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INSTITUTO DE MATEMÁTICA E ESTATÍSTICA
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JEFERSON ARLEY POVEDA CONTRERAS

On the classification of Ricci solitons and Yamabe solitons

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JEFERSON ARLEY POVEDA CONTRERAS

On the classification of Ricci solitons and Yamabe solitons

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Área de concentração: Geometria.

Orientador: Prof. Dr. Benedito Leandro Neto

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ATA DE DEFESA DE TESE

Ata nº **04** da sessão de Defesa de Tese de **Jeferson Arley Poveda Contreras,** que confere o título de Doutor em Matemática, **na área de concentração de Geometria.**

Ao vigésimo dia do mês de março do ano de dois mil e vinte e três, a partir das dez horas, via Web videoconferência, realizou-se a sessão pública de Defesa de Tese intitulada **"On the classification of Ricci solitons and Yamabe solitons."** Os trabalhos foram instalados pelo Orientador e Presidente da banca, Professor Doutor **Benedito Leandro Neto - IME/UFG** com a participação dos demais membros da Banca Examinadora: Professor Doutor **Marcelo Bezerra Barboza - IME/UFG -** membro titular interno, Professor Doutor **Ernani de Sousa Ribeiro Júnior - DMAT/UFC -** membro titular externo, Professora Doutora **Keti Tenenblat - MAT/UnB** membra titular externa e o Professor Doutor **João Paulo dos Santos - MAT/UnB**, membro titular externo. Durante a arguição os membros da banca **não fizeram** sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Tese, tendo sido o candidato **APROVADO** pelos seus membros. Proclamados os resultados pelo Professor Doutor **Benedito Leandro Neto - IME/UFG**, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos Membros da Banca Examinadora, ao vigésimo dia do mês de março do ano de dois mil e vinte e três.

TÍTULO SUGERIDO PELA BANCA

On the classification of Ricci solitons and Yamabe solitons

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JEFERSON ARLEY POVEDA CONTRERAS

On the classification of Ricci solitons and Yamabe solitons

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Prof. Dr. Benedito Leandro Neto
Instituto de Matemática e Estatística – UFG
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Jeferson Arley Poveda Contreras

Bacharel em Matemática pela Universidad Distrital Francisco José de Caldas (2015), mestre em Matemática pela UFG - Universidade Federal de Goiás (2018).

Dedico este trabalho a minha família e amigos .

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A geometria é uma atividade artística intelectual

Henri Poincaré,
La Science et lH́ypothèse.

Resumo

Contreras, Jeferson Poveda. **On the classification of Ricci solitons and Yamabe solitons**. Goiânia, 2023. 76p. Tese de Doutorado . Departamento de Matemática, Instituto de Matemática e Estatística, Universidade Federal de Goiás.

Neste trabalho, estudaremos as soluções autossimilares tanto do fluxo de Ricci quanto do fluxo de Yamabe. Essas soluções também são conhecidas como Ricci e Yamabe soliton, respectivamente. Inspirados na equação da divergência usada por Robinson em sua demonstração da unicidade dos buracos negros estáticos e na classificação de Brendle dos solitons estáveis de Ricci, faremos algumas caracterizações importantes desses solitons. Provamos que solitons de Yamabe gradientes quadridimensionais devem ter uma métrica de Yamabe, desde que uma condição assintótica seja válida. Inspirados na geometria do soliton de charuto, demonstramos que um soliton de Ricci gradiente estável é Ricci-flat com uma função potencial constante ou um quociente do produto soliton estável $N^{n-1} \times \mathbb{R}$, onde N^{n-1} é Ricci-flat, ou isométrico ao soliton de Bryant. No capítulo final, provamos alguns resultados de rigidez para solitons de Ricci contraídos e expandidos.

Palavras–chave

soliton de Ricci, soliton de Yamabe, steady, shrinking, expanding, pinching

Abstract

Contreras, Jeferson Poveda. **On the classification of Ricci solitons and Yamabe solitons**. Goiânia, 2023. 76p. PhD. Thesis . Departamento de Matemática, Instituto de Matemática e Estatística, Universidade Federal de Goiás.

In this work, we will study the self-similar solutions of both Ricci flow and Yamabe flow. These solutions are also known as Ricci and Yamabe soliton, respectively. Inspired by the divergence equation used by Robinson in his demonstration of the uniqueness of static black holes and by Brendle's classification of steady Ricci solitons, we will make some important characterizations of these solitons. We prove that four-dimensional gradient Yamabe solitons must have a Yamabe metric, provided that an asymptotic condition holds. Inspired by the geometry of the cigar soliton, we demonstrate that a gradient steady Ricci soliton is either Ricci flat with a constant potential function or a quotient of the product steady soliton $N^{n-1} \times \mathbb{R}$, where N^{n-1} is Ricci flat, or isometric to the Bryant soliton. In the final Chapter, we prove some rigidity results for shrinking and expanding Ricci solitons.

Keywords

Ricci soliton, Yamabe soliton, steady, shrinking, expanding, pinching.

Contents

Introduction

Natural generalizations of Einstein manifolds are Ricci solitons. They are the fixed points of the Ricci flow in the space of Riemannian metrics and correspond to self-similar Ricci flow solutions. Ricci solitons played an essential role in Perelman's proof of the Poincaré conjecture.

Definition. *An n-dimensional Riemannian manifold M^n with complete Riemannian metric g is a gradient Ricci soliton (M^n, g, f, ρ) if there is a smooth function f on M and a constant ρ such that*

$$Ric + \nabla^2 f = \rho g.$$

Here, $\nabla^2 f$ denotes the Hessian of f and Ric is the Ricci curvature of g. The function f is called a potential function. Also, if $\rho > 0$, $\rho < 0$ or $\rho = 0$, we have a shrinking, expanding, or steady gradient Ricci soliton, respectively.

A gradient Ricci soliton is an Einstein manifold when f is constant (trivial). Thus, Ricci solitons are natural extensions of Einstein metrics.

Hamilton showed that any 2-dimensional compact shrinking Ricci soliton must be Einstein [41]. In [45], the author proved that the three-dimensional compact shrinking Ricci soliton must be Einstein. Any compact steady and expanding Ricci solitons, as well as the compact shrinking Ricci solitons in dimensions two and three, must be trivial. However, in general, this is not true. In fact, $\mathbb{CP}^2\#(-\mathbb{CP}^2)$ is a compact non-Einstein shrinking Ricci soliton (cf. [5, 46]).

In [42], Hamilton conjectured that any compact gradient shrinking Ricci soliton with a positive curvature operator must be Einstein. In [13], the authors proved that a compact gradient shrinking Ricci soliton must be Einstein if it admits a Riemannian metric with a positive curvature operator and satisfies an integral inequality. Petersen and Wylie [61] proved the rigidity of compact shrinkers under a pinched integral condition. In [37], Fernández-López and García-Río gave an important classification for compact shrinking Ricci solitons in terms of the range of the potential function (Theorem 2.4). The works [8, 57] accurately classified the non-compact case under certain geometric hypothesis conditions.

A lot of progress on classifying shrinking Ricci Soliton has been made during the last few years. In [8], Cao and Chen provided a classification for complete Bach-flat gradient shrinking Ricci solitons. In [8, Theorem 1.4] was proved that if the D-tensor (cf. Equation 1-6) is identically zero, then a complete four-dimensional shrinking Ricci soliton is Einstein, or a finite quotient of $\mathbb{R}^4$ or $\mathbb{S}^3\times\mathbb{R}$. Moreover, considering $n\geq 5$ the shrinking soliton must be either Einstein, a finite quotient of the Gaussian shrinking soliton $\mathbb{R}^n$ or a finite quotient of $N^{n-1}\times\mathbb{R}$, where N^{n-1} is Einstein (see also [15]).

Robinson in [62] used an important technique to prove the uniqueness of static vacuum black holes, where the author provided a fundamental divergence formula for the static vacuum equations. In [1], Brendle, based on the work of Robinson, got a classification of steady Ricci solitons. Still, he did not closely follow the procedure of Robinson to construct the divergence formula for the steady Ricci soliton (cf. [48]). In [49], combining [1] and [62], the authors proved that an n-dimensional gradient Yamabe soliton must be a Riemannian manifold with constant scalar curvature.

In this work, we will build a divergence formula (Lemma 3.2) for the shrinking Ricci soliton following the ideas of Robinson and Brendle closely, so with the help of this formula, we classify a compact shrinking Ricci soliton under a natural condition over the scalar curvature.

If a Ricci soliton is trivial (i.e., Einstein), then the scalar curvature satisfies $R=n\rho$. The following result was also inspired by [38]. We will show the triviality of compact gradient shrinking Ricci soliton over a gap on the potential function. This gap on f can also be interpreted as a lower bound on the scalar curvature; see the pinched conditions assumed by Catino in [15].

Theorem. *4.1. Let (M^n,g,f,ρ) be a compact gradient shrinking Ricci soliton such that*

$$f(x)\leq\sqrt{\left(\frac{n+2}{4}-\frac{\delta}{4\rho}\right)^2+\frac{n\delta}{4\rho}}-\left(\frac{n+2}{4}-\frac{\delta}{4\rho}\right),$$

where $\delta=\min_M(R)$ is a constant. Then, (M^n,g,f) is Einstein.

For any trivial (i.e., Einstein) shrinking Ricci soliton with the constant potential function, we have $\rho=\frac{R}{n}$. Thus, if (M^n,g,f,ρ) is a compact gradient shrinking Ricci soliton such that $n\rho=\min_M R$ the condition over f in Theorem 4.1 is equivalent to

$$f(x)\leq\frac{1}{2}(\sqrt{n^2+1}-1).$$

Hence, the condition over f in Theorem 4.1 is trivially satisfied for an Einstein manifold with, for instance, $f=0$. Also, the soliton $\mathbb{CP}^n\#(-\mathbb{CP}^n)$ is not a counter-example for this theorem.

In [53, Theorem 6.1], the authors proved an upper bound for the diameter of a compact shrinking Ricci soliton. This bound depends only on the dimension and the injectivity radius of the manifold. Moreover, in [38], they proved that a compact shrinking Ricci soliton has a diameter bounded below by a constant depending on the geometry of the manifold.

It is important to highlight that the scalar curvature R on a shrinking Ricci soliton is nonnegative and

$$\frac{1}{4}(r(x)-c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x)+c_2)^2,$$

where $r(x)=d(x_0,x)$ is the distance function from some fixed point $x_0\in M$, and c_1 and c_2 are positive constants depending only on n and the geometry of g on the unit ball $B_{x_0}(1)$ (cf. [12]).

For expanding Ricci solitons, we have that on a complete non-compact gradient expanding soliton with nonnegative Ricci curvature (or pinched Ricci curvature), the potential function f satisfies the estimates

$$\frac{1}{4}(r(x)-c_1)^2-c_2 \leq -f(x) \leq \frac{1}{4}(r(x)+2\sqrt{-f(x_0)})^2.$$

Here, $c_1>0$ and $c_2>0$ are constants [11]. Of course, the normalizing of f and the coefficients in the above estimates have to be adjusted accordingly.

Now we prove a rigidity result for complete (non-compact) shrinking Ricci solitons assuming the curvature is pinched, and an asymptotic condition on the gradient of the scalar curvature at infinity holds. Rigidity results for shrinking Ricci solitons assuming pinched conditions was proved by Catino in [15, 16].

Theorem. *4.2. Let (M^n,g,f,ρ) be a complete gradient shrinking Ricci soliton such that*

$$|\nabla R| = o(r^{-n-2}); \quad r\to\infty,$$

and one of the following conditions hold:

(I) $Ric \leq \dfrac{R\left(\frac{2R}{n-1}-\rho\right)}{\left(2\rho(1+f)-\frac{n-3}{n-1}R\right)}.$

(II)

$$\begin{aligned}|\nabla R| &\leq \frac{(n-1)}{(3n-2)}\left[|\nabla f|\sqrt{\left(2\rho(1+f)-\frac{(n-3)}{(n-1)}R\right)^2+\frac{4(3n-2)}{(n-1)}R\left(\rho-\frac{R}{n-1}\right)}\right]\\ &+\frac{(n-1)}{(3n-2)}|\nabla f|\left(2\rho(1+f)-\frac{(n-3)}{(n-1)}R\right).\end{aligned}$$

Then, the D-tensor must be identically zero. Consequently,

- *(i)* (M^4, g) *is either Einstein, or a finite quotient of* $\mathbb{R}^4$ *or* $\mathbb{S}^3 \times \mathbb{R}$*;*
- *(ii)* (M^n, g)*,* $n \geq 5$*, is either Einstein or is a finite quotient of the Gaussian shrinking soliton or is a finite quotient of* $N^{n-1} \times \mathbb{R}$*, where* N^{n-1} *is Einstein.*

The hypothesis considered in the above theorem will be discussed after the proof of this theorem in chapter 4.

By following the same strategy of Theorem 4.2 we can prove our next result for expanding Ricci solitons. The theorem that follows is possibly the most important result of this thesis.

Theorem. 4.3 *Let* (M^3, g, f) *be a complete gradient expanding Ricci soliton with non-positive Ricci curvature such that* $1 \leq -f$ *and* R *goes to zero at infinity. Suppose that*

$$|\nabla f|^2 |\nabla R| = o(r^{-3}),$$

where r *is the distance function. Then,* (M^3, g) *is rotationally symmetric. Moreover,* R *must be constant, particularly* $R = 0$*.*

The above result is consistent with the geometry of the expander cylinders. This soliton will be presented in Chapter 4.

Switching gears, let us now focus on $\rho = 0$. Gradient steady solitons play an important role in Hamilton's Ricci flow [42] as they correspond to translating solutions and often arise as Type II singularity models, thus playing a crucial role in the singularity analysis of the Ricci flow [58].

It is well-known that compact gradient steady Ricci solitons must be Ricci flat. For dimension $n = 2$, Hamilton [41] obtained the *cigar soliton*, i.e., the first example of a complete non-compact gradient steady soliton on $\mathbb{R}^2$, where the explicit metric is given by

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2},$$

and the potential function is

$$f = -\ln(1 + x^2 + y^2).$$

This soliton is also known as Witten's black hole. It has positive curvature and is asymptotic to a cylinder of finite circumference at infinity. The scalar curvature decays exponentially. It is important to highlight that

$$R|\nabla f| = |\nabla R|$$

on the cigar soliton. We can see this naturally by changing to polar coordinates and obtaining

$$R = \frac{1}{1+r^2}.$$

This identity also is trivially satisfied, in higher dimensions, for Ricci flat solitons with a constant potential function, a quotient of the product steady soliton $N^{n-1} \times \mathbb{R}$, where N^{n-1} is Ricci flat and the product of the cigar soliton and any complete Ricci flat manifold.

For dimension $n \geq 3$, Robert Bryant [4] proved that there exists, up to scaling, a unique complete rotationally symmetric gradient Ricci soliton on $\mathbb{R}^n$. Its sectional curvature is positive and the scalar curvature R decays like r^{-1} at infinity. The volume of the geodesic balls $B_r(0)$ grows according to the order of $r^{(n+1)/2}$. Here, r denotes the geodesic distance function. The dichotomic of steady Ricci solitons was studied in [22].

It was conjectured by Perelman [58] that in dimension $n = 3$, the Bryant soliton is the only complete non-compact (κ-noncollapsed) gradient steady soliton with positive sectional curvature. This conjecture was proved by Brendle [2] in 2013 and was extended by himself in 2014 [3] for arbitrary dimensions ($n \geq 4$) under the additional condition that the steady Ricci soliton is asymptotically cylindrical. Moreover, Deng and Zhu [31] proved that any non-compact κ-noncollapsed steady Ricci soliton with a nonnegative curvature operator must be rotationally symmetric if it has a linear curvature decay. As we can see, for $n \geq 4$, it is desirable to find geometrically interesting conditions under which the uniqueness would hold.

In the context of finding interesting geometric conditions to prove classification, Cao and Chen [10] proved that a complete non-compact n-dimensional $(n \geq 3)$ locally conformally flat gradient steady Ricci soliton with positive sectional curvature is isometric to the Bryant soliton. Moreover, they showed that a complete non-compact n-dimensional locally conformally flat gradient steady Ricci soliton is either flat or isometric to the Bryant soliton (see also [17]). For $n = 4$, Chen and Wang [25] showed that any 4-dimensional complete half-conformally flat gradient steady Ricci soliton is either Ricci flat, or locally conformally flat (hence isometric to the Bryant soliton). In [6], Cao, Catino, Chen, Mantegazza, and Mazzieri proved that any n-dimensional $(n \geq 4)$ complete Bach-flat gradient steady Ricci soliton with positive Ricci curvature is isometric to the Bryant soliton. Very recently, Cao and Yu proved that any n-dimensional complete non-compact gradient steady Ricci soliton with vanishing D-tensor is either Ricci flat, or a quotient of the product steady soliton $N^{n-1} \times \mathbb{R}$, where N^{n-1} is Ricci flat, or isometric to the Bryant soliton (up to scalings). Here, the D-tensor is a 3-tensor defined by (1-6), see [11].

In recent years, much progress has been made in understanding the curvature estimate of gradient steady Ricci solitons, see, e.g., [19, 28, 31, 39]. This topic will be important in analyzing and classifying steady Ricci solitons.

It follows from a result by Chen [24] that every complete gradient steady Ricci soliton has nonnegative scalar curvature, i.e., $R \geq 0$. On the other hand, it is well known that a complete gradient steady Ricci soliton satisfies

$$R + |\nabla f|^2 = C_0,$$

where C_0 is a positive constant. In other words, the scalar curvature of a complete gradient steady Ricci soliton is uniformly bounded. In particular, up to normalization, we may consider

$$R + |\nabla f|^2 = 1.$$

This, therefore, implies that a (normalized) gradient steady Ricci soliton satisfies $0 \leq R \leq 1$.

Before proceeding, we recall that Brendle [1] (see also [6, Proposition 5.2]) proved the following result.

Theorem (Brendle, [1])**.** *Let (M^n, g, f), $n \geq 3$, be a complete n-dimensional gradient steady Ricci soliton. Suppose that the scalar curvature R of (M^n, g) is positive and approaches zero at infinity. Denote by $\psi : (0,1) \to \mathbb{R}$ the smooth function such that the vector field*

$$X := \nabla R + \psi(R)\nabla f = 0 \tag{0-1}$$

on the Bryant soliton, and define $u : (0,1) \to \mathbb{R}$ by

$$u(s) = \log \psi(s) + \frac{1}{n-1}\int_{1/2}^{s} \left(\frac{n}{1-t} - \frac{n-1-(n-3)t}{(1-t)\psi(t)} \right) dt.$$

Moreover, assume that there exists an exhaustion of M^n by bounded domains Ω_l such that

$$\lim_{l\to\infty} \int_{\partial\Omega_l} e^{u(R)} \langle \nabla R + \psi(R)\nabla f, \nu \rangle = 0.$$

Then $X = 0$ and $D_{ijk} = 0$. In particular, for $n = 3$, (M^3, g, f) is isometric to the Bryant soliton.

Brendle's theorem given above combined with Proposition 5.1 by Cao et al. [6] implies that a complete gradient steady Ricci soliton $(M^n,\, g,\, f)$, $n \geq 4$, with positive Ricci curvature such that the scalar curvature R approaches zero at infinity so that the asymptotic condition given above is satisfied for some exhaustion of M^n by bounded domains Ω_l is isometric to the Bryant soliton.

We highlight that the proof of Theorem below is partially based on the ideas outlined by Robinson [62] to study the uniqueness of static black holes, which depends essentially of a suitable divergence formula. In this paper, motivated by the result obtained by Brendle [1] and the ideas by Robinson [62], we obtain a divergence formula (Lemma 3.2) for the gradient steady Ricci soliton by following the ideas of Robinson in order to obtain a rigidity result.

On the original paper of Robinson [62], there exists a natural vector field

$$X=\nabla|\nabla f|^2+\frac{8f|\nabla f|^2}{(1-f^2)}\nabla f$$

such that it vanishes identically on the three-dimensional Schwarzschild solution. Based on this vector field, Robinson built a divergence formula which is the core of its demonstration of the uniqueness of static vacuum black holes. However, since the Bryant soliton is not an exact solution like Schwarzschild is, i.e., explicitly given, the construction of a divergence formula can be more trick for steady solitons. This is why the unusual condition (0-1) rises in the above theorem, and we will avoid this condition in our result based on the geometry of the Cigar soliton. More precisely, we have established the following result.

Theorem. *3.1. Let $\left(M^n, g, f\right)$ be a complete non-compact gradient steady Ricci soliton satisfying*

$$\sigma R|\nabla f|\leq|\nabla R|, \tag{0-2}$$

where $\sigma=\frac{(n+1)+\sqrt{(n-1)(7n-13)}}{3n-2}$. Suppose that there exists an exhaustion of M^n by bounded domains Ω_ℓ such that

$$\lim_{\ell\to+\infty}\int_{\partial\Omega_\ell}|\nabla R+R\nabla f|=0. \tag{0-3}$$

Then (M^n, g, f) is either Ricci flat with a constant potential function, or a quotient of the product steady soliton $N^{n-1}\times\mathbb{R}$, where N^{n-1} is Ricci flat, or isometric to the Bryant soliton (up to scalings).

In [39, Lemma 3], the authors proved that a nonnegatively curved steady soliton satisfies the following inequality:

$$|Ric|^2\leq\frac{R^2}{2}. \tag{0-4}$$

Therefore,

$$|\nabla R|^2\leq 2R^2|\nabla f|^2.$$

This shows us that (0-2) can be interpreted as a lower bound for $|\nabla R|$. In fact, $\sigma\leq\frac{1+\sqrt{7}}{3}$, see also [31, Lemma 2.3]. Moreover, the vector field $X = \nabla R + R\nabla f$ vanishes identically on the Cigar soliton. Moreover, for $n = 2$ the constant $\sigma = 1$.

We can note that by combining our hypothesis (0-2) with the inequality (0-4), we arrive at the range of values

$$(0.60)R\cdots \approx \frac{\sigma}{2}R \leq |Ric| \leq \frac{\sqrt{2}}{2}R \approx (0.70)R\cdots$$

Notably, it is crucial to observe that the positivity of the scalar curvature R in (0-4) is a fundamental assumption for our analysis.

One possible conjecture concerning the curvature decay of a gradient steady Ricci soliton is that the decay rate is either linear or exponential [22]. In [19, Corollary 1], the author proved the scalar curvature of a steady Ricci soliton with nonnegative Ricci curvature decays exponentially if an asymptotic condition holds.

In [27, Theorem 9.56], Chow, Lu, and Ni proved that the scalar curvature of complete and non-compact gradient steady Ricci soliton with positively pinched Ricci curvature has exponential decay. In that sense, condition (0-3) can be replaced by an exponential decay for the scalar curvature to get a rigidity of the metric (cf. Corollary 3.1), i.e.,

$$R = o(e^{-r}), \quad r \to \infty,$$

where r stands for the geodesic distance from a fixed point.

In [52], the authors proved that the sectional curvature Rm of a gradient steady Ricci soliton (non-flat) with potential function f bounded from above (or f a proper function, which happens if $Ric > 0$) by a constant and such that $|Rm|r = o(1)$, at infinity, decays like

$$|Rm| \leq c(1+r)^{3(n+1)}e^{-r},$$

where c is a positive constant and r stands for the geodesic distance. It is known that the assumption over f holds true when $Ric > 0$. The exponential decay rate in the theorem is sharp as seen from $M = N \times \Sigma$; where Σ is the cigar soliton and N is a compact Ricci flat manifold. Moreover, Chan and Zhu [22] proved that a gradient steady Ricci soliton (M, g, f) with $|Rm| \to 0$, then either one of the following estimates holds outside a compact set of M:

$$C^{-1}r^{-1} \leq |Rm| \leq Cr^{-1};$$
$$C^{-1}e^{-r} \leq |Rm| \leq Ce^{-r},$$

where C is a positive constant and r is the distance function.

The control of the curvature is an important information in the analysis of a Ricci soliton and there are several results proving that the scalar curvature controls the sectional curvature, see [9, 19]. Here, it is important to emphasize that four-dimensional Ricci solitons must satisfy the following condition:

$$|Rm| \leq A\left(\frac{|\nabla Ric|}{|\nabla f|}+|Ric|\right),$$

where A is an universal positive constant (see [9]). PleaseFor more detail, see also [19, Proposition 2].

Therefore, inspired by the above curvature properties of four-dimensional steady Ricci solitons we will prove that if the Ricci curvature is controlled by the scalar curvature we get the Bryant soliton. Moreover, we will not assume that the steady Ricci soliton is κ-noncollapsed.

Theorem. *3.2. Let $\left(M^n, g, f\right)$ be a complete non-compact gradient steady Ricci soliton with positive Ricci curvature and*

$$|Ric| \leq \frac{1}{4}\left[\frac{3}{2}\frac{|\nabla R|^2}{|\nabla f|^2}-2R^2\right].$$

Suppose that

$$\lim_{r\to\infty} R = 0 \quad \textit{and} \quad \lim_{r\to\infty} |Rm| r = o(1).$$

Then (M^n, g, f) is isometric to the Bryant soliton (up to scalings).

According to Hamilton [40, 42], a Riemannian manifold (M^n, g) is *positively pinched Ricci curvature* if there is a uniform constant $\delta > 0$ such that

$$\delta R g \leq Ric(g).$$

Deng and Zhu [30] proved that any $(n \geq 2)$-dimensional complete non-compact steady Kähler-Ricci soliton (M^n, g, f) with positively pinched Ricci curvature should be Ricci flat (provided that there is a point p so that $\nabla f(p) = 0$). The existence of such a point p is called equilibrium point condition. Under our approach, we shall show that this condition can be removed. To be precise, we have established the following results.

Corollary. *3.1. Let $\left(M^n, g, f\right)$, $n \geq 3$, be a complete non-compact gradient steady Kähler-Ricci soliton. If (M^n, g) has positively pinched Ricci curvature , then (M^n, g, f) is Ricci flat.*

Thus, the above corollary answers a question proposed by Chow, Lu, and Ni in the case of steady Kähler-Ricci solitons (cf. [27, 56]), and proves Corollary 1.5 in [30]. In fact, the above corollary still holds for any gradient steady Ricci soliton (not necessarily Kähler).

Steady Ricci solitons with pinched Ricci curvature have been studied in the past years (cf. [32] and the references therein). In [56], Ni proved that any steady Ricci soliton with pinched Ricci curvature and nonnegative sectional curvature must be flat. Then, Deng and Zhu [30] proved that Ni's result is valid without the assumption on the sectional curvature for steady Kähler-Ricci solitons.

Finally, we will apply our strategy to Yamabe solitons. A Riemannian manifold (M^n, g) is called Yamabe soliton if

$$(R-\lambda)g = \frac{1}{2}\mathfrak{L}_Z g,$$

for some constant λ and vector field Z. Here, R and $\mathfrak{L}_Z g$ stand for the scalar curvature and the Lie derivative for the metric g, respectively. When we take $Z = \nabla f$, where f is some smooth potential function in M, we obtain a gradient Yamabe soliton, i.e.,

$$(R-\lambda)g = \nabla^2 f, \tag{0-5}$$

where ∇^2 stands for the Hessian operator for g. For $\lambda = 0$ the Yamabe soliton is steady, for $\lambda > 0$ it is shrinking and for $\lambda < 0$ expanding [1, 10, 29, 43, 44, 50, 51]. It is important to highlight that any steady and shrinking gradient Yamabe solitons have non-negative scalar curvature. In 2-dimensions Yamabe solitons are equivalent to Ricci solitons. In this case, an important example is the soliton cigar provided by Hamilton.

The Yamabe solitons represent self-similar solutions to the Yamabe flow. This geometric flow behaved like a heat flow and was used in an attempt to solve the Yamabe problem. Many examples of gradient Yamabe solitons can be found in [55].Moreover, the Yamabe soliton equation is related to conformal gradient solitons [18, 23]. This geometric structure is important to classify warped product manifolds in the form:

$$(a, b) \times_f N^{n-1}, \tag{0-6}$$

where $(a, b) \subset \mathbb{R}$.

Following the ideas presented by [23], Cao et al. [10] were able to provide a classification for gradient Yamabe solitons. They proved that every complete nontrivial gradient Yamabe soliton admits a special global warped product structure with an 1-dimensional base and the warping function provided by $|\nabla f|$. Consequently, a nontrivial complete gradient Yamabe soliton with positive Ricci curvature is rotationally symmetric.

Locally conformally flat Yamabe solitons were also classified in this work, inspired by [29]. In [29], the authors proved that any locally conformally flat gradient Yamabe solitons with positive sectional curvature must be rotationally symmetric. Moreover, Daskalopoulos and Sesum provided some examples of gradient yamabe soliton having asymptotic behavior. It is important to remember that any closed gradient Yamabe soliton must have constant scalar curvature; hence trivial since f is harmonic and thus is constant [29, 43, 50].

In this work we combine [1] and [62] to prove that an n-dimensional gradient Yamabe soliton must be a Riemannian manifold with constant scalar curvature, provided that an asymptotic condition holds (see also [63, Theorem 2.3]).

Theorem. *2.1. Let (M^n, g, f), $n \geq 5$, be an n-dimensional gradient Yamabe soliton such that*

$$|Ric|^2 \leq \frac{R^2}{n-1}. \tag{0-7}$$

Assume that there exists an exhaustion of M^n by bounded domains Ω_ℓ such that

$$\lim_{\ell\to+\infty}\int_{\partial\Omega_\ell} R\langle\nabla R + \frac{R}{n(n-1)}\nabla f, \nu\rangle = 0,$$

where ν is the normal vector field. Then, (M^n, g) must have constant scalar curvature. Moreover, the D-tensor (1-6) vanishes identically, and the equality holds in (0-7).

The existence of exhaustion of M is, in fact, a natural assumption. For example, Cao and Chen [7, Proposition 2.3] proved that the potential function is an exhaustion function for the steady gradient Ricci solitons. For the gradient Yamabe solitons, we can quote [63, Theorem 4.1 and Lemma 3.3]. Moreover, in [64, Theorem 1.6], the authors proved that under an asymptotic condition, the potential function is an exhaustion function for the gradient Yamabe soliton.

Theorem. *2.2. Let (M^3, g, f) be a complete three-dimensional gradient steady Yamabe soliton such that*

$$|Ric|^2 \geq \frac{R^2}{2}.$$

Assume that there exists an exhaustion of M^3 by bounded domains Ω_ℓ such that

$$\lim_{\ell\to+\infty}\int_{\partial\Omega_\ell} R\langle\nabla R + \frac{R}{6}\nabla f, \nu\rangle = 0,$$

where ν is the normal vector field. Then, (M^3, g) must be isometric to the standard three-dimensional Euclidean space.

Estimates for the curvature are an essential tool for the classification of solitons. The curvature bounds for three-dimensional gradient steady Ricci solitons were proved by Chen [24]. Recently, bounds for the curvature of four-dimensional gradient Ricci solitons were provided in [26]. For Yamabe solitons, bounds for the scalar curvature considering some asymptotic conditions were provided in [51, 63].

Considering the asymptotic condition (2-4), we prove that any four-dimensional gradient Yamabe soliton is a Riemannian manifold with constant scalar curvature, i.e., a Yamabe metric. In this result, no additional condition on the curvature is necessary, and no restriction for λ is required.

Theorem. 2.3. *Let (M^4, g, f) be a four dimensional gradient Yamabe soliton. Moreover, assume that there exists an exhaustion of M^4 by bounded domains Ω_ℓ such that*

$$\lim_{\ell\to+\infty}\int_{\partial\Omega_\ell} R\langle\nabla R+\frac{R}{12}\nabla f,\nu\rangle=0.$$

Then, (M^4, g, f) has constant scalar curvature.

It is interesting to highlight that this theorem is important to understand the structure of four-dimensional warped product manifolds of the form (0-6).

CHAPTER 1

Preliminary

This chapter introduces the Ricci Solitons, the Yamabe solitons, and the D-tensor, among other valuable concepts for this work. For this reason, we have been guided by texts such as [8, 11, 20, 27]. To that end, we will introduce basic objects of Riemannian geometry that are important for a better understanding of the dissertation; see also [20, 35, 60].

1.1 Hilbert-Schmidt Norm and Kato Inequality

Let us start with a definition that will be very useful for this dissertation. The following definition is also known as the Hilbert-Schmidt norm.

Definition 1.1. *Let T be a $(0,r)$-tensor, the norm of T on a Rimannian manfild (M,g) is given by*

$$|T|^2 = g^{i_1 j_1} \cdots g^{i_r j_r} T_{i_1 \cdots i_r} T_{j_1 \cdots j_r},$$

where $T_{i_1 \cdots i_r}$ are the components of T, and $g^{i_r j_r}$ the components of the inverse of g.

Another important property of tensors used in this work is the Kato inequality.

Proposition 1.1 (Kato's inequality)**.** *Let T be any tensor defined on any Riemannian manifold (M,g). Then, we have*

$$|\nabla |T|| \leq |\nabla T|$$

which holds point-wise wherever $|T| \neq 0$.

Proof: By Cauchy-Schwarz equation, we have

$$|\nabla |T|^2| = |2\langle T, \nabla T\rangle| \leq 2|T||\nabla T|.$$

On the other hand, $|\nabla |T|^2| = 2|T||\nabla |T||$, so it follows that

$$|T||\nabla |T|| \leq |T||\nabla T|,$$

which implies a demonstration of proposition wherever $|T| \neq 0$. □

1.2 Gradient, Divergence, and Laplacian

The Lie derivative allows us to define the Hessian of a smooth function on a Riemannian manifold (M, g) as a $(0,2)$-tensor:

$$\mathrm{Hess} f(X,Y) = \frac{1}{2}(\mathfrak{L}_{\nabla f} g)(X,Y).$$

The Hessian is supposed to measure how the metric changes as we move along the gradient field. To further understand this, consider the divergence of a vector field X to be the function $\mathrm{div}X$ that measures how the volume form changes along the flow for X:

$$\mathfrak{L}_X \mathrm{vol} = (\mathrm{div}X)\mathrm{vol}.$$

We can see that selecting a positively oriented orthonormal frame $\{E_i\}_{i=1}^n$ we have

$$\begin{aligned}
\mathrm{div}X &= (\mathfrak{L}_X \mathrm{vol})(E_1,\cdots,E_n) \\
&= \mathfrak{L}_X(\mathrm{vol}(E_1,\cdots,E_n)) - \sum \mathrm{vol}(E_1,\cdots,\mathfrak{L}_X E_i,\cdots,E_n) \\
&= -\sum g(\mathfrak{L}_X E_i.E_i) \\
&= \frac{1}{2}\sum (L_X(g(E_i,E_i)) - g(\mathfrak{L}_X E_i,E_i) - g(E_i,\mathfrak{L}_X E_i)) \\
&= \sum \frac{1}{2}(\mathfrak{L}_X g)(E_i,E_j).
\end{aligned}$$

The above computation leads us to the classic definition of divergence.

Definition 1.2. *Let X be a smooth vector field in $(M, \langle\cdot,\cdot\rangle)$. The divergence of X, denoted by div X, is defined by*

$$div\, X = \sum_i \langle \nabla_{E_i} X, E_i \rangle.$$

Let us remember some useful properties of the divergence.

Proposition 1.2. *If X, Y are smooth vectors field in $(M, \langle\cdot,\cdot\rangle)$ and $f : M \to \mathbb{R}$ is a smooth function, then:*

a) $div(X+Y) = div(X) + div(Y)$;
b) $div(fX) = f div(X) + \langle \nabla f, X \rangle$.

Now, consider T be a $(1,1)$-tensor in a Riemannian manifold (M^n, g). Thus,

$$\begin{aligned} (divT)(Z) &= \sum_i \langle (\nabla_{E_i} T)(Z), E_i \rangle \\ &= \sum_i \langle \nabla_{E_i}(T(Z)) - T(\nabla_{E_i} Z), E_i \rangle \\ &= \sum_i \langle \nabla_{E_i}(T(Z)), E_i \rangle - \sum_i \langle T(\nabla_{E_i} Z), E_i \rangle \\ &= div(T(Z)) - \sum_i \langle \nabla_{E_i} Z, T^*(E_i) \rangle \\ &= div(T(Z)) - \langle \nabla Z, T^* \rangle. \end{aligned}$$

Therefore,

$$div(T(Z)) = (divT)(Z) + \langle \nabla Z, T^* \rangle, \tag{1-1}$$

where T^* is a adjoint operator of T.

Consequently, the following lemma holds.

Lemma 1.1. *Let T be a symmetric $(0,2)$-tensor on a Riemannian manifold (M^n, g) then*

$$div(T(\phi Z)) = \phi(divT)(Z) + \phi\langle \nabla Z, T \rangle + T(\nabla\phi, Z),$$

for all $Z \in \mathfrak{X}(M)$ and any smooth function ϕ on M.

Let us now define an important operator, which will be very important throughout the dissertation.

Definition 1.3. *The Laplacian Δ of a smooth function on $(M, \langle \cdot, \cdot \rangle)$ is the divergence of its gradient, i.e.,*

$$\Delta f = div(\nabla f).$$

1.3 Curvature

In this section, we will establish the convention for the curvature operators.

Definition 1.4. *Let M^n be an n-dimensional Riemannian manifold with Levi-Civita connection ∇. The Riemannian curvature tensor of M is a $(1,3)$-tensor*

$$R : (X, Y, Z) \longmapsto R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

It is also natural to use Rm for the curvature operator when it is convenient.

Using the metric g we have Rm as an $(0,4)$-tensor on M as follows:

$$R(X,Y,Z,W) = g(R(X,Y)Z,W).$$

The curvature tensor has important and useful symmetries.

Proposition 1.3. *The Riemannian curvature tensor $R(X, Y, Z, W)$ satisfies the following properties:*

1. *Rm is skew-symmetric in the first two and last two entries:*

$$R(X, Y, Z, W) = -R(Y, X, Z, W) = R(Y, X, W, Z).$$

2. *Rm is symmetric between the first two and last two entries:*

$$R(X,Y,Z,W) = R(Z,W,X,Y).$$

3. *Rm satisfies a cyclic permutation property called Bianchi's first identity:*

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0.$$

4. *∇Rm satisfies a cyclic permutation property called Bianchi's second identity:*

$$(\nabla_Z R)(X,Y)W + (\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W = 0.$$

Proof: See [60]. □

The sectional (or Riemannian) curvature is closely related to the curvature tensor, which we define below.

Definition 1.5. *Let M be a Riemannian manifold, a point $p \in M$ and a two-dimensional subspace $\sigma \subset T_pM$ the real number $Rm(u, v) = Rm(\sigma)$, where $\{u, v\}$ is any basis of σ, is called the sectional curvature of σ at p.*

Geometrically the sectional curvature is the Gaussian curvature of the surface generated by the geodesics leaving p that are tangent to σ.

Proposition 1.4. *Let* $\sigma \subset T_pM$ *be a two-dimensional subspace of the tangent space* T_pM *and let* $u, v \in \sigma$ *be two linearly independent vectors. Then,*

$$Rm(u, v) = \frac{g(R(u, v)u, v)}{g(u, u)g(v, v) - g(v, u)^2}$$

does not depend on the choice of vectors $u, v \in \sigma$.

Proof. See [35] and [60]. ☐

1.3.1 Ricci curvature

We will define the following geometric objects: the Ricci tensor and the Ricci curvature.

Definition 1.6. *The Ricci tensor of an n-dimensional Riemannian manifold* M^n*, denoted by Ric, is a symmetric* $(0,2)$*-tensor defined by*

$$Ric(X,Y) = Tr\{X \longmapsto R(Z,X)Y\},$$

or equivalently,

$$Ric(X,Y) = \sum_{l=1}^{n} \langle R(E_l,X)Y, E_l \rangle,$$

where $\{E_1, \cdots, E_n\}$ *is a frame for* M*. In coordinates,*

$$R_{ij} = g^{kl} R_{ikjl}.$$

Thus, Ric is a symmetric bilinear form, and we can also define it as a symmetric $(1,1)$-tensor:

$$Ric(v) = \sum_{i=1}^{n} R(v, E_i)E_i.$$

Definition 1.7. *A Riemannian manifold* (M^n, g) *is called an Einstein manifold if the Ricci tensor is a multiple of the metric g, i.e.,* $Ric = cg$ *for some constant c.*

1.3.2 Scalar curvature

The last curvature quantity we define here is the scalar curvature:

Definition 1.8. *The scalar curvature R of Mis defined by*

$$R = \sum_{i<j} Ric(E_i, E_j),$$

where $E_1,\cdots,E_n$ *is a frame of* M. *In coordinates,*

$$R = g^{ij}R_{ij}.$$

Proposition 1.5. *On any Riemannian manifold, the scalar and Ricci curvature are related by*

$$dR = 2divRic.$$

Proof: See [60]. □

1.4 Ricci soliton and Yamabe soliton

We start this section by defining a Ricci soliton.

Definition 1.9. *We say that a Riemannian manifold* (M^n, g) *is a Ricci soliton if there exists a vector field* X *over* M *and a real number* ρ *satisfying*

$$Ric_g + \frac{1}{2}\mathfrak{L}_X g = \rho g. \tag{1-2}$$

We say that the Ricci soliton is expanding, steady, or shrinking if ρ is less than, equal to, or greater than zero, respectively.

If the vector field X is the gradient of some differentiable function $f : M \to \mathbb{R}$, from the equation (1-2) we have a gradient Ricci soliton.

Definition 1.10. *An n-dimensional Riemannian manifold* M^n *with complete Riemannian metric* g *is a gradient Ricci soliton* (M^n, g, f, ρ) *if there is a smooth function* f *on* M *and a constant* ρ *such that*

$$Ric + Hess(f) = \rho g. \tag{1-3}$$

Here, $\nabla^2 f$ *denotes the Hessian of* f *and* Ric *is the Ricci curvature of* g. *The function* f *is called a potential function. Also, if* $\rho > 0$, $\rho < 0$ *or* $\rho = 0$, *we have a shrinking, expanding, or steady gradient Ricci soliton, respectively.*

In the following chapters, we present some crucial properties of Ricci solitons that will be of much support. To that end, we remember the Ricci equation:

$$\nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f = R_{ijkl}\nabla^l f.$$

Proposition 1.6. *Let* (M^n, g, f, ρ) *a gradient Ricci soliton. Then,*

(1) $\nabla_i R_{jk} - \nabla_j R_{ik} = -R_{ijkl}\nabla^l f.$

(2) $\nabla R = 2Ric(\nabla f)$.

(3) $R + |\nabla f|^2 = 2\rho f$ *(up to scaling).*

(4) $R + \Delta f = n\rho$.

(5) $\Delta R + 2|Ric|^2 = \langle \nabla R, \nabla f \rangle + 2\rho R$.

Proof. We start defining $g = \langle \cdot, \cdot \rangle$. To prove the first item, we combine (1-3) with the Ricci equation. Now, taking the trace and using Bianchi's second identity, we have:

$$\begin{aligned} \nabla_i R_{jk} g^{jk} - g^{jk} \nabla_j R_{ik} &= -g^{jk} R_{ijkl} \nabla^l f; \\ \nabla_i R - \frac{1}{2} \nabla_i R &= R_{il} \nabla^l f; \\ \nabla_i R &= 2R_{il} \nabla^l f, \end{aligned}$$

and this proves Item (2).

For Item (3),

$$\begin{aligned} \nabla_i (R + |\nabla f|^2 - 2\rho f) &= \nabla_i R + 2\nabla_i \nabla_j f \nabla^j f - 2\rho \nabla_i f \\ &= 2R_{ij} \nabla^j f + 2\nabla_i \nabla_j f \nabla^j f - 2\rho \nabla_i f \\ &= 2(-\nabla_i \nabla_j f + \rho g_{ij}) \nabla^j f + 2\nabla_i \nabla_j f \nabla^j f - 2\rho \nabla_i f \\ &= 0. \end{aligned}$$

Contracting (1-3), we get Item (4), i.e.,

$$\begin{aligned} g^{ij} R_{ij} + g^{ij} \nabla_i \nabla_j f &= \rho g^{ij} g_{ij}; \\ R + \Delta f &= n\rho. \end{aligned}$$

Finally, to obtain the Item (6) we compute the Laplacian of the Ricci tensor by using Item (1) and the second Bianchi identity. Hence,

$$\begin{aligned} \Delta R_{ik} = \nabla^j \nabla_j R_{ik} &= \nabla^j \nabla_i R_{jk} + \nabla^j R_{ijkl} \nabla^l f + R_{ijkl} \nabla^j \nabla^l f \\ &= \nabla_i \nabla^j R_{jk} + R^j_{ijl} R^l_k + R^j_{ikl} R^l_j \\ &\quad + \nabla_k R^j_{lij} \nabla^l f - \nabla_s R^j_{kij} \nabla^l f + R_{ijkl} \nabla^j \nabla^l f \\ &= \nabla_i \nabla^j R_{jk} + R_{il} R^l_k + R^j_{ikl} R^l_j \\ &\quad - \nabla_k R_{li} \nabla^l f + \nabla_l R_{ki} \nabla^l f + R_{ijkl} \nabla^j \nabla^l f. \end{aligned}$$

Replacing $\nabla^j\nabla^l f$ by using the soliton's equation we get

$$\begin{aligned} \Delta R_{ik} &= \frac{1}{2}\nabla_i\nabla_k R + R_{il}R^l_k + R^j_{ikl}R^l_j \\ &\quad -\nabla_k R_{li}\nabla^l f + \langle \nabla R_{ik}, \nabla f\rangle - R_{ijkl}R^{jl} + \rho R_{ik}, \\ &= \langle \nabla R_{ik}, \nabla f\rangle + \rho R_{ik} - 2R_{ijkl}R^{jl} + R_{il}R^l_k \\ &\quad +\frac{1}{2}\nabla_k\nabla_i R - \nabla_k R_{il}\nabla^l f. \end{aligned} \tag{1-4}$$

On the other hand, by differentiating the relation on Item (2) we obtain

$$0 = \frac{1}{2}\nabla_k(\nabla_i R - 2R_{il}\nabla^l f).$$

Hence,

$$\frac{1}{2}\nabla_k\nabla_i R - \nabla_k R_{il}\nabla^l f = R_{il}\nabla^l\nabla_k f.$$

Then, by equation (1-4), we conclude

$$\begin{aligned} \Delta R_{ik} &= \langle \nabla R_{ik}, \nabla f\rangle + \rho R_{ik} - 2R_{ijkl}R^{jl} + R_{il}R^l_k + R_{il}\nabla^l\nabla_k f, \\ &= \langle \nabla R_{ik}, \nabla f\rangle + \rho R_{ik} - 2R_{ijkl}R^{jl} + R_{il}R^l_k + \rho R_{il} - R_{il}R^l_k, \\ &= \langle \nabla R_{ik}, \nabla f\rangle + 2\rho R_{ik} - 2R_{ijkl}R^{jl}. \end{aligned}$$

Item (6) follows by contracting the above equation on i and k. ☐

The above results can be found in [36, Proposition 2.1] and they will be very important in the characterization of the solitons at several stages of this dissertation.

Switching gears, we now define the Yamabe solitons and discuss some important and useful results about them.

Definition 1.11. *A Riemannian manifold (M^n, g) of dimension is called a gradient Yamabe soliton if there exists a differentiable potential function $f : M^n \to \mathbb{R}$ and a constant λ such that*

$$(R-\lambda)g = \nabla^2 f. \tag{1-5}$$

where ∇^2 stands for the Hessian operator for f with respect to g. For $\lambda = 0$ the Yamabe soliton is steady, for $\lambda > 0$ it is shrinking, and for $\lambda < 0$ expanding.

In [29], inspired by [10], the authors developed one of the first classification theorems for Yamabe soliton.

Theorem 1.1. *[29, Theorem 1.3] All complete locally conformally flat gradient Yamabe solitons with positive sectional curvature $Rm > 0$ are rotationally symmetric.*

Cao, Sun, and Zhang [10] motivated by the above theorem investigate the structure of gradient Yamabe solitons not necessarily locally conformally flat. It turns out that, every complete nontrivial gradient Yamabe soliton admits a special warped product structure.

The following theorem shows us through the critical points of the potential function and the scalar curvature the warped product morphology of the complete gradient Yamabe Soliton.

Theorem 1.2. *[10, Theorem 1.2] Let* (M^n, g, f) *be a nontrivial complete gradient Yamabe soliton satisfying equation (1-5). Then* $|\nabla f|^2$ *is constant on regular level surfaces of* f*, and either*

1. f *has a unique critical point at some point* $x_0 \in M^n$*, and* (M^n, g, f) *is rotationally symmetric and equal to the warped product*

$$([0,\infty), dr^2) \times_{|\nabla f|} (\mathbb{S}^{n-1}, \bar{g}_{can}),$$

where $\bar{g}$ *can is the round metric on* $\mathbb{S}^{n-1}$*, or*

2. f *has no critical point and* (M^n, g, f) *is the warped product*

$$(\mathbb{R}, dr^2) \times_{|\nabla f|} (N^{n-1}, \bar{g}),$$

where $(N^{n-1}, \bar{g})$ *is a Riemannian manifold of constant scalar curvature, say* $\bar{R}$*. Moreover, if* (M^n, g, f) *has nonnegative Ricci curvature* $Ric \geq 0$ *then* (M^n, g) *is isometric to the Riemannian product* $(\mathbb{R}, dr^2) \times (N^{n-1}, \bar{g})$*; if the scalar curvature* $R \geq 0$ *on* M^n*, then either* $\bar{R} > 0$*, or* $R = \bar{R} = 0$ *and* (M^n, g) *is isometric to the Riemannian product* $(\mathbb{R}, dr^2) \times (N^{n-1}, \bar{g})$*.*

Now we present some of the main properties of the Yamabe soliton.

Lemma 1.2. *[44, Lemma 2.1] Let* $\left(M^n,\, g,\, f\right)$ *be a gradient Yamabe soliton. Then,*

(1) $n(R-\lambda) = \Delta f$.

(2) $\nabla|\nabla f|^2 = 2(R-\lambda)\nabla f$.

(3) $\nabla R + \frac{1}{n-1} Ric(\nabla f) = 0$.

Proof. The first item can be obtained directly by contracting the Yamabe soliton equation (1-5), i.e.,

$$\begin{aligned} (R-\lambda) g^{ij} g_{ij} &= g^{ij} \nabla_i \nabla_j f; \\ n(R-\lambda) &= \Delta f. \end{aligned}$$

On the other hand,

$$\begin{aligned}\nabla_i|\nabla f|^2 &= 2\nabla_i\nabla_j f\nabla^j f\\ &= 2(R-\lambda)g_{ij}\nabla^j f\\ &= 2(R-\lambda)\nabla_i f.\end{aligned}$$

So, we have Item (2).

Using the equation (1-5) again, we obtain

$$\nabla_i(R-\lambda)=\nabla_i\nabla_j\nabla_j f.$$

From the Ricci equation, Item (1) and Item (2), we deduce

$$\begin{aligned}g^{ik}\nabla_i(R-\lambda)g_{jk} &= g^{ik}\nabla_i\nabla_j\nabla_k f\\ &= \nabla_j(\Delta f)+R_{jl}\nabla^l f\\ &= n\nabla_j(R-\lambda)+R_{jl}\nabla^l f.\end{aligned}$$

Reorganizing the equation we have

$$0 = (n-1)\nabla_i(R-\lambda)+R_{ij}\nabla^j f.$$

This concludes the proof of Item (3). ☐

It is important to highlight that any steady and shrinking gradient Yamabe solitons have non-negative scalar curvature. Examples of Yamabe solitons of dimension greater than 2 are not easy to find. However, in [29] we can find the construction of very interesting examples with asymptotic behavior.

1.5 D-tensor

In [7], Cao and Chen classified locally conformally flat gradient steady solitons, they construct an important 3-tensor inspired by Israel's proof of the uniqueness of static vacuum black holes [62], the D-tensor connects the Weyl tensor and the Cotton tensor with the solitons structure. Similarly, in this dissertation, we will use the D-tensor throughout the following three chapters as an important tool for the classification and characterization of solitons.

We remember that the Weyl tensor W is defined by the following formula

$$\begin{aligned} R_{ijkl} &= W_{ijkl} + \frac{1}{n-2}\left(g_{jl}R_{ik} + g_{ik}R_{jl} - g_{jk}R_{il} - g_{il}R_{jk}\right) \\ &\quad - \frac{R}{(n-1)(n-2)}\left(g_{jl}g_{ik} - g_{il}g_{jk}\right), \end{aligned}$$

where R_{ijkl} stands for the Riemannian curvature operator. Moreover, the Cotton tensor C is given according to

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}\left(g_{jk}\nabla_i R - g_{ik}\nabla_j R\right).$$

Proposition 1.7. *[8, Lemma 3.1] Let* (M^n, g, f, ρ) $(n \geq 3)$ *be a complete gradient Ricci soliton satisfying* (1-3). *Then,*

$$D_{ijk} = C_{ijk} + W_{ijkl}\nabla^l f,$$

where

$$\begin{aligned} D_{ijk} &= \frac{1}{n-2}(R_{jk}\nabla_i f - R_{ik}\nabla_j f) \\ &\quad + \frac{1}{2(n-1)(n-2)}[g_{ik}(2R\nabla_j f - \nabla_j R) - g_{jk}(2R\nabla_i f - \nabla_i R)]. \end{aligned} \tag{1-6}$$

Proof. From the soliton equation (1-3), we have

$$\nabla_j R_{jk} - \nabla_j R_{ik} = -\nabla_i\nabla_j\nabla_k f + \nabla_j\nabla_i\nabla_k f = -R_{ijkl}\nabla_l f.$$

Hence, using Proposition 1.6-(3), we can infer that

$$\begin{aligned} C_{ijk} &= \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R) \\ &= -R_{ijkl}\nabla_l f - \frac{1}{(n-1)}(g_{jk}R_{il} - g_{ik}R_{jl})\nabla^l f \\ &= -W_{ijkl}\nabla_l f - \frac{1}{(n-2)}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) \\ &\quad + \frac{1}{2(n-1)(n-2)}(g_{jk}\nabla_i R - g_{ik}\nabla_j R) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{ik}\nabla_j f - g_{jk}\nabla_i f) \\ &= -W_{ijkl}\nabla_l f + D_{ijk}. \end{aligned}$$

□

Remark 1.1. *For the steady Ricci soliton (Chapter 3) we will consider $f := -f$, so the D-tensor gets the form:*

$$\begin{aligned} D_{ijk} &= \frac{1}{n-2}\left(R_{ik}\nabla_j f - R_{jk}\nabla_i f\right) \\ &\quad + \frac{1}{2(n-1)(n-2)}\left[g_{jk}\left(\nabla_i R + 2R\nabla_i f\right) - g_{ik}\left(\nabla_j R + 2R\nabla_j f\right)\right]. \end{aligned}$$

By Proposition 1.7 we can see that D is equal to the Cotton tensor in gradient Ricci soliton of dimension $n = 3$. We can also see that

$$D_{ijk} = -D_{jik} \quad \text{and} \quad g^{ij}D_{ijk} = g^{ik}D_{ijk} = 0.$$

Similarly, for the Yamabe solitons, the relation between the Weyl tensor and the D-tensor is given by the next proposition.

Proposition 1.8. *[44, Proposition 2.2] Let $\left(M^n, g, f\right)$ be a gradient Yamabe soliton. Then,*

$$W_{ijkl}\nabla^l f = D_{ijk},$$

where

$$\begin{aligned} (n-2)D_{ijk} &= (R_{jk}\nabla_i f - R_{ik}\nabla_j f) \\ &+ (\nabla_j R + \frac{R}{n-1}\nabla_j f)g_{ik} - (\nabla_i R + \frac{R}{n-1}\nabla_i f)g_{jk}. \end{aligned} \tag{1-7}$$

Proof. By using Lemma 1.2-(3) and (1-5), we have

$$\begin{aligned} \nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f &= \nabla_i((R-\lambda)g_{kj}) - \nabla_j((R-\lambda)g_{ki}) \\ &= -\frac{1}{n-1}(R_{il}g_{jk}\nabla^l f - R_{jl}g_{ik}\nabla^l f). \end{aligned}$$

Using the Ricci equation, Proposition 1.3-(3) and the definition of Weyl tensor we obtain

$$\begin{aligned} \nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f &= R_{lkji}\nabla^l f \\ &= W_{ijkl}\nabla^l f - \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk})\nabla^l f \\ &\quad + \frac{1}{n-2}(g_{jl}R_{ik} - g_{jk}R_{il} + g_{ik}R_{jl} - g_{il}R_{jk})\nabla^l f \\ &= W_{ijkl}\nabla^l f + \frac{R}{(n-1)(n-2)}(g_{kj}\nabla_i f - g_{ki}\nabla_j f) \\ &\quad - \frac{1}{n-2}(R_{kj}\nabla_i f - R_{ki}\nabla_j f) - \frac{1}{n-2}(g_{jk}R_{il}\nabla^l f - g_{ik}R_{jl}\nabla^l f). \end{aligned}$$

Combining the two equations, by the definition of D_{ijk} we obtain

$$\begin{aligned} W_{ijkl}\nabla^l f &= \frac{1}{n-2}(R_{kj}\nabla_i f - R_{ki}\nabla_j f) + \frac{1}{(n-1)(n-2)}(g_{jk}R_{il}\nabla^l f - g_{ik}R_{jl}\nabla^l f) \\ &\quad - \frac{R}{(n-1)(n-2)}(g_{kj}\nabla_i f - g_{ki}\nabla_j f) \\ &= D_{ijk}. \end{aligned}$$

□

We now present some classification results for solitons via D-tensor. In the following proposition, Cao and Chen [8] proved that the tensor D_{ijk} can provide essential properties of the level set of the potential function. In the main chapters, we will give several theorems proving that D is identically zero under some pinching and asymptotic conditions. Therefore, it is important to understand the consequences of $D = 0$ for the soliton's structure.

Proposition 1.9. *[8, Proposition 3.2] Let (M^n, g, f), $n \geq 3$, be any complete gradient Ricci soliton with $D = 0$, and let c be a regular value of f and $\Sigma_c = \{x \in M; f(x) = c\}$ be the level set of f. Then, for any local orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ with $e_1 = \nabla f/|\nabla f|$ and $\{e_2, \cdots, e_n\}$ tangent to Σ_c, we have*

- *(a) $|\nabla f|^2$ and the scalar curvature R of (M^n, g, f) are constants on Σ_c;*
- *(b) $R_{1a} = 0$ and $e_1 = \nabla f/|\nabla f|$ is an eigenvector for Ric;*
- *(c) The second fundamental form h_{ab} of Σ_c is of the form $h_{ab} = \frac{H}{n-1} g_{ab}$;*
- *(d) The mean curvature H of Σ_c is constant;*
- *(e) On Σ_c the Ricci tensor of (M^n, g, f) either has a unique eigenvalue γ, or has two distinct eigenvalues γ and μ of multiplicity 1 and $n-1$, respectively. Moreover, both ρ and μ are constant on Σ_c.*

Remark 1.2. *The previous proposition holds, in essence, for the Yamabe solitons (cf. [10, 44]).*

The following theorems are important for the classification and characterization of solitons, and they will be crucial in the proof of some main results of this dissertation. The proof of these theorems follows, in part, from the previous proposition.

It is important to note that the compact, steady case, is trivial. We can see this in the following proposition (cf. [7, 42, 45]).

Proposition 1.10. *On a compact manifold M^n, a gradient steady Ricci soliton is necessarily a Ricci-flat Einstein metric.*

Also, the compact gradient Yamabe solitons are characterized by the next theorem.

Theorem 1.3. *[43, Theorem 1] Let (M^n, g) be an n-dimensional compact Yamabe gradient soliton with $n \geq 3$. Then, (M, g) is a manifold of constant scalar curvature, i.e., have a Yamabe metric.*

We have the following results for the non-compact solitons; we need them to prove our main theorems.

Theorem 1.4. *[11, Theorem 1.1] Let (M^n, g, f), $n \geq 5$, be a complete non-compact gradient steady Ricci soliton with vanishing D-tensor. Then, (M^n, g, f) is either Ricci-flat with a constant potential function, or a quotient of the product steady soliton $N^{n-1} \times \mathbb{R}$, where N^{n-1} is Ricci-flat, or isometric to the Bryant solitons (up to scaling).*

The above result also holds, in part, for $n = 4$ and $n = 3$. Let us announce these results below.

Theorem 1.5. *[8, Theorem 1.4] Let (M^4, g, f) be a complete gradient steady Ricci soliton with $D_{ijk} = 0$. Then, (M^4, g, f) is either Ricci-flat or isometric to the Bryant soliton.*

Theorem 1.6. *[7, Theorem 1.2] Let (M^3, g, f) be a complete non-compact gradient steady Ricci soliton with $D_{ijk} = 0$. Then, (M^3, g, f) is either flat or isometric to the Bryant soliton*

In chapter 4 of this work, we will use the classification result for the shrinking Ricci solitons based on the vanishing of the D-tensor.

Theorem 1.7. *[8, Corollary 5.1] Let (M^n, g, f), $n \geq 4$, be a complete gradient shrinking Ricci soliton with $D_{ijk} = 0$. Then,*

- *(M^4, g, f) is either Einstein, or a finite quotient of $\mathbb{R}^4$ or $\mathbb{S}^3 \times \mathbb{R}$;*

- *(M^n, g, f), $n \geq 5$, is either Einstein, or is a finite quotient of the Gaussian shrinking soliton $\mathbb{R}^n$, or is a finite quotient of $N^{n-1} \times \mathbb{R}$, where N^{n-1} is Einstein.*

We must remember that the authors [27, 57, 59] showed that any complete 3-dimensional gradient shrinking Ricci soliton is a finite quotient of either the round sphere $\mathbb{S}^3$, or of the Gaussian shrinking soliton $\mathbb{R}^3$, or of the round cylinder $\mathbb{S}^2 \times \mathbb{R}$.

Another classification theorem through the D-tensor can be found for non-compact gradient expanding Ricci soliton as follows.

Proposition 1.11. *[11, Proposition 3.3] Let (M^n, g, f), $n \geq 5$, be a complete non-compact gradient expanding Ricci soliton with vanishing D-tensor. Then, (M^n, g, f) is either*

1. *Einstein (of negative scalar curvature) with a constant potential function; or*

2. *a quotient of some warped product expanding Ricci soliton of the form*

$$(\mathbb{R}, dr^2) \times_{\varphi} (N^{n-1}, \bar{g}),$$

where $(N^{n-1}, \bar{g})$ is an Einstein manifold.

1.6 Asymptotic behavior of solitons

Brendle [1] uses the D-tensor to classify gradient steady Ricci solitons. To show $D = 0$, Brendle used an exhaustion function to build his asymptotic hypothesis. This technique was of great help in Brendle's work, and we will use part of this idea in the main chapters of this work.

For now, we will show the definition of exhaustion and a result that helped us to pose the theorems of Chapters 3 and 4.

Definition 1.12. *[47] Let $\Omega \subset M$ be any domain of a Riemannian manifold M. A function $\phi : \Omega \to \mathbb{R}$ is called an exhaustion function for Ω if, for any $c \in \mathbb{R}$, the set $\Omega_c \equiv \{x \in \Omega : \phi(x) \leq c\}$ is relatively compact in Ω.*

Definition 1.13. *[47] Let $U \subset M$ be open (bounded or unbounded). An exhaustion of U is a sequence of compacts $\Omega_i \subset U$ such that $U = \cup \Omega_i$ ($\Omega_i \subset int\Omega_{i+1}$), for every $i \in \mathbb{N}$.*

We will present some examples of exhaustion functions for a gradient Ricci Solitons: Proposition 1.8, Proposition 1.9, and Proposition 1.11. Also, in the Yamabes Soliton, it is natural to find exhaustions [63, Theorem 4.1 and Lemma 3.3], and [64, Theorem 1.6].

The characterization of Ricci solitons is very relevant for geometry. Some of the most relevant characteristics are the asymptotic behavior of curvature, potential function, and volume growth rates.

Thus, we will first present some estimates for the potential function in the different Ricci solitons [7, 12, 27], and then present some curvature and volume estimates [12, 21].

Besides being an inspiration, the following results were crucial in demonstrating the results.

We start from the steady case. The following result came from Theorem 3.2 and Corollary 3.4 in [22].

Theorem 1.8. *[22] Let (M^n, g, f) be a complete non-compact gradient steady Ricci soliton. Assume the scalar curvature R goes to zero at infinity (at least linearly) and f is proper (which is true if $Ric > 0$). Then,*

$$r - c\ln(r+1) - c \leq -f(x) \leq r + c,$$

where $r(x) = d(x_0, x)$ is the distance function from $x_0 \in M$, and $c > 0$ is a constant. In particular, M has at most polynomial growth.

In [12], Cao and Zhou proved the following theorem,

Theorem 1.9. *[12, Theorem 1.1] Let (M^n, g, f) be a complete non-compact gradient shrinking Ricci soliton. Then, the potential function f satisfies the estimates*

$$\frac{1}{4}(r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_2)^2.$$

Here, c_1 and c_2 are positive constants depending only on n and the geometry of g_{ij} on the unit ball $B_{x_0}(1)$.

Consequently, we have the following volume growth for geodesic balls.

Theorem 1.10. *[12, Theorem 1.2] Let $(M^n, g f)$ be a complete non-compact gradient shrinking Ricci soliton. Then there exists some positive constant $C_1 > 0$ such that*

$$|B_{x_0}(r)| \leq C_1 r^n$$

for $r > 0$ sufficiently large.

The following estimation was largely inspired by the construction of the main result (Theorem 4.3) of chapter 4.

Theorem 1.11. *[11] Let (M^n, g, f) be a complete non-compact gradient expanding Ricci soliton with nonnegative Ricci curvature (or pinched Ricci curvature). Then, the potential function f satisfies the estimates*

$$\frac{1}{4}(r(x) - c_1)^2 - c_2 \leq -f(x) \leq \frac{1}{4}(r(x) + 2\sqrt{-f(x_0)})^2.$$

Here, $c_1 > 0$ and $c_2 > 0$ are constants.

Remark 1.3. *In Theorem 1.11, we can replace the assumption of nonnegative Ricci curvature $Ric \geq 0$ by $Ric \leq -(\frac{1}{2} - \varepsilon)g$ for any small $\varepsilon > 0$. Of course, the normalizing of f and the coefficients has to be adjusted accordingly.*

We invoke Corollary 4.7 in [21] to infer that the geodesic ball has Euclidean growth for some expanders.

Theorem 1.12. *[21, Corollary 4.7] In particular, any 3-dimensional complete non-compact gradient Ricci expander such that R goes to zero at infinity (f is automatically proper in this case) has Euclidean volume growth, i.e.,*

$$|B_{x_0}(r)| \leq Cr^3,$$

where C is a positive constant, and $x_0 \in M$.

In the study of Ricci solitons, it is natural to consider the qualitative or quantitative properties of a general soliton. For example, Chan and Zhang [21] studied volume estimates for three types of solitons, Cao and Zhou [12] derive optimal growth estimates on the potential functions, and Fernández and García [38] obtained sufficient and necessary conditions for a gradient Ricci soliton be Einstein expressed in terms of upper and lower bounds on the behavior of the Ricci tensor and the potential function.

We now present some results that are of great importance for this dissertation. Remember that there are two generic types of scalar curvature decay for steady Ricci solitons, i.e., the linear decay $R \leq Cr^{-1}$ and the exponential decay $R \leq Ce^{-r}$. The following results prove, in this sense, estimates of curvature decay. The next results about decay on curvature shows to the reader that an asymptotic curvature decay as a hypothesis to classify Ricci solitons is natural to assumption and often used.

Corollary 1.1. *[19, Corollary 1] Let (M^n, g, f) be an n-dimensional complete non-Ricci flat gradient steady Ricci soliton with $Ric \geq 0$. Suppose that $\overline{\lim}_{r\to\infty} rR < \frac{1}{2}$. Then there exists a constant $C > 0$ such that*

$$R \leq Ce^{-r} \qquad on \quad M$$

The following theorem is a result for steady gradient Ricci soliton of dimension 4 and has its basis in the conjecture made by [52], which generalizes the dichotomy of the exponential or linear growth of the gradient steady Ricci soliton.

Theorem 1.13. *[22, Teorema 1.3] Let (M^4, g, f) be a 4-dimensional, complete, non-Ricci flat steady gradient Ricci soliton with proper potential function $\lim_{x\to\infty} f = -\infty$ and linear scalar curvature decay, i.e. $R \leq \overline{C}(r+1)^{-1}$, where $\overline{C}$ is a positive constant. Then there exists a positive constant C such that either one of the following estimates holds near infinity*

$$C^{-1}r^{-1} \leq |Rm| \leq Cr^{-1};$$
$$C^{-1}e^{-r} \leq |Rm| \leq Ce^{-r},$$

where r is the distance function.

An important result of curvature decay in expanding Ricci soliton is:

Theorem 1.14. *[27, Theorem 9.56] If (M^n, g) is a gradient expanding Ricci soliton on a non-compact manifold with pinched Ricci curvature in the sense that $R_{ij} \geq \varepsilon R g_{ij}$ for some $\varepsilon > 0$, where $R \geq 0$, then the scalar curvature R has exponential decay.*

In [52], the authors found that the curvature of a gradient steady Ricci soliton must decay exponentially if it decays faster than linear and the potential function is bounded above. Thus, they obtained the following result.

Theorem 1.15. *[52, Theorem 1.11] Let (M^n, g, f) be a complete gradient steady Ricci soliton with potential f bounded above by a constant. If its Riemann curvature satisfies*

$$|Rm|(x)r(x) = o(1)$$

as $x \to \infty$, then

$$|Rm|(x) \leq c(1 + r(x))^{3(n+1)} e^{-r(x)}.$$

We have for the gradient steady Ricci soliton an estimate for the volume of geodesic balls without any additional assumption.

Theorem 1.16. *[21, Theorem 3.1] Suppose that (M^n, g, f) is a complete non-compact gradient steady Ricci soliton. Then, for all $p \in M$, there exist positive constants a and c such that for all large $r \gg 1$, it holds that*

$$c^{-1} \leq |B_r(p)| \leq c e^{a\sqrt{r}}.$$

An interesting result that should be mentioned is the following one.

Corollary 1.2. *[21, Corollary 3.4] Let (M^n, g, f), $n \geq 4$, be a complete non-compact non-Ricci-flat gradient steady Ricci soliton normalized. If the scalar curvature R of M decays at least linearly and f is proper, then has at most polynomial volume growth.*

CHAPTER 2

Four dimensional gradient Yamabe solitons

2.1 Preliminary Results

Before presenting the main results, we shall present some theorems and lemmas which will be helpful for the establishment of the desired results. This chapter is based on [49].

Lemma 2.1. *[44, Lemma 2.1] Let* $\left(M^n, g, f\right)$ *be a gradient Yamabe soliton. Then,*

1. $n(R-\lambda) = \Delta f$;
2. $\nabla|\nabla f|^2 = 2(R-\lambda)\nabla f$;
3. $\nabla R + \frac{1}{n-1} Ric(\nabla f) = 0.$

The following result can be deduced from the previous lemma.

Proposition 2.1. *Let* $\left(M^n, g, f\right)$ *be a gradient Yamabe soliton. Then,*

$$\Delta R + \frac{1}{2(n-1)}\langle \nabla R, \nabla f\rangle + \frac{1}{n-1}(R-\lambda)R = 0.$$

Proof: Taking the divergence of Lemma 2.1-(3) we get

$$\Delta R + \frac{1}{n-1} div(Ric(\nabla f)) = 0.$$

Changing $T = Ric$, $Z = \nabla f$ and $\phi = 1$ in Lemma 1.1 and using the contracted second Bianchi identity (Proposition 1.5) we obtain

$$\begin{aligned} \Delta R + \frac{1}{(n-1)}[\langle div(Ric), \nabla f\rangle + \langle Ric, \nabla^2 f\rangle] &= 0; \\ \Delta R + \frac{1}{2(n-1)}\langle \nabla R, \nabla f\rangle + \frac{1}{n-1}(R-\lambda)R &= 0. \qquad (2\text{-}1) \end{aligned}$$

□

Remember that, for a Riemannian manifold (M^n, g), $n \geq 3$, the Weyl tensor W is defined by the following decomposition formula

$$\begin{aligned} R_{ijkl} &= W_{ijkl} + \frac{1}{n-2}\left(g_{jl}R_{ik} + g_{ik}R_{jl} - g_{jk}R_{il} - g_{il}R_{jk}\right) \\ &\quad - \frac{R}{(n-1)(n-2)}\left(g_{jl}g_{ik} - g_{il}g_{jk}\right), \end{aligned}$$

where R_{ijkl} and R_{ij} stand for the Riemannian and Ricci curvatures for the metric tensor g, respectively.

We obtain from (1-7) the following key lemma (cf. [1, 62]).

Lemma 2.2. *Let $\left(M^n, g, f\right)$ be a gradient Yamabe soliton. Then,*

$$\begin{aligned} (n-2)^2(n-4)|D|^2 + 2(n-1)n^2|\nabla R|^2 &= 2(n-4)|\nabla f|^2\left(|Ric|^2 - \frac{R^2}{(n-1)}\right) \\ &\quad + 8div\left(n(n-1)R\nabla R + R^2\nabla f\right). \end{aligned}$$

Proof. We start by calculating the norm of the D-tensor (1-7), from Definition 1.1 we get

$$\begin{aligned} (n-2)^2|D|^2 &= 2|R_{jk}\nabla_i f|^2 - 2R_{jk}\nabla_i f R_{ik}\nabla_j f + 2(n-1)|\nabla R + \frac{R}{n-1}\nabla f|^2 \\ &\quad + 4(R_{jk}\nabla_i f - R_{ik}\nabla_j)(\nabla_j R + \frac{R}{n-1}\nabla f)g_{ik}; \\ (n-2)^2|D|^2 &= 2|Ric|^2|\nabla f|^2 - 2\langle Ric(\nabla f), Ric(\nabla f)\rangle + 2(n-1)|\nabla R + \frac{R}{n-1}\nabla f|^2 \\ &\quad + 4\langle Ric(\nabla f) - R\nabla f, \nabla R + \frac{R}{n-1}\nabla f\rangle. \end{aligned}$$

Then, by using Lemma 2.1-(3) we have

$$\begin{aligned} (n-2)^2|D|^2 &= 2|Ric|^2|\nabla f|^2 - 2(n-1)^2|\nabla R|^2 - 2(n-1)|\nabla R + \frac{R}{n-1}\nabla f|^2 \\ &= 2|\nabla f|^2\left(|Ric|^2 - \frac{R^2}{(n-1)}\right) - 2n(n-1)|\nabla R|^2 - 4R\langle\nabla R, \nabla f\rangle. \end{aligned}$$

Hence, from (2-1) we obtain

$$\begin{aligned} (n-2)^2|D|^2 &= 2|\nabla f|^2\left(|Ric|^2 - \frac{R^2}{(n-1)}\right) - 2n(n-1)|\nabla R|^2 \\ &\quad + 8R^2(R-\lambda) + 8(n-1)R\Delta R. \end{aligned} \tag{2-2}$$

On the other hand, from Lemma 2.1-(1) and (2-1), we deduce the following divergence formula:

$$div\left(8n\frac{(n-1)}{(n-4)}R\nabla R+\frac{8}{n-4}R^2\nabla f\right)=8R^2(R-\lambda)+8(n-1)R\Delta R+8n\frac{(n-1)}{(n-4)}|\nabla R|^2.$$

Finally, combining the last two equations, the result follows. □

2.2 Main Results

We will now present the main results of this chapter. The goal is to prove that under certain pinched and asymptotic conditions, the Yamabe soliton metric becomes rigid, i.e., a Yamabe metric (a metric with constant scalar curvature). We will also show that the D-tensor is identically zero under those conditions (cf. Proposition 1.9).

Theorem 2.1. *Let (M^n, g, f), $n \geq 5$, be an n-dimensional gradient Yamabe soliton such that*

$$|Ric|^2 \leq \frac{R^2}{n-1}. \tag{2-3}$$

Assume that there exists an exhaustion of M^n by bounded domains Ω_ℓ such that

$$\lim_{\ell\to+\infty}\int_{\partial\Omega_\ell} R\langle\nabla R+\frac{R}{n(n-1)}\nabla f,\nu\rangle=0, \tag{2-4}$$

where ν is the normal vector field. Then, (M^n, g) must have constant scalar curvature, i.e., $R=\lambda$. Moreover, the D-tensor (1-6) vanishes identically, and the equality holds in (2-3).

Proof. Let Ω be a bounded domain on M with smooth boundary $\partial\Omega$. Using the divergence theorem, we get

$$\begin{aligned}\int_\Omega[(n-2)^2(n-4)|D|^2+2(n-1)n^2|\nabla R|^2] &= 2(n-4)\int_\Omega\left[|\nabla f|^2\left(|Ric|^2-\frac{R^2}{(n-1)}\right)\right]\\ &\quad +8\int_\Omega div\left(n(n-1)R\nabla R+R^2\nabla f\right)\\ &= 2(n-4)\int_\Omega\left[|\nabla f|^2\left(|Ric|^2-\frac{R^2}{(n-1)}\right)\right]\\ &\quad +8\int_{\partial\Omega}\langle n(n-1)R\nabla R+R^2\nabla f,\nu\rangle,\end{aligned}$$

where ν is the normal vector field.

Consider an exhaustion of M by bounded domains Ω_ℓ such that

$$\lim_{\ell\to\infty}\int_{\partial\Omega_\ell} n(n-1)R\langle\nabla R+\frac{R}{n(n-1)}\nabla f,\nu\rangle=0.$$

Therefore,

$$\begin{aligned}\int_{\Omega_\ell}[(n-2)^2(n-4)|D|^2+2(n-1)n^2|\nabla R|^2] &= 2(n-4)\int_{\Omega_\ell}\left[|\nabla f|^2\left(|Ric|^2-\frac{R^2}{(n-1)}\right)\right]\\ &\quad+\int_{\partial\Omega_\ell}8\langle n(n-1)R\nabla R+R^2\nabla f,\nu\rangle. \qquad (2\text{-}5)\end{aligned}$$

Then, making $\ell\to\infty$ we have

$$\begin{aligned}\int_{M}[(n-2)^2(n-4)|D|^2+2(n-1)n^2|\nabla R|^2] &+ 2(n-4)\int_{M}\left[|\nabla f|^2\left(\frac{R^2}{(n-1)}-|Ric|^2\right)\right]=\\ &= 8\lim_{\ell\to\infty}\int_{\partial\Omega_\ell}\langle n(n-1)R\nabla R+R^2\nabla f,\nu\rangle\\ &= 0.\end{aligned}$$

Finally, the D-tensor (for $n\geq 5$) and $|\nabla R|$ are identically zero. Also, observe that there is no open subset Ω of M^n where $\{\nabla f=0\}$ is dense. If f is constant in Ω, since M^n is complete, we have that f is analytic, which implies f is constant everywhere. Therefore, $\frac{R^2}{(n-1)}=|Ric|^2$. ☐

The choice of dimension in Lemma 2.2 leads us to different consequences. Let us see an interesting rigidity result concerning the steady Yamabe solitons with dimension $n=3$.

Theorem 2.2. *Let (M^3,g,f) be a complete three-dimensional gradient steady Yamabe soliton such that*

$$|Ric|^2\geq\frac{R^2}{2}.$$

Assume that there exists an exhaustion of M^3 by bounded domains Ω_ℓ such that

$$\lim_{\ell\to+\infty}\int_{\partial\Omega_\ell}R\langle\nabla R+\frac{R}{6}\nabla f,\nu\rangle=0,$$

where ν is the normal vector field. Then, (M^3,g) must be isometric to the standard three-dimensional Euclidean space.

Important estimates of the curvature in the Yamabe Solitons were recently discovered. In [51] with a natural asymptotic condition (i.e., $\lim_{x\to\infty}R(x)\leq 0$), the authors proved that the sectional curvature is nonnegative. Also, in [63] under an asymptotic condition, it was demonstrated that the scalar curvature R is constant.

Proof of Theorem 2.2. Following the proof of Teorema 2.1, we have

$$\begin{aligned}\int_{\Omega_\ell}[-|D|^2+36|\nabla R|^2] &= -2\int_{\Omega_\ell}\left[|\nabla f|^2\left(|Ric|^2-\frac{R^2}{2}\right)\right]\\ &+ 8\int_{\partial\Omega_\ell}\langle 6R\nabla R+R^2\nabla f,\nu\rangle.\end{aligned}$$

Making $\ell\to\infty$ and taking into account that $n=3$ $(W=0)$ we have $D=0$, then we conclude that $|\nabla R|$ is identically zero.

If $\{\nabla f=0\}$ is not dense and M is complete we obtains that

$$|Ric|^2=\frac{R^2}{2}.$$

Hence, the steady soliton is Ricci-flat if $R=0$. Consequently, from (2-2), (M^3,g,f) with $\lambda=0$ must be isometric to $\mathbb{R}^3$ with standard metric.

□

Considering the asymptotic condition (2-4) we prove that any four-dimensional gradient Yamabe soliton is a Riemannian manifold with constant scalar curvature, i.e., a Yamabe metric. In this result, no additional condition on the curvature is necessary and no restriction for λ is required.

Theorem 2.3. *Let (M^4,g,f) be a four dimensional gradient Yamabe soliton. Moreover, assume that there exists an exhaustion of M^4 by bounded domains Ω_ℓ such that*

$$\lim_{\ell\to+\infty}\int_{\partial\Omega_\ell}R\langle\nabla R+\frac{R}{12}\nabla f,\nu\rangle=0.$$

Then, (M^4,g,f) has constant scalar curvature, i.e., $R=\lambda$.

Proof. Following the proof of Theorem 2.1 in (2-5) we take $n=4$ to obtain

$$\int_{\Omega_\ell}[96|\nabla R|^2] = 8\int_{\partial\Omega_\ell}\langle 12R\nabla R+R^2\nabla f,\nu\rangle.$$

Then, making $\ell\to\infty$ we have

$$\int_M[96|\nabla R|^2]=8\lim_{\ell\to\infty}\int_{\partial\Omega_\ell}\langle 12R\nabla R+R^2\nabla f,\nu\rangle=0.$$

Finally, $|\nabla R|$ are identically zero

□

Remark 2.1. *We can also conclude from the proof of Theorem 2.1, $n\geq 5$, that the D-tensor (1-6) vanishes identically. This fact has a great impact on the structure of gradient Yamabe solitons (Proposition 1.9, see also [44, Proposition 2.4]).*

CHAPTER 3

A comparison theorem for steady Ricci solitons

3.1 Preliminary Results

Similarly, as in the previous Chapter, we will give some preliminary results before presenting our main results and following the same structure argumentation as in Chapter 2.

First, we review some basic facts and present key results that will be useful in proving our main result. To that end, in this section, we consider $f := -f$. Thus, from Proposition 1.6-(2), (3) with $\rho = 0$ we have

$$\nabla R = -2Ric(\nabla f). \tag{3-1}$$

And,

$$R + |\nabla f|^2 = C_0 \tag{3-2}$$

at each any point on M, where C_0 is a positive constant. Up to a normalization of f, we may assume that $C_0 = 1$. We remember that a complete gradient steady Ricci soliton has nonnegative scalar curvature (see [6, Lemma 2.2]), i.e., $R \geq 0$. Therefore, we conclude that a steady (normalized) gradient Ricci soliton must satisfy

$$0 \leq R \leq 1.$$

As a consequence, we have the following key lemma (see [1, Proposition 4]).

Lemma 3.1. *Let $\left(M^n, g, f\right)$ be a gradient steady Ricci soliton. Then we have:*

$$|D|^2 + \frac{|\nabla R + 2R\nabla f|^2}{2(n-1)(n-2)^2} = -\frac{(1-R)}{(n-2)^2}\Delta R - \frac{(1-R)}{(n-2)^2}\langle \nabla R, \nabla f\rangle - \frac{1}{2(n-2)^2}|\nabla R|^2.$$

Where D is the tensor defined in Section 1.5

Proof. Let us write the norm of D only depending on the function f and the scalar curvature R. We begin the computation using (1-6) and Hilbert-Schmidt's norm, Definition 1.1. Therefore, we start with the following equality

$$\begin{aligned}|D|^2 &= \frac{2}{(n-2)^2}|Ric|^2|\nabla f|^2 - \frac{2}{(n-2)^2}R_{ik}\nabla_j f R_{jk}\nabla_i f\\ &+ \frac{1}{2(n-1)(n-2)^2}|\nabla R + 2R\nabla f|^2\\ &+ \frac{2}{(n-1)(n-2)^2}(R_{ik}\nabla_j f - R_{jk}\nabla_i f)(\nabla_i R + 2R\nabla_i f)g_{jk}.\end{aligned}$$

Then, by using Definition 1.10, (3-1) and (3-2) we get

$$\begin{aligned}|D|^2 &= \frac{2}{(n-2)^2}|Ric|^2|\nabla f|^2 - \frac{2}{(n-2)^2}\nabla_i\nabla_k f\nabla_j f\nabla_j\nabla_k f\nabla_i f\\ &+ \frac{1}{2(n-1)(n-2)^2}|\nabla R + 2R\nabla f|^2\\ &- \frac{1}{(n-1)(n-2)^2}(\nabla_i R + 2R\nabla_i f)(\nabla_i R + 2R\nabla_i f)\\ &= \frac{2}{(n-2)^2}|Ric|^2|\nabla f|^2 - \frac{1}{2(n-2)^2}|\nabla R|^2\\ &- \frac{1}{2(n-1)(n-2)^2}(|\nabla R|^2 + 4R\langle\nabla R, \nabla f\rangle + 4R^2|\nabla f|^2). \qquad (3\text{-}3)\end{aligned}$$

On the other hand, calculating $\nabla|\nabla f|^2$ and using Definition 1.10 we get

$$\begin{aligned}\nabla_j|\nabla f|^2 &= \nabla_j\langle\nabla_i f, \nabla_i f\rangle\\ &= 2\left\langle\nabla_j\nabla_i f, \nabla_i f\right\rangle\\ &= 2\langle Hess f, \nabla_i f\rangle\\ &= 2R_{ij}(\nabla_i f).\end{aligned}$$

Thus, we obtain the following property

$$2Ric(\nabla f) = \nabla|\nabla f|^2.$$

Taking the divergence, Lemma 1.1, the above identity and Definition 1.10, we obtain

$$2|Ric|^2 + \langle\nabla R, \nabla f\rangle = \Delta|\nabla f|^2,$$

we can check the above identity in Proposition 1.6 -(5). Thus, from (3-2) we have

$$2|Ric|^2 = -\langle \nabla R, \nabla f\rangle - \Delta R.$$

Therefore, by combining this identities with (3-3) we obtain

$$|D|^2 + \frac{|\nabla R + 2R\nabla f|^2}{2(n-1)(n-2)^2} = \frac{-1}{(n-2)^2}|\nabla f|^2\Delta R - \frac{|\nabla f|^2}{(n-2)^2}\langle \nabla R, \nabla f\rangle - \frac{1}{2(n-2)^2}|\nabla R|^2.$$

The result follows by using one more time (3-2) in the above equation. □

Remark 3.1. *We need to point out that Proposition 4 in [1] follows from the above lemma considering* $n = 3$. *In fact, we can rewrite Lemma 3.1 in the following form*

$$\begin{aligned} |D|^2 &= \frac{-(1-R)\Delta R}{(n-2)^2} - \frac{\langle \nabla R, \nabla f\rangle}{(n-2)^2} - \frac{n|\nabla R|^2}{2(n-1)(n-2)^2} \\ &+ \frac{(n-3)R\langle \nabla R, \nabla f\rangle}{(n-1)(n-2)^2} - \frac{2R^2(1-R)}{(n-1)(n-2)^2}. \end{aligned}$$

The above equation was used in [1], for $n = 3$.

In what follows, we provide a new divergence formula for the gradient steady Ricci solitons. Now, consider smooth functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$ such that

$$Y = \phi(R)\nabla R + \psi(R)\nabla f$$

on M.

Lemma 3.2. *Let* (M^n, g, f) *be a gradient steady Ricci soliton. Then,*

$$(1-R)^{3/2} div\left(\frac{\nabla R}{\sqrt{1-R}} - 2\sqrt{1-R}\nabla f\right) = -(n-2)^2|D|^2 - \frac{|\nabla R + 2R\nabla f|^2}{2(n-1)} - 2R(1-R)^2.$$

Proof. From Proposition 1.6-(4) with $\rho = 0$, we have $\Delta f = R$. So,

$$div(Y) = \phi\Delta R + \phi'|\nabla R|^2 + \psi'\langle \nabla R, \nabla f\rangle + \psi\Delta f = \phi\Delta R + \phi'|\nabla R|^2 + \psi'\langle \nabla R, \nabla f\rangle + R\psi.$$

Thus,

$$\begin{aligned} 2(1-R)div(Y) &= 2(1-R)\phi\Delta R + 2(1-R)\phi'|\nabla R|^2 \\ &\quad + 2(1-R)\psi'\langle \nabla R, \nabla f\rangle + 2(1-R)R\psi. \end{aligned}$$

Combining this with the previous lemma, we get

$$\begin{aligned}2\phi(n-2)^2|D|^2+\frac{\phi|\nabla R+2R\nabla f|^2}{(n-1)} &= -2\phi(1-R)\Delta R-2\phi(1-R)\langle\nabla R,\nabla f\rangle-\phi|\nabla R|^2\\ &= -2(1-R)div(Y)+2(1-R)\phi'|\nabla R|^2+2(1-R)\psi'\langle\nabla R,\nabla f\rangle\\ &\quad +2(1-R)R\psi-2(1-R)\phi\langle\nabla R,\nabla f\rangle-\phi|\nabla R|^2.\end{aligned}$$

Hence,

$$\begin{aligned}2\phi(n-2)^2|D|^2+\frac{\phi|\nabla R+2R\nabla f|^2}{(n-1)} &= -2(1-R)div(Y)+2(1-R)R\psi\\ &+ [2(1-R)\phi'-\phi]|\nabla R|^2+2(1-R)[\psi'-\phi]\langle\nabla R,\nabla f\rangle.\end{aligned}$$

Consider,

$$\phi=\frac{k}{\sqrt{1-R}}\quad\text{and}\quad\psi=-2k\sqrt{1-R},$$

where k is a nonnull constant. Therefore,

$$2\phi(n-2)^2|D|^2+\frac{\phi|\nabla R+2R\nabla f|^2}{(n-1)}+4k(1-R)^{3/2}R=-2(1-R)div(Y).$$

□

3.2 Main Results

We will now present our main results. Our first goal in this chapter is to obtain rigidity results for gradient steady Ricci solitons through the divergence equation (Lemma 3.2). We established the following theorem.

Theorem 3.1. *Let* (M^n,g,f) *be a complete non-compact gradient steady Ricci soliton satisfying*

$$\sigma R|\nabla f|\leq|\nabla R|, \tag{3-4}$$

where $\sigma=\frac{(n+1)+\sqrt{(n-1)(7n-13)}}{3n-2}$. *Suppose that there exists an exhaustion of* M^n *by bounded domains* Ω_ℓ *such that*

$$\lim_{\ell\to+\infty}\int_{\partial\Omega_\ell}|\nabla R+R\nabla f|=0. \tag{3-5}$$

Then (M^n,g,f) *is either Ricci flat with a constant potential function, or a quotient of the product steady soliton* $N^{n-1}\times\mathbb{R}$, *where* N^{n-1} *is Ricci flat, or isometric to the Bryant* *soliton (up to scalings).*

Remark 3.2. *We point out that no sectional curvature bound is assumed. This is important because, in dimension* 4*, the sectional curvature of shrinking and steady Ricci solitons may change sign. Moreover, equality in* (3-4) *holds for the cigar soliton with* $\sigma = 1$ *(*$n = 2$*), and* (3-5) *is trivially satisfied. In that sense, our theorem is a comparison result with the geometry of the cigar soliton.*

Proof of Theorem 3.1. From Lemma 3.2 we can infer that

$$(1-R)^{3/2} div\,(Y) = -(n-2)^2|D|^2 - \frac{|\nabla R + 2R\nabla f|^2}{2(n-1)} - 2R(1-R)^2,$$

where $Y = \frac{\nabla R}{\sqrt{1-R}} - 2\sqrt{1-R}\nabla f$.

Let Ω be a bounded domain on M with smooth boundary. Using the divergence theorem, we get

$$\begin{aligned}
&\int_{\partial\Omega} \langle (1-R)\nabla R - 2(1-R)^2\nabla f, \nu\rangle = \int_{\Omega\cap\{R<1\}} div\left((1-R)^{3/2}Y\right)\\
=\ & \int_{\Omega\cap\{R<1\}} (1-R)^{3/2} div\,(Y) - \frac{3}{2}\int_{\Omega\cap\{R<1\}} \sqrt{1-R}\langle Y, \nabla R\rangle\\
=\ & -\int_{\Omega\cap\{R<1\}} \left[(n-2)^2|D|^2 + \frac{|\nabla R + 2R\nabla f|^2}{2(n-1)} + 2R(1-R)^2\right]\\
-\ & \frac{3}{2}\int_{\Omega\cap\{R<1\}} \sqrt{1-R}\langle Y, \nabla R\rangle\\
=\ & -\int_{\Omega\cap\{R<1\}} \left[(n-2)^2|D|^2 + \frac{|\nabla R + 2R\nabla f|^2}{2(n-1)} + 2R(1-R)^2\right]\\
-\ & \frac{3}{2}\int_{\Omega\cap\{R<1\}} \langle \nabla R - 2(1-R)\nabla f, \nabla R\rangle\\
=\ & -\int_{\Omega\cap\{R<1\}} \left[(n-2)^2|D|^2 + \frac{|\nabla R + 2R\nabla f|^2}{2(n-1)} + 2R(1-R)^2\right]\\
-\ & \frac{3}{2}\int_{\Omega\cap\{R<1\}} [|\nabla R|^2 - 2(1-R)\langle \nabla f, \nabla R\rangle]\\
=\ & -\int_{\Omega\cap\{R<1\}} \left[(n-2)^2|D|^2 + \frac{|\nabla R + 2R\nabla f|^2}{2(n-1)}\right]\\
-\ & \int_{\Omega\cap\{R<1\}} [\frac{3}{2}|\nabla R|^2 + 6(1-R)Ric(\nabla f, \nabla f) + 2R(1-R)^2],
\end{aligned} \tag{3-6}$$

where ν is the normal vector field of ∂M. We used in the above equation the identity

$$-2Ric(\nabla f) = \nabla R. \tag{3-7}$$

Furthermore, using $R = \Delta f$ we get

$$div((1-R)^2\nabla f) - 4(1-R)Ric(\nabla f, \nabla f) = R(1-R)^2.$$

Thus, from the above equations we have

$$\begin{aligned}\int_{\partial\Omega}\langle (1-R)\nabla R - 2(1-R)^2\nabla f, \nu\rangle &= -\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2\right]\\ &\quad -\int_{\Omega\cap\{R<1\}}\left[\frac{|\nabla R + 2R\nabla f|^2}{2(n-1)} + \frac{3}{2}|\nabla R|^2\right.\\ &\quad \left. -2(1-R)Ric(\nabla f, \nabla f) + 2div((1-R)^2\nabla f)\right],\end{aligned}$$

and so

$$\begin{aligned}\int_{\partial\Omega}\langle (1-R)\nabla R, \nu\rangle &= -\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2 + \frac{|\nabla R + 2R\nabla f|^2}{2(n-1)}\right]\\ &\quad -\int_{\Omega\cap\{R<1\}}[\frac{3}{2}|\nabla R|^2 - 2(1-R)Ric(\nabla f, \nabla f)]. \qquad (3\text{-}8)\end{aligned}$$

Moreover, a straightforward computation yields to

$$\begin{aligned}\int_{\partial\Omega}\langle (1-R)\nabla R, \nu\rangle &\leq -\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2 + \frac{|\nabla R + R\nabla f|^2}{2(n-1)}\right]\\ &\quad -\frac{3}{2}\int_{\Omega\cap\{R<1\}}\left[|\nabla R|^2 + \frac{1}{(n-1)}R^2|\nabla f|^2\right]\\ &\quad -\frac{1}{(n-1)}\int_{\Omega\cap\{R<1\}}R\langle\nabla R, \nabla f\rangle + \int_{\Omega\cap\{R<1\}}2(1-R)Ric(\nabla f, \nabla f)\\ &\leq -\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2 + \frac{n|\nabla R + R\nabla f|^2}{2(n-1)}\right]\\ &\quad -\int_{\Omega\cap\{R<1\}}|\nabla R|^2 + \frac{(n-4)}{2(n-1)}\int_{\Omega\cap\{R<1\}}R^2|\nabla f|^2\\ &\quad +\frac{n-2}{(n-1)}\int_{\Omega\cap\{R<1\}}R\langle\nabla R, \nabla f\rangle\\ &\quad +\int_{\Omega\cap\{R<1\}}2(1-R)Ric(\nabla f, \nabla f).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{\partial\Omega}(1-R)\langle\nabla R+R\nabla f,\nu\rangle \leq & -\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2+\frac{n|\nabla R+R\nabla f|^2}{2(n-1)}\right]\\ & -\int_{\Omega\cap\{R<1\}}|\nabla R|^2+\frac{(n-4)}{2(n-1)}\int_{\Omega\cap\{R<1\}}R^2|\nabla f|^2\\ & +\frac{n-2}{(n-1)}\int_{\Omega\cap\{R<1\}}R\langle\nabla R,\nabla f\rangle\\ & -\int_{\Omega\cap\{R<1\}}(1-R)\langle\nabla R,\nabla f\rangle+\int_{\partial\Omega}(1-R)R\langle\nabla f,\nu\rangle.\end{aligned}$$

We also have the following identity:

$$\begin{aligned}\int_{\partial\Omega}(1-R)R\langle\nabla f,\nu\rangle &= \int_{\Omega\cap\{R<1\}}div((1-R)R\nabla f)\\ &= \int_{\Omega\cap\{R<1\}}[(1-R)R\Delta f+(1-R)\langle\nabla R,\nabla f\rangle-R\langle\nabla R,\nabla f\rangle]\\ &= \int_{\Omega\cap\{R<1\}}[R^2|\nabla f|^2+(1-R)\langle\nabla R,\nabla f\rangle-R\langle\nabla R,\nabla f\rangle].\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\partial\Omega}(1-R)\langle\nabla R+R\nabla f,\nu\rangle \leq & -\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2+\frac{n|\nabla R+R\nabla f|^2}{2(n-1)}\right]\\ & -\int_{\Omega\cap\{R<1\}}|\nabla R|^2+\frac{3(n-2)}{2(n-1)}\int_{\Omega\cap\{R<1\}}R^2|\nabla f|^2\\ & +\frac{1}{(n-1)}\int_{\Omega\cap\{R<1\}}2RRic(\nabla f,\nabla f) \qquad (3\text{-}9)\\ = & -\int_{\Omega\cap\{R<1\}}[(n-2)^2|D|^2]\\ & +\int_{\Omega\cap\{R<1\}}\left[-\frac{(3n-2)}{2(n-1)}|\nabla R|^2+\frac{(n-3)}{(n-1)}R^2|\nabla f|^2\right]\\ & +\frac{(n+1)}{(n-1)}\int_{\Omega\cap\{R<1\}}2RRic(\nabla f,\nabla f).\end{aligned}$$

Considering (3-7) we can infer that

$$\begin{aligned}2Ric(\nabla f,\nabla f) &= \langle 2Ric(\nabla f),\nabla f\rangle\\ &\leq |-\nabla R||\nabla f|\\ &\leq |\nabla R||\nabla f|.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\partial\Omega}(1-R)\langle\nabla R+R\nabla f,\nu\rangle \;\leq\; & -\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2\right]\\ & +\int_{\Omega\cap\{R<1\}}\left[-\frac{(3n-2)}{2(n-1)}|\nabla R|^2+\frac{(n+1)R|\nabla f|}{(n-1)}|\nabla R|\right.\\ & \left.+\frac{(n-3)}{(n-1)}R^2|\nabla f|^2\right].\end{aligned}$$

Now, from hypothesis

$$\frac{(n+1)+\sqrt{(n-1)(7n-13)}}{3n-2}R|\nabla f|\leq|\nabla R|.$$

Therefore,

$$-\frac{(3n-2)}{2(n-1)}|\nabla R|^2+\frac{(n+1)R|\nabla f|}{(n-1)}|\nabla R|+\frac{(n-3)}{(n-1)}R^2|\nabla f|^2\leq 0.$$

Consequently,

$$\int_{\Omega_\ell\cap\{R<1\}}\left[(n-2)^2|D|^2\right]\leq-\int_{\partial\Omega_\ell}(1-R)\langle\nabla R+R\nabla f,\nu\rangle.$$

Now, consider an exhaustion of M by bounded domains Ω_ℓ such that

$$\lim_{\ell\to\infty}\int_{\partial\Omega_\ell}|\nabla R+R\nabla f|=0.$$

Then, making $\ell\to\infty$, we have

$$\begin{aligned}&\int_{\{R<1\}}\left[(n-2)^2|D|^2\right]\leq-\lim_{\ell\to\infty}\int_{\partial\Omega_\ell}(1-R)\langle\nabla R+R\nabla f,\nu\rangle\\ &\leq\left|\lim_{\ell\to\infty}\int_{\partial\Omega_\ell}(1-R)\langle\nabla R+R\nabla f,\nu\rangle\right|\\ &\leq\lim_{\ell\to\infty}\int_{\partial\Omega_\ell}(1-R)|\langle\nabla R+R\nabla f,\nu\rangle|\leq\lim_{\ell\to\infty}\int_{\partial\Omega_\ell}|\nabla R+R\nabla f|=0.\end{aligned}$$

Since M is complete, from (3-2) we can infer that $\{R<1\}$ is dense. Otherwise, it would exist $Q\subset\{R<1\}$ such that $R\big|_Q=1$, and so $|\nabla f|^2|_Q=0$, thus f will be a constant function. Therefore, the D-tensor is identically zero on M.

To finish the proof, we need to invoke Theorem 1.4, Theorem 1.5, and Theorem 1.6. □

Remark 3.3. *Let $\{E_1,E_2,\cdots,E_n\}$ be a local orthonormal frame consisting of eigenvec-*

tors of the Ricci tensor. Then, if the sectional curvature is non-negative, we obtain

$$\begin{aligned}\frac{1}{2}R - R_{ii} &= \frac{1}{2}\left(\sum_{j\neq i} R_{jj} - R_{ii}\right)\\ &= \frac{1}{2}\sum_{j\neq i}(R_{jj} - R_{jiij})\\ &= \frac{1}{2}\sum_{j\neq i, k\neq i, k\neq j} R_{kjjk} \leq 0.\end{aligned}$$

Thus, $0 \leq \frac{1}{2}Rg_{ij} - R_{ij}$ *and if* $0 \leq R_{ij}$ *we get* $0 \leq (\frac{1}{2}Rg_{ij} - R_{ij})R_{ij} = \frac{1}{2}R^2 - |Ric|^2$. *Therefore,*

$$|\nabla R|^2 \leq 2R^2|\nabla f|^2.$$

This shows us that (3-4) can be interpreted as a lower bound for $|\nabla R|$. *In fact,* $\sigma \leq \dfrac{1+\sqrt{7}}{3}$, *see also [31, Lemma 2.3].*

One conjecture concerning the curvature decay of a gradient steady Ricci soliton is

Conjecture 3.1 ([22]). *If* (M, g, f) *is a complete non-Ricci flat gradient steady Ricci soliton with* $|Rm| \to 0$, *then either one of the following estimates holds outside a compact set of* M

$$\begin{aligned}C^{-1}r^{-1} \leq |Rm| \leq Cr^{-1};\\ C^{-1}e^{-r} \leq |Rm| \leq Ce^{-r},\end{aligned}$$

where C *is a positive constant and* r *is the distance function.*

In this sense, Chan and Zhu in Theorem 1.13 proved that in a gradient steady Ricci soliton (M^4, g, f) with $\lim_{x\to\infty} f = -\infty$ and linear scalar curvature decay, these estimates hold.

In citechan2019 (see Corollary 1.1), the author proved the scalar curvature of a steady Ricci soliton with nonnegative Ricci curvature decays exponentially if an asymptotic condition holds.

In Theorem 1.14, Chow, Lu, and Ni proved that the scalar curvature of complete and non-compact gradient steady Ricci soliton with positively pinched Ricci curvature has exponential decay. In that sense, condition (3-5) can be replaced by an exponential decay for the scalar curvature (cf. Corollary 3.1), i.e.,

$$R = o(e^{-r}), \quad r \to \infty,$$

where r stands for the geodesic distance from a fixed point. Here, “o” means that exist a positive constant A such that $|R| \leq Ae^{-r}$, for $r \geq r_0$.

In Theorem 1.15, the authors proved that the sectional curvature Rm of a gradient steady Ricci soliton (non-flat) with potential function f bounded from above by a constant and such that $|Rm|r = o(1)$, at infinity, decays like

$$|Rm| \leq c(1+r)^{3(n+1)}e^{-r}, \tag{3-10}$$

where c is a positive constant and r stands for the geodesic distance. It is known that the assumption over f holds true when $Ric > 0$. The exponential decay rate in the theorem is sharp as seen from $M = N \times \Sigma$; where Σ is the cigar soliton and N is a compact Ricci flat manifold.

The control of the curvature is an important issue in the analysis of a Ricci soliton. Several results prove that the scalar curvature controls the sectional curvature, see [9, 19]. Here is important to emphasize that four-dimensional Ricci solitons must satisfy the following condition:

$$|Rm| \leq A\left(\frac{|\nabla Ric|}{|\nabla f|} + |Ric|\right),$$

where A is an universal positive constant (see also [9, 53]). Please, see also [19, Proposition 2].

Therefore, inspired by the above curvature properties of four-dimensional steady Ricci solitons we will prove that if the Ricci curvature is controlled, we get the Bryant soliton. Moreover, we will not assume that the steady Ricci soliton is κ-noncollapsed.

Theorem 3.2. *Let $\left(M^n, g, f\right)$ be a complete non-compact gradient steady Ricci soliton with positive Ricci curvature and*

$$|Ric| \leq \frac{1}{4}\left[\frac{3}{2}\frac{|\nabla R|^2}{|\nabla f|^2} - 2R^2\right].$$

Suppose that

$$\lim_{r\to\infty} R = 0 \quad \textit{and} \quad \lim_{r\to\infty} |Rm|r = o(1).$$

Then (M^n, g, f) is isometric to the Bryant soliton (up to scalings).

Proof. From (3-6) we get

$$\int_{\partial\Omega}(1-R)\langle\nabla R-2(1-R)\nabla f,\nu\rangle=-\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2+\frac{|\nabla R+2R\nabla f|^2}{2(n-1)}\right]$$
$$-\int_{\Omega\cap\{R<1\}}[\frac{3}{2}|\nabla R|^2+6(1-R)Ric(\nabla f,\nabla f)+2R(1-R)^2].$$

Hence, by rearranging the equation, we get

$$\int_{\partial\Omega}(1-R)\langle\nabla R+2R\nabla f,\nu\rangle=-\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2+\frac{|\nabla R+2R\nabla f|^2}{2(n-1)}\right]$$
$$-\int_{\Omega\cap\{R<1\}}[\frac{3}{2}|\nabla R|^2+6(1-R)Ric(\nabla f,\nabla f)+2R(1-R)^2]+2\int_{\partial\Omega}(1-R)\langle\nabla f,\nu\rangle.$$

Thus, we use that $div[2(1-R)\nabla f]=-2\langle\nabla R,\nabla f\rangle+2(1-R)R$ to obtain

$$\int_{\partial\Omega}(1-R)\langle\nabla R+2R\nabla f,\nu\rangle=-\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2+\frac{|\nabla R+2R\nabla f|^2}{2(n-1)}\right]$$
$$-\int_{\Omega\cap\{R<1\}}[\frac{3}{2}|\nabla R|^2+2(1-3R)Ric(\nabla f,\nabla f)-2R^2(1-R)].$$

Now, since $Ric>0$ and $-R\geq-1$ we can infer that

$$-\int_{\partial\Omega}(1-R)\langle\nabla R+2R\nabla f,\nu\rangle\geq\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2+\frac{|\nabla R+2R\nabla f|^2}{2(n-1)}\right]$$
$$+\int_{\Omega\cap\{R<1\}}[\frac{3}{2}|\nabla R|^2-4Ric(\nabla f,\nabla f)-2R^2(1-R)].$$

From Kato's inequality (Proposition 1.1) , for the steady solitons we have

$$|Ric|^2=|\nabla^2 f|^2\geq|\nabla|\nabla f||^2=|\nabla\sqrt{1-R}|^2=\frac{|\nabla R|^2}{4|\nabla f|^2}.$$

Therefore,

$$2Ric(\nabla f,\nabla f)=-\langle\nabla R,\nabla f\rangle\leq|\nabla R||\nabla f|\leq2|Ric||\nabla f|^2. \tag{3-11}$$

So,

$$-\int_{\partial\Omega}(1-R)\langle\nabla R+2R\nabla f,\nu\rangle\geq\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2+\frac{|\nabla R+2R\nabla f|^2}{2(n-1)}\right]$$
$$+\int_{\Omega\cap\{R<1\}}[\frac{3}{2}|\nabla R|^2-4|Ric||\nabla f|^2-2R^2|\nabla f|^2].$$

From the hypothesis, we get

$$\begin{aligned}\int_{\{R<1\}}\left[(n-2)^2|D|^2\right] &\leq -\lim_{\ell\to\infty}\int_{\partial\Omega_\ell}(1-R)\langle\nabla R+2R\nabla f,\nu\rangle\\ &\leq \left|\lim_{\ell\to\infty}\int_{\partial\Omega_\ell}(1-R)\langle\nabla R+2R\nabla f,\nu\rangle\right|\\ &\leq \lim_{\ell\to\infty}\int_{\partial\Omega_\ell}(1-R)|\langle\nabla R+2R\nabla f,\nu\rangle|\\ &\leq \lim_{\ell\to\infty}\int_{\partial\Omega_\ell}|\nabla R+2R\nabla f|.\end{aligned}$$

By Corollary 1.2, we have that $|B_r(p)|\leq wr^{n-1}$ where w is a constant, thus particularly the bounded domains as geodesic balls, $|\partial\Omega_\ell|\leq wr(\ell)^{n-1}$, where w is a constant. Here, $\ell\to\infty$ implies that $r\to\infty$. For a nontrivial gradient steady soliton with $Ric\geq 0$ and $\lim_{r\to\infty}R=0$, we know that $|Ric|^2\leq R^2$ (see [14] and [19, page 12]), and that f is an exhaustion function for M (cf. Section 3 in [21]). Consider

$$\Omega(r)=\{x\in M;\quad f(x)\leq r\}.$$

Thus, from (3-11) and using that $|\nabla R|\leq\sqrt{2}R$, we have

$$\begin{aligned}\int_{\{R<1\}}\left[(n-2)^2|D|^2\right] &\leq \lim_{\ell\to\infty}\int_{\partial\Omega_\ell}(|\nabla R|+2R|\nabla f|)\leq\lim_{\ell\to\infty}\int_{\partial\Omega_\ell}(\sqrt{2}R+2R)|\nabla f|\\ &\leq A(n)\lim_{r\to\infty}(2+\sqrt{2})wr^{n-1}|Rm|.\end{aligned}$$

Here, $c=A(n)(2+\sqrt{2})w$ and $A(n)$ a constant depending on n. From (3-10) we get

$$\begin{aligned}\int_{\{R<1\}}\left[(n-2)^2|D|^2\right] &\leq A(n)\lim_{r\to\infty}(2+\sqrt{2})wr^{n-1}(1+r)^{3(n+1)}e^{-r}\\ &\leq c\lim_{r\to\infty}(1+r)^{2(2n+1)}e^{-r}=0,\end{aligned}$$

Then, $D=0$ and the proof follows like in Theorem 3.1. □

Remark 3.4. *It is easy to see from* (3-3)

$$\begin{aligned}|D|^2 &= \frac{2}{(n-2)^2}|Ric|^2|\nabla f|^2-\frac{1}{2(n-2)^2}|\nabla R|^2\\ &\quad -\frac{1}{2(n-1)(n-2)^2}(|\nabla R|^2+4R\langle\nabla R,\nabla f\rangle+4R^2|\nabla f|^2).\end{aligned}$$

that if we consider that

$$|Ric|^2 \leq \frac{1}{4}\frac{|\nabla R|^2}{|\nabla f|^2},$$

then

$$|D|^2 + \frac{1}{2(n-1)(n-2)^2}|\nabla R + 2R\nabla f|^2 \quad \leq \quad 0.$$

Therefore, $D = 0$ *and* $\nabla R + 2R\nabla f = 0$*. Moreover, it is well-known that if the Ricci curvature is positive and its scalar curvature converges to zero at infinity, then the potential function is an exhaustion function for the steady Ricci soliton.*

According to Hamilton [40, 42], we have the following definition

Definition 3.1. *A Riemannian manifold* (M^n, g) *is positively pinched Ricci curvature if there is a uniform constant* $\delta > 0$ *such that*

$$\delta R g \leq Ric.$$

where R *is the scalar curvature and* Ric *is the Ricci curvature of metric* g*.*

Steady Ricci solitons with pinched Ricci curvature have been studied in the past years (cf. [32] and the references therein). In [56], Ni proved that any steady Ricci soliton with pinched Ricci curvature and nonnegative sectional curvature must be flat. Then, Deng and Zhu [30] proved that Ni's result is true without the assumption over the sectional curvature for steady Kähler-Ricci solitons.

Deng and Zhu [30] proved that any $(n \geq 2)$-dimensional complete non-compact steady Kähler-Ricci soliton (M^n, g, f) with positively pinched Ricci curvature should be Ricci flat (provided that there is a point p so that $\nabla f(p) = 0$). The existence of such a point p is called the equilibrium point condition. Under our approach, we shall show that this condition is not necessary. To be precise, we have established the following results.

Corollary 3.1. *Let* $\left(M^n, g, f\right)$*,* $n \geq 3$*, be a complete non-compact gradient steady Kähler-Ricci soliton. If* (M^n, g) *has positively pinched Ricci curvature , then* (M^n, g, f) *is Ricci-flat.*

Proof. It is well-known that for Kähler manifolds with nonnegative Ricci curvature the following inequality holds:

$$2Ric(\nabla f, \nabla f) \leq R|\nabla f|^2,$$

see Theorem 3.2 in [33]. So, from (3-9) we get

$$\begin{aligned}\int_{\partial\Omega}(1-R)\langle\nabla R+R\nabla f,\nu\rangle \;&\leq\; -\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2+\frac{n|\nabla R+R\nabla f|^2}{2(n-1)}\right]\\ &\quad -\int_{\Omega\cap\{R<1\}}|\nabla R|^2+\frac{3n-4}{2(n-1)}\int_{\Omega\cap\{R<1\}}R^2|\nabla f|^2.\end{aligned}$$

Assuming $\frac{1}{2}\sqrt{\frac{3n-4}{2(n-1)}}Rg\leq Ric$, from (3-7) we have

$$\sqrt{\frac{3n-4}{2(n-1)}}R\leq 2Ric\left(\frac{\nabla f}{|\nabla f|},\frac{\nabla f}{|\nabla f|}\right)=\frac{-1}{|\nabla f|^2}\langle\nabla R,\nabla f\rangle\leq\frac{|\nabla R|}{|\nabla f|}. \tag{3-12}$$

Thus

$$\begin{aligned}\int_{\partial\Omega}(1-R)\langle\nabla R+R\nabla f,\nu\rangle \;&\leq\; -\int_{\Omega\cap\{R<1\}}\left[(n-2)^2|D|^2+\frac{n|\nabla R+R\nabla f|^2}{2(n-1)}\right]\\ &\quad -\int_{\Omega\cap\{R<1\}}|\nabla R|^2+\frac{|\nabla R|^2}{|\nabla f|^2}\int_{\Omega\cap\{R<1\}}|\nabla f|^2.\end{aligned}$$

The result follows the same steps of Theorem 3.2. By Theorem 1.16 and Corollary 1.2, we have that M has exponential growth, i.e., $|B_r(p)|\leq we^{a\sqrt{r}}$ where w, a are constants, thus the bounded domains $|\partial\Omega_\ell|\leq we^{a\sqrt{r}}$, where a and w stand for positive constants. Moreover, it is well-known that if the Ricci curvature is positive and its scalar curvature converges to zero at infinity, the potential function is an exhaustion function for the steady Ricci soliton. We know that if $Ric\geq 0$ we have $|\nabla R|\leq 2R|\nabla f|$, thus from Proposition 1.6-(3) with $\rho=0$ and normalized $C_0=1$ we can conclude that (3-5) satisfies

$$\lim_{r\to+\infty}\int_{\partial\Omega(r)}(|\nabla R|+R|\nabla f|)\leq 3w\lim_{r\to+\infty}e^{a\sqrt{r}}e^{-r}=0,$$

Here, we consider that the scalar curvature has an exponential decay at infinity, i.e.,

$$R=o(e^{-r}),\quad r\to\infty,$$

where r stands for the geodesic distance from a fixed point, see [27, Theorem 9.56].

Therefore, we obtain $\nabla R+R\nabla f=0$ and $D=0$. Moreover,

$$\frac{3n-4}{2(n-1)}R^2|\nabla f|^2=|\nabla R|^2.$$

So, either $n=2$ or (M,g) is Ricci-flat and f a constant function. □

Thus, Corollary 3.1 answers a question proposed by Chow, Lu and Ni in case of steady Kähler-Ricci solitons (cf. [27, 56]), and proves Corollary 1.5 in [30] without the hypothesis of equilibrium point condition. In fact, the above corollary still holds for any gradient steady Ricci soliton (not necessarily Kähler).

Remark 3.5. *We highlight that in the Kähler case, Cao found two examples of complete rotationally symmetric non-compact gradient steady Kähler-Ricci solitons (see [7] and the references therein). These examples are* $U(n)$ *invariant and have positive sectional curvature.*

CHAPTER 4

Rigidity results fot shrinking and expanding Ricci solitons

4.1 Preliminary Results

Before we start, we shall present some preliminaries that will be useful for establishing the desired results. It is well-known that from Proposition 1.6, a gradient Ricci soliton satisfies the equation

$$\nabla R = 2Ric(\nabla f). \tag{4-1}$$

Moreover,

$$R + |\nabla f|^2 - 2\rho f = C.$$

Note that if we normalize f by adding the constant C to it, then we have

$$R + |\nabla f|^2 = \lambda f, \tag{4-2}$$

where $\lambda = 2\rho$.

Now, recalling the covariant 3-tensor D, see (1-6), we have the following important lemma.

Lemma 4.1. *Let (M^n, g, f) be a gradient shrinking (or expanding) Ricci soliton. Then,*

$$\begin{aligned}(n-2)^2|D|^2 + \frac{|2R\nabla f - \nabla R|^2}{2(n-1)} &= |\nabla f|^2\langle\nabla R, \nabla f\rangle - |\nabla f|^2\Delta R\\ &\quad +\lambda R|\nabla f|^2 - \frac{1}{2}|\nabla R|^2.\end{aligned}$$

Proof. Let us write the norm of D only depending on the function f and the scalar curvature R. We begin the computation using (1-6) and Definition 1.1. We start with the following identity:

$$\begin{aligned}|D|^2 &= \frac{2}{(n-2)^2}|Ric|^2|\nabla f|^2 - \frac{2}{(n-2)^2}R_{ik}\nabla_j f R_{jk}\nabla_i f\\ &+ \frac{1}{2(n-1)(n-2)^2}|2R\nabla f-\nabla R|^2\\ &+ \frac{2}{(n-1)(n-2)^2}(R_{jk}\nabla_i f - R_{ik}\nabla_j f)(2R\nabla_j f-\nabla_j R)g_{ik}.\end{aligned}$$

Then, by using equation (1-3), (4-1) and (4-2) we get

$$\begin{aligned}|D|^2 &= \frac{2}{(n-2)^2}|Ric|^2|\nabla f|^2 - \frac{1}{2(n-2)^2}2R_{ik}\nabla_j f 2R_{jk}\nabla_i f\\ &+ \frac{1}{2(n-1)(n-2)^2}|2R\nabla f-\nabla R|^2\\ &- \frac{1}{(n-1)(n-2)^2}(2R\nabla_j f-\nabla_j R)(2R\nabla_j f-\nabla_j R)\\ &= \frac{2}{(n-2)^2}|Ric|^2|\nabla f|^2 - \frac{1}{2(n-2)^2}|\nabla R|^2\\ &- \frac{1}{2(n-1)(n-2)^2}|2R\nabla f-\nabla R|^2.\end{aligned}$$

Now, we need to use the following identity (cf. Proposition 1.6-(5)):

$$2|Ric|^2 = \langle\nabla R,\nabla f\rangle + \lambda R - \Delta R.$$

Therefore, by combining these last two identities, we obtain

$$\begin{aligned}|D|^2 + \frac{|2R\nabla f-\nabla R|^2}{2(n-1)(n-2)^2} &= \frac{|\nabla f|^2}{(n-2)^2}\langle\nabla R,\,\nabla f\rangle - \frac{|\nabla f|^2}{(n-2)^2}\Delta R\\ &+\frac{\lambda R|\nabla f|^2}{(n-2)^2} - \frac{1}{2(n-2)^2}|\nabla R|^2.\end{aligned}$$

□

Our next lemma is a divergence formula inspired by the works of Brendle [1], and Robinson [62]. Those divergence formulas were key in classifying steady Ricci solitons and static vacuum spaces.

Now, consider a smooth function $\psi:\mathbb{R}\to\mathbb{R}$ such that

$$Y = \nabla R + \psi(|\nabla f|^2)\nabla f$$

on M.

Lemma 4.2. *Let* $\left(M^n, g, f\right)$ *be a gradient shrinking (or expanding) Ricci soliton. Then,*

$$\begin{aligned} 2(\lambda f-R)\operatorname{div}(Y) &= -2(n-2)^2|D|^2-\frac{|2R\nabla f-\nabla R|^2}{(n-1)}-|\nabla R|^2 \\ &\quad +2(\lambda f-R)[R^2-\frac{\lambda}{2}(n+2f)R+\frac{\lambda^2}{2}(n+2)f]. \end{aligned}$$

Proof. Remember that $\Delta f=\rho n-R$ and $|\nabla f|^2=\lambda f-R$. Hence,

$$\begin{aligned} \operatorname{div}(Y) &= \Delta R+\lambda\psi|\nabla f|^2-\psi\langle\nabla R,\nabla f\rangle+\psi\Delta f \\ &= \Delta R+\lambda\psi|\nabla f|^2-\psi\langle\nabla R,\nabla f\rangle+(\rho n-R)\psi. \end{aligned}$$

Thus,

$$\begin{aligned} -2(\lambda f-R)\operatorname{div}(Y) &= -2(\lambda f-R)\Delta R-\lambda 2(\lambda f-R)\psi|\nabla f|^2 \\ &\quad +2(\lambda f-R)\psi\langle\nabla R,\nabla f\rangle-2(\lambda f-R)(\rho n-R)\psi. \end{aligned}$$

Combining the above equation with the previous lemma we get

$$\begin{aligned} 2(n-2)^2|D|^2+\frac{|2R\nabla f-\nabla R|^2}{(n-1)} &= -2(\lambda f-R)\Delta R+2(\lambda f-R)\langle\nabla R,\nabla f\rangle \\ &\quad +2\lambda R(\lambda f-R)-|\nabla R|^2. \end{aligned}$$

Hence,

$$\begin{aligned} & 2(n-2)^2|D|^2+\frac{|2R\nabla f-\nabla R|^2}{(n-1)} \\ = \; & -2(\lambda f-R)\operatorname{div}(Y)+2\lambda(\lambda f-R)\psi|\nabla f|^2-2(\lambda f-R)\psi\langle\nabla R,\nabla f\rangle \\ & +2(\lambda f-R)(\rho n-R)\psi+2(\lambda f-R)\langle\nabla R,\nabla f\rangle+2\lambda R(\lambda f-R)-|\nabla R|^2. \end{aligned}$$

Consequently,

$$\begin{aligned} & 2(n-2)^2|D|^2+\frac{|2R\nabla f-\nabla R|^2}{(n-1)}=-2(\lambda f-R)\operatorname{div}(Y)+2\lambda(\lambda f-R)\psi|\nabla f|^2 \\ & +2(\lambda f-R)(\rho n-R)\psi+2\lambda R(\lambda f-R)-|\nabla R|^2+2(\lambda f-R)(1-\psi)\langle\nabla R,\nabla f\rangle. \end{aligned}$$

Consider ψ as an identity function, i.e.,

$$\psi=(\lambda f-R).$$

Therefore,

$$2(\lambda f - R)\mathrm{div}(Y) \;=\; -2(n-2)^2|D|^2 - \frac{|2R\nabla f - \nabla R|^2}{(n-1)} - |\nabla R|^2 \tag{4-3}$$
$$+2(\lambda f - R)^2(\rho n - R) + 2\lambda R(\lambda f - R) + 2\lambda(\lambda f - R)|\nabla f|^2.$$

By a straightforward computation, we get the result.

□

4.2 Main Results

Now, inspired by [38] and Catino [15] we will show that the triviality of compact gradient shrinking Ricci solitons can be characterized by a gap in the potential function (or on the scalar curvature).

Theorem 4.1. *Let* (M^n, g, f, ρ) *be a compact gradient shrinking Ricci soliton such that the potential function satisfies*

$$f(x) \leq \sqrt{\left(\frac{n+2}{4} - \frac{\delta}{4\rho}\right)^2 + \frac{n\delta}{4\rho}} - \left(\frac{n+2}{4} - \frac{\delta}{4\rho}\right),$$

where $\delta = \min_M(R)$ *is a constant. Then,* (M^n, g, f) *is Einstein.*

Proof. Now, from Lemma 4.2 we can infer that

$$2(\lambda f - R)\mathrm{div}(Y) \;=\; -2(n-2)^2|D|^2 - \frac{|2R\nabla f - \nabla R|^2}{(n-1)} - |\nabla R|^2$$
$$+2(\lambda f - R)[R^2 - \frac{\lambda}{2}(n+2f)R + \frac{\lambda^2}{2}(n+2)f].$$

On the other hand, from (4-2) we have $R \leq 2\rho f = \lambda f$, and since M is compact we can assume that $R \geq \delta = \min_M(R)$. Therefore,

$$R^2 - \frac{\lambda}{2}(n+2f)R + (n+2)\frac{\lambda^2}{2}f \leq \lambda^2 f^2 + (n+2)\frac{\lambda^2}{2}f - \frac{\lambda}{2}(n+2f)\delta.$$

Then, considering

$$\frac{\delta}{2\lambda} - \frac{n+2}{4} - \sqrt{\left(\frac{n+2}{4} - \frac{\delta}{2\lambda}\right)^2 + \frac{n\delta}{2\lambda}} \leq f \leq \frac{\delta}{2\lambda} - \frac{n+2}{4} + \sqrt{\left(\frac{n+2}{4} - \frac{\delta}{2\lambda}\right)^2 + \frac{n\delta}{2\lambda}}$$

we can infer that

$$\lambda^2 f^2 + \left[(n+2)\frac{\lambda^2}{2} - \delta\lambda\right] f - \frac{n\lambda\delta}{2} \leq 0$$

Now, since $\lambda f - R \geq 0$ we can conclude that

$$\operatorname{div}(Y) \leq 0.$$

Therefore, $\operatorname{div}(Y) = 0$ and

$$\begin{aligned} 0 &= 2(n-2)^2|D|^2 + \frac{|2R\nabla f - \nabla R|^2}{(n-1)} + |\nabla R|^2 \\ &\quad -2(\lambda f - R)[\lambda^2 f + (\lambda f - R)(n\frac{\lambda}{2} - R)] \geq 0. \end{aligned}$$

Finally, we can conclude that (M, g, f) is Einstein. □

Remark 4.1. *It is well-known there exists a non-Einstein compact shrinking Ricci soliton on $\mathbb{CP}2\#(-\mathbb{CP}2)$. This example does not satisfy the hypothesis assumed in Theorem 4.1, i.e., the maximum of f in $\mathbb{CP}2\#(-\mathbb{CP}2)$ is bigger than the bound assumed for the potential function in the above theorem.*

In [53, Theorem 6.1], the authors proved an upper bound for the diameter of a compact shrinking Ricci soliton. This bound depends only on the dimension and the injectivity radius of the manifold. Moreover, in [38] they proved that a compact shrinking Ricci soliton has a diameter bounded below by a constant depending of the geometry of the manifold.

It is important to highlight that the scalar curvature R on a shrinking Ricci soliton is nonnegative and (Theorem 1.9)

$$\frac{1}{4}(r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_2)^2, \tag{4-4}$$

where $r(x) = d(x_0, x)$ is the distance function from some fixed point $x_0 \in M$, and c_1 and c_2 are positive constants depending only on n and the geometry of g_{ij} on the unit ball $B_{x_0}(1)$ (cf. [12]).

For expanding Ricci solitons we have Theorem 1.11. On a complete non-compact gradient expanding soliton with nonnegative Ricci curvature (or pinched Ricci curvature) the potential function f satisfies the estimates

$$\frac{1}{4}(r(x) - c_1)^2 - c_2 \leq -f(x) \leq \frac{1}{4}(r(x) + 2\sqrt{-f(x_0)})^2. \tag{4-5}$$

Here, $c_1 > 0$ and $c_2 > 0$ are constants.

In what follows, we remember some examples of shrinking and expanding Ricci solitons. More examples can be found in [20].

Example 4.1 (Einstein manifolds)**.** *The first and trivial examples of gradient Ricci solitons are Einstein manifolds. By choosing f to be a constant function, they are endorsed with a soliton structure.*

Example 4.2 (Gaussian solitons)**.** *The flat metric on $\mathbb{R}^n$ with the potential function $f = \frac{\rho}{2}|x|^2$, where ρ is either positive or negative (i.e., either Shrinker or Expander).*

Example 4.3 (Cylinders)**.**

- *(Shrinker:) Consider $M^n = \mathbb{R}^{n-k} \times \mathbb{S}^k$, $k \geq 2$, $(x, y) \in \mathbb{R}^{n-k} \times \mathbb{S}^k$, the potential function $f(x, y) = \frac{(k-1)}{2}|x|^2$ and $\rho = (k-1)$. This soliton has positive constant scalar curvature.*

- *(Expander:) Consider $M^n = \mathbb{R}^{n-k} \times \mathbb{H}^k$, $(x, y) \in \mathbb{R}^{n-k} \times \mathbb{H}^k$, the potential function $f(x, y) = \frac{-(k-1)}{2}|x|^2$ and $\rho = -(k-1)$. This soliton has negative constant scalar curvature.*

Now we prove a rigidity result for complete shrinking Ricci solitons assuming the curvature (Ricci or scalar) is pinched and an asymptotic condition on the gradient of the scalar curvature at infinity holds. Rigidity results for shrinking Ricci solitons assuming pinched conditions was proved by Catino in [15, 16].

Theorem 4.2. *Let (M^n, g, f, ρ) be a complete gradient shrinking Ricci soliton such that*

$$|\nabla R| = o(r^{-n-2}); \quad r \to \infty,$$

and one of the following conditions hold:

(I) $Ric \leq \dfrac{R\left(\frac{2R}{n-1} - \rho\right)}{\left(2\rho(1+f) - \frac{n-3}{n-1}R\right)}.$

(II)

$$\begin{aligned} |\nabla R| \leq& \frac{(n-1)}{(3n-2)}|\nabla f|\left(2\rho(1+f) - \frac{(n-3)}{(n-1)}R\right) \\ +& \frac{(n-1)}{(3n-2)}\left[|\nabla f|\sqrt{\left(2\rho(1+f) - \frac{(n-3)}{(n-1)}R\right)^2 + \frac{4(3n-2)}{(n-1)}R\left(\rho - \frac{R}{n-1}\right)}\right]. \end{aligned}$$

Then, the D-tensor must be identically zero. Consequently, *(i)* *(M^4, g) is either Einstein, or a finite quotient of $\mathbb{R}^4$ or $\mathbb{S}^3 \times \mathbb{R}$;*

(ii) *(M^n, g), $n \geq 5$, is either Einstein or is a finite quotient of the Gaussian shrinking soliton or is a finite quotient of $N^{n-1} \times \mathbb{R}$, where N^{n-1} is Einstein.*

Proof. Let Ω be a bounded domain on M with a smooth boundary. Using that $Y = \nabla R + (\lambda f - R)\nabla f$ and the divergence theorem, from Lemma 4.2 equation (4-3) we get

$$\begin{aligned}
&\int_{\partial\Omega} \langle (\lambda f - R)Y, \nu\rangle = \int_{\Omega} \operatorname{div}\left((\lambda f - R)Y\right) \\
= {} & \int_{\Omega} (\lambda f - R)\operatorname{div}(Y) + \int_{\Omega} \lambda \langle \nabla f, Y\rangle - \int_{\Omega} \langle \nabla R, Y\rangle \\
= {} & \int_{\Omega} \Big[-(n-2)^2|D|^2 - \frac{|2R\nabla f - \nabla R|^2}{2(n-1)} - \frac{|\nabla R|^2}{2} \\
& + (\lambda f - R)^2(\rho n - R) + \lambda R(\lambda f - R) + \lambda(\lambda f - R)|\nabla f|^2 \Big] \\
& + \int_{\Omega} \lambda \langle \nabla f, \nabla R + (\lambda f - R)\nabla f\rangle - \int_{\Omega} \langle \nabla R, \nabla R + (\lambda f - R)\nabla f\rangle \\
= {} & \int_{\Omega} \Big[-(n-2)^2|D|^2 - \frac{|2R\nabla f - \nabla R|^2}{2(n-1)} - \frac{|\nabla R|^2}{2} \\
& + (\lambda f - R)^2(\rho n - R) + \lambda R(\lambda f - R) + \lambda(\lambda f - R)|\nabla f|^2 \Big] \\
& + \int_{\Omega} \lambda(\lambda f - R)|\nabla f|^2 + \lambda \langle \nabla f, \nabla R\rangle - |\nabla R|^2 - (\lambda f - R)\langle \nabla f, \nabla R\rangle \\
= {} & \int_{\Omega} \Big[-(n-2)^2|D|^2 - \frac{|2R\nabla f - \nabla R|^2}{2(n-1)} - \frac{3|\nabla R|^2}{2} \\
& + (\lambda f - R)^2(\rho n - R) + \lambda R(\lambda f - R) + 2\lambda(\lambda f - R)|\nabla f|^2 \\
& + \lambda \langle \nabla f, \nabla R\rangle - (\lambda f - R)\langle \nabla f, \nabla R\rangle \Big].
\end{aligned}$$

Furthermore, using $(\rho n - R) = \Delta f$ we get

$$\begin{aligned}
(\lambda f - R)^2(\rho n - R) &= \operatorname{div}\left((\lambda f - R)^2\nabla f\right) + 2(\lambda f - R)\langle \nabla R, \nabla f\rangle \\
&\quad - 2\lambda(\lambda f - R)|\nabla f|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_{\partial\Omega} \quad & \left\langle (\lambda f - R)\nabla R + (\lambda f - R)^2\nabla f, \nu \right\rangle \\
& = \int_{\Omega} \Big[-(n-2)^2|D|^2 - \frac{|2R\nabla f - \nabla R|^2}{2(n-1)} - \frac{3|\nabla R|^2}{2} \\
& + \operatorname{div}\left((\lambda f - R)^2\nabla f\right) + \lambda R(\lambda f - R) + \lambda \langle \nabla f, \nabla R\rangle + (\lambda f - R)\langle \nabla f, \nabla R\rangle \Big],
\end{aligned}$$

and so

$$\begin{aligned}\int_{\partial\Omega}(\lambda f-R)\left\langle\nabla R,\nu\right\rangle &= \int_{\Omega}\Big[-(n-2)^2|D|^2-\frac{|2R\nabla f-\nabla R|^2}{2(n-1)}-\frac{3|\nabla R|^2}{2}\\ &+\lambda R(\lambda f-R)+\lambda\left\langle\nabla f,\nabla R\right\rangle+(\lambda f-R)\left\langle\nabla f,\nabla R\right\rangle\Big].\end{aligned}$$

Hence, from (4-2) we get

$$\begin{aligned}\int_{\partial\Omega}\left\langle(\lambda f-R)\nabla R,\nu\right\rangle &= \int_{\Omega}\Big[-(n-2)^2|D|^2-\frac{|2R\nabla f-\nabla R|^2}{2(n-1)}-\frac{3|\nabla R|^2}{2}\\ &+\lambda R|\nabla f|^2+\lambda\left\langle\nabla f,\nabla R\right\rangle+|\nabla f|^2\left\langle\nabla f,\nabla R\right\rangle\Big].\end{aligned}$$

By a straightforward computation, we obtain

$$\begin{aligned}\int_{\partial\Omega}\left\langle(\lambda f-R)\nabla R,\nu\right\rangle &= \int_{\Omega}\Big[-(n-2)^2|D|^2\\ &-\left(\frac{2R}{n-1}-\lambda\right)R|\nabla f|^2+\left(\frac{2R}{n-1}+\lambda\right)\left\langle\nabla f,\nabla R\right\rangle\\ &-\left(\frac{3n-2}{2(n-1)}\right)|\nabla R|^2+|\nabla f|^2\left\langle\nabla f,\nabla R\right\rangle\Big].\end{aligned}\tag{4-6}$$

From (4-1) and (4-6) we get

$$\begin{aligned}\int_{\partial\Omega}\left\langle(\lambda f-R)\nabla R,\nu\right\rangle &= \int_{\Omega}\Big[-(n-2)^2|D|^2\\ &-\left(\frac{2R^2}{n-1}-\lambda R\right)|\nabla f|^2+2\left(\frac{2R}{n-1}+\lambda\right)Ric(\nabla f,\nabla f)\\ &-\left(\frac{3n-2}{2(n-1)}\right)|\nabla R|^2+2|\nabla f|^2Ric(\nabla f,\nabla f)\Big].\end{aligned}$$

So, from (4-1) we get

$$\begin{aligned}
\int_{\partial\Omega}\langle(\lambda f-R)\nabla R,\nu\rangle &\leq -\int_{\Omega}(n-2)^2|D|^2-\int_{\Omega}\left(\frac{3n-2}{2(n-1)}\right)|\nabla R|^2\\
&\quad+\int_{\Omega}\left[2\left(\frac{2R}{n-1}+\lambda+|\nabla f|^2\right)Ric(\nabla f,\nabla f)\right.\\
&\quad\left.-\left(\frac{2R}{n-1}-\lambda\right)R|\nabla f|^2\right]\\
&= -\int_{\Omega}(n-2)^2|D|^2-\int_{\Omega}\left(\frac{3n-2}{2(n-1)}\right)|\nabla R|^2\\
&\quad+2\int_{\Omega}\left[\left(2\rho(1+f)-\frac{n-3}{n-1}R\right)Ric(\nabla f,\nabla f)\right.\\
&\quad\left.-\left(\frac{R}{n-1}-\rho\right)R|\nabla f|^2\right]\\
&\leq -\int_{\Omega}(n-2)^2|D|^2-\int_{\Omega}\left(\frac{3n-2}{2(n-1)}\right)|\nabla R|^2\\
&\quad+\int_{\Omega}\left[\left(2\rho(1+f)-\frac{n-3}{n-1}R\right)|\nabla R||\nabla f|\right. \qquad (4\text{-}7)\\
&\quad\left.-2\left(\frac{R}{n-1}-\rho\right)R|\nabla f|^2\right].
\end{aligned}$$

From the first inequality above we conclude the first item (I) of this theorem.

Then, by considering

$$\begin{aligned}
|\nabla R| &\leq \frac{(n-1)}{(3n-2)}\left[|\nabla f|\sqrt{\left(2\rho(1+f)-\frac{(n-3)}{(n-1)}R\right)^2+\frac{4(3n-2)}{(n-1)}R\left(\rho-\frac{R}{n-1}\right)}\right]\\
&- \frac{(n-1)}{(3n-2)}|\nabla f|\left(2\rho(1+f)-\frac{(n-3)}{(n-1)}R\right)
\end{aligned}$$

we conclude that

$$-\left[\frac{3n-2}{2(n-1)}\right]|\nabla R|^2+\left(2\rho(1+f)-\frac{n-3}{n-1}R\right)|\nabla f||\nabla R|-2\left(\frac{R}{n-1}-\rho\right)R|\nabla f|^2\leq 0.$$

Hence,

$$\int_{\Omega_\ell}(n-2)^2|D|^2 \leq -\int_{\partial\Omega_\ell}\langle(\lambda f-R)\nabla R,\nu\rangle. \qquad (4\text{-}8)$$

Therefore, making $\ell\to\infty$ we have

$$\begin{aligned}
\int_M\left[(n-2)^2|D|^2\right] \leq \lim_{\ell\to\infty}\int_{\partial\Omega_\ell}|\nabla f|^2|\nabla R| &= \lim_{\ell\to\infty}\int_{\partial\Omega_\ell}(\lambda f-R)|\nabla R|\\
&\leq \lambda\lim_{\ell\to\infty}\int_{\partial\Omega_\ell}f|\nabla R|\to 0.
\end{aligned}$$

Consider the bounded domains with Euclidean growth, i.e., $|\partial\Omega_\ell| \leq c_3 r(\ell)^{n-1}$, where c_3 is a positive constant. Here, $\ell \to \infty$ implies that $r \to \infty$ (see Theorem 1.10). From (4-4) we get

$$\int_M \left[(n-2)^2|D|^2\right] \leq \lambda \lim_{\ell\to\infty} \int_{\partial\Omega_\ell} f|\nabla R| \leq c_3 \frac{\lambda}{4} \lim_{r\to+\infty} r^{n-1}(r+c_2)^2|\nabla R| \to 0.$$

In fact, by [12] we know that f is an exhaustion function, so $\Omega = \{x \in M, \beta(x) \leq r(x)\}$, where $\beta(x) = 2\sqrt{f(x)}$. Moreover, the authors proved that the geodesic ball has Euclidean growth.

Therefore, D vanishes identically, and R must be constant. Now, we invoke Theorem 1.7. ☐

Remark 4.2.

1. *In the shrinking case we can infer that*

$$0 < \frac{2R}{n-1} + 2\rho + |\nabla f|^2 = 2\rho(1+f) - \frac{n-3}{n-1}R.$$

Observe that condition (I) *is trivially satisfied for the Gaussian shrinking Ricci soliton. Also, considering* $\mathbb{S}^3$ *with* $f = 0$ *we can see that this condition is trivial. Condition* (I) *is compatible with [54, Equation 2.20].*

2. *Moreover, condition* (II) *holds for Cylinders and for Gaussian solitons. For instance,* $\mathbb{S}^3 \times \mathbb{R}$ *have* $\rho = 2$ *and* $R = 6$. *Thus, the bad term is zero, i.e.,* $\rho - R/(n-1) = 0$. *Thus, condition* (II) *is trivially satisfied. Any shrinking Ricci soliton with nonnegative Ricci curvature satisfies*

$$|\nabla R|^2 \leq 4R^2|\nabla f|^2.$$

By following the same strategy of Theorem 4.2 we can prove our next result for expanding Ricci solitons.

Theorem 4.3. *Let* (M^3, g, f) *be a complete gradient expanding Ricci soliton with nonpositive Ricci curvature such that* $1 \leq -f$ *and* R *goes to zero at infinity. Suppose that*

$$|\nabla f|^2|\nabla R| = o(r^{-3}),$$

where r *is the distance function. Then,* (M^3, g) *is rotationally symmetric. Moreover,* R *must be constant, particularly* $R = 0$.

Remark 4.3. *The hypothesis over* f, *i.e.,* $1 \leq -f$, *is reasonable considering the cylinder* $\mathbb{R} \times \mathbb{H}^2$ *and the Gaussian soliton, see also* (4-5). *We also recommend to the reader [20,* *Theorem 30]. In this theorem P.-Y. Chan proved that, an 3-dimensional complete gradient expanding Ricci soliton such that* $\lim_{x\to+\infty} r(x)^2R(x) = 0$ *must be isometric to* $\mathbb{R}^3$.

Proof of Theorem 4.3. It is well-known that if $Ric \geq 0$ (or $Ric \leq 0$) we have (cf. [57, Equation 2.7]):

$$|\nabla R|^2 \leq 4R^2|\nabla f|^2.$$

In fact, consider an orthonormal frame $\{e_1, e_2, e_3, \ldots, e_n\}$ diagonalizing Ric at a regular point p, with associated eigenvalues α_k, $k = 1, \ldots, n$, respectively. That is, $R_{ij} = \alpha_i\delta_{ij}$. From Proposition 1.6-(2), $\nabla R = 2Ric(\nabla f)$ we can infer that

$$|\nabla R|^2 = 4\langle Ric(\nabla f), Ric(\nabla f)\rangle = 4\sum_i \alpha_i^2(\nabla_i f)^2 \leq 4\left(\sum_i \alpha_i\right)^2 \sum_i (\nabla_i f)^2 = 4R^2|\nabla f|^2.$$

Thus, if $R \leq 0$ we can conclude that

$$(|\nabla R| + 2R|\nabla f|)(|\nabla R| - 2R|\nabla f|) \leq 0,$$

i.e.,

$$|\nabla R| \leq -2R|\nabla f|.$$

Considering $\frac{n-3}{n-1}R \leq 2\rho(1+f)$ from (4-2) and (4-7) we have

$$\begin{aligned}
\int_{\partial\Omega}\langle(\lambda f - R)\nabla R, \nu\rangle \leq\; & -\int_\Omega (n-2)^2|D|^2 - \int_\Omega\left(\frac{3n-2}{2(n-1)}\right)|\nabla R|^2 \\
& +2\int_\Omega\left[\left(-2\rho(1+f) + \frac{n-3}{n-1}R\right) - \left(\frac{R}{n-1} - \rho\right)\right]R|\nabla f|^2 \\
=\; & -\int_\Omega (n-2)^2|D|^2 - \int_\Omega\left(\frac{3n-2}{2(n-1)}\right)|\nabla R|^2 \\
& -2\int_\Omega\left(\rho + |\nabla f|^2 + \frac{3R}{n-1}\right)R|\nabla f|^2 \\
=\; & -\int_\Omega (n-2)^2|D|^2 - \int_\Omega\left(\frac{3n-2}{2(n-1)}\right)|\nabla R|^2 \\
& -2\int_\Omega\left(\rho + \frac{6\rho}{n-1}f + \frac{n-4}{n-1}|\nabla f|^2\right)R|\nabla f|^2.
\end{aligned}$$

Assuming $n = 3$ and $\rho = -1/2$, the condition $\frac{n-3}{n-1}R \leq 2\rho(1+f)$ is equivalent to $1 \leq -f$. We can see that

$$-1 \leq 1 \leq -f \leq 4(-f).$$

Hence,

$$\begin{aligned}\int_{\partial\Omega}\langle(\lambda f-R)\nabla R,\nu\rangle &\leq -\int_{\Omega}(n-2)^2|D|^2-\int_{\Omega}\left(\frac{3n-2}{2(n-1)}\right)|\nabla R|^2\\ &\quad -\int_{\Omega}\left(-1+3f-|\nabla f|^2\right)R|\nabla f|^2.\\ &\leq -\int_{\Omega}(n-2)^2|D|^2-\int_{\Omega}\left(\frac{3n-2}{2(n-1)}\right)|\nabla R|^2\\ &\quad -\int_{\Omega}\left(-f-|\nabla f|^2\right)R|\nabla f|^2.\end{aligned}$$

Since $R\leq 0$, from (3-2) we get $-f-|\nabla f|^2\leq 0$. Thus,

$$\int_{\partial\Omega}\langle(\lambda f-R)\nabla R,\nu\rangle\leq-\int_{\Omega}|D|^2-\int_{\Omega}\frac{7}{4}|\nabla R|^2.$$

Now, since $Ric\leq 0$, for some geodesic $\gamma(s)$ of M we have

$$-f''(s)-\frac{1}{2}c_0=Ric(\gamma'(s),\gamma'(s))\leq 0.$$

Hence,

$$-f(x)\leq\frac{c_0}{2}r(x)^2+c_1r(x)+c_2,$$

where c_i is constant ($i=0,\,1,\,2$), and $r(x)$ is the distance function from a fixed point.

Therefore, by the same steps of [12, Equation 3.1] we have

$$\Omega(r)=\{x\in M,\,\rho(x)\leq r(x)\},$$

where $\rho(x)=2\sqrt{-f(x)}$.

Then making, $r\to\infty$ we get

$$\int_M|D|^2+\frac{7}{4}\int_M|\nabla R|^2\leq\lim_{r\to\infty}\int_{\partial\Omega(r)}(\lambda f-R)|\nabla R|=\lim_{r\to\infty}\int_{\partial\Omega(r)}|\nabla f|^2|\nabla R|.$$

We invoke Corollary 1.12 to infer that the geodesic ball has Euclidean growth, i.e.,

$$\lim_{r\to\infty}\int_{\partial\Omega(r)}|\nabla f|^2|\nabla R|\leq C\lim_{r\to\infty}r^2|\nabla f|^2|\nabla R|=0,$$

where C is a constant.

Therefore, D vanishes identically, and R must be constant. Since D is equivalent to the Cotton tensor in three dimensions, the result follows (Lemma 1.7), i.e., $(M^3,\,g,\,f)$ is locally conformally flat. The rotational symmetry now follows from Proposition 1.9. $\square$

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