that each element of the vector space is a linear combination of elements of its basis. This result is also true for free-modules.

$$\begin{tikzcd}
S \arrow[rr, "f"'] \arrow[dr, "i"'] & & F \\
& E \arrow[ur, "t"] &
\end{tikzcd}$$

Figure 1.1: Diagram for Definition 1.1.3

Theorem 1.1.1. *Let E be a free module on the subset S. Then the set S spans E, i.e., each element of E can be written as a linear combination of elements of S.*

Proof. Let D be the set of all linear combinations of elements of S. It is immediate to see that D is a submodule of E since D is closed by addition and scalar multiplication by definition, hence D is a subgroup of E. We need to show that $D = E$. To do so, consider the quotient group $\frac{E}{D}$. Such a group can be turned into a module over R. To do so, consider the element $y \in x + D$. In this way,

$$y - x \in D \Rightarrow \alpha(y - x) = \alpha y - \alpha x \in D.$$

Therefore $y \in x + D \Rightarrow \alpha y \in \alpha x + D$. Consequentely the multiplication

$$\alpha(x + D) = \alpha x + D \tag{1.1}$$

is well-defined. Moreover, for any $\alpha, \beta \in R$, and any $x, y \in E$,

$$\begin{gathered}
\alpha((x + D) + (y + D)) = \alpha((x + y) + D) = (\alpha x + D) + (\alpha y + D); \\
(\alpha + \beta)(x + D) = (\alpha + \beta)x + D = (\alpha x + D) + (\beta x + D); \\
(\alpha\beta)(x + D) = (\alpha\beta)x + D = (\alpha(\beta x)) + D = \alpha(\beta(x + D)); \\
1(x + D) = 1x + D = x + D;
\end{gathered}$$

so this is a scalar multiplication and $\frac{E}{D}$ is a module over R.

Finally consider the linear maps

$$p : E \to \frac{E}{D},$$

the canonical projection, and

$$0 : E \to \frac{E}{D},$$

the mapping taking all elements in E and assigning them the value $0 + D$. Hereby, for the injection $i : S \to E$,

$$p \circ i = 0 \circ i : S \to \frac{E}{D},$$

where we used the fact that $S \subset D \Rightarrow p(S) = \{0\} = 0(S)$. By the unicity of the linear map $t : S \to E$ from Definition 1.1.3, we must have $t = p = 0$, meaning

$$Im(p) = \frac{E}{D} = \{0 + D\}.$$

This is only true if $E = D$, finishing this proof. □

Theorem 1.1.1 tells us that every finite-dimensional vector space is a free-module since, by its definition, the vector space contains at least one finite subset of elements that spans the whole space.

Example 1.1.3. *Consider* $\mathbb{R}^n$ *as the typical n-tuple real vector space, which addition is done coordinate-wise, and scalar multiplication is done by multiplying the scalar over each coordinate:*

$$(x_1, ..., x_i, ..., x_n) + (y_1, ..., y_i, ..., y_n) = (x_1 + y_1, ..., x_i + y_i, ..., x_n + y_n),$$
$$\alpha(x_1, ..., x_i, ..., x_n) = (\alpha x_1, ..., \alpha x_i, ..., \alpha x_n).$$

Then the set $\{e_i \mid 1 \leq i \leq n\}$ *spans* $\mathbb{R}^n$*, where* e_i *is the element with all entries being null except the* i*-th coordinate, which is* 1*. For example, if* $n = 3$*, then*

$$\begin{cases} e_1 & = (1, 0, 0) \\ e_2 & = (0, 1, 0) \\ e_3 & = (0, 0, 1). \end{cases}$$

The fact that such subset spans $\mathbb{R}^n$ *comes immediatly from the addition and scalar multiplication definitions: suppose* $(x_1, ..., x_n) \in \mathbb{R}^n$*. Then*

$$(x_1, ..., x_n) = \sum_{i=1}^{n} x_i e_i.$$

This set is called the ***canonical basis of*** $\mathbb{R}^n$*.*

Free-modules perform a central role in the algebraic construction of the tensor product, as we can see promptly.

Definition 1.1.4. *Let* R *be a commutative ring with unity, and let* $E_1, ..., E_n$ *be modules over* R*; let* M *be the free module generated by the cartesian product* $\prod_{i=1}^{n} E_i$ *and* N *be the submodule generated by the elements of the form*

$$(x_1, ..., x_i + x_i', ..., x_n) - (x_1, ..., x_i, ..., x_n) - (x_1, ..., x_i', ..., x_n), \tag{1.2}$$
$$(x_1, ..., \alpha x_i, ..., x_n) - \alpha(x_1, ..., x_i, ..., x_n), \tag{1.3}$$

for all $x_i, x_i' \in E_i, 1 \leq i \leq n, \alpha \in R$*. The quotient module* $\dfrac{M}{N}$ *over* R *is called the* ***tensor product of*** $E_1, ..., E_n$*.*

The next theorem being presented shows two points: the first is the existence of the tensor product, which justifies the last definition and shows that $\dfrac{M}{N}$ is indeed a module over R; the second is its universal nature.

Theorem 1.1.2 (Universality of the Tensor Product)**.** *Let* $E_1, ..., E_n$ *be* R*-modules. Then there exists a pair (T,g) consisting of an* R*-module* T *and a* R*-multilinear mapping* $g : \prod_{i=1}^{n} E_i \to T$ *with the following property: given any* R*-module* F *and any* R*-multilinear mapping*

$$f : \prod_{i=1}^{n} E_i \to F, \tag{1.4}$$

there exists a unique linear mapping of modules

$$f^* : T \to F \tag{1.5}$$

such that $f = f^* \circ g$*.*

Proof. The existence of $T = \dfrac{M}{N}$ is immediate from Theorem 1.1.1. The existence of the function g is done by taking the composition of the insertion $i : \prod_{i=1}^{n} E_i \to M$ with the canonical projection $p : M \to T = \dfrac{M}{N}$. In symbols, $g = p \circ i$ (see Figure 1.2). To see that g is multilinear, take $(x_1, ..., \alpha x_i + x_i', ..., x_n) = (\alpha x_i + x_i')$, where $\alpha \in R$ (in this part only we are omitting the "fixed" coordinates for the sake of convenience and simplicity). Therefore, for any $1 \leq i \leq n$,

$$\begin{gathered} g(\alpha x_i + x_i') = p(\alpha x_i + x_i') = (\alpha x_i + x_i') + N = \\ = (\alpha x_i + N) + (x_i' + N) = \alpha(x_i + N) + (x_i' + N) = \alpha g(x_i) + g(x_i'), \end{gathered}$$

In addition, since i and p are unique, g is the only multilinear mapping between $\prod_{i=1}^{n} E_i$ and T.

$$\begin{tikzcd} S \arrow[r, "i"'] \arrow[rr, bend right, "g = p \circ i"] & M \arrow[r, "p"'] & F \end{tikzcd}$$

Figure 1.2: First diagram for Theorem 1.1.2

The only point left to prove is the universality of T. To do so, we need to show that f factors over T in such a way that the Diagram 1.3 commutes.

$$\begin{tikzcd} \prod_{i=1}^{n} E_i \arrow[rr, "f"] \arrow[dr, "g"'] & & F \\ & T = \dfrac{M}{N} \arrow[ur] & \end{tikzcd}$$

Figure 1.3: Second diagram for Theorem 1.1.2

First we define the mapping $\overline{f}$ as follows:

$$\begin{aligned} \overline{f} &: M \to F \\ z = \sum_{j=1}^{m} \alpha_j x_j &\mapsto \overline{f}(z) := \sum_{j=1}^{m} \alpha_j f(x_j), \end{aligned} \tag{1.6}$$

where $x_j \in \prod_{i=1}^{n} E_i$ and $\alpha_j \in R$ for all $1 \leq j \leq m$ (notice here x_j is being considered an element of the cartesian product $\prod_{i=1}^{n} E_i$, and not a coordinate of an element of said product). This mapping is a linear extesion of f over M, that is,

$$\overline{f}(x) = \overline{f}(1x) = 1f(x) = f(x),$$

for all $x \in \prod_{i=1}^{n} E_i$. Moreover, $\overline{f}$ is a linear mapping: for any $z_1, z_2 \in M$ and any

$\gamma \in R$,

$$\begin{aligned} z_1 &= \sum_{j=1}^{r} \alpha_j x_j, \\ z_2 &= \sum_{k=1}^{s} \beta_k y_k, \end{aligned}$$

with $x_j, y_k \in \prod_{i=1}^{n} E_i$, and $\alpha_j, \beta_k \in R$; we have

$$\overline{f}(\gamma z_1 + z_2) = \gamma \sum_{j=1}^{r} \alpha_j f(x_j) + \sum_{k=1}^{s} \beta_k f(y_k) = \gamma \overline{f}(z_1) + \overline{f}(z_2),$$

prooving the linearity of $\overline{f}$. It is also worth noting that $\overline{f}$ is unique, a fact rather immediate from its linear property.

Now let's look at how $\overline{f}$ acts on N: for any $y \in N$, y can be written as linear combinations from elements of the form (1.2) and (1.3). Because $\overline{f}$ is linear, we can just look at the action of $\overline{f}$ on (1.2):

$$\begin{gathered} \overline{f}((x_1, ..., x_i + x_i', ..., x_n) - (x_1, ..., x_i, ..., x_n) - (x_1, ..., x_i', ..., x_n)) = \\ = \overline{f}((x_1, ..., x_i + x_i', ..., x_n)) - \overline{f}((x_1, ..., x_i, ..., x_n)) - \overline{f}((x_1, ..., x_i', ..., x_n)) = \\ = f((x_1, ..., x_i + x_i', ..., x_n)) - f((x_1, ..., x_i, ..., x_n)) - f((x_1, ..., x_i', ..., x_n)) = \\ = f((x_1, ..., x_i, ..., x_n)) + f((x_1, ..., x_i', ..., x_n)) \\ -f((x_1, ..., x_i, ..., x_n)) - f((x_1, ..., x_i', ..., x_n)) = \\ = 0; \end{gathered}$$

and on (1.3):

$$\begin{gathered} \overline{f}((x_1, ..., \gamma x_i, ..., x_n) - \gamma(x_1, ..., x_i, ..., x_n)) = \\ = \overline{f}((x_1, ..., \gamma x_i, ..., x_n)) - \overline{f}(\gamma(x_1, ..., x_i, ..., x_n)) = \\ = f((x_1, ..., \gamma x_i, ..., x_n)) - f(\gamma(x_1, ..., x_i, ..., x_n)) = \\ = \gamma f((x_1, ..., x_i, ..., x_n)) - \gamma f((x_1, ..., x_i, ..., x_n)) = \\ = 0; \end{gathered}$$

where, in both cases, we used the multilinearity of f and the fact the restriction of $\overline{f}$ to $\prod_{i=1}^{n} E_i$ is identical to f by construction. This means that all elements of the submodule N are equal to zero, in other words,

$$N \subseteq \ker(\overline{f}). \tag{1.7}$$

With all this “tools” in hand, we can proceed with the last part of our proof. Let

$$\begin{aligned} f^* &: T \to F \\ z + N &\mapsto f^*(z + N) = \overline{f}(z). \end{aligned} \tag{1.8}$$

Such function is well-defined:

$$\begin{gathered} z + N = z' + N \Leftrightarrow z - z' = y \in N \Rightarrow \\ \Rightarrow \overline{f}(z - z') = \overline{f}(y) = 0 = \overline{f}(z) - \overline{f}(z') \Rightarrow \\ \Rightarrow f^*(z) = \overline{f}(z) = \overline{f}(z') = f^*(z'). \end{gathered}$$

Moreover, by the linearity of $\overline{f}$, f^* is also linear.

We can write $\overline{f} = f^* \circ p$. By the unicity of $\overline{f}$, f^* is also unique. Additionally, using the Diagram 1.4, it is easy to see that

$$\begin{tikzcd}
\prod_{i=1}^{n} E_i \arrow[r, "i"] \arrow[dr, "f"'] & M \arrow[r, "p"] \arrow[d, "\overline{f}"] & T = \dfrac{M}{N} \arrow[dl, "f^*"] \\
& F &
\end{tikzcd}$$

Figure 1.4: Third diagram for Theorem 1.1.2

$$f = \overline{f} \circ i = (f^* \circ p) \circ i = f^* \circ (p \circ i) = f^* \circ g,$$

which finishes this proof. □

It's worth noting that, by this theorem, the tensor product $\dfrac{M}{N}$ is not only an R-module, but it's also a free-module over R with $i(\prod_{i=1}^{n} E_i) = Im(i)$ being its basis. This means that, when dealing with linear maps having $\dfrac{M}{N}$ as its domain, we can only study the elements in the basis of $\dfrac{M}{N}$, namely elements of the form $(x_1, ..., x_n) + N$.

The next result is immediate consequence from Theorem 1.1.2.

Corollary 1.1.1. *Let (T, g) and (T', g') be two pairs with the property specified in Theorem 1.1.2. Then there exists an unique isomorphism*

$$h : T \to T',$$

such that $g' = h \circ g$.

Proof. Identifying $F = T'$ and $f = g'$, from Theorem 1.1.2 we can find two linear maps: one being $h : T \to T'$ such that $g' = h \circ g$; the other one being $h' : T' \to T$ such that $g = h' \circ g'$, both cases being illustrated in the Figures 1.5 and 1.6. Thus,

$$g' = h \circ g = h \circ (h' \circ g') = (h \circ h') \circ g',$$

meaning $h \circ h'$ is the identity map for T'. The same reasoning shows that $h' \circ h$ is the identity map for T. This means that h is an isomorphism between T and T'.

The unicity of h is given by Theorem 1.1.2. □

Corollary 1.1.1 shows us the unicity of the tensor product $\dfrac{M}{N}$ (up to an isomorphism), hence we can talk about it as **the** thensor product instead of **a** tensor product.

From this point onwards we shall abandon the notation $\dfrac{M}{N}$ for the tensor product to adopt the typical notation $\otimes_{i=1}^{n} E_i$, while its elements $(x_1, ..., x_n) + N$ will be written as $x_1 \otimes ... \otimes x_n$.

$$\begin{tikzcd}
\prod_{i=1}^{n} E_i \arrow[rr, "g'"] \arrow[dr, "g"'] & & T' \\
& T \arrow[ur, "h"'] &
\end{tikzcd}$$

Figure 1.5: First diagram for Corollary 1.1.1

$$\begin{tikzcd}
\prod_{i=1}^{n} E_i \arrow[rr, "g'"] \arrow[dr, "g"'] & & T \\
& T' \arrow[ur, "h'"'] &
\end{tikzcd}$$

Figure 1.6: Second diagram for Corollary 1.1.1

The last theorem shows the existence of the tensor product, but this doesn't mean every tensor product as a interesting structure, that is, it's not trivial.

Example 1.1.4. *Let* $m, n \in \mathbb{Z}$ *with* $gcd(m, n) = 1$*. Then the tensor product of the* $\mathbb{Z}$*-modules* $\dfrac{\mathbb{Z}}{m\mathbb{Z}}$ *and* $\dfrac{\mathbb{Z}}{n\mathbb{Z}}$ *is trivial, i.e.,*

$$(\frac{\mathbb{Z}}{m\mathbb{Z}}) \otimes (\frac{\mathbb{Z}}{n\mathbb{Z}}) = \{0\} .$$

By Theorem 1.1.2, the tensor product $\frac{\mathbb{Z}}{m\mathbb{Z}} \otimes \frac{\mathbb{Z}}{n\mathbb{Z}}$ *is a* $\mathbb{Z}$*-module genereted by the elements* $x \otimes y$*, where* $x \in \frac{\mathbb{Z}}{m\mathbb{Z}}$ *and* $y \in \frac{\mathbb{Z}}{n\mathbb{Z}}$*. Therefore, by the bilinearity of the tensor product,*

$$m(x \otimes y) = (mx) \otimes y = 0 \otimes y = 0,$$
$$n(x \otimes y) = x \otimes (ny) = x \otimes 0 = 0.$$

Bézout's lemma (see [18]) tells us that we can find $r, s \in \boldsymbol{Z}$ *such that*

$$gcd(m, n) = 1 = rm + sn.$$

So

$$1(x \otimes y) = (rm + sn)(x \otimes y) = (rmx) \otimes y + x \otimes (sny) = 0,$$

concluding that this tensor product has only zero as its element.

A question arises from this example: *when is the tensor product different from the trivial case?* Here we present a sufficient condition answering this question.

Theorem 1.1.3. *Let* E *and* F *be two free-modules over the ring with unit* R*, and let* $\{x_1, ..., x_m\}$ *and* $\{y_1, ..., y_n\}$ *be two basis for* E *and* F *respectively. Then the tensor product* $E \otimes F$ *is a free-module over* R *and the set* $\{x_i \otimes y_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ *is a basis for* $E \otimes F$*.*

Proof. The fact that $E \otimes F$ is a free-module is already known. So let's consider $z \in E \otimes F$. This means we can write z as a linear combination of elements of the form $z_1 \otimes z_2$, where $z_1 \in E$ and $z_2 \in F$. By our hypothesis,

$$\begin{aligned} z_1 &= \sum_{i=1}^{m} \alpha_i x_i \\ z_2 &= \sum_{j=1}^{n} \beta_j y_j, \end{aligned} \tag{1.9}$$

The bilinearity of the tensor product tells us that

$$(\sum_{i=1}^{m} \alpha_i x_i) \otimes (\sum_{j=1}^{n} \beta_j y_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \beta_j (x_i \otimes y_j),$$

meaning the set $S = \{x_i \otimes y_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$ generates the tensor product $E \otimes F$. Now let G be an arbitray R-module and $f : S \to G$ be a function. Then the function

$$\begin{aligned} t : E \times F &\to G, \\ (z_1, z_2) &\mapsto \sum_{i=1}^{m} \sum_{j=1}^{n} \alpha_i \beta_j f(x_i, y_j), \end{aligned} \tag{1.10}$$

is a bilinear mapping by construction. Moreover, Theorem 1.1.2 guarantees that t can be uniquely factor through $E \otimes F$ by a linear map t^*, hence we have $t^*(x_i \otimes y_j) = f(x_i, y_j)$, meaning $t^* \circ i = f$ and S is a basis for $E \otimes F$.

$$\begin{tikzcd}
S \arrow[r] \arrow[dr, "f"'] & E \otimes F \arrow[d, "t^*"] & E \times F \arrow[l] \arrow[dl, "t"] \\
& G &
\end{tikzcd}$$

Figure 1.7: Diagram for Theorem 1.1.3

□

An immediate consequence of Theorem 1.1.3 follows: *if V and W are two vector spaces over a field F with dimensions m and n respectively, then $V \otimes W$ is a vector space over F with dimesion mn.*

Example 1.1.5. *Let's consider the set of square matrices $Mat(m, \mathbb{R})$ and $Mat(n, \mathbb{R})$ over the real numbers. It's a well known fact from linear algebra that both are vector spaces over the real numbers with dimension m^2 and n^2 respectively. This means that the vector space $Mat(m, \mathbb{R}) \otimes Mat(n, \mathbb{R})$ has dimension $m^2 n^2 = (mn)^2$, thus we can conclude that $Mat(m, \mathbb{R}) \otimes Mat(n, \mathbb{R})$ and $Mat(mn, \mathbb{R})$ are isomorphic vector spaces.*

The tensor product between more than two R-modules may be defined more than one way. For example, when we say "consider the tensor product between the R-modules E_1, E_2 and E_3", we could be talking about $(E_1 \otimes E_2) \otimes E_3$, $E_1 \otimes (E_2 \otimes E_3)$, or even $E_1 \otimes E_2 \otimes E_3$. Which one are we talking about or should be used later in the theory?

Theorem 1.1.4. *Let E_1, E_2, and E_3 be R-modules. Then there exist unique isomorphisms*

$$(E_1 \otimes E_2) \otimes E_3 \to E_1 \otimes E_2 \otimes E_3 \to E_1 \otimes (E_2 \otimes E_3) \tag{1.11}$$

such that

$$(x \otimes y) \otimes z \mapsto x \otimes y \otimes z \mapsto x \otimes (y \otimes z), \tag{1.12}$$

for $x \in E_1, y \in E_2$ and $z \in E_3$.

Proof. We begin by showing the first arrow of Equation (1.11).

Let $z \in E_3$ be arbitrary and define

$$\begin{aligned} h_z : E_1 \times E_2 &\to E_1 \otimes E_2 \otimes E_3 \\ (x, y) &\mapsto h_z(x, y) = x \otimes y \otimes z. \end{aligned} \tag{1.13}$$

By the multilinearity of the tensor product, h_z is bilinear. Therefore, by Theorem 1.1.2, it induces an unique linear mapping

$$\begin{aligned} h_z^* : E_1 \otimes E_2 &\to E_1 \otimes E_2 \otimes E_3 \\ x \otimes y &\mapsto h_z^*(x, y) = x \otimes y \otimes z. \end{aligned} \tag{1.14}$$

$$\begin{tikzcd}
E_1 \times E_2 \arrow[rr, "h_z"'] \arrow[dr] & & E_1 \otimes E_2 \otimes E_3 \\
& E_1 \otimes E_2 \arrow[ur, "h_z^*"'] &
\end{tikzcd}$$

Figure 1.8: First diagram for Theorem 1.1.4

Consider the mapping

$$\begin{aligned} f : (E_1 \otimes E_2) \times E_3 &\to E_1 \otimes E_2 \otimes E_3 \\ (t, z) &\mapsto h_z^*(t), \end{aligned}$$

where t is a (finite) linear combination from elements of the form $x_i \otimes y_j$. Then f is a bilinear map. Indeed, for any $\gamma \in R$,

$$f(\gamma t + t', z) = h_z^*(\gamma t + t') = \gamma h_z^*(t) + h_z^*(t') = \gamma f(t, z) + f(t', z),$$

and

$$
\begin{gathered}
f(t,\gamma z+z')=h^*_{\gamma z+z'}(t)=h^*_{\gamma z+z'}(\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i\beta_j x_i\otimes y_j)=\\
=\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i\beta_j h^*_{\gamma z+z'}(x_i\otimes y_j)=\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i\beta_j(x_i\otimes y_j\otimes(\gamma z+z'))=\\
=\gamma\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i\beta_j(x_i\otimes y_j\otimes z)+\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i\beta_j(x_i\otimes y_j\otimes z')=\\
=\gamma\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i\beta_j h^*_z(x_i\otimes y_j)+\sum_{i=1}^{m}\sum_{j=1}^{n}\alpha_i\beta_j h^*_{z'}(x_i\otimes y_j)=\gamma h^*_z(t)+h^*_{z'}(t)=\\
=\gamma f(t,z)+f(t,z'),
\end{gathered}
$$

where we used the linearity of h^*_z for any $z\in E_3$.

By Theorem 1.1.2 again, there exists an unique linear mapping

$$f^*:(E_1\otimes E_2)\otimes E_3\to E_1\otimes E_2\otimes E_3,$$

such that $f^*(t\otimes z)=h^*_z(t)$. In particular, $f^*((x\otimes y)\otimes z)=x\otimes y\otimes z$.

$$\begin{tikzcd}
(E_1\otimes E_2)\times E_3 \arrow[rr, "f"'] \arrow[dr] & & E_1\otimes E_2\otimes E_3 \\
& (E_1\otimes E_2)\otimes E_3 \arrow[ur, "f^*"'] &
\end{tikzcd}$$

Figure 1.9: Second diagram for Theorem 1.1.4

Now consider the mapping

$$
\begin{aligned}
g:E_1\times E_2\times E_3&\to(E_1\otimes E_2)\otimes E_3\\
(x,y,z)&\mapsto(x\otimes y)\otimes z.
\end{aligned}
$$

The multilinearity of the tensor product guarantees g being multilinear. Additionally, g induces an unique linear mapping

$$
\begin{aligned}
g^*:E_1\otimes E_2\otimes E_3&\to(E_1\otimes E_2)\otimes E_3\\
x\otimes y\otimes z&\mapsto(x\otimes y)\otimes z.
\end{aligned}
\tag{1.15}
$$

A reminder for the last equation: since $E_1\otimes E_2\otimes E_3$ is generated by elements of the form $x\otimes y\otimes z$, we can consider g^* acting only over such elements.

Notice that $(x\otimes y)\otimes z=g^*(x\otimes y\otimes z)=g^*\circ f^*((x\otimes y)\otimes z)$ for all elements that generates the modules $(E_1\otimes E_2)\otimes E_3$, so $g^*\circ f^*=Id_{(E_1\otimes E_2)\otimes E_3}$, the identity map of $(E_1\otimes E_2)\otimes E_3$. Similarly,

$$f^*\circ g^*(x\otimes y\otimes z)=x\otimes y\otimes z\Rightarrow f^*\circ g^*=Id_{E_1\otimes E_2\otimes E_3},$$

$$\begin{tikzcd}
E_1 \times E_2 \times E_3 \arrow[rr, "g"'] \arrow[dr] & & (E_1 \otimes E_2) \otimes E_3 \\
& E_1 \otimes E_2 \otimes E_3 \arrow[ur, "g^*"'] &
\end{tikzcd}$$

Figure 1.10: Third diagram for Theorem 1.1.4

meaning f^* has a two-sided inverse, so f^* is invertible and hence a bijection with $(f^*)^{-1} = g^*$. Therefore, $(E_1 \otimes E_2) \otimes E_3 \cong E_1 \otimes E_2 \otimes E_3$.

The proof for the second arrow of Equation (1.11) is a verbatim of the previous case. $\square$

Theorem 1.1.4 shows us that we can write the tensor product as $x \otimes y \otimes z$ without considering the "order" of operation, so there's no ambiguity in saying "the tensor product between the E_1, E_2 and E_3".

The next theorem brings some more isomorphisms between tensor products.

Theorem 1.1.5. *Let E, E_1, E_2 and E_3 be R-modules. Then the following isomorphims hold true:*

$$E_1 \otimes E_2 \cong E_2 \otimes E_1; \tag{1.16}$$

$$(E_1 \oplus E_2) \otimes E_3 \cong (E_1 \otimes E_3) \oplus (E_2 \otimes E_3); \tag{1.17}$$

$$E_1 \otimes (E_2 \oplus E_3) \cong (E_1 \otimes E_2) \oplus (E_1 \otimes E_3); \tag{1.18}$$

$$R \otimes E \cong E. \tag{1.19}$$

The core idea in proving each isomorphism is the same done in Theorem 1.1.4. Here we'll present the proof for Equation (1.17), the other cases being omitted since they are almost identical to the following one, as well as what was done in Theorem 1.1.4.

Proof. For Equation (1.17), we define

$$\begin{aligned} f : (E_1 \oplus E_2) \times E_3 &\to (E_1 \otimes E_3) \oplus (E_2 \otimes E_3) \\ ((x, y), z) &\mapsto (x \otimes z, y \otimes z). \end{aligned} \tag{1.20}$$

First notice f is bilinear: indeed, for any $(x, y), (x', y') \in E_1 \oplus E_2$, any $z, z' \in E_3$, and any $c \in R$,

$$\begin{aligned} f(c(x, y) + (x', y'), z) = f((cx + x', cy + y'), z) = ((cx + x') \otimes z, (cy + y') \otimes z) = \\ = c(x \otimes z, y \otimes z) + (x' \otimes z, y' \otimes z) = cf((x, y), z) + f((x', y'), z); \end{aligned}$$

$$\begin{aligned} f((x, y), cz + z') = (x \otimes (cz + z'), y \otimes cz + z') = c(x \otimes z, y \otimes z) + (x \otimes z', y \otimes z') = \\ = cf((x, y), z) + f((x, y), z'). \end{aligned}$$

Theorem 1.1.2 tells us about the existence of the following unique linear mapping:

$$\begin{aligned} f^* : (E_1 \oplus E_2) \otimes E_3 &\to (E_1 \otimes E_3) \oplus (E_2 \otimes E_3) \\ (x, y) \otimes z &\mapsto (x \otimes z, y \otimes z). \end{aligned} \tag{1.21}$$

$$\begin{tikzcd}
(E_1 \oplus E_2) \times E_3 \arrow[rr, "f"'] \arrow[dr] & & (E_1 \otimes E_3) \oplus (E_2 \otimes E_3) \\
& (E_1 \oplus E_2) \otimes E_3 \arrow[ur, "f^*"'] &
\end{tikzcd}$$

Figure 1.11: First diagram for Theorem 1.1.5

Our objective now is to construct the inverse of f^*. To do so, let's consider the functions

$$\begin{aligned} \psi_1 : E_1 \times E_3 &\to (E_1 \otimes E_3) \oplus (E_2 \otimes E_3) \\ (x, z) &\mapsto (x, 0) \otimes z, \end{aligned} \tag{1.22}$$

and

$$\begin{aligned} \psi_2 : E_2 \times E_3 &\to (E_1 \otimes E_3) \oplus (E_2 \otimes E_3) \\ (y, z) &\mapsto (0, y) \otimes z. \end{aligned} \tag{1.23}$$

By the nature of the tensor product and of the direct sum $E_1 \oplus E_2$, ψ_1 and ψ_2 are bilinear:

$$\psi_1(cx + x', z) = (cx + x', 0) \otimes z = (c(x, 0) + (x', 0)) \otimes z =$$
$$= c(x, 0) \otimes z + (x', 0) \otimes z = c\psi_1(x, z) + \psi_1(x, z');$$

$$\psi_1(x, cz + z') = (x, 0) \otimes (cz + z') =$$
$$= c((x, 0) \otimes z) + (x, 0) \otimes z' = c\psi_1(x, z) + \psi_1(x, z');$$

the case for ψ_2 being trivial to see by replacing 1 for 2 and x for y in the last reasoning.

Since the tensor product is universal (Theorem 1.1.2), we have ψ_1^* and ψ_2^* such that

$$\begin{aligned} \psi_1^*(x \otimes z) &= (x, 0) \otimes z, \\ \psi_2^*(y \otimes z) &= (0, y) \otimes z. \end{aligned} \tag{1.24}$$

This is illustrated in Figure 1.12.

$$\begin{tikzcd}
E_1 \times E_3 \arrow[rr, "\psi_1"'] \arrow[dr] & & (E_1 \oplus E_2) \otimes E_3 & & E_2 \times E_3 \arrow[ll, "\psi_2"] \arrow[dl] \\
& E_1 \otimes E_3 \arrow[ur, "\psi_1^*"'] & & E_2 \otimes E_3 \arrow[ul, "\psi_2^*"] &
\end{tikzcd}$$

Figure 1.12: Second diagra for Theorem 1.1.5

Let's also consider the functions

$$\begin{aligned} h_1 : E_1 \times E_3 &\to (E_1 \otimes E_3) \oplus (E_2 \otimes E_3) \\ (x, z) &\mapsto (x \otimes z, 0); \end{aligned} \tag{1.25}$$

$$\begin{aligned} h_2 : E_2 \times E_3 &\to (E_1 \otimes E_3) \oplus (E_2 \otimes E_3) \\ (y, z) &\mapsto (0, y \otimes z). \end{aligned} \tag{1.26}$$

The two of them are bilinear since they are defined based on the tensor product and the direct sum. So again, there exist unique linear maps $h_1^*(x \otimes z) = (x \otimes z, 0)$ and $h_2^*(y \otimes z) = (0, y \otimes z)$ which make the diagram in Figure 1.13 commute.

$$\begin{tikzcd}
E_1 \times E_3 \arrow[rr, "h_1"'] \arrow[dr] & & (E_1 \oplus E_3) \oplus (E_2 \otimes E_3) & & E_2 \times E_3 \arrow[ll, "h_2"] \arrow[dl] \\
& E_1 \otimes E_3 \arrow[ur, "h_1^*"] & & E_2 \otimes E_3 \arrow[ul, "h_2^*"'] &
\end{tikzcd}$$

Figure 1.13: Third diagram for Theorem 1.1.5

Finally, let's establish the function

$$\begin{aligned} \psi : (E_1 \otimes E_3) \oplus (E_2 \otimes E_3) &\to (E_1 \oplus E_2) \otimes E_3 \\ (x \otimes z_1, y \otimes z_2) &\mapsto (x, 0) \otimes z_1 + (0, y) \otimes z_2. \end{aligned} \tag{1.27}$$

Notice that $\psi \circ h_1^* = \psi_1^*$ and $\psi \circ h_2^* = \psi_2^*$. Moreover, ψ is a linear map by construction.

$$\begin{tikzcd}
& & (E_1 \oplus E_3) \oplus (E_2 \otimes E_3) \arrow[dd, "\psi"] & & \\
E_1 \times E_3 \arrow[r] \arrow[urr, "h_1"] \arrow[drr, "\psi_1"'] & E_1 \otimes E_3 \arrow[ur, "h_1^*"] \arrow[dr, "\psi_1^*"] & & E_2 \otimes E_3 \arrow[ul, "h_2^*"'] \arrow[dl, "\psi_2^*"'] & E_2 \times E_3 \arrow[l] \arrow[ull, "h_2"'] \arrow[dll, "\psi_2"] \\
& & (E_1 \oplus E_2) \otimes E_3 & &
\end{tikzcd}$$

Figure 1.14: Forth diagram for Theorem 1.1.5

Finally we argue that ψ is the inverse of f^*:

$$\begin{gathered} f^* \circ \psi(x \otimes z_1, y \otimes z_2) = f^*((x, 0) \otimes z_1 + (0, y) \otimes z_2) = \\ = f^*((x, 0) \otimes z_1) + f^*((0, y) \otimes z_2) = (x \otimes z_1, 0 \otimes z_1) + (0 \otimes z_2, y \otimes z_2) = \\ = (x \otimes z_1, y \otimes z_2) \Rightarrow f^* \circ \psi = Id_{(E_1 \otimes E_3) \oplus (E_2 \otimes E_3)}; \end{gathered}$$

$$\begin{gathered} \psi \circ f^*((x, y) \otimes z) = \psi(x \otimes z, y \otimes z) = (x, 0) \otimes z + (0, y) \otimes z = \\ = ((x, 0) + (0, y)) \otimes z = (x, y) \otimes z \Rightarrow \psi \circ f^* = Id_{(E_1 \oplus E_2) \otimes E_3}. \end{gathered}$$

Hence ψ is letf and right-inverse of f^*, showing that f^* is an invertible linear transformation, *i.e.*, an isomorphism. □

So far we look at how the tensor product interact with different R-modules. Now we shall inspect how it affects linear mappings between R-modules.

Theorem 1.1.6. *Let E, E', G and G' be R-modules, and $f : E \to E'$ and $g : G \to G'$ be linear maps. Then:*

(i) there exists an unique linear mapping $\phi : E \otimes G \to E' \otimes G'$ such that $\phi(x \otimes y) = f(x) \otimes g(y)$;

(ii) if E'' and G'' are also R-modules, and $f' : E' \to E''$ and $g' : G' \to G''$ are two linear maps, then there exist unique linear maps α, β and γ such that

$$\begin{aligned} \alpha : E \otimes G &\to E' \otimes G' \\ x \otimes y &\mapsto f(x) \otimes g(y); \end{aligned} \tag{1.28}$$

$$\begin{aligned} \beta : E' \otimes G' &\to E'' \otimes G'' \\ x' \otimes y' &\mapsto f'(x') \otimes g'(y'); \end{aligned} \tag{1.29}$$

$$\begin{aligned} \gamma : E \otimes G &\to E'' \otimes G'' \\ x \otimes y &\mapsto f' \circ f(x) \otimes g' \circ g(y); \end{aligned} \tag{1.30}$$

and

$$\gamma = \beta \circ \alpha; \tag{1.31}$$

(iii) there exists a bilinear map between $\mathcal{L}(E, E') \times \mathcal{L}(G, G')$ and $\mathcal{L}(E \otimes G, E' \otimes G')$. Moreover, such map is a functor.

Note. In this work, we won't delve too much in the theory of Categories. With that being said, the functorial property of the tensor product is quite proeminent in the theory to be developed, becoming very hard to avoid it. Therefore, here is presented a basic definition for functors without going into many details. The interested reader can refer to [13]-[14] for a more complete study of such structures.

Let $\mathcal{R}$ be the collection of all R-modules. A functor F is any rule that associates to each $E \in \mathcal{R}$ a new R-morphism $F(E) \in \mathcal{R}$, and to each morphism $f : E \to G$ beteween R-modules a new morphism $F(f) : F(E) \to F(G)$ such that:

(i) for all $E \in \mathcal{R}$, $F(Id_E) = Id_{F(E)}$;

(ii) if $f : E \to G$ and $g : G \to H$ are two morphims between R-modules, then $F(g \circ f) = F(g) \circ F(f)$.

Proof. Here we'll make use of the following convention: for the linear maps $f : E \to E'$ and $g : G \to G'$, define

$$\begin{aligned} (f, g) : E \times G &\to E' \times G' \\ (x, y) &\mapsto (f(x), g(y)). \end{aligned}$$

The existence and unicity of ϕ is given by noticing that, for $t' : E' \times G' \to E' \otimes G'$ being the bilinear map defined in Theorem 1.1.2, $t' \circ (f, g)$ is bilinear:

$$\begin{aligned} t' \circ (f, g)(cx_1 + x_2, y) &= t'(f(cx_1 + x_2), g(y)) = (cf(x_1) + f(x_2)) \otimes g(y) = \\ &= c(f(x_1) \otimes g(y)) + f(x_2) \otimes g(y) = ct' \circ (f, g)(x_1, y) + t' \circ (f, g)(x_2, y); \end{aligned}$$

the linearity over the second coordinate being proved identically to the first one. Therefore $\phi(x \otimes y) = f(x) \otimes g(y)$ does exist and it's unique according to Theorem 1.1.2.

$$\begin{tikzcd}
E \times G \arrow[r, "{(f,g)}"'] \arrow[dr] & E' \times G' \arrow[r, "t'"'] & E' \otimes G' \\
& E \otimes G \arrow[ur, "\phi"] &
\end{tikzcd}$$

Figure 1.15: First diagram for Theorem 1.1.6.

The existence and unicity of α, β and γ are given by the first point of this theorem, just notice that $f' \circ f$ and $g' \circ g$ are linear, so the composition with $t'' : E'' \times G'' \to E'' \otimes G''$ is bilinear, namely $t'' \circ (f' \circ f, g' \circ g)$. We're left to show that $\gamma = \beta \circ \alpha$.

By construction, $\beta(x' \otimes y') = f'(x') \otimes g'(y')$. In particular, $x' = f(x)$ and $y' = g(y)$, so $\beta(f(x) \otimes g(y)) = f' \circ f(x) \otimes g' \circ g(y) = \gamma(x \otimes y)$. But we also have $f(x) \otimes g(y) = \alpha(x \otimes y)$ by construction, hence $\beta(f(x) \otimes g(y)) = \beta \circ \alpha(x \otimes y) = \gamma(x \otimes y)$.

$$\begin{tikzcd}
E \times G \arrow[r, "{(f,g)}"'] \arrow[d] & E' \times G' \arrow[r, "{(f',g')}"'] \arrow[d] & E'' \times G'' \arrow[d] \\
E \otimes G \arrow[r, "\alpha"] \arrow[rr, bend right, "\gamma"] & E' \otimes G' \arrow[r, "\beta"] & E'' \otimes G''
\end{tikzcd}$$

Figure 1.16: Second diagram for Theorem 1.1.6.

For the last point, let T be defined as

$$\begin{aligned} T : \mathcal{L}(E; E') \times \mathcal{L}(G; G') &\to \mathcal{L}(E \otimes G; E' \otimes G') \\ (f, g) &\mapsto T(f, g), \end{aligned} \tag{1.32}$$

where $T(f, g)(x \otimes y) := f(x) \otimes g(y)$, or, using the notation of (ii), $T(f, g) = \alpha$. We need to show that T is bilinear and functorial. For any $f, f' \in \mathcal{L}(E; E')$, any $c \in R$, and any $x \otimes y \in E \otimes G$,

$$\begin{gathered} T(cf + f', g)(x \otimes y) = ((cf + f')(x)) \otimes g(y) = (cf(x) + f'(x)) \otimes g(y) = \\ = c(f(x) \otimes g(y)) + f'(x) \otimes g(y) = cT(f, g)(x \otimes y) + T(f', g)(x \otimes y) = \\ = (cT(f, g) + T(f', g))(x \otimes y) \Rightarrow \\ \Rightarrow T(cf + f', g) = cT(f, g) + T(f', g). \end{gathered}$$

The proof for the second coordinate is analogous to the first, which improves T bilinear.

Lastly, let the R-module E be fixed. Then $G \mapsto E \otimes G$ and $g \mapsto T(Id_E, g)$ is a functor "tensor product on the left by E". Being more specific, let E be a fixed R-module and consider the collection $\mathcal{R}$ of all R-modules. We define

$$\begin{aligned} \tau_E : \mathcal{R} &\to \mathcal{R} \\ G &\mapsto \tau_E(G) := E \otimes G. \end{aligned}$$

Then τ_E defines a linear mapping

$$\tau_E : \mathcal{L}(G, G') \to \mathcal{L}(\tau_E(G); \tau_E(G'))$$

for each pair of R-modules E and G by the formula

$$\tau_E(g) = T(Id_E, g).$$

This finishes this proof. □

One can easily generalize all previous theorems and corollaries of this section to the cartesian product of n R-modules by using the same reasoning developed in previous proofs together with the application of the Finite Induction Theorem. The principal generalizations are written in the following corollary.

Corollary 1.1.2. *Let $E_1, ..., E_n, G_1, ..., G_n$ and F be a collection of R-modules; let $f_i \in \mathcal{L}(E_i, G_i)$ where $1 \leq i \leq n$. Then the following assertives are true:*

(i) for any combination of parenthesis placed in the tensor product $E_1 \otimes ... \otimes E_n$, the resulting R-module will be isomorphic to $\otimes_{i=1}^n E_i$;

(ii) $(\oplus_{i=1}^n E_i) \otimes F \cong \oplus_{i=1}^n (E_i \otimes F)$ and $F \otimes (\oplus_{i=1}^n E_i) \cong \oplus_{i=1}^n (F \otimes E_i)$;

(iii) there exist an unique multilinear mapping

$$\begin{gathered} T : \prod_{i=1}^{n} \mathcal{L}(E_i, G_i) \to \mathcal{L}(\otimes_{i=1}^n E_i, \otimes_{i=1}^n G_i) \\ (f_1, ..., f_n) \to T(f_1, ..., f_n), \end{gathered} \tag{1.33}$$

such that

$$T(f_1, ..., f_n)(x_1 \otimes ... \otimes x_n) = f_1(x_1) \otimes ... \otimes f_n(x_n). \tag{1.34}$$

1.2 The Tensor Algebra

We've stabilished some of the main properties of the tensor product, making the foundation of the theory to appear. Now we start to build upon this foundation.

Definition 1.2.1. *Let E_n be an R-module for each $n \in \mathbb{N}$. The sequence of modules*

$$(E_n)_{n\in\mathbb{N}} = (E_1, E_2, ..., E_n, ...) \tag{1.35}$$

is called a ***graded module*** *with addition being coordinate-wise, and scalar multiplication being done to all coordinates. The elements of the graded module E_n are called the* ***homogeneous element of degree*** *n. Additionally, if $(E_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ are two graded modules, and $f_n : E_n \to G_n$ is a linear mapping between modules, than we define*

$$\begin{gathered} \prod f_n : (E_n)_{n\in\mathbb{N}} \to (G_n)_{n\in\mathbb{N}} \\ (x_1, x_2, ..., x_n, ...) \mapsto (f_1(x_1), f_2(x_2), ..., f_n(x_n), ...). \end{gathered} \tag{1.36}$$

It's immediate from the definition that $\prod f_n$ is a linear mapping between modules.

Let's turn our attetion to a special case of graded modules: *the graded tensor product.*

Definition 1.2.2. *Let $(E_n)_{n\in\mathbb{N}}$ and $(G_n)_{n\in\mathbb{N}}$ be two graded modules. We define their **graded tensor product** to be the graded module whose the k-th term is*

$$\begin{aligned}((E_n)_{n\in\mathbb{N}} \otimes (G_n)_{n\in\mathbb{N}})_k &= \oplus_{k=m+n}(E_m \otimes G_n)\\ &= (E_0 \otimes G_k) \oplus \ldots \oplus (E_k \otimes G_0).\end{aligned} \tag{1.37}$$

Theorem 1.2.1. *Let $(E_n)_{n\in\mathbb{N}}$, $(G_n)_{n\in\mathbb{N}}$ and $(F_n)_{n\in\mathbb{N}}$ be graded modules. Then, given the bilinear functions*

$$f_{m,n} : E_m \times G_n \to F_{m+n}$$

for all $m, n \in \mathbb{N}$, there is exactly one multilinear mapping of modules

$$\begin{aligned}f : ((E_n)_{n\in\mathbb{N}}) \otimes ((G_n)_{n\in\mathbb{N}}) &\to (F_n)_{n\in\mathbb{N}}\\ f(x \otimes y) &\mapsto f_{m,n}(x, y),\end{aligned} \tag{1.38}$$

for all $x \in \oplus_{n\in\mathbb{N}}E_n$ and all $y \in \oplus_{n\in\mathbb{N}}G_n$.

Proof. By the universality of the tensor product (Theorem 1.1.2) there exsits an unique R-multilinear mapping

$$\begin{aligned}f^*_{m,n} : E_m \otimes G_n &\to F_{m+n}\\ f^*_{m,n}(x \otimes y) &\mapsto f_{m,n}(x, y),\end{aligned}$$

for each $m, n \in \mathbb{N}$. Moreover, the injection

$$i_{m+n} : (\oplus_{n\in\mathbb{N}}E_n) \otimes (\oplus_{n\in\mathbb{N}}G_n) \to E_m \otimes G_n$$

is also unique, hence the composition $f^*_{m,n} \circ i_{m+n}$ is an unique multilinear mapping. Therefore the mapping defined in Equation 1.38 is the desired multilinear mapping, as stated. □

In order to define the tensor algebra, one needs to understand the meaning of the word "algebra" used in this context.

Definition 1.2.3. *Let E be an R-module. We say that E is an **algebra**, or an **R-algebra** whenever we want to make explicit the commutative ring R, when:*

(i) *E is a ring;*

(ii) *for all $x, y \in E$ and all $c \in R$, $c(x.y) = (cx).y = x.(cy)$.*

*If E is a ring with unity, i.e., $1 \in E$, then we say E is an **algebra with unity**.*

What follows are some examples of algebras that will appear later in the text.

Example 1.2.1. *Consider* $E = \mathbb{C}$*, the set of complex numbers with the usual operations of addition and multiplication, i.e.,*

$$(x + iy) + (z + iw) = (x + z) + i(y + w) \tag{1.39a}$$

and

$$(x + iy)(z + iw) = (xz - yw) + i(xw + yz) \tag{1.39b}$$

respectively, with $i^2 = -1$*. Thus, if we consider the scalar multiplication as just the usual multiplication between a real number and a complex number, that is*

$$\alpha(x + iy) = (\alpha x) + i(\alpha y), \tag{1.39c}$$

then $\mathbb{C}$ *becomes an algebra over the real field. Additionally, as a vector space,* $\{1, i\}$ *is a linearly independent set that spans* $\mathbb{C}$*, hence* $dim(\mathbb{C}) = 2$*, thus* $\mathbb{C}$ *is isomorphic to* $\mathbb{R}^2$ *as a vector space.*

Example 1.2.2. *Consider* $E = \mathbb{R} \times \mathbb{R}$ *and define the following operations:*

$$(x, y) + (z, w) = (x + z, y + w) \tag{1.40a}$$

and

$$(x, y)(z, w) = (xz + yw, xw + yz). \tag{1.40b}$$

It's straightforward to prove that E *is a ring with* $(1, 0)$ *being its unity, thus this won't be done here. If we define the scalar multiplication as*

$$\alpha(x, y) = (\alpha x, \alpha y) \tag{1.40c}$$

with $\alpha \in \mathbb{R}$*, then* E *becomes an algebra over* $\mathbb{R}$*. This algebra shall be denoted as* $\mathbb{R} \oplus \mathbb{R}$*. Moreover, as a vector space,* $\{(1, 0), (0, 1)\}$ *is a linearly independent set that spans* $\mathbb{R} \oplus \mathbb{R}$*, thus it has dimension* 2*, meaning* $\mathbb{R} \oplus \mathbb{R}$ *is isomorphic to* $\mathbb{R}^2$ *as a vector space.*

An interesting comparison can be made between Examples 1.2.1 and 1.2.2. In Example 1.2.2, we know that $(1, 0)$ is the unity of the algebra, while $(0, 1)^2 = (1, 0)$. If we define $(1, 0) = 1$ and $(0, 1) = i$, then we can write all elements of $\mathbb{R} \oplus \mathbb{R}$ as

$$z = x(1, 0) + y(0, 1) = x + iy, \tag{1.41}$$

with $x, y \in \mathbb{R}$. Like in $\mathbb{C}$, i is called the imaginary unit of $\mathbb{R} \oplus \mathbb{R}$. Notice that, in this new notation, $i^2 = 1$ for $i \in \mathbb{R} \oplus \mathbb{R}$. This idea of "changing the minus sign in the square of the imaginary unity" will appear again when dealing with the algebras of $\mathbb{R}^4$.

Example 1.2.3. *Consider* $E = \mathbb{H} = \{x_0 + ix_1 + jx_2 + kx_3 \mid x_0, x_1, x_2, x_3 \in \mathbb{R}\}$*, the set of quartenions with the usual operations:*

$$\begin{aligned}(x_0 + ix_1 + jx_2 + kx_3) + (y_0 + iy_1 + jy_2 + ky_3) = (x_0 + y_0) + i(x_1 + y_1) \\ + j(x_2 + y_2) + k(x_3 + y_3)\end{aligned} \tag{1.42a}$$

and

$$\begin{aligned}(x_0 + ix_1 + jx_2 + kx_3)(y_0 + iy_1 + jy_2 + ky_3) = {} & (x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3) \\ & + i(x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2) \\ & + j(x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1) \\ & + k(x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0).\end{aligned} \tag{1.42b}$$

From them we can write the classical equations

$$\begin{cases} i^2 & = j^2 = k^2 = -1 \\ ijk & = -1 \end{cases} \tag{1.43}$$

Similar to $\mathbb{R} \oplus \mathbb{R}$ *and* $\mathbb{C}$, $\mathbb{H}$ *is an algebra over the field of the real numbers, where the scalar multiplication is just the usual multiplication between a real number and a quartenionic number. Furthermore, the set* $\{1, i, j, k\}$ *is a linearly independent set that spans* $\mathbb{H}$, *so, as a vector space over* $\mathbb{R}$, $\mathbb{H}$ *has dimension* 4, *therefore the algebra of quartenions is isomorphic to* $\mathbb{R}^4$ *as a vector space.*

Definition 1.2.4. *Let* A *be an algebra over* R *and* X *be a subset of* A. *We say that* A *is* ***generated by*** X *if, for any* $z \in A$, *we can write*

$$z = \sum_{i=1}^{m} y_i \tag{1.44a}$$

of any $m > 0$ *elements, each* y_i *being the product*

$$y_i = x_{i1} * \dots * x_{ik_i}, \tag{1.44b}$$

with k_i *elements and each* x_{ij_i} *being either an element of* X *or the opposite of an element of* X.

Example 1.2.4. *Let* $A = \mathbb{C}$, *the algebra of complex numbers over the real numbers. This algebra is generated by the subset*

$$X = \{ix \mid x \in \mathbb{R}\} = i\mathbb{R}.$$

Indeed, let $x + iy \in \mathbb{C}$. *Thus*

$$x + iy = \underbrace{(i(-x))}_{\in i\mathbb{R}} \underbrace{(i1)}_{\in i\mathbb{R}} + \underbrace{(iy)}_{\in i\mathbb{R}}.$$

By the same argument we have that the algebra $\mathbb{R} \oplus \mathbb{R}$ *is generated by the set* $X = \{ix \mid x \in \mathbb{R} \text{ and } i^2 = 1\}$.

Finally, consider the algebra of quartenions. This algebra is generated by

$$X = \{x + yj \mid x, y \in \mathbb{C}\} = \mathbb{C} \oplus j\mathbb{C}.$$

This is true since

$$\begin{aligned} x + iy + jz + kw &= (x + iy) + (jz + kw) \\ &= \underbrace{(x + iy)}_{\in \mathbb{C} \oplus j\mathbb{C}} + \underbrace{(z + iw)}_{\in \mathbb{C} \oplus j\mathbb{C}} \underbrace{(j1)}_{\in \mathbb{C} \oplus j\mathbb{C}}. \end{aligned}$$

The next theorem gives us a more abstract algebra when compared to the previous examples.

Theorem 1.2.2. *Let E be an R-module and define the following graded module:*

$$\begin{aligned} T(E) &= (T_n(E))_{n\in\mathbb{N}} \\ &= (R, E, E\otimes E, E\otimes E\otimes E, ...), \end{aligned} \tag{1.45}$$

where

$$\begin{aligned} T_0(E) &= R, \\ T_n(E) &= T_{n-1}(E)\otimes E, \end{aligned} \tag{1.46}$$

for all $n \geq 1$. Then this graded module is, in fact, an algebra over R with multiplication been the graded tensor product (Definition 1.2.2) and the element $(1, 0, 0, ...)$ as its unit element.

Proof. Here we shall ripe what we sow in the last section. Firstly, we already know that each $T_n(E)$ is an R-module, hence $T(E)$ is graded module by definition. Additionally, the functions

$$\begin{gathered} f^*_{m,n} : T_m(E)\otimes T_n(E) \to T_{m+n}(E) \\ (x_1\otimes ...\otimes x_m)\otimes(y_1\otimes ...\otimes y_n) \mapsto x_1\otimes ...\otimes x_m\otimes y_1\otimes ...\otimes y_n, \end{gathered}$$

are the unique multilinear maps multiplying two elements that generates $T_m(E)$ and $T_n(E)$ (Theorem 1.2.1). In practical terms, this means that the multiplication between elements of the graded tensor product is essentially a problem of "removing the parenthesis between elements of the basis".

To prove the associativity of this product, remember that in Corollary 1.1.2 we showed that the module generated from the tensor product of more than one R-module is the same (up to an isomorphism) for any combination of paranthesis placed between the simbols. Therefore the product is associative.

The element $(1, 0, 0, 0, ...)$ being the unit of $T(E)$ is a direct consequence of the definition of graded tensor product (Definition 1.37) together with the Theorem 1.1.5 and Equation 1.19. ☐

The graded module defined in the previous theorem is called the **(graded) tensor algebra of** E. Notice that the Tensor Algebra of E being an algebra generated by E is an immediate consequence from the definition of a tensor algebra and Definition 1.2.4.

Tensor algebras are in the core of what will be later defined as the Clifford Algebra.

Before proceeding, here follows one classical example that may hit close to the heart of some readers.

Example 1.2.5. *Let $E = R$ be a free module with one generator, namely 1. Then the tensor product $T_n(R) = \otimes_{i=1}^n K \cong K$ is also a free module. Moreover,*

$$a_1\otimes ...\otimes a_n = (a_1...a_n)1\otimes ...\otimes 1,$$

so $T_n(R)$ is being generated by $1\otimes ...\otimes 1 = x_n$. Additionally, we also have

$$x_{m+n} = x_m\otimes x_n \Rightarrow x_n = x_1\otimes ...\otimes x_1 = (x_1)^n,$$

mening each generator x_n is the power of x_1. Therefore, if we identify $x_1 = x$, we can conclude that

$$T(R) = (R, Rx, Rx^2, Rx^3, Rx^4, \dots) = R[x],$$

i.e., we can identify the polynomial ring $R[x]$ as a graded tensor algebra.

So far we looked at tensor product over modules. However our interest lies in understanding how it behaves if, instead of a module, we were considering E and G to be two algebras over R.

Theorem 1.2.3. *Let E and G be R-algebras. Then the tensor product $E \otimes G$ with the multiplication between elements being defined as*

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = (x_1x_2) \otimes (y_1y_2), \tag{1.47}$$

is an algebra over R.

Proof. E and G being algebras means they are R-modules, so $E \otimes G$ is also an R-module by Theorem 1.1.2. We just need to show the multiplication defined in Equation 1.47 turns $E \otimes G$ into an algebra.

Consider the mapping

$$\begin{aligned} M : E \times G \times E \times G &\to E \otimes G \\ (x_1, y_1, x_2, y_2) &\mapsto (x_1x_2) \otimes (y_1y_2). \end{aligned} \tag{1.48}$$

Such mapping is multilinear: indeed, for any $\alpha \in R$,

$$\begin{aligned} M(\alpha x_1 + x_1', y_1, x_2, y_2) &= ((\alpha x_1 + x_1')x_2) \otimes (y_1y_2) \\ &= (\alpha x_1x_2 + x_1'x_2) \otimes (y_1y_2) \\ &= (\alpha x_1x_2) \otimes (y_1y_2) + (x_1'x_2) \otimes (y_1y_2) \\ &= \alpha(x_1x_2) \otimes (y_1y_2) + (x_1'x_2) \otimes (y_1y_2) \\ &= \alpha M(x_1, y_1, x_2, y_2) + M(x_1', y_1, x_2, y_2); \end{aligned}$$

M being linear in the other coordinates is a verbatim from the first case.

By Theorem 1.1.2 again, the following linear mapping

$$\begin{aligned} M^* : E \otimes G \otimes E \otimes G &\to E \otimes G \\ x_1 \otimes y_1 \otimes x_2 \otimes y_2 &\mapsto (x_1x_2) \otimes (y_1y_2). \end{aligned} \tag{1.49}$$

does exist and it is unique.

Now consider the canonical projection

$$\begin{aligned} N : (E \otimes G) \times (E \otimes G) &\to (E \otimes G) \otimes (E \otimes G) \\ (x_1 \otimes y_1, x_2 \otimes y_2) &\mapsto (x_1 \otimes y_1) \otimes (x_2 \otimes y_2) \end{aligned} \tag{1.50}$$

and the isomorphism

$$\begin{aligned} K : (E \otimes G) \otimes (E \otimes G) &\to E \otimes G \otimes E \otimes G \\ (x_1 \otimes y_1) \otimes (x_2 \otimes y_2) &\mapsto x_1 \otimes y_1 \otimes x_2 \otimes y_2. \end{aligned} \tag{1.51}$$

Thus the composition mapping

$$P = M^* \circ K \circ N : (E \otimes G) \times (E \otimes G) \to E \otimes G$$

is the unique bilinear mapping satisfying

$$P(x_1 \otimes y_1, x_2 \otimes y_2) = (x_1 x_2) \otimes (y_1 y_2)$$

by Theorem 1.1.2. But that's exaclty Equation 1.47, hence this multiplication is well-defined. We're left to show this multiplication is associative and left and right-distributive over addition. To do so, let $x_1, x_2, x_3 \in E$ and $y_1, y_2, y_3 \in G$ be arbitrary. Consequently,

$$\begin{aligned}
(x_1 \otimes y_1)[(x_2 \otimes y_2)(x_3 \otimes y_3)] &= (x_1 \otimes y_1)(x_2 x_3 \otimes y_2 y_3) \\
&= (x_1(x_2 x_3)) \otimes (y_1(y_2 y_3)) \\
&= ((x_1 x_2) x_3) \otimes ((y_1 y_2) y_3) \\
&= (x_1 x_2 \otimes y_1 y_2)(x_3 \otimes y_3) \\
&= [(x_1 \otimes y_1)(x_2 \otimes y_2)](x_3 \otimes y_3),
\end{aligned}$$

proving the multiplication associative; whilst

$$\begin{aligned}
(x_1 \otimes y_1)(x_2 \otimes y_2 + x_3 \otimes y_3) &= P(x_1 \otimes y_1, x_2 \otimes y_2 + x_3 \otimes y_3) \\
\text{(P bilinear)} &= P(x_1 \otimes y_1, x_2 \otimes y_2) + P(x_1 \otimes y_1, x_3 \otimes y_3) \\
&= (x_1 \otimes y_1)(x_2 \otimes y_2) + (x_1 \otimes y_1)(x_3 \otimes y_3),
\end{aligned}$$

proves multiplication is left-distributive over addition. Right-distribution is proven analogously. □

We close this section with a theorem and some definitions from linear algebra. Such theorem shall not be proved here but the interested reader can check [19] for a formal proof.

Definition 1.2.5. *Let V be a vector space and let g be a bilinear form over V. The pair (V, g) is said to be a* ***quadratic vector space****. For the particullar case where*

$$g(y, x) = g(x, y), \tag{1.52}$$

for all $x, y \in V$, then we say that g is ***symmetric****. Additionally, if V has dimension n, then the* ***nullity*** *of g is defined as*

$$null(g) = dim(ker(g)), \tag{1.53}$$

while the ***rank*** *of g is defined as*

$$rank(g) = n - null(g). \tag{1.54}$$

Definition 1.2.6. *Let (V, g) be a quadratic vector space over the field F with g being symmetric. Then*

$$\begin{aligned} Q : V &\to F \\ x &\mapsto g(x, x) \end{aligned} \tag{1.55}$$

is defined as the ***Quadratic form associated with*** *g.*

Theorem 1.2.4. *Let V be a finite-dimensional vector space over $\mathbb{R}$, and let g be a symmetric bilinear form over V with rank r. Then there exists an ordered basis $\{e_1, ..., e_n\}$ for V such that*

$$g(e_i, e_j) = \pm\delta_{ij}, \text{ for } 1 \leq i, j \leq r, \tag{1.56}$$

where δ_{ij} is the Kronecker delta function. Furthermore, the number p of basis vectors e_i for which $g(e_i, e_i) = 1$ is independent of the choice of the basis. As such, the signature of g, defined as

$$sig(g) = r - p \tag{1.57}$$

is also independent of the choice of the basis.

Example 2.1.3, together with the previous theorem, motivates the following definition.

Definition 1.2.7. *Let V be vector space over $\mathbb{R}$ with $dim(V) = n$, g be an bilinear form over V such that $rank(g) = n$ and $sig(g) = m$. We define*

$$\begin{aligned} p &= \frac{n+m}{2}, \\ q &= \frac{n-m}{2}. \end{aligned} \tag{1.58}$$

In this case, since $V \simeq \mathbb{R}^n$, we write $\mathbb{R}^{p+q}$ to represent the quadratic vector space (V, g).

Chapter 2

Clifford Algebra

In this text we shall only work with Clifford Algebras over real vector spaces, although the definition would be the same if we were considering complex vector spaces. For a treatment using the complex numbers, see References [4] and [16]. For a more in-depth look into the structure of the Clifford Algebras, the specific isomorphisms and how they relate to some specific groups, like orthogonal transformations, the Pin Group and the Spin Group, see References [2], [3], [4], [5] and [6].

2.1 Definition of a Clifford Algebra

Definition 2.1.1. *Let V be a real vector space and g be a non-degenerate symmetric bilinear form over V, that is, for $x \in V$,*

$$g(x,x) = 0 \Leftrightarrow x = 0.$$

Also let A be an algebra with unity 1_A and let $\gamma : V \to A$ be a linear mapping. The pair (A, γ) is a ***Clifford Algebra*** *for the quadratic space (V, g) when A is generated as an algebra by $\{\gamma(x) | x \in V\}$ and $\{\alpha 1_A | \alpha \in \mathbb{R}\}$, and γ satisfies*

$$\gamma(x) * \gamma(y) + \gamma(y) * \gamma(x) = 2g(x,y)1_A \tag{2.1}$$

for all $x, y \in V$, where the asterisk is used to represent the product between elements of the algebra A. In this case, the mapping γ is said to be a ***Clifford mapping****.*

Equation (2.1) can be rewriten by using the quadratic form Q associated with g:

$$Q(x) = g(x,x). \tag{2.2}$$

Thus, using the identity

$$Q(x+y) - Q(x) - Q(y) = 2g(x,y), \tag{2.3}$$

Equation (2.1) becomes

$$(\gamma(x))^2 = Q(x)1_A = g(x,x)1_A. \tag{2.4}$$

We'll make use of both equations, depending of which is more convinient at the time. For now, we begin with two simple but important examples of Clifford Algebras associated with quadratic real vector spaces.

Example 2.1.1. *Let $V \simeq \mathbb{R}$, $\{e\}$ be a vector generating V, i.e., $e \neq 0$, and let*

$$g(e,e) = 1$$

be the bilinear form associated with V. This means, for any $x \in V$,

$$g(x,x) = \alpha^2, \tag{2.5}$$

where $x = \alpha e$.

Consider the set $\{1, e\}$, and let A be a vector space generated by this set, that is,

$$A = \mathbb{R} \oplus V. \tag{2.6}$$

Suppose also the following set of equations are observed in A:

$$(\alpha_1 + \beta_1 e) + (\alpha_2 + \beta_2 e) = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)e, \tag{2.7a}$$

$$(\alpha_1 + \beta_1 e) * (\alpha_2 + \beta_2 e) = (\alpha_1\alpha_2 + \beta_1\beta_2) + (\alpha_1\beta_2 + \alpha_2\beta_1)e, \tag{2.7b}$$

$$\gamma(\alpha_1 + \beta_1 e) = (\gamma\alpha_1) + (\gamma\beta_1)e, \tag{2.7c}$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{R}$. Consequently, 1 is the unity of A and $e * e = e^2 = 1 = g(x,x)$ is observed.

Clearly A is a Clifford Algebra for V: it's straightforward from the set of Equations 2.7 to prove A is an algebra with unity. Moreover, consider the insertion mapping

$$\begin{aligned} i : V &\to A \\ x &\mapsto x. \end{aligned}$$

For any $x \in V$, i satisfies Equation 2.4 by construction of A, meaning i is a Clifford mapping. In addition, the mapping

$$\phi : A \to \mathbb{R} \oplus \mathbb{R},$$

where $\phi(1) = (1,0)$ and $\phi(e) = (0,1)$, is an algebra isomorphism between A and the algebra $\mathbb{R} \oplus \mathbb{R}$ (see Example 1.2.2), thus $\mathbb{R} \oplus \mathbb{R}$ is algebraicaly isomorphic to a Clifford algebra.

Example 2.1.2. *Let's consider the same vector space in Example 2.1.1, but here we take the bilinear form*

$$g(e,e) = -1.$$

This means Equation 2.5 becomes

$$g(x,x) = -\alpha^2 \tag{2.8}$$

for $x = \alpha e \in V$.

Once again, let A be the vector space defined by Equation 2.6 and satisfying

$$(\alpha_1 + \beta_1 e) + (\alpha_2 + \beta_2 e) = (\alpha_1 + \alpha_2) + (\beta_1 + \beta_2)e, \tag{2.9a}$$

$$(\alpha_1 + \beta_1 e) * (\alpha_2 + \beta_2 e) = (\alpha_1\alpha_2 - \beta_1\beta_2) + (\alpha_1\beta_2 + \alpha_2\beta_1)e, \tag{2.9b}$$

$$\gamma(\alpha_1 + \beta_1 e) = (\gamma\alpha_1) + (\gamma\beta_1)e, \tag{2.9c}$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{R}$. Thus, by the same reasoning done in the previous example, the pair (A, j), with j being the insertion of V into A, is a Clifford Algebra for V. Moreover, the mapping

$$\phi : A \to \mathbb{C}$$
$$\phi(1) = 1 \text{ and } \phi(e) = i,$$

with i being the imaginary unity of $\mathbb{C}$, is an isomorphism of algebras. Hence, $\mathbb{C}$ is also algebraicaly isomorphic to a Clifford Algebra.

At a first glance, with this two examples, one may think the prodcut in a Clifford Algebra is commutative. Such is not the case in general. This can be immediatly seen from its definition: suppose $g(x, y) = 0$. Then, by Equation 2.1,

$$\gamma(y) * \gamma(x) = -\gamma(x) * \gamma(y). \tag{2.10}$$

This brings to attention an interesting point: suppose V is a finite-dimensional vector space over $\mathbb{R}$ and $g = \langle ., . \rangle$ is a inner-product for V. Consequently, $g(x, y) = 0 = \langle x, y \rangle$ (i.e., x perpendicular to y) means that $\gamma(x)$ and $\gamma(y)$ anticommute. So we have a geometric property in V (perpendicularity) being reflected as an algebraic property in the Clifford Algebra A (anticommutation). Although this relation between "geometry in V, algebra in A" often appears when treating with Clifford Algebras, it's not a property we'll make frequent use in this work. For the intrested reader, see [20].

This first two examples gave us a glance of an important point: knowing the rank of the bilinear form g is not sufficient to know the isomorphism being related to the Clifford Algebra. In both cases, Example 2.1.1 and Example 2.1.2, the rank in question is the same, namely 1. What changes is a second property associated with the quadratic form: its *signature*. But how this properties relate to each other will be explored in more details in the next section. For now, we come back to the foundations and present a theorem showing how we can construct a Clifford Algebra from a given quadratic space over $\mathbb{R}$.

Theorem 2.1.1. *Let (V, g) be a quadratic vector space over $\mathbb{R}$. Then there exists a pair $(Cl_g(V), \gamma)$ consisting of a Clifford Algebra $Cl_g(V)$ for V and a Clifford mapping $\gamma : V \to Cl_g(V)$ with the following property: given any Clifford Algebra A for V with a Clifford mapping*

$$\rho : V \to A, \tag{2.11}$$

there exists an unique homomorphism of algebras

$$\psi : Cl_g(V) \to A \tag{2.12}$$

such that $\rho = \psi \circ \gamma$.

This is the theorem of existence and universality of a particular set of Clifford Algebras. The proof is done by constructing a Clifford Algebra from the quotient of the tensor algebra $T(V)$ over an specific ideal. But, before starting its proof, we need to stablish said ideal.

Definition 2.1.2. *Let (V, g) be a quadratic space over $\mathbb{R}$ and let $T(V)$ be its tensor algebra. We define the two-sided ideal I_g as the ideal generated by elements of the form*

$$x \otimes x - g(x, x)1, \tag{2.13}$$

where here 1 is the neutral element for the product of the algebra $T(V)$.

Proof of Theorem 2.1.1. We begin by studying the two-sided ideal I_g, which is being spanned by elements of the form (2.13). But, by using the bilinearity of g, we also have

$$\begin{aligned}(x+y)\otimes(x+y)-g(x+y,x+y)1 &= x\otimes x+x\otimes y+y\otimes x+y\otimes y\\ &\quad -g(x,x)-g(x,y)-g(y,x)-g(y,y)\\ &= (x\otimes x-g(x,x))+(y\otimes y-g(y,y))\\ &\quad +(x\otimes y+y\otimes x-2g(x,y)).\end{aligned}$$

The first two parenthesis are elements of I_g, hence $x\otimes y+y\otimes x-2g(x,y)\in I_g$. Moreover, for every $x\in V$,

$$(\frac{1}{2}x)\otimes x+x\otimes(\frac{1}{2}x)-2g(\frac{1}{2}x,x)=x\otimes x-g(x,x).$$

Thus, we can write the elements of I_g as being generated by elements of the form (2.13), or of the form

$$x\otimes y+y\otimes x-2g(x,y)1. \tag{2.14}$$

With this in mind, we define the following equivalence relation:

$$A\sim B\Leftrightarrow A=B+y, y\in I_g, \tag{2.15}$$

where $A,B\in T(V)$. We write $[A]$ to denote the equivalent classes of A under this equivalence relation, and define the following operations:

$$[A]+[B]=[A+B], \tag{2.16a}$$

$$\alpha[A]=[\alpha A], \tag{2.16b}$$

$$[A]*[B]=[A\otimes B], \tag{2.16c}$$

where $\alpha\in\mathbb{R}$.

The addition defined by Equation 2.16a is well-defined since $T(V)$ is an algebra, therefore an abelian group over addition, hence its quotient is also an abelian group over the two-sided ideal I_g by the First Isomorphism Theorem of Groups [21]. Scalar multiplication (Equation (2.16b)) is also well-defined because, for any $\alpha\in\mathbb{R}$ and any $C\in[A]$,

$$\begin{aligned}&C\sim A\Leftrightarrow C=A+y, y\in I_g\\ &\qquad\Rightarrow \alpha C=\alpha A+\alpha y\Leftrightarrow \alpha C\sim\alpha A \quad\Rightarrow \alpha C\in[\alpha A].\end{aligned}$$

Lastly, the multiplication (Equation (2.16c)) is well-defined as well: suppose $C\in[A]$ and $D\in[B]$. Then we can find $y,z\in I_g$ such that $C=A+y$ and $D=B+z$. Therefore,

$$C\otimes D=(A+y)\otimes(B+z) \quad = A\otimes B+A\otimes z+y\otimes B+y\otimes z.$$

The last three terms are elements of the ideal I_g, hence $C\otimes D\sim A\otimes B$, i.e., $[C\otimes D]=[A\otimes B]$.

We've just proved that $Cl_g(V)$ is an algebra. To prove that it is a Clifford Algebra, take $x, y \in V$. We can write

$$\begin{aligned} x \otimes y = &\frac{1}{2}(x \otimes y - y \otimes x) + g(x,y) \\ &+ \frac{1}{2}((x+y) \otimes (y+x) - g(x+y, x+y)) \\ &- \frac{1}{2}(x \otimes x - g(x,x)) - \frac{1}{2}(y \otimes y - g(y,y)). \end{aligned}$$

The last three terms are elements of I_g, meaning we have

$$x \otimes y \sim \frac{1}{2}(x \otimes y - y \otimes x) + g(x,y).$$

Thus, if we consider the composition mapping

$$\gamma = p \circ i, \tag{2.17}$$

where $i : V \to T(V)$ is the insertion of V into $T(V)$ and $p : T(V) \to \dfrac{T(V)}{I_g}$ is the canonical projection, then

$$\begin{aligned} \gamma(x) * \gamma(y) + \gamma(y) * \gamma(x) &= [x] * [y] + [y] * [x] \\ &= [x \otimes y] + [y \otimes x] \\ &= [\frac{1}{2}(x \otimes y - y \otimes x) + g(x,y)] \\ &+ [\frac{1}{2}(y \otimes x - x \otimes y) + g(y,x)] \\ &= 2g(x,y)[1]. \end{aligned}$$

So γ satisfies Equation (2.1), hence it's a Clifford mapping. Additionally, since the elements of V generates its tensor algebra, the algebra $Cl_g(V)$ is generated by $\{[x] \mid x \in V\}$ and $\{\alpha[1] \mid \alpha \in \mathbb{R}\}$, and so it's a Clifford Algebra.

So far, we've proved the existence. The next part is proving its universality, that is, Equation (2.12). To do so, by using the fact that $\rho \in \mathcal{L}(V, A)$, we construc the mapping

$$\begin{aligned} \rho_k = \prod_{i=1}^{k} \rho : \prod_{i=1}^{k} V &\to A \\ (x_1, ..., x_k) &\mapsto \rho(x_1) * ... * \rho(x_k), \end{aligned} \tag{2.18}$$

which is a linear mapping for any $k \geq 1$: indeed, for any $\alpha \in \mathbb{R}$ and any $1 \leq j \leq k$,

$$\begin{aligned} \rho_k(x_1, ..., \alpha x_j + x'_j, ..., x_k) &= \rho(x_1) * ... * \rho(\alpha x_j + x'_j) * ... * \rho(x_k) \\ &= \rho(x_1) * ... * (\alpha\rho(x_j) + \rho(x'_j)) * ... * \rho(x_k) \\ &= \alpha\rho(x_1) * ... * \rho(x_j) * ... * \rho(x_k) \\ &+ \rho(x_1) * ... * \rho(x'_j) * ... * \rho(x_k) \\ &= \alpha\rho_k(x_1, ..., x_j, ..., x_k) + \rho_k(x_1, ..., x'_j, ...x_k). \end{aligned}$$

Thus, by the Universality of the Tensor Product (Theorem 1.1.2), for each $k \geq 1$, there exists an unique linear mapping satisfying

$$\begin{aligned} \rho^*_k : T_k(V) &\to A \\ x_1 \otimes ... \otimes x_k &\mapsto \rho(x_1) * ... * \rho(x_k). \end{aligned} \tag{2.19}$$

For the Clifford Algebra A, we can define the graded module $(A_n)_{n\in\mathbb{N}} = (A, A, ..., A, ...)$. Thus, we've just shown that

$$\rho' = \prod \rho_n^* : \begin{aligned} T(V) &\to (A_n)_{n\in\mathbb{N}} \\ (x_n)_{n\in\mathbb{N}} &\mapsto (\rho_n(x_n))_{n\in\mathbb{N}} \end{aligned} \tag{2.20}$$

is a multilinear mapping between graded modules as defined by Equation (1.36), where ρ_n^* is define by Equation (2.19) for the cases $n \geq 1$, while $\rho_0^* = i : \mathbb{R} \to A$ is the insertion of $\mathbb{R}$ in A.

Take the Quotient module $\dfrac{T(V)}{ker(\rho')}$. By the First Isomorphism Theorem of Groups [21], there exists an unique linear mapping $\phi : \dfrac{T(V)}{ker(\rho')} \to (A_n)_{n\in\mathbb{N}}$, such that

$$\rho' = \phi \circ \pi, \tag{2.21}$$

with $\pi : T(V) \to \dfrac{T(V)}{ker(\rho')}$ being the canonical projection. Therefore, if we consider an element of the form $x \otimes x - Q(x)1 \in T(V)$, we have

$$\begin{aligned} \rho'(x \otimes x - Q(x)1) &= \rho'(x \otimes x) - Q(x)\rho'(1) \\ &= \rho(x) * \rho(x) - Q(x)1_A \\ &= 0 = \phi((x \otimes x - Q(x)1) + ker(\rho')). \end{aligned}$$

Since elements of this form spans the ideal I_g, we have $I_g \subseteq ker(\rho')$. Once again, by First Isomorphism Theorem of Groups [21], there exists an unique linear mapping $\rho^* : \dfrac{T(V)}{I_g} \to \dfrac{T(V)}{ker(\rho')}$ such that the diagram in Figure 2.1 commutes.

$$\begin{tikzcd}
T(V) \arrow[rr, "\pi"] \arrow[dr, "p"'] & & \frac{T(V)}{ker(\rho')} \\
& Cl_g(V) = \frac{T(V)}{I_g} \arrow[ur, "\rho^*"'] &
\end{tikzcd}$$

Figure 2.1: First diagram for Theorem 2.1.1.

We summarize all the linear maps used and defined so far in Figure 2.2 in order to visualize what's done and what's left. In said diagram, $i : V \to T(V)$ is the insertion of V into $T(V)$. Also notice the lack of a solid line between $(A_n)_{n\in\mathbb{N}}$ and A. In order to "close this gap", we define: $(a_n)_{n\in\mathbb{N}}$ has *degree* k if $(a_n)_{n\in\mathbb{N}} \neq 0$ and if there exist $k \in \mathbb{N}$ such that $a_k \neq 0$ and $a_n = 0$ if $n > k$.

With the last definition at hand, we can construct the mapping $j : (A_n)_{n\in\mathbb{N}} \to A$ as

$$j(a_n)_{n\in\mathbb{N}} = \begin{cases} \sum_{i=0}^{k} a_k & \text{, if } (a_n)_{n\in\mathbb{N}} \text{ as degree } k; \\ 0 & \text{, otherwise.} \end{cases} \tag{2.22}$$

$$\begin{tikzcd}
V \arrow[r, "\rho"'] \arrow[d, "i"'] & A \\
T(V) \arrow[r, "\rho'"] \arrow[d, "p"'] \arrow[dr, "\pi"] & (A_n)_{n\in\mathbb{N}} \arrow[u, dashed] \\
Cl_g(V) = \dfrac{T(V)}{I_g} \arrow[r, "\rho^*"] & \dfrac{T(V)}{ker(\rho')} \arrow[u, "\phi"']
\end{tikzcd}$$

Figure 2.2: Second diagram for Theorem 2.1.1.

Such mapping is not linear: if we consider $(1, 0, 0, 0, ...)$ and $(0, 1, 1, 1, ...)$, then

$$\begin{aligned} j((1,0,0,0,...) + (0,1,1,1,...)) &= j(1,1,1,1,...) = 0 \\ &\neq 1 + 0 = j(1,0,0,0,...) + j(0,1,1,1,...). \end{aligned}$$

But, the composition mapping

$$\psi = j \circ \phi \circ \rho^* \tag{2.23}$$

is linear: for any $x, y \in V$ and any $\alpha \in \mathbb{R}$,

$$\begin{aligned} \psi(\alpha[x] + [y]) &= \psi([\alpha x + y]) = j \circ \phi \circ \rho^*([\alpha x + y]) \\ &= j \circ \phi \circ (\rho^* \circ p)(\alpha x + y) = j \circ (\phi \circ \pi)(\alpha x + y) \\ &= j \circ \rho'(\alpha x + y) = j(\underbrace{\rho'(\alpha x + y)}_{\text{degree } 1}) \\ &= \rho'(\underbrace{\alpha x + y}_{\in T_1(V)}) = \rho_1^*(\alpha x + y) \\ &= \alpha \rho_1^*(x) + \rho_1^*(y) \\ &= \alpha \rho'(x) + \rho'(y) = \alpha j \circ \rho'(x) + j \circ \rho'(y) \\ &= \alpha \psi([x]) + \psi([y]). \end{aligned}$$

Furthermore, ψ also preserves the multiplication:

$$\begin{aligned} \psi([x] * [y]) &= \psi([x \otimes y]) = j \circ \phi \circ \rho^*([x \otimes y]) \\ &= j \circ \rho'(x \otimes y) = j(\underbrace{\rho'(x \otimes y)}_{\text{degree } 2}) \\ &= \rho'(\underbrace{x \otimes y}_{\in T_2(V)}) = \rho_2^*(x \otimes y) \\ &= \rho(x) * \rho(y) = \rho_1^*(x) * \rho_1^*(y) \\ &= \rho'(x) * \rho'(y) = j \circ \rho'(x) * j \circ \rho'(y) \\ &= \psi(x) * \psi(y). \end{aligned}$$

This proves that ψ an homomorphism between algebras. Finally, using the fact that $\gamma(x) = [x]$, we have

$$\psi([x]) = \rho_1^*(x) = \rho(x) = \psi \circ \gamma(x),$$

i.e., Equation (2.12) is satisfied, finishing this proof. □

By its universal nature, we can say "**the** Clifford Algebra of V" instead of "**a** Clifford Algebra of V" when treating with $Cl_g(V)$, similar with what was done with the tensor algebra $T(V)$.

It's worth noticing that V is embedded in $Cl_g(V)$, by which we mean we can find a subspace of $Cl_g(V)$ isomorphic to V. Indeed, we know that $W = Im(\gamma)$ is a subspace of $Cl_g(V)$. Morevover, if $\gamma(x) \in ker(\gamma)$, then

$$\gamma(x) = 0 \Rightarrow \gamma(x) * \gamma(x) = g(x,x)1_{Cl_g(V)} = 0 \Rightarrow \\ \Rightarrow g(x,x) = 0 \Rightarrow x = 0,$$

since the bilinear form g is considered to be non-singular. Therefore γ is an injective mapping and $V \simeq W$. This tells us we can "identify" W as V and say $V \subset Cl_g(V)$, which allows us to write the elements $\gamma(x)$ as just x. Moreover, instead of saying "$Cl_g(V)$ is generated by $\{\gamma(x) \mid x \in V\} \cup \{\alpha 1 \mid \alpha \in \mathbb{R}\}$", we can say "$Cl_g(V)$ is generated by $V \cup \{\alpha 1 \mid \alpha \in \mathbb{R}\}$". This identifications shall be considered from this point onwards.

In this work we'll be interested in finite-dimensional vector spaces. It turns out the Clifford algebra $Cl_g(V)$ constructed in Theorem 2.1.1 is also finite-dimensional if V is finite-dimensional itself. Not only that but we can calculate the dimesion of $Cl_g(V)$ quite easly given the dimension of V. To do so, we'll need some definitions, properties and notations relating to $Cl_g(V)$.

Definition 2.1.3. *Let (V,g) and (V',g') be two pairs of quadratic vector spaces. A linear map $f : V \to V'$ is said to* ***preserve the bilinear forms*** *if*

$$g'(f(x), f(y)) = g(x,y) \tag{2.24}$$

for any $x, y \in V$. In particular, if both forms are inner products for their respective vector spaces, and if f satisfies Equation (2.24), then f is said to be an ***isometry****.*

Definition 2.1.4. *Let V be a quadratic vector space. We denote by $O(V)$ the set of all linear automorphims of V that preserve the the bilinear form g. In particular, if g is an inner-product of V, then $O(V)$ is the set of* ***orthogonal maps*** *of V.*

Theorem 2.1.2. *Let (V,g) and (V',g') be two quadratic vector spaces; let γ and γ' be the Cliffor mappings for V and V' respectively; and let $f : V \to V'$ be a linear mapping preserving the bilinear forms. Then there exists an unique algebra mapping*

$$\theta_f : Cl_g(V) \to Cl_{g'}(V')$$

such that

$$\theta_f \circ \gamma = \gamma' \circ f \tag{2.25}$$

is observed.

Proof. Consider the composition of linear mappings

$$\gamma' \circ f : V \to Cl_{g'}(V').$$

Such composition is itself a linear mapping. Moreover, $W = Im(f)$, the image of f, is a subspace of V', thus we can restrict γ' and g' to W, and construct $Cl_{g'}(W)$, the Clifford algebra for W. It's easy to see this Clifford algebra is a subalgebra of $Cl_{g'}(V')$. Therefore, by Theorem 2.1.1, there exists an algebra homomorphism

$$(f \circ \gamma')^* : Cl_g(V) \to Cl_{g'}(W)$$

such that $(f \circ \gamma')^*(\gamma(x)) = \gamma' \circ f(x)$ for any $x \in V$.

Since $Cl_{g'}(W)$ is a subalgebra of $Cl_{g'}(V')$, we can use the insertion mapping $i : Cl_{g'}(W) \to Cl_{g'}(V')$ to define

$$\theta_f = i \circ (f \circ \gamma')^*. \tag{2.26}$$

Notice that θ_f satisfies Equation 2.25 and it is an algebra homomorphism. ☐

Corollary 2.1.1. *Each linear transformation $f \in O(V)$ extends uniquely to define an algebraic automorphism θ_f of the Clifford Algebra $Cl_g(V)$.*

This automorphism is referred as the **Bogoliubov automorphism** of $Cl_g(V)$ induced by f.

Corollary 2.1.2. *Suppose $f, h \in O(V)$. Then*

$$\theta_{f\circ h} = \theta_f \circ \theta_h. \tag{2.27}$$

In fact, the function

$$\theta : O(V) \to \mathcal{A}ut(Cl_g(V)), \tag{2.28}$$

which maps each linear mapping of $O(V)$ to its Bogoliubov automorphism, is a group homomorphism.

Proof. If $f, h \in O(V)$, then $f \circ h \in O(V)$, so each of $\theta_{f\circ h}$ and $\theta_f \circ \theta_h$ is an automorphism of $Cl_g(V)$ extending $f \circ h$. By its unicity, it must follow that $\theta_{f\circ h} = \theta_f \circ \theta_h$. Finally, θ being a group homomorphism is immediate consequence of Equation 2.27. ☐

For the next part, we shall make use of the following notations: let $m \in \mathbb{N}$. We denote

$$\mathbf{m} = \{1, ..., m\}.$$

Moreover, let S be the non-empty set

$$S = \{s_1 < ... < s_p\} \subset \mathbf{m};$$

in this case, we write the following product in $Cl_g(V)$:

$$v_S = v_{s_1} * v_{s_2} * ... * v_{s_p}.$$

For convinence sake, we alse write

$$v_\emptyset = 1.$$

With this notations in mind, along with Theorem 1.2.4, we begin presenting some properties the generators of $Cl_g(V)$ hold.

Theorem 2.1.3. *Let (V, g) be a quadratic vector space over $\mathbb{R}$. Let $\{v_1, ..., v_m\}$ be a set of V satisfying*

$$g(v_i, v_j) = 0$$

whenever $i \neq j$. If $S, T \subset \boldsymbol{m}$, then

$$v_T * v_S = (-1)^{pq+r} v_S * v_T, \tag{2.29}$$

where p, q and r are the cardinality of the sets S, T and $S \cap T$ respectively.

Proof. Let $R = S \cap T$. We begin by considering the particular case $R = \emptyset$. By hypothesis, we know that

$$g(v_i, v_j) = 0 \Leftrightarrow v_j * v_i = -v_i * v_j \tag{2.30}$$

whenever $i \neq j$. Suppose $q = 1$ and fix $p \in \mathbb{N}$. In this case,

$$\begin{aligned} v_T * v_S &= v_{t_1} * \underbrace{v_{s_1} * ... * v_{s_p}}_{\text{p factors}} \\ &= (-1)^p v_{s_1} * ... * v_{s_p} * v_{t_1} = (-1)^p v_S * v_T, \end{aligned}$$

where we used the fact S and T are disjoint, thus $t_1 \neq s_i$ for all $1 \leq i \leq p$; and the anticommutative property p-times in the product. Thus Equation 2.29 is observed. By induction, suppose Equation 2.29 also holds for $q = k$. So, for $q = k + 1$,

$$\begin{aligned} v_T * v_S &= v_{t_1} * ... * v_{t_k} * \underbrace{v_{t_{k+1}} * v_{s_1} * ... * v_{s_p}}_{\text{p permutations}} \\ &= (-1)^p \underbrace{v_{t_1} * ... * v_{t_k} * v_{s_1} * ... * v_{s_p}}_{\text{Induction Hypothesis}} * v_{t_{k+1}} \\ &= (-1)^p (-1)^{pk} (v_{s_1} * ... * v_{s_p}) * (v_{t_1} * ... v_{t_k} * v_{t_{k+1}}) \\ &= (-1)^{p(k+1)} v_S * v_T. \end{aligned}$$

Therefore, Equation 2.29 is valid for all $q \in \mathbb{N}$. Since $p \in \mathbb{N}$ is arbitrary, this equations is valid whenever S and T are disjoint.

Next suppose $R \neq \emptyset$. We define $S' = S - R$ and $T' = T - R$. Notice that the cardinality of S' and T' are $p - r$ and $q - r$ respetively. Moreover, by using Equation 2.29, we can permute the elements of S and T to write

$$v_S = (-1)^a v_{S'} * v_R \tag{2.31a}$$

and

$$v_T = (-1)^b v_R * v_{T'}, \tag{2.31b}$$

where a and b are the number of transpositions performed to arrive in Equations 2.31a and 2.31b respectively. Consequently, by multiple aplications of Equation

2.29,

$$\begin{aligned}
v_T * v_S &= (-1)^{a+b} v_R * \underbrace{v_{T'} * v_{S'}}_{\text{Equation 2.29}} * v_R \\
&= (-1)^{a+b}(-1)^{(p-r)(q-r)} \underbrace{v_R * v_{S'}}_{\text{Equation 2.29}} * \underbrace{v_{T'} * v_R}_{\text{Equation 2.29}} \\
\text{(Rearranging factors)} &= (-1)^{a+b+(p-r)(q-r)}(-1)^{(p-r)r}(-1)^{(q-r)r} v_{S'} * v_R * v_R * v_{T'} \\
&= (-1)^{(p-r)(q-r)+(p-r)r+(q-r)r} \underbrace{\left((-1)^a v_{S'} * v_R\right)}_{\text{Equation 2.31a}} * \underbrace{\left((-1)^b v_R * v_{T'}\right)}_{\text{Equation 2.31b}} \\
&= (-1)^{(p-r)(q-r)+(p-r)r+(q-r)r} v_S * v_T \\
&= (-1)^{pq-pr-qr+r^2+pr-r^2+qr-r^2} v_S * v_T \\
&= (-1)^{pq-r^2} v_S * v_T.
\end{aligned}$$

Notice that

$$(-1)^{-r^2} = (-1)^{r^2} = \begin{cases} (-1)^{(2c)^2} = 1 = (-1)^{2c} = (-1)^r \ (r \text{ even}); \\ (-1)^{(2c+1)^2} = -1 = (-1)^{2c+1} \ (r \text{ odd}). \end{cases}$$

So either way (r even or odd), we can always write

$$v_T * v_S = (-1)^{pq-r^2} v_S * v_T = (-1)^{pq+r} v_S * v_T,$$

which finishes this proof. □

The next corollary is a trivial consequence of the previous theorem and of Equation 2.4.

Corollary 2.1.3. *Let $j \in \boldsymbol{m}$ and $S \subset \boldsymbol{m}$. Then, under the conditions of Theorem 2.1.3,*

$$v_j * v_S * v_j = (-1)^{p+r} g(v_j, v_j) v_S, \tag{2.32}$$

where p is the cardinality of S and r is the cardinality of $S \cap \{j\}$.

Lemma 2.1.1. *Let $i_1, ..., i_k \in \boldsymbol{m}$ and let*

$$v_{i_1} * ... * v_{i_k}$$

be a product of elements of $\{v_1, ..., v_m\}$ like stated in Theorem 2.1.3. Then there exists $\beta \in \mathbb{R}$ and $S = \{s_1 < ... < s_p\} \subset \boldsymbol{m}$ such that

$$v_{i_1} * ... * v_{i_k} = \beta_{i_1 ... i_k} v_S. \tag{2.33}$$

Proof. Let σ be a permutation function for $\{1, ..., k\}$ such that

$$i_{\sigma(j_1)} \leq ... \leq i_{\sigma(j_k)},$$

where $j_1, ..., j_k \in \{1, ..., k\}$. Consequently, we can write

$$v_{i_1} * ... * v_{i_k} = sgn(\sigma) v_{i_{\sigma(j_1)}} * ... * v_{i_{\sigma(j_k)}}, \tag{2.34}$$

with *sgn* being the sign function for the permutation σ:

$$sgn(\sigma) = \begin{cases} +1 \text{ if } \sigma \text{ is even;} \\ -1 \text{ if } \sigma \text{ is odd.} \end{cases}$$

If $i_{\sigma(j_r)} = i_{\sigma(j_{r+1})}$, then, by Equation 2.4,

$$v_{i_{\sigma(j_r)}} * v_{i_{\sigma(j_{r+1})}} = g(v_{i_{\sigma(j_r)}}, v_{i_{\sigma(j_r)}})1_{Cl_g(V)}. \tag{2.35}$$

In this case, the Equation 2.34 becomes

$$v_{i_1} * ... * v_{i_k} = \\ sgn(\sigma)g(v_{i_{\sigma(j_r)}}, v_{i_{\sigma(j_r)}})v_{i_{\sigma(j_1)}} * ... * v_{i_{\sigma(j_{r-1})}} * v_{i_{\sigma(j_{r+2})}} * ... * v_{i_{\sigma(j_k)}}. \tag{2.36}$$

We can "take out" identical indexes two-by-two by using the same reasoning done in Equation 2.36. The resulting scalar shall be identified as $\beta_{i_1...i_k}$.

If all factors are removed by the reapeted uses of Equation 2.4, then

$$v_{i_1} * ... * v_{i_k} = \beta_{i_1...i_k}1_{Cl_g(V)} = \beta v_{\emptyset}.$$

If some factors were not removed by the use of Equation 2.4, then the remaining factors have distinct indexes in an ascending order. Let S be the set of said indexes. Thus

$$v_{i_1} * ... * v_{i_k} = \beta_{i_1...i_k}v_S.$$

In both cases, Equation 2.33 is satisfied. □

Lemma 2.1.2. *Let (V, g) be a quadratic space with dimension n, and $\{e_1, ..., e_n\}$ be a basis that diagonilizes g, as stated in Theorem 1.2.4. Then, for any $y_1, ..., y_k \in V$, the product*

$$y_1 * ... * y_k \in Cl_g(V)$$

can be written as a linear combination of elements from the set $\{e_S \mid S \subset \boldsymbol{n}\} \subset Cl_g(V)$.

Proof. Since $y_j \in V$ for each $1 \leq j \leq k$, we can write

$$y_j = \sum_{i_j=1}^{n} \alpha_{i_j}e_{i_j}.$$

So, by the distributive property,

$$y_1 * ... * y_k = \sum_{i_1=1}^{n} ... \sum_{i_k=1}^{n} \alpha_{i_1}...\alpha_{i_k}e_{i_1} * ... * e_{i_k}.$$

By Lemma 2.1.1,

$$e_{i_1} * ... * e_{i_k} = \beta_{i_1...i_k}e_S, \tag{2.37}$$

thus the initial product becomes

$$y_1 * ... * y_k = \sum_{i_1=1}^{n} ... \sum_{i_k=1}^{n} \alpha_{i_1}...\alpha_{i_k}\beta_{i_1...i_k}e_S,$$

which proves our assertion. □

Next we have the first relation between the dimensions of V and $Cl_g(V)$.

Lemma 2.1.3. *Let* (V, g) *be a quadratic vector space over* $\mathbb{R}$. *If* $dim(V) = n$, *then* $dim(Cl_g(V)) \leq 2^n$.

Proof. In here, the set $\{e_1, ..., e_n\}$ shall be a basis as stated in Theorem 1.2.4.

The Clifford algebra $Cl_g(V)$, by its definition, is generated by the set

$$\mathcal{B} = V \cup \{\alpha 1 \mid \alpha \in \mathbb{R}\}.$$

This means any $z \in Cl_g(V)$ can be written in the form

$$z = \sum_{i=1}^{m} z_i \tag{2.38}$$

of any $m > 0$ elements, each z_i being the product

$$z_i = y_{i1} * ... * y_{ik_i}, \tag{2.39}$$

with k_i elements and each y_{ij_i} being either an element of $\mathcal{B}$ or the opposite of an element of $\mathcal{B}$ (see Definition 1.2.4). But, since V is a vector space, if y_{ij_i} is the opposite of an element in V, then $y_{ij_i} \in V$; additionally, if y_{ij_i} is the opposite of an element in $\{\alpha 1 \mid \alpha \in \mathbb{R}\}$, then

$$y_{ij_i} = -(\alpha_{ij_i} 1) = (-\alpha_{ij_i})1,$$

which is itself an element of $\{\alpha 1 \mid \alpha \in \mathbb{R}\}$; thus, in all possible cases, each y_{ij_i} is an element of $\mathcal{B}$.

For each $1 \leq i \leq m$, if in the product

$$y_{i1} * ... * y_{ik_i}$$

there is a factor

$$y_{ij_i} = \alpha_{ij_i} 1 \in \{\alpha 1 \mid \alpha \in \mathbb{R}\},$$

then we can rewrite the previous product as

$$y_{i1} * ... * y_{ik_i} = \alpha_{ij_i} y_{i1} * ... * y_{ij_{i-1}} * y_{ij_{i+1}} * ... * y_{ik_i}.$$

This process can be repeated until the last element of $\{\alpha 1 \mid \alpha \in \mathbb{R}\}$ is "taken out" of the multiplication in Equation 2.39. If all factors are elements of $\{\alpha 1 \mid \alpha \in \mathbb{R}\}$, then

$$y_{i1} * ... * y_{ik_i} = \alpha_{i1} ... \alpha_{ik_i} 1 = \alpha_{i1} ... \alpha_{ik_i} e_{\emptyset}.$$

Otherwise, the remaining factors of the multiplication are all elements of V, which, by Lemma 2.1.2, can be written as a linear combination of elements from the set $\{e_S \mid S \subset \mathbf{n}\}$. Both cases show us that, for $1 \leq i \leq m$, each z_i is a linear combination of elements from the set $\{e_S \mid S \subset \mathbf{n}\}$. Moreover, since z in Equation 2.38 is a sum of elements in the algebra, z can also be written as a linear combination of elements from the set $\{e_S \mid S \subset \mathbf{n}\}$. Hence this set generates $Cl_g(V)$ as a vector space.

The set $\{e_S \mid S \subset \mathbf{n}\}$ has cardinality

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} = 2^n,$$

thus $dim(Cl_g(V)) \leq 2^n$. □

For the next theorem we shall consider the particular Bogoliubov automorphism θ_{-I}, where $-I$ takes each element of V and maps it to its opposite.

Theorem 2.1.4. *Let* (V, g) *be a quadratic vector space over* $\mathbb{R}$. *If* $dim(V) = n$, *then* $dim(Cl_g(V)) = 2^n$.

Proof. We already know that $dim(Cl_g(V)) \leq 2^n$ (Lemma 2.1.3). To show the equality, let $m \leq n$ be a fixed natural number and let $\{v_1, ..., v_n\}$ be a basis for V such that $g(v_i, v_j) = 0$ whenever $i \neq j$. Suppose

$$\sum_{S \subset \mathbf{m}\ ,\ \alpha_S \neq 0} \alpha_S v_S = 0 \tag{2.40}$$

is a non-trivial relation of elements in $Cl_g(V)$ involving as few nonzero coefficients as possible, *i.e.*, the vectors v_S are linearly dependent and none has a zero coefficient. If this summation only has one $\alpha_S \neq 0$, then

$$\alpha_S v_S = 0 \Rightarrow \alpha_S v_S * v_S = 0.$$

Notice that,

$$\begin{aligned} v_S * v_S &= (v_{s_1} * ... * v_{s_p}) * (v_{s_1} * ... * v_{s_p}) \\ &= (-1)^{(p-1)+(p-2)+...+1} v_{s_1}^2 * ... * v_{s_p}^2 \\ &= (-1)^{\frac{1}{2}p(p-1)} g(v_{s_1}, v_{s_1})...g(v_{s_p}, v_{s_p}). \end{aligned}$$

Thus

$$\alpha_S v_S = 0 \Rightarrow \alpha_S (-1)^{\frac{1}{2}p(p-1)} g(v_{s_1}, v_{s_1})...g(v_{s_p}, v_{s_p}) = 0.$$

This means one of the $g(v_{s_i}, v_{s_i})$ must be zero since $\alpha_S \neq 0$ by hypothesis. This contradicts the non-singularity of the bilinear g, therefore we can not have a summation with only one $\alpha_S \neq 0$. So Equation (2.40) must have at least two terms.

Next consider the automorphism

$$\frac{1}{2}(Id + \theta_{-I}) \in \mathcal{A}ut(Cl_g(V)),$$

where Id is the identity map. So,

$$\begin{aligned} \frac{1}{2}(Id + \theta_{-I})(\sum_{S \subset \mathbf{m}} \alpha_S v_S) &= \frac{1}{2}(\sum_{S \subset \mathbf{m}} \alpha_S v_S + \sum_{S \subset \mathbf{m}} \alpha_S \theta_{-I}(v_S)) \\ &= \frac{1}{2}(\sum_{S \subset \mathbf{m}} \alpha_S v_S + \sum_{S \subset \mathbf{m}} (-1)^{\text{card}(S)} \alpha_S v_S) \\ &= \frac{1}{2}(\sum_{S \subset \mathbf{m}} \alpha_S (1 + (-1)^{\text{card}(S)}) v_S) \\ &= \sum_{S \subset \mathbf{m},\ \text{card}(S)\ \text{even}} \alpha_S v_S \end{aligned}$$

This automorphism is "filtering" the summation by eliminating the sets S with an odd cardinality. Since our hypothesis asks Equation (2.40) to have has few nonzero coeficients as possible, either all sets have even cardinality, which would make the automorphism act as the identity, or all sets must have an odd cardinality, meaning the automorphism would act as the 0 automorphism. Either way, the important fact we can extract is all sets S in the Equation (2.40) have either an even or odd cardinality simultaneously.

Finally, let $1 \leq j \leq m$ and consider the multiplication

$$\begin{aligned}
0 &= v_j * (\sum_{S \subset \mathbf{m}\ ,\ \alpha_S \neq 0} \alpha_S v_S) * v_j \\
&= \sum_{S \subset \mathbf{m}\ ,\ \alpha_S \neq 0} \alpha_S v_j * v_S * v_j \\
\text{(Corollary 2.1.3)} &= \sum_{S \subset \mathbf{m}\ ,\ \alpha_S \neq 0} \alpha_S (-1)^{\mathrm{card}(S)+\mathrm{card}(S\cap\{j\})} g(v_j, v_j) v_S \\
&= g(v_j, v_j) \sum_{S \subset \mathbf{m}\ ,\ \alpha_S \neq 0} (-1)^{\mathrm{card}(S)+\mathrm{card}(S\cap\{j\})} \alpha_S v_S \\
(g(v_j, v_j) \neq 0) &= \sum_{S \subset \mathbf{m}\ ,\ \alpha_S \neq 0} (-1)^{\mathrm{card}(S)+\mathrm{card}(S\cap\{j\})} \alpha_S v_S.
\end{aligned}$$

Therefore we obtain by adding Equation (2.40)

$$0 = \sum_{S \subset \mathbf{m}\ ,\ \alpha_S \neq 0} (1 + (-1)^{\mathrm{card}(S)+\mathrm{card}(S\cap\{j\})}) \alpha_S v_S.$$

Suppose $card(S)$ is even. Then we must have $S \cap \{j\} = \emptyset$, otherwise we would have a non-trivial summation with fewer terms than Equation 2.40, a contradiction. But j is arbitrary, hence $S \cap \{j\} = \emptyset$ must be true for each $1 \leq j \leq m$. The only possible case is $S = \emptyset$, meaning the summation in Equation 2.40 has only one term, an absurd as proved previously in the first paragraph of this proof. Consequently $card(S)$ must be odd and $S \cap \{j\} = \{j\}$. (If the intersection was the empty set, we would arrive again at a summation with fewer terms than Equation 2.40, an absurd as stated previously.) Once again, by the arbitrarity of j, $S \cap \{j\} = \{j\}$ for each $1 \leq j \leq m$, meaning $S = \{1, 2, ..., m\} = \mathbf{m}$. There is only one such set, thus the summation in Equation 2.40 has, yet again, one term, giving us another absurd.

In conclusion, all possible scenarios for Equation 2.40 being a non-trivial linear combination with as few nonzero coefficients as possible lead to a contradiction, thus proving the set

$$\mathcal{S}_m = \{v_S \mid S \subset \mathbf{m}\}$$

is linearly independet for each $1 \leq m \leq n$. Notice that, for $m = n$,

$$\mathrm{card}(\mathcal{S}_n) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} = 2^n,$$

hence $dim(Cl_g(V)) \geq 2^n$. By Lemma 2.1.3, we conclude that $dim(Cl_g(V)) = 2^n$. □

So far, the examples here presented were the only possible cases of Clifford Algebras related to an unidimensional vector spaces. The next example looks into two cases for spaces with dimension 2.

Example 2.1.3. *Let* $V \simeq \mathbb{R}^2$ *with basis* $\{e_1, e_2\}$, *and let* g *be the bilinear form*

$$g(e_1) = g(e_2) = -1.$$

By the previous theorem, $dim(Cl_g(V)) = 2^2 = 4$ *with basis*

$$\mathcal{S}_4 = \{v_S \mid S \subset \mathit{4}\} = \{1, e_1, e_2, e_1 * e_2\}.$$

Hence the following set of equations are observed:

$$\begin{cases} 1 * 1 & = 1; \\ e_1 * e_1 & = g(e_1, e_1)1 = -1; \\ e_2 * e_2 & = g(e_2, e_2)1 = -1; \\ e_1 * e_2 & = -e_2 * e_1. \end{cases}$$

For $(e_1 * e_2)^2$*, we have*

$$\begin{aligned} (e_1 * e_2)^2 &= (e_1 * e_2) * (e_1 * e_2) = e_1 * (e_2 * e_1) * e_2 \\ &= -e_1 * (e_1 * e_2) * e_2 = -(e_1 * e_1) * (e_2 * e_2) \\ &= -(-1) * (-1) = -1. \end{aligned}$$

Consider the mapping

$$\phi : Cl_g(V) \to \mathbb{H}$$

where

$$\begin{cases} \phi(1) = 1; \\ \phi(e_1) = i; \\ \phi(e_2) = j; \\ \phi(e_1 * e_2) = k. \end{cases}$$

Thus it is immediate to see that ϕ *is an algebra isomorphism between* $Cl_g(V)$ *and* $\mathbb{H}$*, hence* $Cl_g(V) \simeq \mathbb{H}$*.*

Example 2.1.4. *Under the same conditions of the previous example, if we were to change the bilinear form into*

$$g(e_1) = g(e_2) = +1,$$

then we would have

$$\begin{cases} 1 * 1 & = 1; \\ e_1 * e_1 & = g(e_1, e_1)1 = +1; \\ e_2 * e_2 & = g(e_2, e_2)1 = +1; \\ e_1 * e_2 & = -e_2 * e_1. \end{cases}$$

(Notice how e_1 *and* e_2 *still anticommute.) Meanwhile,* $(e_1 * e_2)^2$ *would remain the same:*

$$\begin{aligned} (e_1 * e_2)^2 &= (e_1 * e_2) * (e_1 * e_2) = e_1 * (e_2 * e_1) * e_2 \\ &= -e_1 * (e_1 * e_2) * e_2 = -(e_1 * e_1) * (e_2 * e_2) \\ &= -(1) * (1) = -1. \end{aligned}$$

Consider the mapping

$$\phi : Cl_g(V) \to Mat(2, \mathbb{R})$$

where

$$\begin{cases} \phi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \\ \phi(e_1) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \\ \phi(e_2) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \\ \phi(e_1 * e_2) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}; \end{cases} \tag{2.41}$$

which is an algebra isomorphism between $Cl_g(V)$ *and* $Mat(2, \mathbb{R})$*, ergo* $Cl_{g'}(V) \simeq Mat(2, \mathbb{R})$*.*

One last comment before closing this section. There exists another algebra having similar properties to the Clifford Algebra $Cl_g(V)$:

(i) this algebra can be constructed as a quotient of the tensor algebra $T(V)$;

(ii) this algebra has an universal nature;

(iii) this algebra has V embedded into it;

(iv) if $dim(V) = n$, then the algebra has dimension 2^n.

This is the case for the called *Grassman Algebras*, usually notated by $\bigwedge(V)$. For finite-dimensional vector spaces, we have $Cl_g(V) \simeq \bigwedge(V)$ as vector spaces since both have the same dimension. But they are not algebraicaly isomorphic. This is pretty immediate by the basic property they hold: while in Clifford Algebras we want the square of any element in V to be a multiple of the unity ($x * x = g(x,x)1 \neq 0$), in Grassman Algebras the square of such elements are, by construction, equal to zero ($x \wedge x = 0$). Although not explored in this text, this algebras share more similarities then the ones just stated. The interested reader can check References [4][16].

2.2 Classification of Clifford Algebras

We're now in place to begin constructing the building blocks that shall classify the Clifford Algebras of finite-dimensional real vector spaces. Such classifications will be done by a series of theorems and corollaries, with the so called *Periodicity theorem* being in the core of said classifications.

Before begining, an observation. Let $V \simeq \mathbb{R}^{p+q}$ and g be the bilinear form associated with V having rank $p + q$ and signature $p - q$ (see Definition 1.2.7). Then the Clifford Algebra $Cl_g(V)$ associated with the quadratic space (V, g) will be written as $Cl_{p,q}$. Moreover, as a reminder, the bilinear form is non-singular (see Definition 2.1.1).

We'll start with the first "building block" of our construction.

Theorem 2.2.1. *Let* $Cl_{2,0}$ *and* $Cl_{1,1}$ *be the two Clifford Algebras associated with the vector spaces* $\mathbb{R}^{2+0}$ *and* $\mathbb{R}^{1+1}$ *respectively. Then the following isomorphism is verified:*

$$Cl_{2,0} \simeq Cl_{1,1}. \tag{2.42}$$

Proof. Let $\{1, e_1, e_2, e_1 * e_2\}$ be the generators of $Cl_{2,0}$, such that $e_1^2 = 1$, $e_2^2 = 1$ and, as consequence, $(e_1 * e_2)^2 = (e_1 * e_2) * (e_1 * e_2) = -1$; and $\{1, v_1, v_2, v_1 * v_2\}$ the generators of $Cl_{1,1}$, such that $v_1^2 = 1$, $v_2^2 = -1$ and $(v_1 * v_2)^2 = 1$.

Then the linear mapping

$$\begin{gathered}\phi : Cl_{2,0} \to Cl_{1,1}\\ \phi(1) = 1,\ \phi(e_1) = v_1,\ \phi(e_2) = v_1 * v_2,\ \phi(e_1 * e_2) = v_2;\end{gathered}$$

is an isomorphism between algebras. Indeed, notice that

$$\begin{gathered}\phi(e_1^2) = \phi(1) = 1 = (v_1)^2 = (\phi(e_1))^2;\\ \phi(e_2^2) = \phi(1) = 1 = (v_1 * v_2)^2 = (\phi(e_1) * \phi(e_2))^2;\\ \phi(e_2^2) = \phi(-1) = -1 = (v_2)^2 = (\phi(e_2))^2;\end{gathered}$$

thus ϕ is preserving the product between the genertors of $Cl_{2,0}$, therefore ϕ preserves multiplication of the algebra elements. Since $dim(Cl_{2,0}) = 2^2 = 4 = dim(Cl_{1,1})$, we can conclude that ϕ is an algebra isomorphism. □

So far, we've seen the first five "classification" isomorphisms: one in our previous result, Theorem 2.2.1, and four being presentend in Examples 2.1.1, 2.1.2, 2.1.3 and 2.1.4. The next corollary is being stated for simplicity sake for when such isomorphims are referenced later in the text.

Corollary 2.2.1. *The following algebraic isomorphisms hold true:*

$$\begin{aligned}\text{(Example 2.1.1)}\ Cl_{1,0} &\simeq \mathbb{R} \oplus \mathbb{R};\\ \text{(Example 2.1.2)}\ Cl_{0,1} &\simeq \mathbb{C};\\ \text{(Theorem 2.2.1 and Example 2.1.4)}\ Cl_{1,1} &\simeq Cl_{2,0} \simeq Mat(2, \mathbb{R});\\ \text{(Example 2.1.3)}\ Cl_{0,2} &\simeq \mathbb{H}.\end{aligned} \tag{2.43}$$

Theorem 2.2.1 is the first example we have of two Clifford Algebras associated with different values of signatures being isomorphic to one another. In other words, the Clifford Algebra $Cl_{2,0}$ is associated with $\mathbb{R}^{2+0}$, a vector space with quadratic form Q having signature 2, while $Cl_{1,1}$ is associated with $\mathbb{R}^{1+1}$, a vector space with quadratic form Q having signature 0.

Our next example of isomorphisms between Clifford Algebras is more general then the previous cases and starts to make use of our tensor product knowledge.

Theorem 2.2.2. *Let $Cl_{p,q}$ be a Clifford Algebra associated with the the quadratic space $\mathbb{R}^{p+q}$. Then there exists the following isomorphism:*

$$Cl_{p+1,q+1} \simeq Cl_{1,1} \otimes Cl_{p,q}, \tag{2.44}$$

where either $p > 0$ or $q > 0$.

Proof. Let $\{e_i | 1 \leq i \leq p + q + 2\}$ be an orthonormal basis for the quadratic space $\mathbb{R}^{p+1,q+1}$ with quadratic form Q, such that we have the following relations:

$$Q(e_i) = \begin{cases} 1, & 1 \leq i \leq p + 1\\ -1, & p + 2 \leq i \leq p + q + 2. \end{cases} \tag{2.45}$$

Similarly, we define $\{v_i | 1 \leq i \leq p+q\}$, an orthonormal basis for the quadratic space $\mathbb{R}^{p,q}$ with quadrtic form Q' and

$$Q'(v_i) = \begin{cases} 1, & 1 \leq i \leq p \\ -1, & p+2 \leq i \leq p+q; \end{cases} \tag{2.46}$$

and $\{r, s\}$, an orthonormal basis for the quadratic space $\mathbb{R}^{1,1}$ with quadrtic form H, and

$$\begin{aligned} H(r) &= 1 \\ H(s) &= -1. \end{aligned} \tag{2.47}$$

Consider the linear mapping $f : \mathbb{R}^{p+1,q+1} \to Cl_{p,q} \otimes Cl_{1,1}$ defined as

$$f(e_i) = \begin{cases} v_i \otimes (r * s),\ 1 \leq i \leq p \\ 1 \otimes r,\ i = p+1 \\ 1 \otimes s,\ i = p+2 \\ v_i \otimes (r * s),\ p+3 \leq i \leq p+q+2. \end{cases} \tag{2.48}$$

We want to show that this linear mapping is a Clifford mapping. To do so, we must calculate

$$f(e_i) * f(e_j) + f(e_j) * f(e_i)$$

for each value of $1 \leq i, j \leq p+q+2$, therefore giving us ten cases to study.

1st. $1 \leq i, j \leq p$:

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) = &(v_i \otimes r * s) * (v_j \otimes r * s) \\ &+ (v_j \otimes r * s) * (v_i \otimes r * s) = \\ &= (v_i * v_j + v_j * v_i) \otimes (r * s)^2 = \\ &= (v_i * v_j + v_j * v_i) \otimes (-r^2 * s^2) = \\ &= 2\delta_{i,j} 1 \otimes 1, \end{aligned}$$

where we used the identities $v_i * v_j + v_j * v_i = 2\delta_{i,j}$, $r^2 = 1$, $s^2 = -1$ and $s * r = -r * s$. Thus

$$f(e_i) * f(e_j) + f(e_j) * f(e_i) = 2\delta_{i,j} 1 \otimes 1. \tag{2.49}$$

2nd. $1 \leq i \leq p$ and $j = p+1$:

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) = &(v_i \otimes r * s) * (1 \otimes r) \\ &+ (1 \otimes r) * (v_i \otimes r * s) = \\ &= v_i \otimes (r * s * r + r * r * s) = \\ &= v_i \otimes (-r^2 * s + r^2 * s) = \\ &= v_i \otimes 0 = 0. \end{aligned}$$

So

$$f(e_i) * f(e_j) + f(e_j) * f(e_i) = 0. \tag{2.50}$$

3rd. $1 \leq i \leq p$ and $j = p+2$:

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (v_i \otimes r * s) * (1 \otimes s) \\ &\quad + (1 \otimes s) * (v_i \otimes r * s) = \\ &= v_i \otimes (r * s * s + s * r * s) = \\ &= v_i \otimes (r * s^2 - s^2 * r) = \\ &= v_i \otimes 0 = 0. \end{aligned}$$

Hence

$$f(e_i) * f(e_j) + f(e_j) * f(e_i) = 0. \tag{2.51}$$

4th. $1 \leq i \leq p$ and $p+3 \leq j \leq p+q+2$:

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (v_i \otimes r * s) * (v_j \otimes r * s) \\ &\quad + (v_j \otimes r * s) * (v_i \otimes r * s) = \\ &= (v_i * v_j + v_j * v_i) \otimes (r * s)^2 = \\ &= (v_i * v_j + v_j * v_i) \otimes (-r^2 * s^2) = \\ &= 2\delta_{i,j} 1 \otimes 1 = 0, \end{aligned}$$

since $i \neq j$ for all its values in this case. Thus

$$f(e_i) * f(e_j) + f(e_j) * f(e_i) = 0. \tag{2.52}$$

5th. $i = j = p+1$:

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (1 \otimes r) * (1 \otimes r) + (1 \otimes r) * (1 \otimes r) = \\ &= 2(1 \otimes r^2) = \\ &= 2(1 \otimes 1). \end{aligned}$$

Then

$$f(e_i) * f(e_j) + f(e_j) * f(e_i) = 21 \otimes 1. \tag{2.53}$$

6th. $i = p+1$ and $j = p+2$:

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (1 \otimes r) * (1 \otimes s) + (1 \otimes s) * (1 \otimes r) = \\ &= 1 \otimes (r * s + s * r) = 0. \end{aligned}$$

Therefore

$$f(e_i) * f(e_j) + f(e_j) * f(e_i) = 0. \tag{2.54}$$

7th. $i = p+1$ and $p+3 \leq j \leq p+q+2$:

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (1 \otimes r) * (v_j \otimes r * s) \\ &\quad + (v_j \otimes r * s) * (1 \otimes r) = \\ &= v_j \otimes (r^2 * s + r * s * r) = \\ &= v_j \otimes (r^2 * s - r^2 * s) = 0. \end{aligned}$$

Because of this we have

$$f(e_i) * f(e_j) + f(e_j) * f(e_i) = 0. \tag{2.55}$$

8th. $i = j = p + 2$:

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (1 \otimes s) * (1 \otimes s) + (1 \otimes s) * (1 \otimes s) = \\ &= 2(1 \otimes s^2) = \\ &= 2(1 \otimes (-1)). \end{aligned}$$

So

$$f(e_i) * f(e_j) + f(e_j) * f(e_i) = -2(1 \otimes 1). \tag{2.56}$$

9th. $i = p + 2$ and $p + 3 \leq j \leq p + q + 2$:

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (1 \otimes s) * (v_j \otimes r * s) \\ &\quad + (v_j \otimes r * s) * (1 \otimes s) = \\ &= v_j \otimes (s * r * s + r * s^2) = \\ &= v_j \otimes (-r * s^2 + r * s^2) = 0. \end{aligned}$$

Thus

$$f(e_i) * f(e_j) + f(e_j) * f(e_i) = 0. \tag{2.57}$$

10th. $p + 3 \leq i, j \leq p + q + 2$:

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (v_i \otimes r * s) * (v_j \otimes r * s) \\ &\quad + (v_j \otimes r * s) * (v_i \otimes r * s) = \\ &= (v_i * v_j + v_j * v_i) \otimes (r * s)^2 = \\ &= (v_i * v_j + v_j * v_i) \otimes (-r^2 * s^2) = \\ &= -2\delta_{i,j} 1 \otimes 1, \end{aligned}$$

where we used the identity $v_i * v_j + v_j * v_i = -2\delta_{i,j}$. Hence

$$f(e_i) * f(e_j) + f(e_j) * f(e_i) = -2\delta_{i,j} 1 \otimes 1. \tag{2.58}$$

Therefore, if we consider any element $x \in \mathbb{R}^{p+1,q+1}$ as

$$x = \sum_{i=1}^{p+q+2} \alpha_i e_i,$$

we have

$$\begin{aligned} (f(x))^2 &= \sum_{i=1}^{p+q+2} \sum_{j=1}^{p+q+2} \alpha_i \alpha_j f(e_i) * f(e_j) = \\ &= 2(\sum_{i=1}^{p+1} (\alpha_i)^2 - \sum_{i=p+2}^{p+q+2} \alpha_i)^2) 1 \otimes 1 = \\ &= 2Q(x) 1 \otimes 1. \end{aligned}$$

The factor 2 appearing multiplying Q at the end is irrelevant since we could have defined f by multiplying a factor of one half. Doing so would give us

$$(f(x))^2 = Q(x) 1 \otimes 1.$$

$$\begin{tikzcd}
\mathbb{R}^{p+1,\,q+1} \arrow[rr, "f"'] \arrow[dr, "Q"'] & & Cl_{p,q} \otimes Cl_{1,1} \\
& Cl_{p+1,\,q+1} \arrow[ur, "f^*"'] &
\end{tikzcd}$$

Figure 2.3: Diagram for Theorem 2.2.2.

Thus f is a Clifford mapping. By the universality of the Clifford Algebras (Theorem 2.1.1), the diagram in Figure 2.3 commutes.

By construction, the image of f generates the algebra $Cl_{p,q} \otimes Cl_{1,1}$. Indeed, notice that $Cl_{p,q} \otimes Cl_{1,1}$ is generated by elements of the form

$$\begin{aligned}
v_1^{\varepsilon_1} * \ldots * v_{p+q}^{\varepsilon_{p+q}} \otimes 1 &= \prod_{i=1}^{p+q} (v_i^{\varepsilon_i} \otimes 1), \\
v_1^{\varepsilon_1} * \ldots * v_{p+q}^{\varepsilon_{p+q}} \otimes r &= \prod_{i=1}^{p+q} (v_i^{\varepsilon_i} \otimes r), \\
v_1^{\varepsilon_1} * \ldots * v_{p+q}^{\varepsilon_{p+q}} \otimes s &= \prod_{i=1}^{p+q} (v_i^{\varepsilon_i} \otimes s), \\
v_1^{\varepsilon_1} * \ldots * v_{p+q}^{\varepsilon_{p+q}} \otimes r * s &= \prod_{i=1}^{p+q} (v_i^{\varepsilon_i} \otimes r * s),
\end{aligned} \tag{2.59}$$

where $\varepsilon_i \in \{0, 1\}$ for $1 \leq i, j \leq p + q + 2$, giving us the following cases:

(a) $1 \leq i \leq p + q$ and $\varepsilon_i = 0$:

$$\begin{aligned}
1 \otimes 1 &= f(e_{p+1}) * f(e_{p+1}), \\
1 \otimes r &= f(e_{p+1}), \\
1 \otimes s &= f(e_{p+2}), \\
1 \otimes r * s &= f(e_{p+1}) * f(e_{p+2});
\end{aligned} \tag{2.60}$$

(b) $1 \leq i \leq p$ and $\varepsilon_i = 1$:

$$\begin{aligned}
v_i \otimes 1 &= f(e_i) * f(e_{p+1}) * f(e_{p+2}), \\
v_i \otimes r &= -f(e_i) * f(e_{p+2}), \\
v_i \otimes s &= -f(e_i) * f(e_{p+1}), \\
v_i \otimes r * s &= f(e_i);
\end{aligned} \tag{2.61}$$

(c) $p + 2 \leq i \leq p + q$ and $\varepsilon_i = 1$:

$$\begin{aligned}
v_i \otimes 1 &= f(e_{p+2+i}) * f(e_{p+1}) * f(e_{p+2}), \\
v_i \otimes r &= -f(e_{p+2+i}) * f(e_{p+2}), \\
v_i \otimes s &= -f(e_{p+2+i}) * f(e_{p+1}), \\
v_i \otimes r * s &= f(e_{p+2+i}).
\end{aligned} \tag{2.62}$$

Since, by construction, f is taking $\{e_i|1 \leq i \leq p+q+2\}$, a basis of $\mathbb{R}^{p+1,q+1}$, into the set $\{v_i \otimes (r * s), 1 \otimes r, 1 \otimes s \mid 1 \leq i \leq p+q\}$, a basis for $Cl_{p,q} \otimes Cl_{1,1}$, we have f a surjective linear mapping between vector spaces. Moreover, by Theorem 2.1.1,

$$Im(f) = Im(f^*),$$

thus f^* is a surjective homomorphism of algebras. Additionally,

$$rank(f^*) = dim(Cl_{p,q} \otimes Cl_{1,1}) \underbrace{=}_{\text{Theorem 1.1.3}} dim(Cl_{p,q})dim(Cl_{1,1}) = 2^{p+q+2}$$

and

$$dim(Cl_{p+1,q+1}) = 2^{p+q+2},$$

meaning that $ker(f^*) = \{0\}$, so f^* is injective.

In conclusion, f^* is an algebra isomorphism between $Cl_{p+1,q+1}$ and $Cl_{p,q} \otimes Cl_{1,1}$, the latter being isomorphic to $Cl_{1,1} \otimes Cl_{p,q}$. □

The next example shows us the reasoning used through out the classification of the Clifford Algebras and how theorems like Theorem 2.2.2 are applied.

Example 2.2.1. *Consider the Clifford Algebra* $Cl_{2,2}$. *By Theorem 2.2.2 and Corollay 2.2.1, we know that*

$$Cl_{2,2} = Cl_{1+1,1+1} \simeq Cl_{1,1} \otimes Cl_{1,1} \simeq Mat(2, \mathbb{R}) \otimes Mat(2, \mathbb{R}) \simeq Mat(4, \mathbb{R}).$$

Theorem 2.2.3. *Let* $Cl_{p,q}$ *be a Clifford Algebra associated with the the quadratic space* $\mathbb{R}^{p+q}$. *Then the following isomorphisms hold:*

$$\begin{aligned} Cl_{q+2,p} &\simeq Cl_{2,0} \otimes Cl_{p,q} \\ Cl_{q,p+2} &\simeq Cl_{0,2} \otimes Cl_{p,q}, \end{aligned} \tag{2.63}$$

where either $p > 0$ *or* $q > 0$.

Proof. The proof for $Cl_{q,p+2}$ is similar, if not almost identical, to the case $Cl_{q+2,p}$, so we shall proof only the later. Moreover, this proof is quite close to Theroem 2.2.2's proof, as we shall see. Let $\{e_i|1 \leq i \leq p+q+2\}$ be an orthonormal basis for the quadratic space $\mathbb{R}^{p+1,q+1}$ with quadratic form Q, such that we have the following relations:

$$Q(e_i) = \begin{cases} 1, & 1 \leq i \leq q+2 \\ -1, & q+3 \leq i \leq p+q+2. \end{cases} \tag{2.64}$$

Similarly, we define $\{v_i|1 \leq i \leq p+q\}$, an orthonormal basis for the quadratic space $\mathbb{R}^{p,q}$ with quadrtic form Q' and

$$Q'(v_i) = \begin{cases} 1, & 1 \leq i \leq p \\ -1, & p+2 \leq i \leq p+q, \end{cases} \tag{2.65}$$

and $\{r, s\}$, an orthonormal basis for the quadratic space $\mathbb{R}^{2,0}$ with quadrtic form H, and

$$\begin{aligned} H(r) &= 1 \\ H(s) &= 1. \end{aligned} \tag{2.66}$$

Consider the linear mapping $f : \mathbb{R}^{q+2,p} \to Cl_{p,q} \otimes Cl_{2,0}$ defined as

$$f(e_i) = \begin{cases} v_i \otimes (r * s), 1 \leq i \leq q \\ 1 \otimes r, i = q+1 \\ 1 \otimes s, i = q+2 \\ v_i \otimes (r * s), q+3 \leq i \leq p+q+2. \end{cases} \tag{2.67}$$

Once again we want to show that this linear mapping is a Clifford mapping. So, for $1 \leq i, j \leq p+q+2$, we once again have 10 cases to study:

1st. $1 \leq i, j \leq p$: $f(e_i) * f(e_j) + f(e_j) * f(e_i) = -2\delta_{i,j} 1 \otimes 1$.

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (v_i \otimes r * s) * (v_j \otimes r * s) \\ &+ (v_j \otimes r * s) * (v_i \otimes r * s) = \\ &= (v_i * v_j + v_j * v_i) \otimes (r * s)^2 = \\ &= (v_i * v_j + v_j * v_i) \otimes (-r^2 * s^2) \\ &= -2\delta_{i,j} 1 \otimes 1, \end{aligned}$$

2nd. $1 \leq i \leq p$ and $j = p+1$: $f(e_i) * f(e_j) + f(e_j) * f(e_i) = 0$.

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (v_i \otimes r * s) * (1 \otimes r) \\ &+ (1 \otimes r) * (v_i \otimes r * s) = \\ &= v_i \otimes (r * s * r + r * r * s) = \\ &= v_i \otimes (-r^2 * s + r^2 * s) = v_i \otimes 0 = 0. \end{aligned}$$

3rd. $1 \leq i \leq p$ and $j = p+2$: $f(e_i) * f(e_j) + f(e_j) * f(e_i) = 0$.

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (v_i \otimes r * s) * (1 \otimes s) \\ &+ (1 \otimes s) * (v_i \otimes r * s) = \\ &= v_i \otimes (r * s * s + s * r * s) = \\ &= v_i \otimes (r * s^2 - s^2 * r) = 0. \end{aligned}$$

4th. $1 \leq i \leq p$ and $p+3 \leq j \leq p+q+2$: $f(e_i) * f(e_j) + f(e_j) * f(e_i) = 0$.

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (v_i \otimes r * s) * (v_j \otimes r * s) \\ &+ (v_j \otimes r * s) * (v_i \otimes r * s) = \\ &= (v_i * v_j + v_j * v_i) \otimes (r * s)^2 = \\ &= (v_i * v_j + v_j * v_i) \otimes (-r^2 * s^2) = \\ &= -2\delta_{i,j} 1 \otimes 1 = 0. \end{aligned}$$

5th. $i = j = p+1$: $f(e_i) * f(e_j) + f(e_j) * f(e_i) = 2(1 \otimes 1)$.

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (1 \otimes r) * (1 \otimes r) + (1 \otimes r) * (1 \otimes r) = \\ &= 2(1 \otimes r^2) = 2(1 \otimes 1). \end{aligned}$$

6th. $i = p+1$ and $j = p+2$: $f(e_i) * f(e_j) + f(e_j) * f(e_i) = 0$.

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (1 \otimes r) * (1 \otimes s) + (1 \otimes s) * (1 \otimes r) = \\ &= 1 \otimes (r * s + s * r) = 0. \end{aligned}$$

7th. $i = p+1$ and $p+3 \leq j \leq p+q+2$: $f(e_i) * f(e_j) + f(e_j) * f(e_i) = 0$.

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (1 \otimes r) * (v_j \otimes r * s) \\ &+ (v_j \otimes r * s) * (1 \otimes r) = \\ &= v_j \otimes (r^2 * s + r * s * r) = \\ &= v_j \otimes (r^2 * s - r^2 * s) = 0. \end{aligned}$$

8th. $i = j = p+2$: $f(e_i) * f(e_j) + f(e_j) * f(e_i) = -2(1 \otimes 1)$.

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (1 \otimes s) * (1 \otimes s) + (1 \otimes s) * (1 \otimes s) = \\ &= 2(1 \otimes s^2) = 2(1 \otimes (-1)) = -2(1 \otimes 1). \end{aligned}$$

9th. $i = p+2$ and $p+3 \leq j \leq p+q+2$: $f(e_i) * f(e_j) + f(e_j) * f(e_i) = 0$.

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (1 \otimes s) * (v_j \otimes r * s) + (v_j \otimes r * s) * (1 \otimes s) = \\ &= v_j \otimes (s * r * s + r * s^2) = \\ &= v_j \otimes (-r * s^2 + r * s^2) = 0. \end{aligned}$$

10th. $p+3 \leq i, j \leq p+q+2$: $f(e_i) * f(e_j) + f(e_j) * f(e_i) = 2\delta_{i,j} 1 \otimes 1$.

$$\begin{aligned} f(e_i) * f(e_j) + f(e_j) * f(e_i) &= (v_i \otimes r * s) * (v_j \otimes r * s) \\ &+ (v_j \otimes r * s) * (v_i \otimes r * s) = \\ &= (v_i * v_j + v_j * v_i) \otimes (r * s)^2 = \\ &= (v_i * v_j + v_j * v_i) \otimes (-r^2 * s^2) = \\ &= -(-2\delta_{i,j}) \otimes 1 = 2\delta_{i,j}(1 \otimes 1). \end{aligned}$$

Therefore, if we consider any element $x \in \mathbb{R}^{q+2,p}$ as

$$x = \sum_{i=1}^{p+q+2} \alpha_i e_i,$$

we have

$$\begin{aligned} (f(x))^2 &= \sum_{i=1}^{p+q+2} \sum_{j=1}^{p+q+2} \alpha_i \alpha_j f(e_i) * f(e_j) = \\ &= 2(-\sum_{i=1}^{q} (\alpha_i)^2 + \sum_{i=q+1}^{p+q+2} \alpha_i)^2) 1 \otimes 1 = \\ &= 2Q(x) 1 \otimes 1, \end{aligned}$$

where, once again, the factor 2 appearing multiplying Q at the end is irrelevant and will be disconsidered for the next part of this proof.

We've just proven f is a Clifford mapping. By Theorem 2.1.1, the diagram in Figure 2.4 commutes.

Finally, we can repeat the argument done at the end of Theorem 2.2.2's proof to conclude that $Im(f^*) = Cl_{p,q} \otimes Cl_{2,0}$, thus $rank(f^*) = dim(Cl_{2,0} \otimes Cl_{p,q}) = dim(Cl_{q+2,p})$, which proves that f^* is an algebra isomorphism between $Cl_{q+2,p}$ and $Cl_{2,0} \otimes Cl_{p,q}$. $\square$

$$\begin{tikzcd}
\mathbb{R}^{q+2,p} \arrow[rr, "f"] \arrow[dr, "Q"'] & & Cl_{2,0} \otimes Cl_{p,q} \\
& Cl_{q+2,p} \arrow[ur, "f^*"'] &
\end{tikzcd}$$

Figure 2.4: Diagram for Theorem 2.2.3.

Once again, we group toghether the previous two theorems into one corollary.

Corollary 2.2.2. *The following isomorphims hold true:*

$$\begin{aligned} Cl_{p+1,q+1} &\simeq Cl_{1,1} \otimes Cl_{p,q} \\ Cl_{q+2,p} &\simeq Cl_{2,0} \otimes Cl_{p,q} \\ Cl_{q,p+2} &\simeq Cl_{0,2} \otimes Cl_{p,q}, \end{aligned} \tag{2.68}$$

where either $p \geq 0$ and $q \geq 0$, with the particular case $p = q = 0$ being identified as the real field, i.e.,

$$Cl_{0,0} = \mathbb{R}. \tag{2.69}$$

Corollary 2.2.2 allows us to expand on our isomorphism done in Example 2.2.1.

Example 2.2.2. *Consider the Clifford Algebra $Cl_{4,0}$. By Corollarie 2.2.2 and Corollary 2.2.1, we can conclude that*

$$\begin{aligned} Cl_{4,0} = Cl_{2+2,0} &\simeq Cl_{2,0} \otimes Cl_{0,2} \simeq Mat(2, \mathbb{R}) \otimes \mathbb{H} \simeq Mat(2, \mathbb{H}) \\ &\simeq Cl_{2,0} \otimes Cl_{0,2} \simeq Cl_{0,2+2} = Cl_{0,4}. \end{aligned}$$

Similarly, if we consider $Cl_{8,0}$, we have

$$\begin{aligned} Cl_{8,0} = Cl_{4+4,0} &\simeq Cl_{4,0} \otimes Cl_{0,4} \simeq Mat(2, \mathbb{R}) \otimes \mathbb{H} \otimes \mathbb{H} \otimes Mat(2, \mathbb{R}) \\ &\simeq Mat(2, \mathbb{R}) \otimes Mat(4, \mathbb{R}) \otimes Mat(2, \mathbb{R}) \\ &\simeq Mat(16, \mathbb{R}) \simeq Cl_{4,0} \otimes Cl_{0,4} \simeq Cl_{0,4+4} = Cl_{0,8}. \end{aligned}$$

The following corollary is an immediate consequence of Corollary 2.2.2 and by using the reasoning done in Examples 2.2.1 and 2.2.2.

Corollary 2.2.3. *The following isomorphims hold true:*

$$\begin{aligned} Cl_{p,q} &\simeq Cl_{p,p} \otimes Cl_{0,p-q}, \text{ for } p \geq q \\ Cl_{p,q} &\simeq Cl_{q,q} \otimes Cl_{p-q,0}, \text{ for } p \leq q. \end{aligned} \tag{2.70}$$

Theorem 2.2.2 tells us how the Clifford Algebra $Cl_{p,q}$ "behaves" when we add 2 into either p or q. The next corollary is similar to this case but now considering adding 4 and 8.

Corollary 2.2.4. *The following isomorphims hold:*

$$\begin{aligned} Cl_{p,q+4} &\simeq Cl_{0,4} \otimes Cl_{p,q} \\ Cl_{p,q+8} &\simeq Cl_{0,8} \otimes Cl_{p,q}. \end{aligned} \tag{2.71}$$

Proof. The first isomorphism is a direct consequence of Corollary 2.2.2 and Example 2.2.2:

$$\begin{aligned} Cl_{p,q+4} &\simeq Cl_{q+2,p} \otimes Cl_{0,2} \simeq Cl_{p,q} \otimes Cl_{2,0} \otimes Cl_{0,2} \\ &\simeq Cl_{p,q} \otimes Cl_{0,4}, \end{aligned}$$

while the second isomorphism is a consequence of the first one:

$$\begin{aligned} Cl_{p,q+8} &\simeq Cl_{p,p+4} \otimes Cl_{0,4} \simeq Cl_{p,q} \otimes Cl_{0,4} \otimes Cl_{0,4} \\ &\simeq Cl_{p,q} \otimes Cl_{4,0} \otimes Cl_{0,4} \simeq Cl_{p,q} \otimes Cl_{0,8}. \end{aligned}$$

☐

We could keep going and exploring similar results for cases like $Cl_{10,0}, Cl_{12,0}, Cl_{16,0}$, and so on. For example, for any $k \geq 1$,

$$Cl_{8k,0} \simeq \otimes_{i=1}^{k} Cl_{8,0} \simeq \otimes_{i=1}^{k} Cl_{0,8} \simeq Cl_{0,8k}.$$

However, such scrutiny is unnecessary. If we look back at the case $Cl_{0,8}$ in Example 2.2.2, we notice that such algebra is isomophic to the algebra $Mat(16, \mathbb{R})$. Such result, together with our previous corollary, is called *the Periodicity Theorem*, which we state bellow for completionism sake.

Theorem 2.2.4 (The Periodicity Theorem). *Let $Cl_{p,q}$ be a Clifford Algebra associated with the the quadratic space $\mathbb{R}^{p+q}$. Then the following isomorphism is true:*

$$Cl_{p,q+8} \simeq Cl_{p,q} \otimes Mat(16, \mathbb{R}). \tag{2.72}$$

This theorem, as the name sugests, stablishes a periodic nature for the Clifford Algebras. This means we can study only the cases where $1 \leq dim(V) = p + q \leq 7$, and the rest becomes a consequence of each of this particular cases together with the Periodicity Theorem. Take, for example, the case $Cl_{2,10}$. Using both, the Periodicity Theorem and Example 2.2.1, we have

$$\begin{aligned} Cl_{2,10} = Cl_{2,2+8} &\underbrace{\simeq}_{\text{Theorem 2.2.4}} Cl_{2,2} \otimes Mat(16, \mathbb{R}) \\ \text{(Example 2.2.1)} &\simeq Mat(4, \mathbb{R}) \otimes Mat(16, \mathbb{R}) \\ \text{(Example 1.1.5)} &\simeq Mat(64, \mathbb{R}). \end{aligned}$$

One question may arrive: Corollary 2.2.4 also stablishes a possible periodicity by adding 4 in q. The reason why 8 is chosen instead of 4 is a practical one. Notice that $Cl_{0,4} \simeq Mat(2, \mathbb{H})$, i.e., the isomorphism happens between the algebra of matrices with quartenion entries; meanwhile the isomorphism between $Cl_{0,8}$ happens with the algebra of matrices with real entries. And, while it is possible to write the quartenions as a subalgebra of $Mat(4, \mathbb{R})$ via an algebraic isomorphism (see Section 3.4), the notation for the classification starts to become cumbersome, speacially when we start to get cases more complexes like $Cl_{3,0}$ and $Cl_{5,1}$. When we present the table classifying the Clifford Algebras, this notion of why 8 over 4 will become clearer.

Example 2.2.3. *Consider the Clifford Algebras $Cl_{3,0}$, $Cl_{0,3}$ and $Cl_{5,1}$. By Corollaries 2.2.1, 2.2.4 and Example 2.2.2, we have*

$$Cl_{3,0} \simeq Cl_{0,1} \otimes Cl_{2,0} \simeq \mathbb{C} \otimes Mat(2, \mathbb{R}) \simeq Mat(2, \mathbb{C}),$$

$$Cl_{0,3} \simeq Cl_{1,0} \otimes Cl_{0,2} \simeq (\mathbb{R} \oplus \mathbb{R}) \otimes \mathbb{H} \simeq (\mathbb{R} \otimes \mathbb{H}) \oplus (\mathbb{R} \otimes \mathbb{H}) \simeq \mathbb{H} \oplus \mathbb{H}$$

and

$$\begin{aligned} Cl_{5,1} &\simeq Cl_{1,1} \otimes Cl_{0,4} \simeq Mat(2,\mathbb{R}) \otimes \mathbb{H} \otimes Mat(2,\mathbb{R}) \\ &\simeq \mathbb{H} \otimes Mat(2,\mathbb{R}) \otimes Mat(2,\mathbb{R}) \\ &\simeq \mathbb{H} \otimes Mat(4,\mathbb{R}) \simeq Mat(4,\mathbb{H}). \end{aligned}$$

With the Periodicity Theorem, we can easily obtain the next corollary, which is the formal proposition of our conclusions elaborated in the previous paragraph.

Corollary 2.2.5.

(i) *Suppose that* $p > q$ *and take* $p-q = 8k+r$, *where* $k \in \mathbb{N}$ *and* $0 \leq r \leq 7$. *Then*

$$Cl_{p,q} \simeq Mat(2^{4k+q}, \mathbb{R}) \otimes Cl_{r,0}. \tag{2.73}$$

(ii) *Suppose that* $p < q$ *and take* $q-p = 8k+r$, *where* $k \in \mathbb{N}$ *and* $0 \leq r \leq 7$. *Then*

$$Cl_{p,q} \simeq Mat(2^{4k+p}, \mathbb{R}) \otimes Cl_{0,r}. \tag{2.74}$$

(iii) *Suppose* $p = q$. *Then*

$$Cl_{p,p} \simeq Mat(2^p, \mathbb{R}). \tag{2.75}$$

Proof. For the case $p = q$, we have

$$Cl_{p,p} \simeq \otimes_{i=1}^{p} Cl_{1,1} \simeq \otimes_{i=1}^{p} Mat(2,\mathbb{R}) \simeq Mat(2^p, \mathbb{R}).$$

For $p > q$,

$$\begin{aligned} Cl_{p,q} &\simeq Cl_{q,q} \otimes Cl_{p-q,0} \simeq Cl_{q,q} \otimes Cl_{8k+r,0} \\ &\simeq Cl_{q,q} \otimes Cl_{8(k-1)+r+6+2,0} \\ &\simeq Cl_{q,q} \otimes Cl_{2,0} \otimes Cl_{0,8(k-1)} \otimes Cl_{0,r+6} \\ &\simeq Mat(2^q,\mathbb{R}) \otimes Mat(2,\mathbb{R}) \otimes Mat(16^{k-1},\mathbb{R}) \otimes Cl_{0,r+6} \\ &\simeq Mat(2^{q+1+4(k-1)},\mathbb{R}) \otimes Cl_{0,r+6} \simeq Mat(2^{q+4k-3},\mathbb{R}) \otimes Cl_{0,r+6} \\ &\simeq Mat(2^{q+4k-3},\mathbb{R}) \otimes Cl_{0,2} \otimes Cl_{2,0} \otimes Cl_{0,2} \otimes Cl_{r,0} \\ &\simeq Mat(2^{q+4k-3},\mathbb{R}) \otimes Cl_{0,4} \otimes Cl_{2,0} \otimes Cl_{r,0} \\ &\simeq Mat(2^{q+4k-3},\mathbb{R}) \otimes Mat(4,\mathbb{R}) \otimes Mat(2,\mathbb{R}) \otimes Cl_{r,0} \\ &\simeq Mat(2^{q+4k-3},\mathbb{R}) \otimes Mat(8,\mathbb{R}) \otimes Cl_{r,0} \\ &\simeq Mat(2^{q+4k},\mathbb{R}) \otimes Cl_{r,0}. \end{aligned}$$

The case $p < q$ is analogous to the previous one. □

This is it. The last corollary was the last "building block" for our classification problem. Once we stablished the isomorphisms for the "simpler" cases $0 \leq p+q \leq 7$, we can apply the Corollary 2.2.5 for the more "complex" cases $p+q \geq 8$, allowing us to classify all Clifford Algebras associated with finite dimensional vector spaces over the real field.

Throughout this and the last section, each example given has been done so to start our classification of the simpler cases. The idea behind each case left is analogous, if not identical, to what was done so far in said examples. As a final example to illustrate this similarity, take the cases $Cl_{5,0}$ and $Cl_{0,7}$. Using everything done so far, we have

$$\begin{aligned} Cl_{5,0} &\simeq Cl_{2,0} \otimes Cl_{0,3} \simeq Mat(2,\mathbb{R}) \otimes (\mathbb{H} \oplus \mathbb{H}) \\ &\simeq (Mat(2,\mathbb{R}) \otimes \mathbb{H}) \oplus (Mat(2,\mathbb{R}) \otimes \mathbb{H}) \\ &\simeq Mat(2,\mathbb{H}) \oplus Mat(2,\mathbb{H}), \end{aligned} \tag{2.76}$$

and

$$\begin{aligned} Cl_{0,7} &\simeq Cl_{0,2} \otimes Cl_{5,0} \simeq \mathbb{H} \otimes (\mathbb{H} \oplus \mathbb{H}) \otimes Mat(2,\mathbb{R}) \\ &\simeq [(\mathbb{H} \otimes \mathbb{H}) \oplus (\mathbb{H} \otimes \mathbb{H})] \otimes Mat(2,\mathbb{R}) \\ &\simeq (Mat(4,\mathbb{R}) \oplus Mat(4,\mathbb{R})) \otimes Mat(2,\mathbb{R}) \\ &\simeq (Mat(4,\mathbb{R}) \otimes Mat(2,\mathbb{R})) \oplus (Mat(4,\mathbb{R}) \otimes Mat(2,\mathbb{R})) \\ &\simeq Mat(8,\mathbb{R}) \oplus Mat(8,\mathbb{R}). \end{aligned} \tag{2.77}$$

The theorem that follows is the classification of all simpler cases, i.e., the combinations of $1 \leq p+q \leq 7$.

Theorem 2.2.5. *The following isomorphisms hold:*

- $p+q=0$:

$$Cl_{0,0} = \mathbb{R}; \tag{2.78}$$

- $p+q=1$:

$$\begin{aligned} Cl_{1,0} &\simeq \mathbb{R} \oplus \mathbb{R}, \\ Cl_{0,1} &\simeq \mathbb{C}; \end{aligned} \tag{2.79}$$

- $p+q=2$:

$$\begin{aligned} Cl_{2,0} &\simeq Cl_{1,1} \simeq Mat(2,\mathbb{R}), \\ Cl_{0,2} &\simeq \mathbb{H}; \end{aligned} \tag{2.80}$$

- $p+q=3$:

$$\begin{aligned} Cl_{3,0} &\simeq Cl_{1,2} \simeq Mat(2,\mathbb{C}), \\ Cl_{0,3} &\simeq \mathbb{H} \oplus \mathbb{H}, \\ Cl_{2,1} &\simeq Mat(2,\mathbb{R}) \oplus Mat(2,\mathbb{R}); \end{aligned} \tag{2.81}$$

- $p+q=4$:

$$\begin{aligned} Cl_{4,0} &\simeq Cl_{1,3} \simeq Cl_{0,4} \simeq Mat(2,\mathbb{H}), \\ Cl_{3,1} &\simeq Cl_{2,2} \simeq Mat(4,\mathbb{R}); \end{aligned} \tag{2.82}$$

- $p+q=5$:

$$\begin{aligned} Cl_{5,0} &\simeq Cl_{1,4} \simeq Mat(2,\mathbb{H}) \oplus Mat(2,\mathbb{H}), \\ Cl_{0,5} &\simeq Cl_{4,1} \simeq Cl_{2,3} \simeq Mat(4,\mathbb{C}), \\ Cl_{3,2} &\simeq Mat(4,\mathbb{R}) \oplus Mat(4,\mathbb{R}); \end{aligned} \tag{2.83}$$

- $p+q=6$:

$$\begin{aligned} Cl_{6,0} &\simeq Cl_{2,4} \simeq Cl_{5,1} \simeq Cl_{1,5} \simeq Mat(4,\mathbb{H}), \\ Cl_{0,6} &\simeq Cl_{4,2} \simeq Cl_{3,3} \simeq Mat(8,\mathbb{R}); \end{aligned} \tag{2.84}$$

- $p+q=7$:

$$\begin{aligned}
&Cl_{7,0} \simeq Cl_{3,4} \simeq Cl_{1,6} \simeq Cl_{5,2} \simeq Mat(8,\mathbb{C}),\\
&Cl_{0,7} \simeq Cl_{4,3} \simeq Mat(8,\mathbb{R}) \oplus Mat(8,\mathbb{R}),\\
&Cl_{6,1} \simeq Cl_{2,5} \simeq Mat(4,\mathbb{H}) \oplus Mat(4,\mathbb{H}).
\end{aligned} \tag{2.85}$$

We group all simple cases into Table 2.2 to help the visualisation. Notice that the isomorphism between the matrix set is not presented just yet.

$p+q=0$	$Cl_{0,0}$	$Cl_{1,1}$	$Cl_{2,2}$	$Cl_{3,3}$	
$p+q=1$	$Cl_{1,0}$	$Cl_{2,1}$	$Cl_{3,2}$	$Cl_{4,3}$	$Cl_{0,7}$
$p+q=2$	$Cl_{2,0}$	$Cl_{3,1}$	$Cl_{4,2}$	$Cl_{0,6}$	
$p+q=3$	$Cl_{3,0}$	$Cl_{4,1}$	$Cl_{5,2}$	$Cl_{0,5}$	$Cl_{1,6}$
$p+q=4$	$Cl_{4,0}$	$Cl_{5,1}$	$Cl_{0,4}$	$Cl_{1,5}$	
$p+q=5$	$Cl_{5,0}$	$Cl_{6,1}$	$Cl_{0,3}$	$Cl_{1,4}$	$Cl_{2,5}$
$p+q=6$	$Cl_{6,0}$	$Cl_{0,2}$	$Cl_{1,3}$	$Cl_{2,4}$	
$p+q=7$	$Cl_{7,0}$	$Cl_{0,1}$	$Cl_{1,2}$	$Cl_{2,3}$	$Cl_{3,4}$

Table 2.1: Real Clifford Algebra Classification for dimension $n < 8$.

We now couple together the previous theorem with Corollary 2.2.5 to start the classification for the cases $p+q \geq 8$. To do so, we need to consider the cases $p \geq q$ and $p < q$, as well as $r = p - q$ mod(8). This gives us 16 cases to be studied:

Case 1: $r = 0$ and $p \geq q$ ($0 = p - q$ mod(8)).

$$Cl_{p,q} \simeq Mat(2^{q+4k}, \mathbb{R});$$

Case 2: $r = 0$ and $p < q$ ($0 = p - q$ mod(8)).

$$Cl_{p,q} \simeq Mat(2^{p+4k}, \mathbb{R});$$

Case 3: $r = 1$ and $p > q$ ($1 = p - q$ mod(8)).

$$\begin{aligned}
Cl_{p,q} &\simeq Mat(2^{q+4k}, \mathbb{R}) \otimes Cl_{1,0} \simeq Mat(2^{q+4k}, \mathbb{R}) \otimes (\mathbb{R} \oplus \mathbb{R})\\
&\simeq Mat(2^{q+4k}, \mathbb{R}) \oplus Mat(2^{q+4k}, \mathbb{R});
\end{aligned}$$

Case 4: $r = 1$ and $p < q$ ($7 = p - q$ mod(8)).

$$\begin{aligned}
Cl_{p,q} &\simeq Mat(2^{p+4k}, \mathbb{R}) \otimes Cl_{0,1} \simeq Mat(2^{p+4k}, \mathbb{R}) \otimes \mathbb{C})\\
&\simeq Mat(2^{p+4k}, \mathbb{C}).
\end{aligned}$$

As a parenthesis, for the previous cases, we can rewrite $q+4k$ as $q+4k = [\frac{2q+8k+r}{2}] = [\frac{p+q}{2}] = [\frac{n}{2}]$, where $[s]$ denotes the integer part of the number s. The same is true for $p+4k$.

Case 5: $r = 2$ and $p > q$ ($2 = p - q$ mod(8)).

$$\begin{aligned}
Cl_{p,q} &\simeq Mat(2^{q+4k}, \mathbb{R}) \otimes Cl_{2,0} \simeq Mat(2^{q+4k}, \mathbb{R}) \otimes Mat(2, \mathbb{R})\\
&\simeq Mat(2^{q+4k+1}, \mathbb{R}).
\end{aligned}$$

Case 6: $r = 2$ and $p < q$ ($6 = p - q \bmod(8)$).

$$\begin{aligned} Cl_{p,q} &\simeq Mat(2^{p+4k}, \mathbb{R}) \otimes Cl_{0,2} \simeq Mat(2^{p+4k}, \mathbb{R}) \otimes \mathbb{H}) \\ &\simeq Mat(2^{p+4k}, \mathbb{H}). \end{aligned}$$

Case 7: $r = 4$ and $p > q$ ($3 = p - q \bmod(8)$).

$$\begin{aligned} Cl_{p,q} &\simeq Mat(2^{q+4k}, \mathbb{R}) \otimes Cl_{3,0} \simeq Mat(2^{q+4k}, \mathbb{R}) \otimes Mat(2, \mathbb{C}) \\ &\simeq Mat(2^{q+4k+1}, \mathbb{C}). \end{aligned}$$

Case 8: $r = 4$ and $p < q$ ($5 = p - q \bmod(8)$).

$$\begin{aligned} Cl_{p,q} &\simeq Mat(2^{p+4k}, \mathbb{R}) \otimes Cl_{0,3} \simeq Mat(2^{p+4k}, \mathbb{R}) \otimes (\mathbb{H} \oplus \mathbb{H}) \\ &\simeq Mat(2^{p+4k}, \mathbb{H}) \oplus Mat(2^{p+4k}, \mathbb{H}). \end{aligned}$$

For the cases $r = 2, 3$, if $p > q$, then $q + 4k + 1 = [\frac{2q+8k+r}{2}] = [\frac{n}{2}]$; and if $p < q$, then $p + 4k + 1 = [\frac{2p+8k+r}{2}] - 1 = [\frac{n}{2}] - 1$.

Case 9: $r = 4$ and $p > q$ ($4 = p - q \bmod(8)$).

$$\begin{aligned} Cl_{p,q} &\simeq Mat(2^{q+4k}, \mathbb{R}) \otimes Cl_{4,0} \simeq Mat(2^{q+4k}, \mathbb{R}) \otimes Mat(2, \mathbb{H}) \\ &\simeq Mat(2^{q+4k+1}, \mathbb{H}). \end{aligned}$$

Case 10: $r = 4$ and $p < q$ ($4 = p - q \bmod(8)$).

$$\begin{aligned} Cl_{p,q} &\simeq Mat(2^{p+4k}, \mathbb{R}) \otimes Cl_{0,4} \simeq Mat(2^{p+4k}, \mathbb{R}) \otimes Mat(2, \mathbb{H}) \\ &\simeq Mat(2^{p+4k+1}, \mathbb{H}). \end{aligned}$$

Case 11: $r = 5$ and $p > q$ ($5 = p - q \bmod(8)$).

$$\begin{aligned} Cl_{p,q} &\simeq Mat(2^{q+4k}, \mathbb{R}) \otimes Cl_{5,0} \simeq Mat(2^{q+4k}, \mathbb{R}) \otimes (Mat(2, \mathbb{H}) \oplus Mat(2, \mathbb{H})) \\ &\simeq Mat(2^{q+4k+1}, \mathbb{H}) \oplus Mat(2^{q+4k+1}, \mathbb{H}). \end{aligned}$$

Case 12: $r = 5$ and $p < q$ ($3 = p - q \bmod(8)$).

$$\begin{aligned} Cl_{p,q} &\simeq Mat(2^{p+4k}, \mathbb{R}) \otimes Cl_{0,5} \simeq Mat(2^{p+4k}, \mathbb{R}) \otimes Mat(4, \mathbb{C}) \\ &\simeq Mat(2^{p+4k+2}, \mathbb{C}). \end{aligned}$$

In the cases where $r = 5, 6$, if $p > q$ or $r = 4$, then $q+4k+1 = [\frac{2q+8k+r}{2}]-1 = [\frac{n}{2}]-1$; and if $p < q$ and $r = 5$, then $p + 4k + 2 = [\frac{2p+8(k+1)+r}{2}] = [\frac{n}{2}]$.

Case 13: $r = 6$ and $p > q$ ($6 = p - q \bmod(8)$).

$$\begin{aligned} Cl_{p,q} &\simeq Mat(2^{q+4k}, \mathbb{R}) \otimes Cl_{6,0} \simeq Mat(2^{q+4k}, \mathbb{R}) \otimes Mat(4, \mathbb{H}) \\ &\simeq Mat(2^{q+4k+2}, \mathbb{H}). \end{aligned}$$

Case 14: $r = 6$ and $p < q$ ($2 = p - q \bmod(8)$).

$$\begin{aligned} Cl_{p,q} &\simeq Mat(2^{p+4k}, \mathbb{R}) \otimes Cl_{0,6} \simeq Mat(2^{p+4k}, \mathbb{R}) \otimes Mat(8, \mathbb{R}) \\ &\simeq Mat(2^{p+4k+3}, \mathbb{R}). \end{aligned}$$

Case 15: $r = 7$ and $p > q$ ($7 = p - q \text{ mod}(8)$).

$$\begin{aligned} Cl_{p,q} &\simeq Mat(2^{q+4k}, \mathbb{R}) \otimes Cl_{7,0} \simeq Mat(2^{q+4k}, \mathbb{R}) \otimes Mat(8, \mathbb{C}) \\ &\simeq Mat(2^{q+4k+3}, \mathbb{C}). \end{aligned}$$

Case 16: $r = 7$ and $p < q$ ($1 = p - q \text{ mod}(8)$).

$$\begin{aligned} Cl_{p,q} &\simeq Mat(2^{p+4k}, \mathbb{R}) \otimes Cl_{0,7} \simeq Mat(2^{p+4k}, \mathbb{R}) \otimes (Mat(8, \mathbb{R}) \oplus Mat(8, \mathbb{R})) \\ &\simeq Mat(2^{p+4k+3}, \mathbb{R}) \oplus Mat(2^{p+4k+3}, \mathbb{R}). \end{aligned}$$

Lastly, in cases $r = 6, 7$, if $p > q$ and $r = 6$, then $q + 4k + 2 = [\frac{n}{2}] - 1$; whilst, if $p < q$, then $p + 4k + 3 = [\frac{n}{2}]$.

This results can be neatly organized into two tables, one for $n < 8$, and the other for $n \geq 8$.

$p - q = 0 \text{ mod}(8)$	$Mat(2^{[\frac{n}{2}]}, \mathbb{R})$
$p - q = 1 \text{ mod}(8)$	$Mat(2^{[\frac{n}{2}]}, \mathbb{R}) \oplus Mat(2^{[\frac{n}{2}]}, \mathbb{R})$
$p - q = 2 \text{ mod}(8)$	$Mat(2^{[\frac{n}{2}]}, \mathbb{R})$
$p - q = 3 \text{ mod}(8)$	$Mat(2^{[\frac{n}{2}]}, \mathbb{C})$
$p - q = 4 \text{ mod}(8)$	$Mat(2^{[\frac{n}{2}]-1}, \mathbb{H})$
$p - q = 5 \text{ mod}(8)$	$Mat(2^{[\frac{n}{2}]-1}, \mathbb{H}) \oplus Mat(2^{[\frac{n}{2}]-1}, \mathbb{H})$
$p - q = 6 \text{ mod}(8)$	$Mat(2^{[\frac{n}{2}]-1}, \mathbb{H})$
$p - q = 7 \text{ mod}(8)$	$Mat(2^{[\frac{n}{2}]}, \mathbb{C})$

Table 2.2: Real Clifford Algebra Classification for dimension n.

To finish this section, we present some examples to illustrate the use of the previous tables.

Example 2.2.4. *Consider the case $Cl_{2,10}$. This means the dimension of the vector space is $n = 2 + 10 = 12$ and*

$$p - q = -8 \equiv 0 \text{ mod}(8).$$

So, by looking at Table 2.2,

$$[\frac{n}{2}] = [\frac{12}{2}] = [6] = 6,$$

and

$$Cl_{2,10} \simeq Mat(2^6, \mathbb{R}) = Mat(64, \mathbb{R}).$$

Example 2.2.5. *Let V be a quadratic space with dimension 31 and its quadratic form having signature 19. By using Equations 1.58, we have*

$$\begin{aligned} p &= \frac{31 + 19}{2} = 25; \\ q &= \frac{31 - 19}{2} = 6. \end{aligned}$$

Thus $p - q = 19$ and

$$19 \equiv 3 \text{ mod}(8).$$

By Table 2.2, we have

$$Cl_{25,6} \simeq Mat(2^{25}, \mathbb{C}),$$

where we used the result

$$[\frac{n}{2}] = [\frac{50}{2}] = [25.0] = 25.$$

Notice how we needed to know the dimension of the vector space and the signature of the quadratic form in the previous example.

Example 2.2.6. *Consider the Clifford Algebra* $Cl_{5,2}$*. By looking at Table* **??***, we can conclude immediatly that*

$$[\frac{n}{2}] = [\frac{5+2}{2}] = [3.5] = 3 \Rightarrow$$
$$\Rightarrow Cl_{5,1} \simeq Mat(8, \mathbb{C}),$$

which agrees with Theorem 2.2.5.

Chapter 3

Classification of Polynomials over Algebras of $\mathbb{R}^4$

We begin this section by presenting some basic properties related to the Quartenions and other algebras of $\mathbb{R}^4$. For more details for this chapter as a whole, the reader is invited to check References [22], [23], [24], [25], [26], [27], [28].

3.1 Quartenions and Algebras of $\mathbb{R}^4$

In the paragraph following Example 1.2.2, we saw the first idea that $\mathbb{R} \oplus \mathbb{R}$ is an algebra of $\mathbb{R}^2$. Here we formalize and flesh out this idea.

Definition 3.1.1. *Let A be a real algebra. If we can find $n \in \mathbb{N}$ such that A is isomorphic to $\mathbb{R}^n$ as a vector space, then we say A is an* ***algebra of*** *$\mathbb{R}^n$.*

Examples 1.2.1 and 1.2.2 tell us that $\mathbb{C}$ and $\mathbb{R} \oplus \mathbb{R}$ are algebras of $\mathbb{R}^2$. Example 1.2.3 informs us that $\mathbb{H}$ is an algebra of $\mathbb{R}^4$. But how can we find an algebra of $\mathbb{R}^4$? To answer this question let's look into the vector space $\mathbb{R}^4$. Consider the canonical basis $\mathcal{C} = \{(1,0,0,0),(0,1,0,0),(0,0,1,0),(0,0,0,1)\}$. If $\mathbb{R}^4$ was suppose to be an algebra, then we know the basis $\mathcal{C}$ generates $\mathbb{R}^4$ as an algebra, then we just need to consider a multiplicative table for the elements in the basis $\mathcal{C}$. Here we shall use the following notation: $1 = (1,0,0,0)$, $i = (0,1,0,0)$, $j = (0,0,1,0)$, $k = (0,0,0,1)$.

$.$	1	i	j	k
1	1^2	$1.i$	$1.j$	$1.k$
i	$i.1$	i^2	$i.j$	$i.k$
j	$j.1$	$j.i$	j^2	$j.k$
k	$k.1$	$k.i$	$k.j$	k^2

Table 3.1: Multiplication table for the elements in $\mathcal{C}$.

For each combination of values for the product a new algebra is generated. Take, for example, Table 3.2. If we look at the products between i, j and k, we notice the product is anticommutative, the square elements vanish, and it's cyclical in the sense that $i.j = k$, $j.k = i$, $k.i = j$. If we consider the subspace V generated by $\{i, j, k\}$, the product so defined is closed in this subspace, i.e., the product of elements in V always give an element of V as a result, meaning V is an algebra. This algebra's product is commmunly known as the cross product (or vector product) of $\mathbb{R}^3$.

.	1	i	j	k
1	1	i	j	k
i	i	0	k	$-j$
j	j	$-k$	0	i
k	k	j	$-i$	0

Table 3.2: The cross product of $\mathbb{R}^3$.

Another example is the cited Quartenions. Similar to the cross product, the multiplication between i, j and k is cyclical, but differently from the last case, $\{i, j, k\}$ doesn't spans an algebra V of $\mathbb{R}^3$ since $i^2 = j^2 = k^2 = -1$, which is not an element of V. The multiplication table for the Quartenions can be seen in Table 3.3.

.	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Table 3.3: The Quartenion's multiplication table.

In this work we're interested in cases similar to the Quartenions: 1 is the unity of the algebra, $i.j = k$, $i^2 \pm 1$, $j^2 \pm 1$ and $k^2 \pm 1$. For each choice of sign of i, j and k, we generate a new algebra of $\mathbb{R}^4$. This algebras are named **Quartenions**, **Coquartenions**, **Tessarines**, **Cotessarines**, **Nectarines**, **Conectarines**, **Tangerines** and **Cotangerines**. The multiplication tables of these 8 algebras can be seen in Table 3.4. By using this table, we can write the multiplication between any two elements x and y from any of these eight algebras.

Algebra	**Notation**	i^2	j^2	k^2	ij	jk	ki
Quartenions	$\mathbb{H}$	-1	-1	-1	k	i	j
Coquartenions	$\mathbb{H}_{\text{coq}}$	-1	1	1	k	$-i$	j
Tessarines	$\mathbb{H}_{\text{tes}}$	-1	1	-1	k	i	$-j$
Cotessarines	$\mathbb{H}_{\text{cot}}$	1	1	1	k	i	j
Nectarines	$\mathbb{H}_{\text{nec}}$	1	-1	1	k	i	$-j$
Conectarines	$\mathbb{H}_{\text{con}}$	1	1	-1	k	$-i$	$-j$
Tangerines	$\mathbb{H}_{\text{tan}}$	1	-1	-1	k	i	j
Cotangerines	$\mathbb{H}_{\text{cotan}}$	-1	-1	1	k	$-i$	$-j$

Table 3.4: The Algebras of $\mathbb{R}^4$.

Example 3.1.1. *Consider algebra of Coquartenions. We want to write the multiplication between any two elements $x, y \in \mathbb{H}_{coq}$. To do so, first we need to find the* *values $j.i$, $k.j$ and $i.k$.*

$$
\begin{aligned}
j.i &= 1.j.i = -i^2.j.i = -i.\underbrace{(i.j)}_{=k}.i = -i.\underbrace{k.i}_{=j} = -i.j = -k,\\
k.j &= k.1.j = k.(-i^2).j = -\underbrace{(k.i)}_{=j}.\underbrace{(i.j)}_{=k} = -j.k = -i,\\
i.k &= i.k.1 = i.k.j^2 = i.\underbrace{(k.j)}_{=-i}.j = i.(-i).j = -i^2.j = -(-1).j = j.
\end{aligned}
$$

With this results, we can multiply $x = x_0 + ix_1 + jx_2 + kx_3$ by $y = y_0 + iy_1 + jy_2 + ky_3$, which gives

$$
\begin{aligned}
x.y = {} & x_0y_0 - x_1y_1 + x_2y_2 + x_3y_3 + i(x_0y_1 + x_1y_0 - x_2y_3 + x_3y_2)\\
& + j(x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1) + k(x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0).
\end{aligned}
$$

An analogous reason can be done for each of the other six algebras. This laborious work returns the following results for the remaining algebras, which are gathered together with the quartenions and coquartenions for completionism sake:

- ***Quartenions:***

$$
\begin{aligned}
x.y = {} & x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 + i(x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2)\\
& + j(x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1) + k(x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0)
\end{aligned}
\tag{3.1a}
$$

- ***Coquartenions:***

$$
\begin{aligned}
x.y = {} & x_0y_0 - x_1y_1 + x_2y_2 + x_3y_3 + i(x_0y_1 + x_1y_0 - x_2y_3 + x_3y_2)\\
& + j(x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1) + k(x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0)
\end{aligned}
\tag{3.1b}
$$

- ***Tessarines:***

$$
\begin{aligned}
x.y = {} & x_0y_0 - x_1y_1 + x_2y_2 - x_3y_3 + i(x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2)\\
& + j(x_0y_2 - x_1y_3 + x_2y_0 - x_3y_1) + k(x_0y_3 + x_1y_2 + x_2y_1 + x_3y_0)
\end{aligned}
\tag{3.1c}
$$

- ***Cotessarines:***

$$
\begin{aligned}
x.y = {} & x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 + i(x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2)\\
& + j(x_0y_2 + x_1y_3 + x_2y_0 + x_3y_1) + k(x_0y_3 + x_1y_2 + x_2y_1 + x_3y_0)
\end{aligned}
\tag{3.1d}
$$

- ***Nectarines:***

$$
\begin{aligned}
x.y = {} & x_0y_0 + x_1y_1 - x_2y_2 + x_3y_3 + i(x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2)\\
& + j(x_0y_2 + x_1y_3 + x_2y_0 - x_3y_1) + k(x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0)
\end{aligned}
\tag{3.1e}
$$

- ***Conectarines:***

$$
\begin{aligned}
x.y = {} & x_0y_0 + x_1y_1 + x_2y_2 - x_3y_3 + i(x_0y_1 + x_1y_0 - x_2y_3 + x_3y_2)\\
& + j(x_0y_2 + x_1y_3 + x_2y_0 - x_3y_1) + k(x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0)
\end{aligned}
\tag{3.1f}
$$

- ***Tangerines:***

$$\begin{aligned} x.y = {} & x_0y_0 + x_1y_1 - x_2y_2 - x_3y_3 + i(x_0y_1 + x_1y_0 + x_2y_3 + x_3y_2) \\ & + j(x_0y_2 + x_1y_3 + x_2y_0 + x_3y_1) + k(x_0y_3 + x_1y_2 + x_2y_1 + x_3y_0) \end{aligned} \tag{3.1g}$$

- ***Cotangerines:***

$$\begin{aligned} x.y = {} & x_0y_0 - x_1y_1 - x_2y_2 + x_3y_3 + i(x_0y_1 + x_1y_0 - x_2y_3 - x_3y_2) \\ & + j(x_0y_2 - x_1y_3 + x_2y_0 - x_3y_1) + k(x_0y_3 + x_1y_2 + x_2y_1 + x_3y_0). \end{aligned} \tag{3.1h}$$

Like the Complex numbers, the conjugate and the absolute value of an element in one of these algebras can also be defined.

Definition 3.1.2. *Let $\mathcal{A}$ be one of the eight algebras of $\mathbb{R}^4$ defined in Table 3.4, and suppose $z = z_0 + iz_1 + jz_2 + kz_3 \in \mathcal{A}$. We define the* ***conjugate of z*** *as*

$$\overline{z} = z_0 - iz_1 - jz_2 - kz_3. \tag{3.2}$$

Definition 3.1.3. *Let $z = z_0 + iz_1 + jz_2 + kz_3 \in \mathcal{A}$, where $\mathcal{A}$ is one of the eight algebras of $\mathbb{R}^4$. We define*

$$abs(z) = \overline{z}z \tag{3.3}$$

as the ***absolute value of z****. Moreover, we say that $\mathcal{R}(z) = z_0$ is the* ***real part of*** *z.*

It's worth noticing the absulute value defined above doesn't necessarily retuns a positive number depending on the algebra in question. The only case where $abs(z) > 0$ for any $z \in \mathcal{A} - \{0\}$ is when $\mathcal{A} = \mathbb{H}$, as it can be seen in the next example.

Example 3.1.2. *Consider the products show in the set of Equations 3.1. Then, for each of the algebras, we have*

$$abs(z) = \begin{cases} z_0^2 + z_1^2 + z_2^2 + z_3^2, \text{ for } \mathbb{H} \\ z_0^2 + z_1^2 - z_2^2 - z_3^2, \text{ for } \mathbb{H}_{coq} \\ z_0^2 + z_1^2 - z_2^2 + z_3^2 - 2(iz_2z_3 - jz_1z_3 + kz_1z_2), \text{ for } \mathbb{H}_{tes} \\ z_0^2 - z_1^2 - z_2^2 - z_3^2 - 2(iz_2z_3 + jz_1z_3 + kz_2z_3), \text{ for } \mathbb{H}_{cot} \\ z_0^2 - z_1^2 + z_2^2 - z_3^2, \text{ for } \mathbb{H}_{nec} \\ z_0^2 - z_1^2 - z_2^2 + z_3^2, \text{ for } \mathbb{H}_{con} \\ z_0^2 - z_1^2 + z_2^2 + z_3^2 - 2(iz_2z_3 + jz_1z_3 + kz_1z_2), \text{ for } \mathbb{H}_{tan} \\ z_0^2 + z_1^2 + z_2^2 - z_3^2 - 2(-iz_2z_3 - jz_1z_3 + kz_1z_2), \text{ for } \mathbb{H}_{cotan} \end{cases} \tag{3.4}$$

The following properties follow immediatly by using the the set of Equations 3.1.

Lemma 3.1.1. *Let $\mathcal{A}$ be any of the eight algebras presented in Table 3.4. Then, for any $x, y, z \in \mathcal{A}$,*

(i) $\overline{z}z = z\overline{z}$;

(ii) $abs(xy) = abs(yx) = abs(x)abs(y)$;

(iii) $\mathcal{R}(xy) = \mathcal{R}(yx)$*;*

(iv) $\overline{xy} = \bar{y}\bar{x}$*;*

(v) $\alpha z = z\alpha$ *for any* $\alpha \in \mathbb{R}$*.*

Moreover, the Quartenions, Coquartenions, Nectarines and Conectarines are noncommutative algebras, whilst the Tessarines, Cotessarines, Tangerines and Cotangerines are commutative algebras.

The next result is an immediate consequence of Equation 3.4.

Lemma 3.1.2. *Let* $\mathcal{A}$ *be any of the four noncommutative algebras from Table 3.4. Then, for any* $z \in \mathcal{A}$*,* $abs(z)$ *is a real number and*

$$z^{-1} = \frac{1}{abs(z)}\bar{z} \tag{3.5}$$

whenever $abs(z) \neq 0$*.*

Corollary 3.1.1. *Let* $\mathcal{A}$ *be any of the four noncommutative algebras. Then, for any* $z \in \mathcal{A}$*,* z *is invertile if, and only if,* $abs(z) \neq 0$*.*

Proof. Suppose z is invertible, then we can find $x \in \mathcal{A}$ such that

$$zx = 1 \Rightarrow \bar{z}zx = abs(z)x = \bar{z}.$$

If $abs(z) = 0$, then $\bar{z} = 0$, contradicting z being invertible, thus we must have $abs(z) \neq 0$. The rest of this theorem follows from Equation 3.5. □

From now on, whenever we use the symbol $\mathcal{A}$, unless stated otherwise, we are refering to one of the four noncommutative algebras from Table 3.4.

We now present the key idea behind the classification of zeros presented in the next sections.

Definition 3.1.4. *Let* $x, y \in \mathcal{A}$*, one of the four noncommutative algebras. We say that* x ***and*** y ***are equivalent****, in symbols* $x \sim y$*, if there exists some invertible element* $h \in \mathcal{A} - \{0\}$ *such that* $y = h^{-1}xh$*. Additionally, we denote by* $[x]$ *the equivalent class of* x*:*

$$[x] = \{y \in \mathbb{H} \mid y = h^{-1}xh \text{ for some } h \in \mathbb{H} - \{0\}\}. \tag{3.6}$$

It's straightforward to prove that $\sim$ is an equivalence relation.

Example 3.1.3. *Let* $\alpha \in \mathbb{R} \subset \mathcal{A}$*. We wish to calculate its equivalence class. Suppose* $y \in \mathcal{A}$ *is equivalent to* α*. Therefore we can find* $h \in \mathcal{A}$ *such that*

$$y = h^{-1}\alpha h = \alpha h^{-1}h = \alpha.$$

In conclusion, $[\alpha] = \{\alpha\}$*.*

The next result is specific for the algebra of Quartenions. Its proof can be seen in Reference [25].

Lemma 3.1.3. *Let $x, y \in \mathbb{H}$. Then x and y are equivalent if, and only if,*

$$\mathcal{R}(x) = \mathcal{R}(y) \text{ and } abs(x) = abs(y). \tag{3.7}$$

Lemma 3.1.3 presents a necessary and sufficient condition in order to two quartenions be equivalent. But such is not the case for the other noncommutative algebras.

Example 3.1.4. *Let $\alpha \in \mathbb{R}$ and $z = \alpha + i + j$ be two coquartenions. Since α is real, we have $[\alpha] = \{\alpha\}$, i.e., its equivalence class consists of a single element, thus α and z are not equivalent. However,*

$$\mathcal{R}(\alpha) = \alpha = \mathcal{R}(z),$$

and

$$abs(\alpha) = \alpha^2 = abs(z),$$

meaning Equation 3.7 is still satisfied. The same result holds for $z = \alpha + j + k \in \mathbb{H}_{nec}$ and $z = \alpha + j + k \in \mathbb{H}_{con}$.

Example 3.1.4 provided us an important result: for a general noncommutative algebra of $\mathbb{R}^4$, Equation 3.7 is not sufficient to guarantee similarity. This motivates another equivalence class to be used in such algebras.

Definition 3.1.5. *Let $x, y \in \mathcal{A}$. We say that x* ***and*** *y* ***are quasi-similar****, in symbols $x \sim^q y$, if Equation 3.7 is observed.*

So, from Example 3.1.4, we can conclude that $\alpha \sim^q z$.

Similar to $\sim$, the relation $\sim_q$ is also an equivalence relation, which can be proven by its definition.

Before closing this section, one point of observation must be made: in Example 2.1.3, we proved the quartenions as one of the real Clifford Algebras with dimension 4. Specifically, $\mathbb{H} \sim Cl_{0,2}$, as stated in Corollary 2.2.1. This same corollary bring us another algebraic isomorphism, $Cl_{2,0} \sim Cl_{1,1}$. Let's take a closer look into $Cl_{2,0}$. Such Clifford Algebra is generated by the set $\{1, e_1, e_2, e_1 * e_2\}$ such that

$$\begin{cases} 1^2 = 1 \\ e_1^2 = 1 \\ e_2^2 = 1 \\ e_1 * e_2 = -e_2 * e_1 \\ (e_1 * e_2)^2 = -1 \end{cases} \tag{3.8}$$

as it was seen in Example 2.1.4. By using 3.8, we can also derive

$$\begin{cases} e_2 * (e_1 * e_2) = -e_1 * e_2^2 = -e_1 \\ (e_1 * e_2) * e_1 = -e_1^2 * e_2 = -e_2. \end{cases} \tag{3.9}$$

By using Table 3.4, Equation 3.8 and Equation 3.9, it's immediate to conclude that the mapping $\phi : Cl_{2,0} \to \mathbb{H}_{con}$, where

$$\begin{cases} \phi(1) = 1 \\ \phi(e_1) = i \\ \phi(e_2) = j \\ \phi(e_3) = k, \end{cases} \tag{3.10}$$

is an algebra isomorphism. Thus the Conectarines is also a Clifford Algebra. By using same reasoning, one can see that $\mathbb{H}_{coq} \sim \mathbb{H}_{nec} \sim Cl_{1,1}$, all algebraic isomorphisms.

In conclusion, all noncommutative algebras of $\mathbb{R}^4$ are algebraicly isomorphic to a Clifford Algebra!

3.2 One-sided Quaternionic Polynomials

In this section we only consider the case $\mathcal{A} = \mathbb{H}$, the general case for the other noncommutative algebras being presented in Section 3.3.

Definition 3.2.1. *We define a* ***one-sided (or simple) quaternionic polynomial*** *as*

$$p(z) = \sum_{k=0}^{n} a_k z^k, \tag{3.11}$$

where $a_k, z \in \mathbb{H}$ *for all* $0 \leq k \leq n$*, and* $a_0 a_n \neq 0$*.*

The condition $a_n a_0 \neq 0$ is here to exclude the cases where 0 is a root of the polynomial ($a_0 = 0$) and so we can say p has degree n ($a_n \neq 0$). Moreover, by *zero* of the polynomial p, also called a *root*, we mean a value $z_0 \in \mathbb{H}$ such that

$$p(z_0) = 0. \tag{3.12}$$

One question may arise: why do we explicitly say *one-sided polynomial*? Why not just *polynomial*, similarl to what is done to the real and complexe cases? To answer this, let's look at the polynomials

$$p_L(z) = iz - k$$

and

$$p_R(z) = zi - k.$$

both with quaternionic entries, that are similarly the same polynomial. But, if we apply the case $z = j$ to both polynomials, then we have

$$p_L(j) = i.j - k = k - k = 0 \neq -2k = -k - k = j.i - k = p_R(j).$$

This shows us that, just because j is a zero for the left case (p_L), it doesn't mean it will be a zero for the right case (p_R). That's why such a explicit distinction between *left* and *right* must be made. A similar argument can be done for the two-sided case. All of this happens because all algebras we shall study in this text are not commutative, different from the real and complex polynomials, which are commutative and thus no distiction needs to be made. With that being said, all results (Lemmas, Theorems and Corollaries) for the left case are also valid to the right case, which can immediately be seen by the fact that all properties we use for the left-side case are also valid to the right-side case.

Before starting, let's look at a simple example to further differ the quaternionic case (and consequently all the other three cases as well) to the complex case.

Example 3.2.1. *Consider the quaternionic polynomial*

$$p(z) = z^2 + 1.$$

We want to find the zeros of this polynomial. This means to calculate $p(z) = 0$.

$$p(z) = 0 \Rightarrow z^2 = -1 \Rightarrow z \in \{i, -i, j, -j, k, -k\},$$

which can be immediately seen by looking at Table 3.3. Thus, for the algebras of $\mathbb{R}^4$*, we may have more zeros associated to a given polynomial then in the complex case.*

Later we shall come back to this example to see the relation the zeroes have between each other.

Now we begin the study proper, and the first step to do so is to simplify the way we can write the polynomial p in Equation 3.2.1.

Theorem 3.2.1 (Iteration Process - Quartenions)**.** *Let* $z \in \mathbb{H}$*. Then, for any* $n \in \mathbb{N}$*, the following two term iteration process is valid:*

$$z^n = \alpha_n x + \beta_n, \tag{3.13a}$$

where $\alpha, \beta \in \mathbb{R}$ *and*

$$\begin{cases} \alpha_0 = 0,\ \beta_0 = 1, \\ \alpha_n = 2\mathcal{R}(z)\alpha_{n-1} + \beta_{n-1},\ \beta_n = -abs(z)^2\alpha_{n-1} \text{ for } n \geq 1. \end{cases} \tag{3.13b}$$

Proof. Before proving the recursion, notice that any quartenion $h = h_0 + ih_1 + jh_2 + kh_3$ is a root of the polynomial

$$z^2 - pz + q,$$

where $p = 2\mathcal{R}(h) = 2h_0$ and $q = abs(h)^2 = h_0^2 + h_1^2 + h_2^2 + h_3^2$. To see this is straightforward:

$$\begin{aligned} h^2 - ph + q &= (h_0 + ih_1 + jh_2 + kh_3)^2 - 2h_0(h_0 + ih_1 + jh_2 + kh_3) \\ &\quad + h_0^2 + h_1^2 + h_2^2 + h_3^2 \\ &= h_0^2 - h_1^2 - h_2^2 - h_3^2 + i(h_0h_1 + h_1h_0 + h_2h_3 - h_3h_2) \\ &\quad + j(h_0h_2 - h_1h_3 + h_2h_0 + h_3h_1) + k(h_0h_3 + h_1h_2 - h_2h_1 + h_3h_0) \\ &\quad - 2h_0^2 - i2h_0h_1 - j2h_0h_2 - k2h_0h_3 + h_0^2 + h_1^2 + h_2^2 + h_3^2 \\ &= 0. \end{aligned}$$

Thus we can write

$$z^2 = 2\mathcal{R}(z)z - abs(z)^2. \tag{3.14}$$

We can now begin the proof, which is done by induction. The case $n = 0$ is satisfied by definition of $z^0 = 1$. Thus suppose it is also true for $n = k$. For $n = k + 1$ we have

$$\begin{aligned} z^{k+1} &= zz^k = z(\alpha_k z + \beta_k) \\ &= \alpha_k z^2 + \beta_k z \\ &\underbrace{=}_{\text{Eq. } 3.14} \alpha_k(2\mathcal{R}(z)z - abs(z)^2) + \beta_k z \\ &= \underbrace{(2\mathcal{R}(z)\alpha_k + \beta_k)}_{=\alpha_{k+1}} z + \underbrace{(-abs(z)\alpha_k)}_{=\beta_{k+1}} \\ &= \alpha_{k+1}z + \beta_{k+1}. \end{aligned}$$

□

Since the Quartenions are being considered, we don't need to differentiate the equivalent class case to the quasi-equivalent class case since both are the same by Lemma 3.1.3.

Corollary 3.2.1. *Let $x \sim y$. Then, for any $n \in \mathbb{N}$, x and y have the same coefficients α_n and β_n defined by Equation 3.13b.*

Proof. By Lemma 3.2.1, $x \sim y$ means $\mathcal{R}(x) = \mathcal{R}(y)$ and $abs(x) = abs(y)$. Thus, by Theorem 3.2.1, x and y have the same coefficients α_n and β_n for all $n \in \mathbb{N}$. ☐

Corollary 3.2.2. *Let $p \in \mathbb{H}[x]$ be a simple polynomial with degree n. Then we can write $p(z)$, with $z \in \mathbb{H}$, as a linear polinomial function with real coefficients, i.e.,*

$$p(z) = Az + B, \tag{3.15}$$

with $A, B \in \mathbb{H}$.

Proof. Let

$$p(z) = \sum_{k=0}^{n} a_k z^k$$

be a simple quartenionic polynomial. Theorem 3.2.1 gives us

$$\begin{aligned} p(z) &= \sum_{k=0}^{n} a_k(\alpha_k z + \beta_k) \\ &= \underbrace{(\sum_{k=0}^{n} a_k \alpha_k)}_{=A(z)} z + \underbrace{(\sum_{k=0}^{n} a_k \beta_k)}_{=B(z)} \\ &= A(z)z + B(z). \end{aligned}$$

☐

Corollary 3.2.2 is interesting: given any quaternionic polynomial p of degree n, we can reduce it into a linear polynomial by using the recursive iteration presented in Theorem 3.2.1. This is surprising since, for example, in the real case, this doesn't happen.

Theorem 3.2.2. *Let $z_0 \in \mathbb{H}$ be fixed and p be a simple polynomial over the quartenions. Then both $A(z)$ and $B(z)$ are constants for all $z \in [z_0]$. Moreover, if z_0 is a non-real zero of p, then the quantities $A(z_0)$ and $B(z_0)$ can vanish only simultaneously. Finally, if $A(z_0) = 0$, then all elements of its equivalence class are also zeros of p; otherwise, z_0 is the only zero of p in $[z_0]$.*

Proof. The first statment of $A(z)$ and $B(z)$ being constant for all $z \in [z_0]$ is immediate from Corollaries 3.2.1 and 3.2.2. Next suppose z_0 is a root of p. This means

$$p(z_0) = Az_0 + B = 0 \Rightarrow Az_0 = -B,$$

where $A = A(z_0)$ and $B = B(z_0)$. Thus $B = 0$ means $A = 0$ since we're assuming $z_0 \neq 0$ (see comment just after Definition 3.2.1). On the other hand, if $A = 0$, then trivially $B = 0$, thus A and B can only vanish simutaneously.

Finally, suppose $z \in [z_0]$. For the case $A(z) = A = 0$, we also have $B(z) = B = 0$ and

$$p(z) = 0z + 0 = 0,$$

i.e., z is a zero of p. For the case $A \neq 0$, then $B \neq 0$ and

$$z_0 = A^{-1}B,$$

is the only possible solution, thus only z_0 is a zero of p in $[z_0]$. □

Example 3.2.2. *Let's go back to the polynomial in Example 3.2.1 and use the results we have so far. By definition of A and B given in Corollary 3.2.2's proof, we just need to calculate α_k and β_k up to $n = 2$:*

$$\begin{aligned} \alpha_0 &= 0, \beta_0 = 1, \\ \alpha_1 &= 1, \beta_1 = 0, \\ \alpha_2 = 2z_0, \beta_2 &= -abs(z)^2 = z_0^2 + z_1^2 + z_2^2 + z_3^2, \end{aligned}$$

where $z_0 = \mathcal{R}(z)$. This gives us

$$A = 2z_0, B = 1 - (z_0^2 + z_1^2 + z_2^2 + z_3^2).$$

Suppose $A \neq 0$. Then the equation $p(z) = Az + B = 0$ gives us

$$\begin{cases} 2z_0^2 & = z_0^2 + z_1^2 + z_2^2 + z_3^2 - 1 \\ z_1 = z_2 = z_3 = 0. \end{cases}$$

Meaning

$$2z_0^2 = z_0^2 - 1 \Rightarrow z_0^2 = -1,$$

which is impossible since $z_0 \in \mathbb{R}$ by hypothesis. Therefore we must have $A = 0$, giving us $z_0 = 0$. Moreover, by Theorem 3.2.2, we know both A and B can only vanish simultaneosly, thus

$$z_1^2 + z_2^2 + z_3^2 = 1.$$

The pair of zeros $\{i, -i\}$, $\{j, -j\}$ and $\{k, -k\}$ happen when $z_2 = z_3 = 0$, $z_1 = z_3 = 0$ and $z_1 = z_2 = 0$ respectively. Additionally, notice that all this zeros satisfy Equation 3.7, therefore they are equivalent (Lemma 3.1.3).

Corollary 3.2.3. *Let p be a simple quartenionic polynomial.*

(i) If $z_0, z_1 \in \mathbb{H}$ are two different zeros of p with $z_1 \in [z_0]$, then all elements of $[z_0]$ are zeros of p.

(ii) Suppose $z_0 \in \mathbb{H} - \mathbb{R}$ is a zero of p. If $A(z_0) \neq 0$, then z_0 is the only zero of p in the equivalence classe $[z_0]$.

Proof. Theorem 3.2.2 tell us that $A(z) = A$ and $B(z) = B$ are constant for all $z \in [z_0]$, thus

$$\begin{aligned} p(z_0) = Az_0 + B = 0 &= Az_1 + B = p(z_1) \\ A(z_0 - z_1) = 0 &\Rightarrow A = 0, \end{aligned}$$

since $z_0 \neq z_1$ by hypothesis. By Theorem 3.2.2, we must have $B = 0$, therefore all $z \in [z_0]$ are zeros of p, which proves the first item.

The second case is an immediate consequence of Theorem 3.2.2. □

Example 3.1.3 and Corollary 3.2.3 motivates the following classification for the zeros of a quaternionic polynomial.

Definition 3.2.2. *Let $z_0 \in \mathbb{H}$ be a zero of the one-sided quaternionic polynomial p. Then:*

*(i) if $z_0 \in \mathbb{R}$, or if $z_0 \in \mathbb{H} - \mathbb{R}$ is such that $A(z_0) \neq 0$, then z_0 is said to be an **isolated zero** of p.*

*(ii) if $z_0 \in \mathbb{H} - \mathbb{R}$ is such that $A(z_0) = 0$, then z_0 is said to be an **spherical zero** of p.*

The name *spherical* comes from the following reasoning: suppose $x = x_0 + ix_1, z \in \mathbb{H}$ such that $x \sim z$. This means $x_0 = z_0$ and

$$abs(x)^2 = abs(z)^2 \Rightarrow x_1^2 = z_1^2 + z_2^2 + z_3^2,$$

which is the equation of a sphere.

As a closing remark for this section, one can calculate all of the zeros from the polynomial $p = z^2 - c$, with $c \in \mathbb{H}$, by using everything presented in this section. Although this polynomial doesn't seem particularly difficult, the amount of cases that need to be analyzed while calculating the zeros show that, even for more simple cases, finding the zeros is no easy task. The interested reader can check Reference [28] for this and more complex examples.

3.3 One-sided Polynomials over $\mathcal{A}$

Here we shall denote by $\mathcal{A}$ one of the three noncommutative algebras of $\mathbb{R}^4$ different from the Quartenions, namely the Coquartenions, the Nectarines and the Conectarines.

Definition 3.3.1. *We define a **one-sided (or simple) polynomial over the algebra $\mathcal{A}$** as*

$$p(z) = \sum_{k=0}^{n} a_k z^k, \tag{3.16}$$

where $a_k, z \in \mathcal{A}$ for all $0 \leq k \leq n$, and a_0 and a_n are invertibles in $\mathcal{A}$.

Equivalently to the Quartenions, such algebras also verify the Iteration Process of Theorem 3.2.1.

Theorem 3.3.1 (Iteration Process - General case)**.** *Let $z \in \mathcal{A}$. Then, for any $n \in \mathbb{N}$, the following two term iteration process is valid:*

$$z^n = \alpha_n z + \beta_n, \tag{3.17a}$$

where $\alpha, \beta \in \mathbb{R}$ and

$$\begin{cases} \alpha_0 = 0,\ \beta_0 = 1, \\ \alpha_n = 2\mathcal{R}(z)\alpha_{n-1} + \beta_{n-1},\ \beta_n = -abs(z)^2\alpha_{n-1} \text{ for } n \geq 1. \end{cases} \tag{3.17b}$$

Proof. For any noncommutative algebra $\mathcal{A}$, the relation

$$h^2 - 2\mathcal{R}(h)z + abs(h) = 0$$

is still valid for any $h \in \mathcal{A}$. To see this, we first define

$$\begin{cases} i^2 &= \epsilon_i \\ j^2 &= \epsilon_j \\ k^2 &= \epsilon_k, \end{cases} \tag{3.18}$$

where ϵ is either $+1$ or -1 depending of the algebra $\mathcal{A}$ in question (see Table 3.4). Keeping in mind that $\mathcal{A}$ is (up to isomorphism) one of the Clifford Algebras, we know that $\{i, j, k\}$ anticommute (see Equation 2.10, or directly from the set of Equations 3.1), therefore

$$\begin{aligned} h^2 - ph + q = {} & (h_0 + ih_1 + jh_2 + kh_3)^2 - 2h_0(h_0 + ih_1 + jh_2 + kh_3) \\ & + h_0^2 - \epsilon_i h_1^2 - \epsilon_j h_2^2 - \epsilon_k h_3^2 \\ = {} & h_0^2 + \epsilon_i h_1^2 + \epsilon_j h_2^2 + \epsilon_k h_3^2 \\ & + i(h_0h_1 + h_1h_0) + \underbrace{ijh_1h_2 + jih_2h_1}_{=0} \\ & + j(h_0h_2 + h_2h_0) + \underbrace{jkh_2h_3 + kjh_3h_2}_{=0} \\ & + k(h_0h_3 + h_3h_0) + \underbrace{ikh_1h_3 + kih_3h_1}_{=0} \\ & - 2h_0^2 - i2h_0h_1 - j2h_0h_2 - k2h_0h_3 \\ & + h_0^2 - \epsilon_i h_1^2 - \epsilon_j h_2^2 - \epsilon_k h_3^2 \\ = {} & 0. \end{aligned}$$

Thus the itereation is valid and the rest of the demonstration follows verbatim from Theorem 3.2.1's proof. ☐

Corollary 3.3.1. *Let p be a polynomial with degree n over $\mathcal{A}$. Then we can write $p(z)$, with $z \in \mathcal{A}$, as a linear polinomial function, i.e.,*

$$p(z) = Az + B, \tag{3.19}$$

with $A, B \in \mathcal{A}$.

Proof. Verbatim from Corollary 3.2.2's proof. ☐

It's worth reminding both A and B depend of the element $z \in \mathcal{A}$, which is explicitly stated by writing $A = A(z)$ and $B = B(z)$.

Corollary 3.3.2. *Let $x \sim_q y$. Then, for any $n \in \mathbb{N}$, x and y have the same coefficients α_n and β_n defined by Equation 3.13b.*

Proof. By definition, $x \sim_q y$ means $\mathcal{R}(x) = \mathcal{R}(y)$ and $abs(x) = abs(y)$. Thus, by Theorem 3.3.1, x and y have the same coefficients α_n and β_n for all $n \in \mathbb{N}$. ☐

Corollary 3.3.3. *Let $z_0 \in \mathcal{A}$ be fixed and p be a polynomial over the algebra $\mathcal{A}$. Then both $A(z)$ and $B(z)$ are constant for all $z \in [z_0]_q$.*

Proof. Immediate from Corollary 3.3.2. ☐

The following results start to shape what kind of properties we need to classify the zeros of p. Although they may look identical to the quaternionic case in a first glance, they will contain some fundamental differences with the previous case.

Theorem 3.3.2. *Let A be invertible on the given class $[z]_q$ and let $[z]_q$ contain a zero z_0 of p. Then*

$$z_0 = A(z_0)^{-1}B(z_0) \tag{3.20}$$

is the only zero in $[z]_q$. If $A(z_0) = B(z_0) = 0$, then all elements in $[z]_q$ are zeros of p.

Proof. First suppose the case where $A(z_0) = A$ is invertible. Then, for $B(z_0) = B$,

$$z_0 = A^{-1}B$$

is a zero of p. To show its uniqueness in $[z]_q$, suppose $z_1 \in [z]_q$ is also a zero of p. By Corollary 3.3.3, $A(z_1) = A$ and $B(z_1) = B$, therefore

$$Az_0 = -B = Az_1 \Rightarrow A(z_0 - z_1) = 0 \Rightarrow z_0 = z_1.$$

Lastly, suppose $A(z_0) = B(z_0) = 0$. By Corollary 3.3.3, $A(z_1) = B(z_1) = 0$ for all $z_1 \in [z]$, thus

$$p(z_1) = 0.$$

☐

Compare the previous theorem with Theorem 3.2.2, its quaternionic counterpart. Even though they are almost identical (both stablish if A is invertible, then z_0 is the only zero in its quasi-equivalence class, and both state if $A = B = 0$, then all elements in z_0 are zeros), the latter present to us A and B can only vanish simultaneously, while the former makes no mention of this being the case. This result shouldn't be surprising since the Coquartenions, Nectarines and Conectarines have *zero divisors*, that is, we can find $x, y \in \mathcal{A} - \{0\}$ such that $xy = 0$. For example, let $\mathcal{A} = \mathbb{H}_{coq}$. Then

$$x = i + j \Rightarrow x^2 = \underbrace{i^2}_{=-1} + \underbrace{ij + ji}_{=0} + \underbrace{j^2}_{=1} = 0.$$

Thus we can conclude Theorem 3.2.2 is a stronger case of Theorem 3.3.2.

Whilst for the algebras $\mathcal{A}$ the values A and B may now vanish simultaneously, we still have a weaker result, which is trivially seen from $Az_0 + B = 0$.

Corollary 3.3.4. *Let $z_0 \in \mathcal{A}$ be a zero of p. If $A(z_0) = 0$, then $B(z_0) = 0$.*

Such fundamental difference between the algebras $\mathcal{A}$ and $\mathbb{H}$ gives rise to a new type of zero.

Theorem 3.3.3. *Let $A \neq 0$, but let A be noninvertible on the given class $[z]_q$, and let $[z]_q$ contain a zero of p. Assume that there is a real constant γ such that*

$$\gamma A + B = 0.$$

Then, for all real α, the quantity

$$z_0 = \alpha \bar{A} + \gamma \tag{3.21}$$

is a zero of p, provided that $z_0 \in [z]_q$.

Proof. First remember that A being noninvertible in $\mathcal{A}$ means that

$$\bar{A}A = A\bar{A} = 0$$

(see Corollary 3.1.1). Thus,

$$\begin{aligned} p(z_0) &= Az_0 + B = A(\alpha\bar{A} + \gamma) + B \\ &= \alpha\underbrace{A\bar{A}}_{=0} + \underbrace{\gamma A + B}_{=0} = 0 \end{aligned}$$

□

Observe the fact that Theorem 3.3.3 doesn't make sense if we were considering $\mathcal{A} = \mathbb{H}$: if $A \neq 0$, then its inverse exists and is $A^{-1} = abs(A)^{-1}\overline{A}$.

Corollary 3.3.5. *Suppose Theorem 3.3.3 is valid and, in addition, $\mathcal{R}(A) \neq 0$ is also observed. Then there is at most one α which defines a zero z_0 which is contained in the quasi similarity class $[z]_q$.*

Proof. The vaule of A is invariant for all elements in $[z]_q$ (Corollary 3.3.3), therefore so is $\mathcal{R}(A) = \mathcal{R}(\bar{A})$. Thus

$$\mathcal{R}(z_0) = \mathcal{R}(\alpha\bar{A} + \gamma) = \alpha\mathcal{R}(\bar{A}) + \gamma = \alpha\mathcal{R}(A) + \gamma$$

tells us that $\mathcal{R}(z_0)$ is fixed if, and only if, α is fixed (γ is fixed by hypothesis). This means that there is at most one value of α which allows z_0 to be quasi similar to z, otherwise $A = A(z_0)$ would vary, a contradiction. □

It's worth pointing out how both Theorem 3.3.3 and Corollary 3.3.5 don't tell us if a given quasi-similar class $[z]_q$ has a zero of the form given by Equation 3.21, they only state that, if such a zero is found in $[z]_q$, and if $\mathcal{R}(A) \neq 0$, then the value of α is unique. Moreover, they don't tell us if there exists another zero of p in $[z]_q$ besides z_0.

With all these results at hand, we can classify the zeros of simple polynomials over $\mathcal{A}$.

Definition 3.3.2. *Let $z_0 \in \mathcal{A}$ be a zero of the one-sided polynomial p over $\mathcal{A}$. Then:*

*(i) if $z_0 \in \mathcal{A}$ is such that $A(z_0)$ is invertible, or if z_0 is the only zero of p in its quasi-similar class, then z_0 is said to be an **isolated zero** of p.*

*(ii) if $z_0 \in \mathcal{A}$ is such that $A(z_0) = B(z_0) = 0$, or if all elements of $[z_0]_q$ are zeros of p, then z_0 is said to be an **hyperbolic zero** of p.*

*(iii) if z_0 is computed by Equation 3.21, then z_0 is said to be an **unexpected zero** of p.*

This convention of names can be found in [22].

The name *hyperbolic* comes from the following reasoning: suppose $x = x_0 + ix_1, z \in \mathcal{A}$ is such that $x \sim_q z$. This means $x_0 = z_0$ and $abs(x)^2 = abs(z)^2$ results in

$$x_1^2 = \begin{cases} +z_1^2 - z_2^2 - z_3^2, \text{ for } \mathcal{A} = \mathbb{H}_{coq}, \\ -z_1^2 + z_2^2 - z_3^2, \text{ for } \mathcal{A} = \mathbb{H}_{nec}, \\ -z_1^2 - z_2^2 + z_3^2, \text{ for } \mathcal{A} = \mathbb{H}_{con}. \end{cases}$$

which all are equations of a hyperbole.

One observation must be made: similarity classes $[z]$ either contain infinitely many elements in case $[z]$ does not contain real elements or $[z]$ consist of a single element, which is possible only if $z \in \mathbb{R}$: suppose $[z] = \{z\}$. Then, for any invertible $h \in \mathcal{A}$,

$$z \sim z \Rightarrow z = h^{-1}zh \Rightarrow hz = zh,$$

which is true only if $z \in \mathbb{R}$ since $\mathcal{A}$ is always noncommutative. Meanwhile, quasi-similarity classes in $\mathcal{A}$ can not contain only one element: take $z = z_0 + iz_1 + jz_2 + kz_3 \in \mathbb{H}_{coq}$. In this case, $z' = z_0 + iabs(z) + jz_0$ has the same real part of z and

$$abs(z')^2 = z_0^2 - \underbrace{i^2}_{=-1} abs(z)^2 - \underbrace{j^2}_{=1} z_0^2 = abs(z)^2,$$

thus $z \sim_q z'$. Similarly, for the Nectarines we have $z \sim_q z' = \mathcal{R}(z) + i\mathcal{R}(z) + j\ abs(z)$, whilst for the Conectarines, $z \sim_q z' = \mathcal{R}(z) + i\mathcal{R}(z) + k\ abs(z)$.

3.4 Two-sided Quaternionic Polynomials

In this section we come back to the quaternionic algebra, but now we study what is called a two-sided polynomial.

Definition 3.4.1. *We define a* ***two-sided quaternionic polynomial*** *as*

$$p(z) = \sum_{k=0}^{n} a_k z^k b_k, \tag{3.22}$$

where $a_k, b_k z \in \mathbb{H}$ *for all* $0 \leq k \leq n$, $a_0 b_0 \neq 0$ *and* $a_n b_n \neq 0$.

Since the Quartenions is noncommutative, we can not rewrite Equation 3.22 as

$$p(z) = \sum_{k=0}^{n} a_k b_k z^k,$$

but later we'll see a method to permutate z with b_k. In order to do so, we need to introduce the column operator.

Definition 3.4.2. *Let* $z = z_0 + iz_1 + jz_2 + kz_3 \in \mathbb{H}$. *Then we define the* ***column operator*** *as*

$$\begin{aligned} col : \mathbb{H} &\to \mathbb{R}^{4\times 1} \\ z &\mapsto col(z), \end{aligned}$$

where $\mathbb{R}^{4\times 1}$ *is the set of real matrices with* 4 *lines and* 1 *column, and*

$$col(z) = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix}. \tag{3.23}$$

The following result is an immediate consequence from the column operator's definition.

Lemma 3.4.1. *The column operator is a linear transformation between* $\mathbb{H}$ *and* $\mathbb{R}^{4\times 1}$ *over* $\mathbb{R}$*.*

In the paragraph just before Example 2.2.3, we observed the algebra of Quaternions could be written as an subalgebra of the algebra $Mat(4, \mathbb{R})$. What follows is the linear transformation that makes such subalgebra possible.

Definition 3.4.3. *Let* $z = z_0 + iz_1 + jz_2 + kz_3 \in \mathbb{H}$*. We define* $\omega_1 : \mathbb{H} \to Mat(4, \mathbb{R})$ *as*

$$\omega_1(z) = \begin{bmatrix} z_0 & -z_1 & -z_2 & -z_3 \\ z_1 & z_0 & -z_3 & z_2 \\ z_2 & z_3 & z_0 & -z_1 \\ z_3 & -z_2 & z_1 & z_0 \end{bmatrix}. \tag{3.24}$$

Theorem 3.4.1. *The function* ω_1 *in Definition 3.4.3 is an injective linear mapping between the real vector spaces* $\mathbb{H}$ *and* $Mat(4, \mathbb{R})$*. Moreover it preserves multiplication, that is,*

$$\omega_1(xy) = \omega_1(x)\omega_1(y) \tag{3.25}$$

for any $x, y \in \mathbb{H}$*.*

Proof. First we prove injectiviness. Suppose $x = y$. This means $x_i = y_i$ for all $0 \leq i \leq 3$, thus $\omega_1(x) = \omega_1(y)$ is observed. Second, linearity, which can be seen from the difinition of ω_1: for any $x, y \in \mathbb{H}$, and any $\alpha \in \mathbb{R}$,

$$\begin{aligned} \omega_1(\alpha x + y) &= \begin{bmatrix} \alpha x_0 + y_0 & -(\alpha x_1 + y_1) & -(\alpha x_2 + y_2 & -(\alpha x_3 + y_3) \\ \alpha x_1 + y_1 & -(\alpha x_0 + y_0) & -(\alpha x_3 + y_3 & -(\alpha x_2 + y_2) \\ \alpha x_2 + y_2 & \alpha x_3 + y_3 & \alpha x_0 + y_0 & -(\alpha x_1 + y_1) \\ \alpha x_3 + y_3 & -(\alpha x_2 + y_2) & \alpha x_1 + y_1 & \alpha x_0 + y_0 \end{bmatrix} \\ &= \alpha \begin{bmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & -x_0 & -x_3 & -x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{bmatrix} + \begin{bmatrix} y_0 & -y_1 & -y_2 & -y_3 \\ y_1 & -y_0 & -y_3 & -y_2 \\ y_2 & y_3 & y_0 & -y_1 \\ y_3 & -y_2 & y_1 & y_0 \end{bmatrix} \\ &= \alpha\omega_1(x) + \omega_1(y) \end{aligned}$$

Lastly, by using Equation 3.1a, we can write

$$\omega_1(xy) =$$

$$= \begin{bmatrix} x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 & -(x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2) & -(x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1) & -(x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0) \\ x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2 & x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 & -(x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0) & x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1 \\ x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1 & x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0 & x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 & -(x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2) \\ x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0 & -(x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1) & x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2 & x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & -x_0 & -x_3 & -x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{bmatrix} \begin{bmatrix} y_0 & -y_1 & -y_2 & -y_3 \\ y_1 & -y_0 & -y_3 & -y_2 \\ y_2 & y_3 & y_0 & -y_1 \\ y_3 & -y_2 & y_1 & y_0 \end{bmatrix} = \omega_1(x)\omega_1(y)$$

□

Theorem 3.4.1 show us $Im(\omega_1)$ is isomorphic to $\mathbb{H}$, not only as a vector space, but as an algebra. This subalgebra of $Mat(4, \mathbb{R})$ is here denoted $\mathbb{H}_{\mathbb{R}}$ [29].

Another important linear mapping is ω_2, which was first introduced by L. I. Aramanovich [30].

Definition 3.4.4. *Let $z = z_0 + iz_1 + jz_2 + kz_3 \in \mathbb{H}$. We define $\omega_2 : \mathbb{H} \to Mat(4, \mathbb{R})$ as*

$$\omega_2(z) = \begin{bmatrix} z_0 & -z_1 & -z_2 & -z_3 \\ z_1 & z_0 & z_3 & -z_2 \\ z_2 & -z_3 & z_0 & z_1 \\ z_3 & z_2 & -z_1 & z_0 \end{bmatrix}. \tag{3.26}$$

Theorem 3.4.2. *The function ω_2 in Definition 3.4.4 is an injective linear mapping between the real vector spaces $\mathbb{H}$ and $Mat(4, \mathbb{R})$. Moreover it has the following property:*

$$\omega_2(xy) = \omega_2(y)\omega_2(x) \tag{3.27}$$

for any $x, y \in \mathbb{H}$.

Proof. Identical to Theorem 3.4.1's proof. ☐

Corollary 3.4.1. *Let $x, y, z \in \mathbb{H}$ be arbitrary. Then the following properties are observed:*

$$col(xy) = \omega_1(x)col(y) = \omega_2(y)col(x), \tag{3.28a}$$

$$col(xyz) = \omega_1(x)\omega_2(z)col(y). \tag{3.28b}$$

Proof. Immediate from Definitions 3.4.3 and 3.4.4. ☐

Equation 3.28b shows us the product between ω_1 and ω_2. Such product plays an important role in the theory to be presented, therefore it seems fitting to define ω_3 as this product:

$$\omega_3(x, y) = \omega_1(x)\omega_2(y). \tag{3.29}$$

All these three mappings have other properties, for instance, ω_3 is normal and orthogonal. Unfortunally such properties are not going to be used or explored here in more depth. The interested reader can check [23] [24].

Similar to Corollaries 3.2.2 and 3.3.1, two-sided quaternionic polynomials have the next corollary.

Corollary 3.4.2. *Let p be a two-sided polynomial over the quartenions as stablished in Definition 3.4.1. Then we can write $col(p(z)) \in \mathbb{R}^4$ as*

$$col(p(z)) = A(z)col(z) + col(B(z)), \tag{3.30}$$

where $B(z) \in \mathbb{H}$ and $A(z) \in Mat(4, \mathbb{R})$.

Proof. By using the Theorem 3.2.1, we can write

$$\begin{aligned} p(z) = \sum_{k=0}^{n} a_k z^k b_k &= \sum_{k=0}^{n} a_k(\alpha_k z + \beta_k) b_k \\ &= \sum_{k=0}^{n} \alpha_k a_k z b_k + \sum_{k=0}^{n} \beta_k a_k b_k. \end{aligned}$$

So, by applying the column operator,

$$\begin{aligned} col(p(z)) &= \sum_{k=0}^{n} \alpha_k \underbrace{col(a_k z b_k)}_{=\omega_3(a_k,b_k)col(z)} + \sum_{k=0}^{n} \beta_k col(a_k b_k) \\ &= \underbrace{(\sum_{k=0}^{n} \alpha_k \omega_3(a_k,b_k)}_{=A(z)})col(z) + \underbrace{\sum_{k=0}^{n} \beta_k col(a_k b_k)}_{=B(z)} \\ &= A(z)col(z) + B(z), \end{aligned}$$

where we used Corollary 3.4.1. □

Similar to Theorems 3.2.2 and 3.3.2, the following theorem is observed.

Theorem 3.4.3. *Let $z_0 \in \mathbb{H}$ be fixed and p be a two-sided polynomial over the quartenions. Then both $A(z)$ and $B(z)$ are constants for all $z \in [z_0]$. Moreover, if z_0 is a non-real zero of p, then $A(z_0)$ is singular (non-invertible) whenever $B(z_0) = 0$.*

Proof. Corollary 3.2.1 inform us α_k and β_k invariant for all $z \in [z_0]$, thus, by their definition, $A(z)$ and $B(z)$ are also invariant. Additionally, if z_0 is a zero of p, then

$$col(p(z_0)) = A(z_0)col(z_0) + B(z_0) = 0 \Rightarrow A(z_0)col(z_0) = -B(z_0).$$

Finally, suppose $B(z_0) = 0$. Then z_0 is an element of $A(z_0)$'s kernel, meaning $A(z_0)$ is singular. □

And finally, similar to the first and second items of Corollary 3.2.3, the two-sided quaternionic polynomial case also satisfies the next two assertions.

Corollary 3.4.3. *Let $z_0, z_1 \in \mathbb{H}$ be two distinct, but equivalent, zeros of the two-sided quaternionic polynomial p. Then $A(z_0)$ is singular.*

Proof. Theorem 3.4.3 inform us $A(z_0) = A(z_1)$ and $B(z_0) = B(z_1)$ since $z_0 \sim z_1$, therefore

$$col(p(z_0)) - col(p(z_1)) = 0 \Rightarrow A(z_0)col(z_0) - A(z_0)col(z_1) = A(z_0)col(z_0 - z_1) = 0.$$

Because we assumed $z_0 \neq z_1$, we can conclude $A(z_0)$ is singular. □

Corollary 3.4.4. *Let $z_0 \in [z]$ be a non-real zero of the two-sided quaternionic polynomial p. If $A(z_0)$ is invertible, then z_0 is the only zero of p in the equivalence class $[z]$.*

Proof. If $z_1 \in [z]$ is a distinc zero of p, then $A(z_0)$ would be singular by Corollary 3.4.3, contradicting the hypothesis of it being invertible. □

Corollaries 3.4.3 and 3.4.4 motivates the classification for the zeros of p.

Definition 3.4.5. *Let $z_0 \in \mathbb{H}$ be a zero of the two-sided quaternionic polynomial p. Then we say z_0 is a* ***zero of type*** *k, $0 \leq k \leq 4$, if*

$$rank(A(z_0)) = 4 - k. \tag{3.31}$$

In particular,

(i) if z_0 real or if it's of type 0*, then it's called an* ***isolated zero****;*

(ii) if z_0 is of type 4*, then it's called an* ***spherical zero****.*

The reason why the particular cases of rank equals 0 or 4 boils down to the simillarity with the one-sided case, which can be seen in the next corollary.

Corollary 3.4.5. *Let z_0 be a zero of the two-sided polynomial p.*

(i) If z_0 is a spherical zero, then z is also a zero of p for all $z \in [z_0]$.

(ii) If z_0 is an isolated zero, then it is the only zero in $[z_0]$.

Proof. For the first case, z_0 being spherical means $rank(A(z_0)) = 0$, i.e., $A(z_0) = 0$, the matrix with all entries equal to zero, thus

$$col(p(z_0)) = \underbrace{A(z_0)}_{=0} col(z_0) + B(z_0) = 0 \Rightarrow B(z_0) = 0.$$

Since $B(z) = B(z_0) = 0$ and $A(z) = A(z_0) = 0$ for all $z \in [z_0]$, we have

$$col(p(z)) = A(z)col(z) + B(z) = 0.$$

For the second case, $z_0 \in \mathbb{R}$ means $[z_0] = \{z_0\}$ as shown in Example 3.1.3. Otherwise, $rank(A(z_0)) = 4$ means the matrix is invertible, thus this is the statement of Corollary 3.4.4. □

Final Considerations

This work here presented had proposed two objectives: the classification of Clifford Algebras for finite-dimensional real vector spaces and the classification of one and two-sided polynomials over noncommutative algebras of $\mathbb{R}^4$. And, as it was seen, a lot can be derived from their study.

For Clifford Algebras, one can make the classification of all the finite-dimensional complex vector spaces. The argument is similar to the real case whilst is utilized a method called *complexificaion of an algebra* (see References [4] and [16]). Additionally, as stated in Chapter 2, in the paragraph immediately after Equation 2.10, Clifford Algebras have a “tendency” to transform geometrical properties into algebraically ones. This can be seen between perpendicularity and anticommutativity (Equation 2.10), or in the Bogoliubov automorphism (Corollary 2.1.1). Moreover, although we know some Clifford Algebras are isomorphic to one another, we never explicitely write the isomorphism, the exception being the simpler cases with dimension 2 and 4. Depending on the isomorphism chosen, the Clifford Algebra may be associated to a group, like the groups Pin, Spin and the theory of Spinors. The curious reader can go to the References [2], [4], [5], [6] and [20].

Finally, for the subject of polynomials, a similar study can be made for the one-sided polynomials with coefficients over the four commutative algebras of $\mathbb{R}^4$, namely the Tessarines, Cotessarines, Tangerines and Cotangerines. Moreover, there exsit some algorithms showing how to find a zero inside a class of quasi-equivalence, as well as the relation of the polynomial with its called *companion*, and how they relate with the solution of linear systems. Although not presented here, there number of zeros a polynomial can have is always finite, be it one-sided or two-sided. All of this can be seen in References [22], [23], [24], [25], [26], [27], [28], [29] and [30].

Bibliography

[1] Profesor Clifford. “Applications of Grassmann’s Extensive Algebra’”. In: *American Journal of Mathematics* 1.4 (1878), pgs.350–358.

[2] Ian Porteous. *Clifford Algebras and the Classical Groups.* 1st ed. Cambridge University Press, 1995.

[3] D. J. H. Garling. *Clifford Algebras: An Introduction.* 1st ed. Cambridge University Press, 2011.

[4] Jayme Vaz Jr., Roldão da Rocha Jr. *An Introduction to Clifford Algebras and Spinors.* 1st ed. Oxford University Press, 2016.

[5] Pertti Lounesto. *Clifford Algebra and Spinors.* 2nd ed. Cambridge University Press, 2001.

[6] Claude C. Chevalley. *The Algebraic Theory of Spinors.* 1st ed. Columbia University Press, 1954.

[7] Paul Dirac. “The Quantum Theory of the Electron”. In: *Proceedings of the Royal Society* 117(778) (1928), pgs.610–624. DOI: `10.1098/rspa.1928.0023`.

[8] C. J. L. Doran, A. N. Lasenby. *Geometric Algebra for Physicists.* Cambridge University Press, 2003.

[9] E. Arcaute, A. Lasenby, C. J. L. Doran. “Twistors in Geometric Algebra”. In: *Adv. Appl. Clifford Algebr.*18 (2008), pgs.373–394.

[10] Kanta Tachibana, Eckhard Hitzer, Stéphane Breuils. “New Applications of Clifford’s Geometric Algebra”. In: *Advances in Applied Clifford Algebras* 32.1 (2022), pgs.17. DOI: `{10.1007/s00006-021-01196-7}`.

[11] Mukundan, R. *Quaternions: From Classical Mechanics to Computer Graphics, and Beyond.* Department of Computer Science, University of Canterbury, Christchurch, New Zealand.

[12] Girard, P. R. “The Quaternion Group and Modern Physics”. In: *Eur. J. Phys.* 5.25 (1984).

[13] Saunders Mac Lane, Garret Birkhoff. *Algebra.* 3rd ed. AMS Chelsea Publishing, 1999.

[14] Serge Lang. *Algebra.* Revised 3rd ed. Department of Mathematics, Yale University, New Haven, CT 96520, USA: Springer, 2005.

[15] Kenneth Hoffman, Ray Kunze. *Linear Algebra.* 2nd ed. Prentice-Hall, Inc., 1971.

[16] Ana Paula da Cunda Corrêa da Silva. *Tese de Mestrado: Álgebras de Clifford: uma construção alternativa.* UFSC, 1999.

[17] Kenneth Hoffman, Ray Kunze. *Linear Algebra.* 2nd ed. Chapter 5, Section 2. Prentice-Hall, Inc., 1971.

[18] Saunders Mac Lane, Garret Birkhoff. *Algebra.* 3rd ed. Chapter 3, Section 11. AMS Chelsea Publishing, 1999.

[19] Kenneth Hoffman, Ray Kunze. *Linear Algebra.* 2nd ed. Chapter 10, Section 5. Prentice-Hall, Inc., 1971.

[20] Roger Plymen, Paul Robinson. *Spinors in Hilbert Space.* 1st ed. Cambridge University Press, 1995.

[21] Saunders Mac Lane, Garret Birkhoff. *Algebra.* 3rd ed. Chapter 2, Section 9. AMS Chelsea Publishing, 1999.

[22] Falcão, M. I., Miranda, F., Severino, R., Soares, M. J. "The number of zeros of unilateral polynomials over coquaternions revisited". In: *Linear and Multilinear Algebra* 67.6 (2018), pgs.1231–1249. DOI: `10.1080/03081087.2018.1450828`.

[23] Janovská, D., Opfer, G. "The classification and the computation of the zeros of quaternionic, two-sided polynomials". In: *Numer. Math.* 115. (2010), pgs.81–100. DOI: `10.1007/s00211-009-0274-y`.

[24] Janovská, D., Opfer, G. "Linear Equations in Quaternionic Variables". In: *Mitt. Math. Ges. Hamburg* 27. (2008)., pgs.223–234.

[25] Janovská, D., Opfer, G. "Givens' Transformation Applied to Quaternion Valued Vectors". In: *BIT Numerical Mathematics* 43. (2003), pgs.991–1002. DOI: `10.1023/B:BITN.0000014561.58141.2c`.

[26] Janovská, D., Opfer, G. "A note on the Computation of all Zeros of Simple Quaternionic Polynomials". In: *SIAM Journal on Numerical Analysis* 48.1 (2010). http://www.jstor.org/stable/41062744, pgs.244–256.

[27] Pogorui, A., Shapiro, M. "On the Structure of the Set of Zeros of Quaternionic Polynomials". In: *Complex Variables, Theory and Application: An International Journal* 49.6 (2004), pgs.379–389. DOI: `10.1080/0278107042000220276`.

[28] Janovská, D., Opfer, G. "Zeros and Singular points for one-sided Coquaternionic Polynomials with an Extension to other $\mathbb{R}^4$ Algebras". In: *Electronic Transactions on Numerical Analysis* 41. (2014)., pgs.133–158.

[29] Praha Drahoslava Janovská, Hamburg Gerhard Opfer. "A Survey of some Recent Results on Clifford Algebras in $\mathbb{R}^4$". In: *Applications of Mathematics* 68.5 (2023), pgs.571–592.

[30] Aramanovitch, L.I. "Quaternion non-linear filter for estimation of rotating body attitude." In: *Math. Meth. Appl. Sci.* 18. (1995), pgs.1239–1255. DOI: `10.1002/mma.1670181504`.