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Instituto de Matemática e Estatística
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Róbson Lousa dos Santos

Electrostatic system and divergence formulas

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UNIVERSIDADE FEDERAL DE GOIÁS
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Róbson Lousa dos Santos

Electrostatic system and divergence formulas

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Advisor: Prof. Dr. Benedito Leandro

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Ata nº **01** da sessão de Defesa de Tese de **Róbson Lousa dos Santos,** que confere o título de Doutor em Matemática, **na área de concentração de Geometria Diferencial.**

Ao décimo sexto dia do mês de fevereiro do ano de dois mil e vinte e três, a partir das dez horas, via Web videoconferência, realizou-se a sessão pública de Defesa de Tese intitulada **"Electrostatic system and divergence formulas."** Os trabalhos foram instalados pelo Orientador e Presidente da banca, Professor Doutor **Benedito Leandro Neto - IME/UFG** com a participação dos demais membros da Banca Examinadora: Professora Doutora **Rafaela Carla Deborah Cederbaum - Universidade de Tübingen-Alemanha** - membro titular externa, Professora Doutora **Maria de Andrade Costa e Silva - DMAT/UFS -** membro titular externa, Professor Doutor **Halyson Irene Baltazar - DMAT/UFPI** membro titular externo e o Professor Doutor **João Paulo dos Santos - MAT/UnB**, membro titular externo. Durante a arguição os membros da banca **não fizeram** sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Tese, tendo sido o candidato **APROVADO** pelos seus membros. Proclamados os resultados pelo Professor Doutor **Benedito Leandro Neto - IME/UFG**, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos Membros da Banca Examinadora, ao décimo sexto dia do mês de fevereiro do ano de dois mil e vinte e três.

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Electrostatic system and divergence formulas

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Róbson Lousa dos Santos

Bachelor (2017) and master (2019) in Mathematics from the Federal University of Goiás. Degree in Mathematics (2019) from Cruzeiro do Sul University. Mathematics Teaching Specialist (2022) from Federal Institute of Maranhão.

This dissertation is dedicate to all the victims of Covid-19 and especially to all Brazilians who lost their lives due to the delay of the vaccine.

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The beauty of mathematics only shows itself to more patient followers.

Maryam Mirzakhani,
The first woman to win the Fields Medal prize.

Resumo

Santos, Róbson. **Sistema eletrostático e fórmulas divergentes**. Goiânia, 2023. 92p. Tese de Doutorado . Programa de Pós-Graduação em Matemática, Instituto de Matemática e Estatística, Universidade Federal de Goiás.

Uma questão clássica em relatividade geral é a classificação de soluções de buracos negros regulares estáticos das equações Eisntein-Maxwell (ou sistema eletrovácuo). Nós provamos alguns resultados de classificação para um sistema eletrovácuo tal que o potencial elétrico é uma função diferenciável da função lapso. Nós, particularmente, mostramos que um espaço n-dimensional eletrovácuo localmente conformemente plano satisfazendo algumas condições deve estar na classe Majumdar-Papapetrou. Além disso, nós provamos que qualquer espaço eletrovácuo de dimensão 3 ou 4 em que algumas condições são satisfeitas deve ser localmente conformemente plano. Mais ainda, nós demonstramos que um espaço electrovácuo n-dimensional satisfazendo algumas condições, sem divergência de quarta ordem do tensor de Weyl e curvatura radial de Weyl zero tal que o potencial elétrico está na classe Reissner-Nordström é localmente uma variedade produto torcido com fibra Einstein de dimensão $n-1$. Finalmente, um espaço electrovácuo tridimensional satisfazendo algumas condições, sem divergência de terceira ordem do tensor de Cotton, também é classificado. Nós também provamos que variedades eletrostáticas (ou eletrovácuos) tridimensional com constante cosmológica não nula e tensor de Bach livre de divergência são localmente conformemente planos, desde que o campo elétrico e o gradiente da função lapso sejam linearmente dependentes. Consequentemente, uma variedade eletrostática tridimensional admite uma estrutura local de produto torcido com uma base unidimensional e fibra uma superfície de curvatura constante.

Palavras–chave

Teoria da Relatividade, Sistema eletrostático, Variedades conformemente planas, Fórmulas divergentes, Tensor de Bach.

Abstract

Santos, Róbson. **Electrostatic system and divergence formulas**. Goiânia, 2023. 92p. PhD. Thesis . Programa de Pós-Graduação em Matemática, Instituto de Matemática e Estatística, Universidade Federal de Goiás.

A classical question in general relativity is about the classification of regular static black hole solutions of the static Einstein-Maxwell equations (or electrovacuum system). We prove some classification results for an electrovacuum system such that the electric potential is a smooth function of the lapse function. We particularly show that an n-dimensional locally conformally flat electrovacuum space satisfying some conditions must be in the Majumdar-Papapetrou class. We also prove that any three or four-dimensional electrovacuum space that some conditions are satisfied must be locally conformally flat. Moreover, we prove that an n-dimensional electrovacuum space satisfying some condition with fourth-order divergence-free Weyl tensor and zero radial Weyl curvature such that the electric potential is in the Reissner-Nordström class is locally a warped product manifold with $(n-1)$-dimensional Einstein fibers. Finally, a three-dimensional electrovacuum space satisfying some conditions with a third-order divergence-free Cotton tensor is also classified. We also prove that three-dimensional electrostatic (or electrovacuum) manifolds with a non-null cosmological constant and divergence-free Bach tensor are locally conformally flat, provided that the electric field and the gradient of the lapse function are linearly dependent. Consequently, a three-dimensional electrostatic manifold admits a local warped product structure with a one-dimensional base and a constant curvature surface fiber.

Keywords

Relativity Theory, Electrostatic System, Conformally flat manifolds, Divergence formulas, Bach tensor.

Contents

Introduction

The electrostatic system, in short, is Einstein's Field Equation – formulated by Albert Einstein in his studies of General Relativity – coupled to the Faraday tensor. This system is also called Einstein-Maxwell, and it will be the research object of this dissertation. This system has a lot of relevance to physics since it describes the relationship between Relativity and Electromagnetism. Furthermore, some well-known solutions for the electrostatic system are standard models for electrically charged static black holes or stars [15, 19, 22, 23]. It is also relevant because it is a set of Partial Differential Equations on an abstract Riemannian manifold. It generalizes the well-known static vacuum Einstein space system, whose most important model is the Schwarzschild space (cf. [1], and the references in there).

There are some important works about the research object of this dissertation and we can quote a few: Cederbaum and Galloway [15] who established the uniqueness of suitably defined subextremal photon spheres in an asymptotically flat electrovacuum spacetime; Chruściel and Delay [19] constructed infinite-dimensional families of non-singular static spacetimes, solutions for the vacuum Einstein-Maxwell equations with a negative cosmological constant; Cruz, Lima, and Sousa [23] connected the electrostatic system with the min-max theory; Hartle and Hawking [28] analyzed some of the stationary solutions of the electrostatic equations; Jahns [33] showed a uniqueness result for the n-dimensional spatial electrovacuum manifold using similar techniques used by [15]; Kunduri and Lucietti [36] proved that any asymptotically flat static spacetime in a higher dimensional electrostatic system must have no magnetic field; Coutinho and Leandro [22] proved that the lapse function must be identically zero at the horizon boundary of an electrostatic system with null cosmological constant. The horizon boundary is closely related to the event horizon, the edge of a black hole.

Specifically, in this work, the electrostatic system will be constructed in Chapter 1. Moreover, we will present essential definitions, examples, and results for our work in this chapter. Also, we will establish the notation used throughout this dissertation. The deduction of the electrostatic system is based on Chruściel and Delay [19, Appendix]. A brief discussion about conformal geometry is also made in this chapter, along with the presentation of some models for the electrostatic system, such as the Reissner-Nordström, Majumdar-Papapetrou, and Charged Nariai space.

We must discuss the cosmological constant briefly. Einstein introduced the constant in his equation to solve some inconsistencies in General Relativity Theory. The cosmological constant is fundamental to describe more precisely some astronomic phenomena, for instance, the expansion of the universe discovery by Hubble (cf. [51]). In this dissertation, the cosmological constant performs a significant role. We will call the electrostatic system with the null cosmological constant by *electrovacuum system*, which will be studied in Chapter 2; the electrostatic system with a non-null cosmological constant will be reviewed in Chapter 3.

More specifically, in Chapter 2, inspired by the Reissner-Nordström and Majumdar–Papapetrou solutions, some results concerning the local geometric structure of the electrovacuum system are proved (see [3]). As a fundamental hypothesis in this chapter, we supposed that the electric potential is a smooth function of the lapse function. We prove a characterization of the solutions for the electrovacuum system satisfying the above condition. Thus, one of the goals of this dissertation is to investigate this class of solutions and to prove some results about their geometric structure.

It is important to highlight that the results in Chapter 2 follow from the conformal structure of the metric. Locally conformally flat manifolds are very important for many research works in recent years (cf. [9, 10, 13, 32, 37, 44]). Observing this, we can state the main question of this work: *what are the curvature conditions we need to guarantee that an electrovacuum (or electrostatic) space is locally conformally flat?* In other words, we are interested in finding some natural conditions for an n-dimensional electrovacuum system to be locally conformally flat.

In Chapter 2, based on Andrade, Leandro, and Lousa [3], we find some sufficient conditions for the electrovacuum system (with a null cosmological constant) to be a locally conformally flat manifold. In higher dimensions, this condition is on the nullity fourth-order divergence of the Weyl tensor. In dimension three, the condition corresponds to the third-order divergence of the Cotton tensor being identically zero.

These studies about conformally flat manifolds are inspired by: Cao and Chen [9], in which the authors classified n-dimensional ($n > 3$) complete Bach-flat gradient shrinking Ricci solitons, showing that any 4-dimensional Bach-flat gradient shrinking Ricci soliton is either Einstein, or locally conformally flat; Catino [12], where the author proved a local characterization for locally conformally flat quasi-Einstein manifolds; Catino, Mastrolia, and Maticelli [13] classified complete gradient Ricci solitons satisfying a fourth-order vanishing condition on the Weyl tensor, making use of the conformally flat results; Hwang and Yun [32] studied static vacuum spaces with the complete divergence of the Bach tensor and Weyl tensor, implying in conformal flatness; Leandro [37], in which similar results were found for Einstein-type manifolds with fourth-order divergence-free Weyl tensor.

In Chapter 3, we study the electrostatic system with a non-null cosmological constant in the three-dimensional case. This chapter is based on the work of Leandro, and Lousa [39], where the authors found a sufficient condition for a three-dimensional electrostatic space to be a locally conformally flat manifold. We will prove that an electrostatic system with a divergence-free Bach tensor must be, locally, a warped product space with a one-dimensional base and fiber being a constant curvature surface. Chapter 3 is, in a sense, a continuation of Chapter 2. The most crucial difference lies in the fact that we are considering a non-null cosmological constant and no direct dependence on the electric function of the lapse function. We recommend to the reader to see [23] as a recent and good overview of the system considered in Chapter 3.

Furthermore, the strategies used in Chapter 3 make use of the differential forms theory. Consequently, the condition used in Chapter 2 of the electric potential be a function of the lapse function is not used in Chapter 3. We will assume an analogous, however, weaker condition. The electric field and the gradient of the lapse function must be linearly dependent. This is always true for an electrostatic space at the horizon boundary. Also, a straightforward computation shows that this assumption implies the assumption used in Chapter 2 about the electric potential and the lapse function. These conditions will be presented in that chapter, along with an in-depth discussion.

We will discuss the theme studied in each chapter, presenting the main results and demonstrations. Moreover, we will establish the connection between the results and the standard models stated in the preliminary section (Chapter 1).

CHAPTER 1

Conformally flat manifolds and electrostatic system

Our main goal in this chapter is to present some results about the locally conformally flat manifolds and deduce the electrostatic system from the warped product structure. To that end, we need first to present the definition of some tensors that will be fundamental through this work. We also will describe other well-known elements from differential geometry as well as some of its properties. Thus, this chapter is important to fix our notation and remember some widely known facts and results in the literature. Therefore, we refer to great works such as [5], [16], [26], [41], [43], among others that will be referenced throughout the chapter, and the entire dissertation.

An n-dimensional *Riemannian manifold* (M^n, g) is a smooth manifold equipped with a Riemannian metric g. Moreover, throughout this work, Einstein's convention will be used. We will consider $\mathscr{X}(M)$ and $\mathscr{D}(M)$ as the sets of all C^∞ vector fields on M and C^∞ real functions defined on M, respectively. Let $f : M^n \to \mathbb{R}$ be a smooth function of M, that is, $f \in \mathscr{D}(M)$ and $(x_1, \ldots, x_n)$ be a local coordinate system, then

$$\nabla_i f = g^{ij}\partial_{x_j} f,$$

denotes the gradient of f. Here, $\partial_{x_j} f$ stands for the partial derivative of f with respect to x_j. Moreover, g^{ij} are the components of the inverse metric g^{-1} of the metric g, with components g_{ij}.

In what follows, we will remember some important well-known definitions to fix notation. These definitions can be found in [26, Chapter 3]. Similar definitions can be found in [5, Chapter 1], [16, Section 2.5], and [35, Section 6C].

Definition 1.1 ([26, Chapter 3]) *The gradient of $f \in \mathscr{D}(M)$ in a point $p \in M$ is the vector field $\nabla f \in T_pM$ given by*

$$\langle \nabla f, v \rangle = df_p(v),$$

for all $v \in T_pM$.

Definition 1.2 ([26, Chapter 3]) *The Hessian tensor of a function* $f \in \mathscr{D}(M)$, *denoted by* $\nabla^2 f$, *is given by*

$$\nabla^2 f(X,Y) = \langle \nabla_X(\nabla f), Y \rangle,$$

for all $X, Y \in \mathscr{X}(M)$.

Definition 1.3 ([26, Chapter 3]) *The Laplacian operator* Δf *of a smooth function* $f \in \mathscr{D}(M)$ *is defined by*

$$\Delta f = \operatorname{div}(\nabla f),$$

where

$$\operatorname{div} X(p) = \text{trace of the linear application}\{Y(p) \mapsto \nabla_Y X(p)\}.$$

Here, $p \in M$ *and* $X, Y \in \mathscr{X}(M)$. *Moreover,* div *stands for the divergence of a vector field with respect to the metric defined over* M.

1.1 Conformally flat manifolds

In this section, we discuss some properties of (locally) conformally flat manifolds.

Definition 1.4 ([16, Chapter 1]) *A Riemannian manifold* (M^n, g) *is said to be locally conformally flat if for a point* $p \in M$, *there exists a local coordinate system* $\{x_i\}$ *in a neighborhood* U *of* p *such that*

$$g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = f\delta_{ij},$$

for some function $f \in \mathscr{D}(U)$. *Here,* δ_{ij} *stands for the Kronecker delta.*

For an important characterization of (locally) conformally flat manifolds, we will present some special tensors that play a fundamental role in this work. The first tensor that we will present is the *Riemann (Curvature) tensor* R_m. This tensor is named in honor of G. Riemann, and it is also known as Riemann–Christoffel tensor (see [49, 50, 51]).

From now on, we will consider a Riemannian manifold (M^n, g), $n \geq 3$, with metric tensor $g = \langle \cdot, \cdot \rangle$. The Riemann curvature tensor is defined by

$$\begin{array}{rcl} R_m : \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) \times \mathscr{X}(M) & \to & \mathscr{D}(M) \\ (X,Y,Z,W) & \mapsto & R_m(X,Y,Z,W) = \langle R(X,Y)Z, W \rangle \end{array}$$

where $R(X,Y)Z : \mathscr{X}(M)\times\mathscr{X}(M)\times\mathscr{X}(M)\to\mathscr{X}(M)$ is

$$R(X,Y)Z = \nabla_X\nabla_y Z - \nabla_Y\nabla_X Z + \nabla_{[X,Y]}Z, \quad X,Y,Z\in\mathscr{X}(M).$$

Here, ∇ and $[\cdot,\cdot]$ stand for the Levi-Civita connection and the Lie bracket, respectively.

In a local coordinate system $(x_1,\ldots,x_n)$ with associated base $\{e_1,\ldots,e_n\}$ for the tangent space of M,the Riemannian curvature tensor is $R_{ijkl} = \langle R(e_i,e_j)e_k,e_l\rangle$. Remember that this tensor has the following symmetries:

- $R_{ijks}+R_{jkis}+R_{kijs} = 0$ (First Bianchi Identity);
- $R_{ijks} = -R_{jiks} = -R_{ijsk} = R_{ksji}$.

Moreover, the Riemannian curvature tensor can be related to $f\in\mathscr{D}(M)$ by using the Ricci identity

$$\nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f = R_{ijkl}\nabla^l f. \tag{1-1}$$

Remember that we are assuming Einstein's convention for sum. Furthermore, the second Bianchi identity is

$$\nabla_m R_{ijkl} + \nabla_k R_{ijlm} + \nabla_l R_{ijmk} = 0. \tag{1-2}$$

Contracting the curvature tensor over the first and last indices, we obtain the *Ricci tensor*, Ric, which in coordinates is given by

$$R_{ij} = g^{kl}R_{ikjl}.$$

Taking the trace one more time, now over the remained indices, the *scalar curvature* R is obtained, that is,

$$R = g^{ij}R_{ij}.$$

The covariant derivatives of the Ricci tensor and the scalar curvature are related by the contracted second Bianchi identity. Note that we can contract the second Bianchi identity (1-2) over m and i, to obtain

$$g^{mi}\nabla_m R_{ijkl} + \nabla_k R_{jl} - \nabla_l R_{jk} = 0.$$

Now over j and k, we have

$$g^{mi}\nabla_m R_{il} + g^{jk}\nabla_k R_{jl} - \nabla_l R = 0,$$

reordering the indices, we get

$$g^{ij}\nabla_i R_{kj} = \frac{1}{2}\nabla_k R. \tag{1-3}$$

We can rewrite the above equation as

$$\mathrm{div}\,\mathrm{Ric} = \frac{1}{2}dR,$$

where dR is the differential form associated to R.

Let us now present the *Weyl tensor*, which is related to the Riemannian curvature tensor. This tensor was first defined in the year of 1918 by the German mathematician Hermann Weyl in [52]. The Weyl tensor is defined, in a local coordinate system, by

$$\begin{aligned} W_{ijkl} &= R_{ijkl} - \frac{1}{n-2}(R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il}) \\ &\quad + \frac{R}{(n-1)(n-2)}(g_{ik}g_{jl} - g_{il}g_{jk}). \end{aligned} \tag{1-4}$$

From straightforward computation (cf. [9]), it is possible to observe that the Weyl tensor has the same symmetries of the curvature tensor, i.e.,

$$W_{ikjl} = -W_{kijl}, \quad W_{ikjl} = -W_{iklj}, \quad \text{and} \quad W_{ikjl} = W_{jlik}.$$

Also, the Weyl tensor is totally trace-free, in other words, the trace of the Weyl tensor vanishes over any two indices, i.e.,

$$g^{ts}W_{ijkl} = 0 \quad \forall\, t, s \in \{i, j, k, l\} \quad \text{and} \quad t \neq s.$$

Still dealing with the Weyl tensor, we have some useful definitions for our work. When the divergence of the Weyl tensor is identically zero, i.e.,

$$\mathrm{div} W = 0,$$

we say that the manifold has *a harmonic Weyl curvature*. In a local coordinate system, we have

$$\mathrm{div} W = g^{lp}\nabla_p W_{ijkl} = \nabla^l W_{ijkl}.$$

It is well-known that if the scalar curvature is constant, then harmonic Weyl curvature implies harmonic curvature, and this follows from (1-3) and (1-4). Therefore, it is possible to calculate the divergence of the Weyl tensor at most in its fourth order, i.e.,

$$\operatorname{div}^4 W = \nabla^{ijkl} W_{ijkl}.$$

Moreover, a Riemannian manifold (M^n, g) has zero radial Weyl curvature if

$$W(\cdot,\cdot,\cdot,\nabla f) = 0,$$

where ∇f is the gradient of $f \in \mathscr{D}(M)$. This condition was used in [12] and [37] in the study of Einstein-type manifolds. It is important to say that Catino [12] proved that this condition can not be removed in the classification of quasi-Einstein manifolds having harmonic Weyl tensor. This condition over the Weyl tensor will be important in the classification of the Electrovacuum system having a fourth-order divergence-free Weyl tensor, i.e., $\operatorname{div}^4 W = 0$.

The curvature tensor is determined by the Ricci tensor in dimension three since the Weyl tensor is identically zero (see [35, Corollary 8.25]).

Theorem 1.5 ([35, Corollary 8.25]) *Let (M^3, g) be an 3-dimensional Riemannian manifold, then*

$$W = 0.$$

In 1899, Émile Cotton [21] defined a third-order tensor that we will discuss ahead, the Cotton tensor. This tensor is written as

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}). \tag{1-5}$$

The tensor is totally trace-free. Moreover, it is skew-symmetric under the two first indices, i.e.,

$$C_{ijk} = -C_{jik}.$$

Furthermore, from a straight computation, we have

$$\nabla^k C_{kij} = \nabla^k C_{kji}.$$

The Cotton tensor also satisfies a Bianchi-type identity, i.e.,

$$C_{ijk} + C_{jki} + C_{kij} = 0. \tag{1-6}$$

Thus, from the contracted second Bianchi identity (1-3), and the commutation formulas, we have

$$\nabla^i C_{jki} = 0. \tag{1-7}$$

In [12], Catino proved that the Cotton tensor and the derivative of the Weyl tensor are related by the identity

$$C_{ijk} = -\frac{n-2}{n-3}\nabla^l W_{ijkl}, \tag{1-8}$$

for $n > 3$. The equation (1-8) provides that the Cotton tensor vanishes if, and only if, the Weyl tensor is harmonic.

Assuming that $n \geq 4$, we can define the Bach tensor

$$B_{ij} = \frac{1}{n-3}\nabla^k\nabla^l W_{ikjl} + \frac{1}{n-2}R^{kl}W_{ikjl}. \tag{1-9}$$

It was defined in 1921 by Rudolf Bach in [4]. Combining (1-8) and (1-9), we deduce a relationship between the previous tensors, that is,

$$B_{ij} = -\frac{1}{n-2}\nabla^k C_{ikj} + \frac{1}{n-2}R^{kl}W_{ikjl}. \tag{1-10}$$

It is natural (see [9]) to define the 3-dimensional Bach tensor by

$$B_{ij} = \nabla^k C_{kij}. \tag{1-11}$$

This definition is natural due to the identity (1-10), and the fact the Weyl tensor is identically zero in dimension $n=3$ (Theorem 1.5). We can also express the divergence of the Bach tensor as a function of the Cotton tensor

$$\nabla^j B_{ij} = \frac{n-4}{(n-2)^2}C_{ijk}R^{jk}. \tag{1-12}$$

The Bach tensor appeared naturally from studies of Huyghens's principle and has some psychical significance mainly about wave propagation (see for instance [48] and the references therein). It is easy to see that this tensor is also trace-free since the Weyl tensor and the Cotton tensor are totally trace-free.

Finally, we describe the last tensor important for this dissertation. This tensor was first defined by Schouten in 1921 [46]. The Schouten tensor can be described by

$$S_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{R}{2(n-1)}g_{ij}\right). \tag{1-13}$$

From this definition, we obtain

$$(n-2)C_{ijk} = \nabla_i S_{jk} - \nabla_j S_{ik}.$$

We can rewrite the other tensors in terms of the Schouten tensor (cf. [9]).

Taking into account these tensors, we can return to the theme under discussion and characterize the locally conformally flat manifolds. The characterization is given by Theorem 1.6 which follows below (see [16, Proposition 1.31]). Any locally conformally flat manifold of dimension $n \geq 4$ is characterized by the Weyl tensor. For dimension three, since $W = 0$ (Theorem 1.5), the locally conformally flat manifolds are characterized by the Cotton tensor. For dimension $n = 2$, it is well-known that every two-dimensional Riemannian metric is locally conformally flat (see [16, Chapter 2], and [35, Corollary 8.29]).

Theorem 1.6 ([16, Proposition 1.31]) *A Riemannian manifold (M^n, g) is locally conformally flat if and only if*

- *(for $n \geq 4$) the Weyl tensor vanishes,*
- *(for $n = 3$) the Cotton tensor vanishes.*

For $n = 3$ we can see that the result holds when the Schouten tensor S is Codazzi, i.e.,

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z)$$

for all $X, Y, Z \in \mathscr{X}(M)$ (see [35, Theorem 8.31]).

There are a lot of examples of conformally flat manifolds in the literature: $\mathbb{S}^n$, $\mathbb{R}^n$, and $\mathbb{H}^n$. Moreover, Chow, Lu, and Ni [16, Corollary 1.33] showed that if a Riemannian manifold has constant sectional curvature, then it is locally conformally flat. Other conditions are well-known in the literature. In the next section, we will provide an important criterion [7, Theorem 1] showing when a warped product is locally conformally flat. This result is very important for our work.

1.2 The electrostatic system

In this section, our goal is to deduce the electrostatic system from Einstein's field equation coupled to the Faraday tensor. The main ideas of this computation can be found in [19, Appendix], see also [23] and the references therein. For this purpose, it is fundamental to bring forward some well-known discussions about the warped product of a Riemannian manifold. The demonstrations of these results concerning warped product structure can be found in [5, Chapter 9] and [43, Chapter 7]. We start with the following definition.

Definition 1.7 ([43, Definition 33]) *Let (B, g_B) and (F, g_F) be Riemannian manifolds furnished with their respective metrics. Moreover, let $f : B \to \mathbb{R}$ be a positive smooth* *mapping. A warped product manifold $M = B \times_f F$ is defined by the product manifold $B \times F$ furnished with metric tensor*

$$g = \pi^*(g_B) + (f \circ \pi)^2 \sigma^*(g_F),$$

where $\pi : M \to B$ and $\sigma : M \to F$ projections of $B \times F$ onto B and F, respectively.

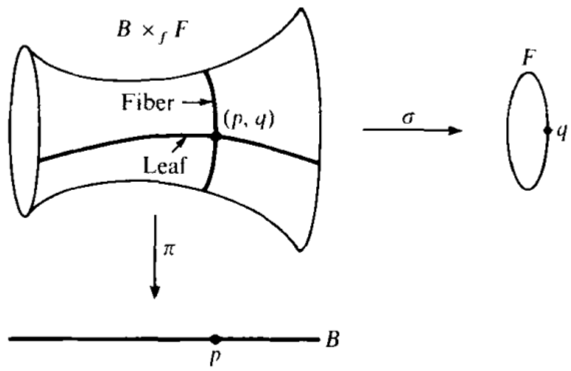

Figure 1.1: Warped product [43, Page 205].

The manifolds B and F are called the base and the fiber of the warped product manifold M, respectively. The function f is called *warping function*. Moreover, we can verify quickly that if the warped function $f \equiv 1$, then the warped manifold is just the standard product manifold. The functions $\widetilde{h} = h \circ \pi \in \mathscr{D}(M)$ and $\widetilde{\psi} = \psi \circ \sigma \in \mathscr{D}(M)$ are called *lifts* of $h \in \mathscr{D}(B)$ and $\psi \in \mathscr{D}(F)$ to M. Also, we denote by $\mathfrak{L}(B)$ and $\mathfrak{L}(F)$ the sets of all lifts $\widetilde{X}$ and $\widetilde{V}$ of $X \in \mathscr{X}(B)$ and of $V \in \mathscr{X}(F)$. Now we can state some well-known results about warped product manifolds.

Theorem 1.8 ([43, Chapter 7]) *Let $M = B \times_f F$ be a warped product manifold, $h \in \mathscr{D}(B)$ and $\psi \in \mathscr{D}(F)$. Then,*

i) $\nabla_g \widetilde{h} = \widetilde{\nabla_{g_B} h}$,

ii) $\nabla_g \widetilde{\psi} = \dfrac{\widetilde{\nabla_{g_f} \psi}}{\widetilde{f}^2}$,

iii) $\Delta_g \widetilde{h} = \Delta_{g_B} h - d \dfrac{g_B(\nabla_{g_B} h, \nabla_{g_B} f)}{f}$, *where d is the dimension of the fiber F,*

iv) $\Delta_g \widetilde{\psi} = \Delta_{g_F} \psi$.

Let us remember some important terms of the Ricci tensor and scalar curvature for a warped product manifold that will be important for this dissertation.

Theorem 1.9 ([43, Chapter 7]) *Let $(M, g) = (B, g_B) \times_f (F, g_F)$ be a warped product and $X, Y \in \mathfrak{L}(B)$. Then,* *i)* $\mathrm{Ric}_g(X,Y) = \mathrm{Ric}_{g_B}(X,Y) - \frac{d}{f}\nabla^2_{g_B} f(X,Y)$;

ii) $R_g = R_{g_B} + \frac{R_{g_F}}{f^2} - 2d\frac{\Delta_{g_B} f}{f} - d(d-1)\frac{|\nabla f|^2_{g_B}}{f^2}$,

where d performs the dimension of the fiber F, R_{g_B} and R_{g_F} are the scalar curvature of B and F, respectively.

Remark 1 *It is convenient to notice that [43, Corollary 43] takes the dimension of F more than 1 in item i) of the Theorem 1.9, but the result also follows if the dimension is equal to 1.*

To relate the warped product structure with conformally flat manifolds we have the following well-known result, see [7, Theorem 1] and the references therein.

Theorem 1.10 *Let $M = B \times_f F$ be a semi-Riemannian warped product.*

- *If $\dim B = 1$, then $M = B \times_f F$ is locally conformally flat if and only if (F, g_F) is a space of constant curvature;*
- *If $\dim B > 1$ and $\dim F > 1$, then $M = B \times_f F$ is locally conformally flat if and only if*
  - *(F, g_F) is a space of constant curvature c_F.*
  - *The function $f : B \to \mathbb{R}^+$ defines a global conformal deformation on B such that $(B, (1/f^2)g_B)$ is a space of constant curvature $\widetilde{c}_B = -c_F$.*
- *If $\dim F = 1$, then $M = B \times_f F$ is locally conformally flat if and only if the function $f : B \to \mathbb{R}^+$ defines a conformal deformation on B such that $(B, (1/f^2)g_B)$ is a space of constant curvature.*

To finish our considerations about the warped product structure we will consider the following scenario (see [10] and [5, Chapter 9]). Let M be a warped product manifold given by

$$(M^n, g) = (I, dr^2) \times_\phi (N^{n-1}, \overline{g}),$$

where $I \subseteq \mathbb{R}$ is a interval and $g = dr^2 + \phi(r)^2\overline{g}$, and let

$$\theta = (\theta_2, \theta_3, \cdots, \theta_n)$$

be a local coordinate system on N^{n-1}, in which $(x_1, x_2, x_3, \cdots, x_n) = (r, \theta_2, \cdots, \theta_n)$. Let also $a, b, c, \cdots$ be the range from 2 to n, thus the Riemannian curvature tensor of (M^n, g) is given by

$$R_{1a1b} = \phi\phi''\overline{g}_{ab}, \qquad R_{1abc} = 0$$

and

$$R_{abcd}=\phi^2\overline{R}_{abcd}-(\phi\phi')^2(\overline{g}_{ac}\overline{g}_{bd}-\overline{g}_{ad}\overline{g}_{bc}).$$

Therefore, contracting the above equations, we obtain the following Ricci tensor formulas for (M^n, g):

$$R_{11}=-(n-1)\frac{\phi''}{\phi},\qquad R_{1a}=0$$

and

$$R_{ab}=\overline{R}_{ab}-[(n-2)(\phi')^2+\phi\phi'']\overline{g}_{ab}.$$

Finally,

$$R=\phi^{-2}\overline{R}-(n-1)(n-2)\left(\frac{\phi'}{\phi}\right)^2-2(n-1)\frac{\phi''}{\phi}.$$

Now, to construct the electrostatic system following the ideas in [19, Appendix], it is necessary to consider (M^n,g) an $n-$dimensional Riemannian manifold with $n\geq 3$, and $f:M\to\mathbb{R}$ a positive function. Now, let

$$\mathscr{M}=M\times_f\mathbb{R}$$

be a warped product with metric

$$\widetilde{g}=g+\varepsilon f^2dt^2$$

where $\varepsilon=\pm1$. Hence, suppose that $x^i=(x^1,\cdots,x^n)$, $x^0=t$, and $x^\mu=(x^0,x^1,\cdots,x^n)$ are ordinary vectors of M, $\mathbb{R}$ and $\mathscr{M}$, respectively. To maintain the same notation used by [19] the greek letters range from 0 to n.

Remember (see [26, Chapter 2]) that Christoffel's symbols are given by

$$\widetilde{\Gamma}^\lambda_{\alpha\beta}=\frac{1}{2}\widetilde{g}^{\lambda m}\left(\frac{\partial\widetilde{g}_{m\alpha}}{\partial x^\beta}+\frac{\partial\widetilde{g}_{m\beta}}{\partial x^\alpha}-\frac{\partial\widetilde{g}_{\alpha\beta}}{\partial x^m}\right).\tag{1-14}$$

Therefore, observe that the Christoffel's symbols for $(\mathscr{M}^{n+1},\widetilde{g})$ are given by

$$\widetilde{\Gamma}^0_{00}=\widetilde{\Gamma}^0_{ij}=\widetilde{\Gamma}^k_{i0}=0,\quad \widetilde{\Gamma}^0_{i0}=\frac{\partial_i f}{f},\quad \widetilde{\Gamma}^k_{00}=-\varepsilon f\nabla_k f,\quad\text{and}\quad \widetilde{\Gamma}^k_{ij}=\Gamma^k_{ij},$$

where ∇ stands for the Levi-Civita derivative operator associated with the metric tensor g.

We conclude that the components of Riemann curvature of $(\mathscr{M},\widetilde{g})$ are

$$\widetilde{R}_{ijkl}=R_{ijkl},\quad \widetilde{R}_{0j0l}=-\varepsilon f\nabla_j\nabla_l f,\quad \text{and}\quad \widetilde{R}_{ijk0}=R_{ij0l}=\widetilde{R}_{0jkl}=0.$$

For the Ricci tensor of $(\mathscr{M},\widetilde{g})$ we have

$$\widetilde{R}_{ik}=R_{ik}-\frac{\nabla_k\nabla_i f}{f},\quad \widetilde{R}_{0k}=0,\quad \text{and}\quad \widetilde{R}_{00}=-\varepsilon f\Delta f. \tag{1-15}$$

Thus, from Theorem 1.9 the scalar curvature of $\mathscr{M}$ is

$$\widetilde{R}=R-2\frac{\Delta f}{f}. \tag{1-16}$$

We point out that the Einstein field equation is given by

$$\widetilde{R}_{\alpha\beta}-\frac{1}{2}\widetilde{R}\widetilde{g}_{\alpha\beta}+\Lambda\widetilde{g}_{\alpha\beta}=T_{\alpha\beta}. \tag{1-17}$$

Here, we will consider T as the Faraday tensor, i.e.,

$$T=F\circ F-\frac{1}{4}|F|^2\widetilde{g},$$

where the Hadamard-Schur product is given by

$$(F\circ F)_{\alpha\beta}:=g^{\mu\nu}F_{\alpha\mu}F_{\beta\nu},$$

and the Hilbert-Schmidt norm of F is given by

$$|F|^2=g^{\alpha\beta}g^{\mu\nu}F_{\alpha\mu}F_{\beta\nu}.$$

Moreover, we can observe that the trace of T is

$$\mathrm{Tr}_{\widetilde{g}}T\equiv\widetilde{g}^{\alpha\beta}T_{\alpha\beta}=|F|^2-\frac{(n+1)}{4}|F|^2=-\frac{n-3}{4}|F|^2. \tag{1-18}$$

Remark 2 *Here it is important to point out that in the three-dimensional case, T is trace-free.*

Now, from the Maxwell equations (see [18] and [19, Equation 1.2]) we can infer that

$$\mathrm{div}_{\widetilde{g}}F=0,\quad F=\mathrm{d}(\psi\mathrm{d}t),\quad \text{and}\quad \partial_t\psi=0,$$

where $\psi \in \mathscr{D}(M)$. Here, we are assuming the existence of a hypersurface-orthogonal globally timelike Killing vector $X = \partial/\partial t$ such that, over $\widetilde{g}$ (with $\varepsilon = -1$), we get

$$\partial_t f = \partial_t g = 0.$$

In what follows, we will consider only $\varepsilon = -1$. Contracting (1-17) over $\widetilde{g}$ we obtain the scalar curvature of $(\mathscr{M}, \widetilde{g})$, given by

$$\widetilde{R} = \frac{-2}{n-1}\left(\mathrm{Tr}_{\widetilde{g}} T - (n+1)\Lambda\right). \tag{1-19}$$

where $\mathrm{Tr}_{\widetilde{g}} T$ represents the trace of T over $\widetilde{g}$. Replacing the curvature in (1-17) we have the following equation

$$\widetilde{R}_{\alpha\beta} = \frac{2\Lambda - \mathrm{Tr}_{\widetilde{g}} T}{n-1}\widetilde{g}_{\alpha\beta} + T_{\alpha\beta}. \tag{1-20}$$

Now, replacing (1-20) in (1-15) we get

$$R_{ij} = \frac{\nabla_i \nabla_j f}{f} + \frac{2\Lambda}{n-1} g_{ij} + T_{ij} - \frac{\mathrm{Tr}_{\widetilde{g}} T}{n-1} g_{ij}. \tag{1-21}$$

Contracting this equation over metric g, we have the scalar curvature of M, i.e.,

$$R = \frac{\Delta f}{f} + \frac{n}{n-1}(2\Lambda - \mathrm{Tr}_{\widetilde{g}} T) + \mathrm{Tr}_g T.$$

Also, we can combine (1-19) with (1-16) to obtain

$$f\Delta f = f^2\left(\frac{R}{2} + \frac{\mathrm{Tr}_{\widetilde{g}} T - (n+1)\Lambda}{n-1}\right).$$

Replacing the scalar curvature R in the above equation we have

$$f\Delta f = f^2\left(T_{\alpha\beta}N^{\alpha}N^{\beta} + \frac{\mathrm{Tr}_{\widetilde{g}} T - 2\Lambda}{n-1}\right), \tag{1-22}$$

where $N^{\alpha}\partial_{\alpha}$ is the unit timelike normal to the level sets of t, i.e., $T_{\alpha\beta}N^{\alpha}N^{\beta} = \mathrm{Tr}_g T - \mathrm{Tr}_{\widetilde{g}} T$.

From the Maxwell equations, we get

$$|F|^2 = -2\frac{|\nabla\psi|^2}{f^2},$$

thus, (1-18) is given by

$$\mathrm{tr}_{\widetilde{g}} T = \frac{n-3}{2}\frac{|\nabla\psi|^2}{f^2}. \tag{1-23}$$

Hence, from the Faraday tensor we have

$$T_{ij} = -\frac{\nabla_i\psi\nabla_j\psi}{f^2} + \frac{1}{2}\frac{|\nabla\psi|^2}{f^2}g_{ij}. \tag{1-24}$$

Contracting (1-24) we have

$$g^{ij}T_{ij} = \frac{(n-2)}{2}\frac{|\nabla\psi|^2}{f^2}.$$

Considering $T_{\alpha\beta}N^{\alpha}N^{\beta} = \dfrac{|\nabla\psi|^2}{2f^2}$ in (1-22) gives us

$$f\Delta f = \frac{n-2}{n-1}|\nabla\psi|^2 - \frac{2\Lambda f^2}{n-1}. \tag{1-25}$$

Furthermore, combing (1-21), (1-23) and (1-24), we obtain

$$fR_{ij} = \nabla_i\nabla_j f + 2\frac{f\Lambda}{n-1}g_{ij} - \frac{\nabla_i\psi\nabla_j\psi}{f} + \frac{1}{n-1}\frac{|\nabla\psi|^2}{f}g_{ij}. \tag{1-26}$$

Since $F = d\psi\wedge dt$, from the Maxwell equations (i.e., $\mathrm{div}_{\widetilde{g}}(F) = 0$ and $\partial_t\psi = 0$) and the Laplacian formula for a warped product metric (cf. Theorem 1.8) we have

$$\Delta_g\psi - \frac{g(\nabla\psi, \nabla f)}{f} = 0. \tag{1-27}$$

On the other hand,

$$\mathrm{div}\left(\frac{\nabla\psi}{f}\right) = \frac{1}{f}\left[\Delta_g\psi - \frac{g(\nabla\psi, \nabla f)}{f}\right].$$

Thus, from (1-27) we obtain

$$\mathrm{div}\left(\frac{\nabla\psi}{f}\right) = 0. \tag{1-28}$$

Finally, taking $\psi := \sqrt{2}\psi$ (for the convention assumed in this work) in (1-25), (1-26), and (1-28) we obtain the electrostatic system that can be defined as follows.

Definition 1.11 *Let (M^n, g) be an n-dimensional smooth Riemannian manifold with*

$n \geq 3$ and $f, \psi : M \to \mathbb{R}$ be smooth functions satisfying

$$
\begin{aligned}
f\mathrm{Ric} &= \nabla^2 f - \frac{2}{f} d\psi \otimes d\psi + 2\frac{f\Lambda}{n-1} g + \frac{2}{(n-1)f}|\nabla\psi|^2 g, \\
\Delta f &= 2\left(\frac{n-2}{n-1}\frac{|\nabla\psi|^2}{f} - \frac{f\Lambda}{n-1}\right), \\
0 &= \operatorname{div}\left(\frac{\nabla\psi}{f}\right),
\end{aligned}
\tag{1-29}
$$

where ∇^2 and Δ stand for the Hessian tensor and the Laplacian operator with respect to metric g, respectively. We refer (M^n, g, f, ψ) as an electrostatic system (or space).

Moreover, the smooth functions f, ψ, and the manifold M^n are called lapse function, electric potential, and spatial factor for the electrostatic system, respectively. Furthermore, $f > 0$ on M, and $f^{-1}(0) = \partial M$ (see [3, 19, 18, 23]).

Now, note that by taking the contraction of the first equation of Definition 1.11, we obtain

$$
fR = \Delta f + 2\frac{n}{n-1} f\Lambda + \frac{2}{(n-1)f}|\nabla\psi|^2.
$$

Then, combining it with the second equation in (1-29), we get a useful equation that does not depend on the dimension of M and relates the scalar curvature R with the lapse function and the electric potential, i.e.,

$$
f^2 R = 2\left(|\nabla\psi|^2 + f^2\Lambda\right). \tag{1-30}
$$

We can assume

$$
\frac{\nabla\psi}{f} = E.
$$

Hence, from a straightforward computation, analogous to the deduction of Definition 1.11, the electrostatic system can be rewritten in the following form:

$$
\begin{aligned}
\nabla^2 f &= f\left(\mathrm{Ric} - \frac{2}{n-1}\Lambda g + 2E^\flat \otimes E^\flat - \frac{2}{n-1}|E|^2 g\right), \\
\Delta f &= 2f\left(\frac{n-2}{n-1}|E|^2 - \frac{1}{n-1}\Lambda\right), \\
0 &= \operatorname{div}(E) \quad \text{and} \quad \operatorname{curl}(fE) = 0,
\end{aligned}
\tag{1-31}
$$

where $E \in \mathscr{X}(M)$ is called electric field, $f \in C^\infty(M)$ is the lapse function, and $E^\flat$ is the one-form metrically dual to E. The above system will be considered in Chapter 3. In the three-dimensional case, the above system was studied in [23].

It is worth highlighting that both systems (1-29) and (1-31) are equivalent if M is simply connected. In fact, if $curl(fE) = 0$ and M is simply connected, we obtain that the field fE is a path-independent vector field, that is, there exists a smooth function ψ called potential function (that we called it electric potential) such that $fE = \nabla\psi$.

The contraction of the first equation and combining it with the second equation in (1-31) we obtain

$$R = 2(|E|^2 + \Lambda). \tag{1-32}$$

Furthermore, we can observe that (see [23, Lemma 4]) over $\partial M = f^{-1}(0)$, the electric field E and the gradient of the lapse function are linearly dependent, i.e., there exists a smooth function $\rho : M \to \mathbb{R}$ such that $E = \rho\nabla f$. In fact, since $curl(fE) = 0$ we obtain

$$df \wedge E^\flat + f dE^\flat = 0. \tag{1-33}$$

Since f is identically zero over ∂M, we get

$$df \wedge E^\flat = 0.$$

More properties of the electrostatic system will be presented in the following chapters.

1.3 Solutions for the electrostatic system

In this section, we aim to present some well-known solutions for the electrostatic system.

Reissner-Nordström solution (RN). *In 1918 G. Nordström and H. Reissner, independently, found a class of exact solutions to the Einstein equation for the gravitational field of a spherical charged mass (see [47] for a wide-ranging discussion about this solution). The Reissner-Nordström (RN) electrostatic spacetime is one of the most important solutions to the electrostatic system, and it can be thought of as a model for a static black hole with electric charge* q *and mass* m*. It is called subextremal, extremal, or superextremal depending on if* $m^2 > q^2$, $m^2 = q^2$ *or* $m^2 < q^2$*, respectively. For instance, we have the following RN solution given by the Riemannian manifold* $M^n = \mathbb{S}^{n-1} \times (r^+, +\infty)$ *with metric tensor*

$$g = \frac{dr^2}{1 - 2mr^{2-n} + q^2 r^{2(2-n)}} + r^2 g_{\mathbb{S}^{n-1}},$$

where r represents the radial coordinate. Here, $m^2 \geq q^2$ are constants, and $r^+ > (m+\sqrt{m^2-q^2})^{1/(n-2)}$. Moreover, the outer horizon for the Reissner-Nordström spacetime is located at $(m+\sqrt{m^2-q^2})^{1/(n-2)}$, which corresponds to the zero set of the lapse function of the RN manifold. The static horizon is defined as the set where the lapse function for a static manifold is identically zero. This set is physically related to the event horizon, the boundary of a black hole. The RN space is locally conformally flat (see [17, 33] for instance).

It is well-known that the lapse function f and the electric potential ψ of the solution Reissner–Nordström satisfies the following relationship (see [15, Equation A.1] and [36, Lemma 3]):

$$f^2 = 1 + 2\frac{n-2}{n-1}\psi^2 - 2\frac{m}{q}\sqrt{2\frac{n-2}{n-1}}\psi. \tag{1-34}$$

Majumdar–Papapetrou solution (MP). *Another important electrovacuum solution is the Majumdar–Papapetrou (see [17, 28, 42]), which is related to an extremal RN solution. The Majumdar-Papapetrou (MP) solution to the electrostatic system represents the static equilibrium of an arbitrary number of charged black holes whose mutual electric repulsion exactly balances their gravitational attraction. A spacetime will be called a standard MP spacetime if the metric tensor is given by*

$$\hat{g} = -f^2dt^2 + f^{-2/(n-2)}(dx_1^2 + \ldots + dx_n^2),$$

in Cartesian coordinates $x = (x_1, \ldots, x_n)$ and $\widehat{M}^{n+1} = (\mathbb{R}^n \backslash \{a_i\}_{i=1}^l) \times \mathbb{R}$, for a finite set of points $a_i \in \mathbb{R}^n$, where

$$\frac{1}{f(x)} = 1 + \sum_{i=1}^{l} \frac{m_i}{r_i^{n-2}}; \quad r_i = |x - a_i|, \tag{1-35}$$

for some positive constants m_i, and the electric potential

$$\pm\sqrt{\frac{2(n-2)}{(n-1)}}\psi = 1 - f,$$

(see [28, Equation 2.3] and [36, Lemma 1]).

Solution invariant by translation. *In [38], the authors found a family of examples for a non-complete n-dimensional electrostatic space distinct from the Reissner-Nordström solution. This example is locally conformally flat and invariant under the action of an $(n-1)$-dimensional translation group. Moreover, the lapse function f and the electric potential ψ are related (see more details in [38, Theorem 1.5]). It* *is not understood if this solution has any physical meaning.*

The classification problem of an electrovacuum spacetime can be stated as follows. Suppose that

$$q_i q_j \geq 0, \quad \forall\, i, j,$$

where q_i is the charge of the i-th connected degenerate component of the electric-charged black hole. Then, the black hole is either an RN black hole or an MP black hole. There are some important and recent results in the literature concerning the classification of electrovacuum spaces (see for instance [20, 36, 42] and their references).

Furthermore, considering the null cosmological constant and the electric potential constant everywhere (or electric field identically zero), we have that the system (1-29) is a generalization of the static vacuum Einstein spacetime. The static vacuum Einstein spacetime is broadly explored in the literature. Furthermore, the most important solution for this system is the Schwarzschild solution.

Now we present some solutions for the system (1-31), with non-null cosmological constant and dimension 3. The following three examples can be found in [23]. These examples are locally conformally flat (see Theorem 1.10).

Unit hemisphere. *The n-dimensional unit hemisphere $\mathbb{S}^n_+ \subset \mathbb{R}^{n+1}$ equipped with the standard metric $g_{\mathbb{S}^n}$,*

$$M = (\mathbb{S}^n_+, g_{\mathbb{S}^n}).$$

For this example, we have that the lapse function is $f = x_{n+1}$, and electric potential constant everywhere (cf. [1]). In other words, the unit hemisphere is an example of a static metric and an electrostatic system with a non-null cosmological constant.

Charged Nariai system. *The charged Nariai system is the 3-dimensional space*

$$\left[0, \frac{\pi}{\alpha}\right] \times \mathbb{S}^2$$

with metric tensor $g = dr^2 + \varphi^2 g_{\mathbb{S}^2}$, where φ is a constant and $g_{\mathbb{S}^2}$ is the standard metric of the sphere $\mathbb{S}^2$ with radius 1. The electric field and the lapse function are given by

$$E = \frac{q}{\varphi^2}\partial_r \quad \textit{and} \quad f(r(x)) = \sin(\alpha r(x)),$$

where $r(x)^2 = x_1^2 + x_2^2 + x_3^2$ such that (x_1, x_2, x_3) are Cartesian coordinates,

$$\alpha = \sqrt{\Lambda - \frac{q^2}{\varphi^4}} \quad \textit{and} \quad \frac{1}{2\Lambda} < \varphi^2 < \frac{1}{\Lambda}.$$

Moreover,

$$0 < m^2 = \frac{1}{18\Lambda}\left[1 + 12q^2\Lambda + \sqrt{(1-4q^2\Lambda)^3}\right]$$

and

$$0 < |q| \leq \varphi^2\sqrt{\Lambda}.$$

The following two examples are pretty similar to the above one. However, there are essential differences that we need to highlight.

Cold Black Hole. *The cold black hole is the* 3*-dimensional space*

$$[0,\infty)\times\mathbb{S}^2$$

with metric tensor $g = dr^2 + \varphi^2 g_{\mathbb{S}^2}$, *where* φ *is a constant and* $g_{\mathbb{S}^2}$ *is the standard metric of the sphere* $\mathbb{S}^2$ *with radius* 1*. The electric field and the lapse function are given by*

$$E = \frac{q}{\varphi^2}\partial_r \quad \text{and} \quad f(r(x)) = \sinh(\beta r(x)),$$

where $r(x)^2 = x_1^2 + x_2^2 + x_3^2$, $\beta = \sqrt{\frac{q^2}{\varphi^4} - \Lambda}$ *and* $0 < \varphi^2 < \frac{1}{2\Lambda}$.

Moreover,

$$0 < m^2 = \frac{1}{18\Lambda}\left[1 + 12q^2\Lambda + \sqrt{(1-4q^2\Lambda)^3}\right]$$

and

$$\varphi^2\sqrt{\Lambda} \leq |q|.$$

Ultracold Black Hole. *The ultracold black hole is the* 3*-dimensional space*

$$[0,\infty)\times\mathbb{S}^2$$

with metric tensor $g = dr^2 + \varphi^2 g_{\mathbb{S}^2}$, *where* $\varphi^2 = \frac{1}{4\Lambda} = q^2$*. The electric field and the lapse function are given by*

$$E = \sqrt{\Lambda}\partial_r \quad \text{and} \quad f(r) = r.$$

Moreover,

$$m = \frac{1}{3}\sqrt{\frac{2}{\Lambda}}.$$

Our last example is a generalization of the Reissner-Nordström solution with *a non-null cosmological constant, and we will provide the n-dimensional case.*

Reissner-Nordström-de Sitter solution (RNdS). *In some recent works (see [11, 23, 24, 27, 29, 30, 34]) we can find a well-known example for the electrostatic system in which the cosmological constant is non zero. In fact, the Reissner-Nordström-de Sitter space is the Reissner-Nordström system with a positive cosmological constant. Topologically, the Reissner-Nordström-de Sitter space is the product of the $\mathbb{R}$ with the unit radius sphere $\mathbb{S}^{n-1}$. The Reissner-Nordström-de Sitter spacetime with mass m and charge q (cf. [24, Equation 2.2] and [34, Equation 1]) is given by*

$$g = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the standard metric of $\mathbb{S}^{n-1}$. The lapse function $f(r)$ is

$$f(r) = 1 - \frac{2\Lambda r^2}{n(n-1)} - \frac{2m}{r^{n-2}} + \frac{q^2}{r^{2(n-2)}},$$

the electric field is given by

$$E = \frac{q}{r^{n-1}} f(r)\partial r$$

and the cosmological constant and the mass are positive, i.e., $\Lambda > 0$ and $m > 0$ (cf. [11, 23]). Hence from (1-30) we have $R > 0$.

CHAPTER 2

The electrostatic system with a null cosmological constant

In this chapter, we prove some results concerning the geometric structure of an electrostatic system with a null cosmological constant (electrovacuum system) such that the electric potential is a smooth function of the lapse function. We will show that an n-dimensional locally conformally flat electrovacuum space satisfying (2-2) with null cosmological constant must be in the Majumdar-Papapetrou class, and we also prove that any three or four-dimensional electrovacuum space satisfying (2-3) with null cosmological constant must be locally conformally flat.

Static electrovacuum spacetimes model exterior regions of static configurations of electrically charged stars or black holes (see [15, 20, 28] and the references therein). Equations of motion for an $(n+1)$-dimensional reduced Einstein-Maxwell spacetime are given by

$$(\widehat{\mathrm{Ric}})_{ij} = 2\left(F_{il}F_j^l - \frac{1}{2(n-1)}|F|^2\widehat{g}_{ij}\right); \quad 1 \leq i, j \leq n+1,$$

where F represents the electromagnetic field (the Faraday tensor) and $\widehat{\mathrm{Ric}}$ is the Ricci tensor for the metric $\widehat{g}$.

Our main ground is the static spacetime $(\widehat{M}^{n+1}, \hat{g}) = M^n \times_f \mathbb{R}$ such that

$$\widehat{g}(x,t) = g(x) - f^2(x)dt^2; \quad x \in M,$$

where (M^n, g) is an open, connected, and oriented Riemannian manifold, and f is a smooth warped function [15, 17, 36]. Considering as electromagnetic field

$$F = d\psi \wedge dt + B,$$

for some smooth function ψ on M. Here, $\xi = \frac{\partial}{\partial t}$ is the static Killing field and the magnetic field B is a 2-tensor on M such that $i_\xi B = 0$. We will be considering n-dimensional spatial slices, i.e., the Riemannian manifold M is orthogonal to the static Killing field. Therefore, it is more convenient to use the dimensionally reduced Einstein-Maxwell equations, i.e., $B = 0$ (see [33, Definition 6] and the references therein). If ξ is strictly timelike in the spacetime then M is a complete manifold. If ξ is null anywhere the above coordinate system breaks down at the level set $\{f = 0\}$. In this case, we extend M to a manifold with a smooth boundary ∂M containing $\{f = 0\}$, which could correspond to an event horizon or an ergosurface if the hypersurface is null or timeline, respectively. We say that (M, g) is complete away from the horizon ∂M if for any sequence of points $\{p_i\}$ such that $p_i \to p \in \partial M$ on the metric of M one has $f(p_i) \to 0$. Conversely, if $\{p_i\}$ is a bounded sequence of M such that $f(p_i) \to 0$, then the definition of (M, g) implies that a subsequence of $\{p_i\}$ converges to a point $p \in \partial M$ [2].

The most common assumption in the analysis and classification of an electrovacuum space is to consider that such space is asymptotically flat (see [15, 17, 36, 42]). Then, we can use classical results to prove that the solution for the electrovacuum system is either MP or RN (cf. [17, Theorem 3.6]). Even though these asymptotic conditions are restrictive in the topological sense, it is physically reasonable in the study of isolated gravitational systems. Usually, in differential geometry, we often assume some conditions over the curvature in the analysis and classification of a Riemannian manifold. In this work, considering just a condition over the curvature for the classification of the electrovacuum space seems to be not enough, since *a priori* we do not have any additional information about the electric potential and the lapse function.

Usually, in differential geometry, we often have some conditions over the metric or curvature (or both) in the attempt to classify an arbitrary space. Locally geometric conditions over the curvatures (Riemannian, Ricci, or scalar) and Weyl tensor have been used in the study and classification of static vacuum spaces (cf. [1, 32, 37]). For instance, it is well known that if a Riemannian manifold has constant scalar curvature and harmonic Weyl curvature, then its curvature tensor should be harmonic (but not necessarily flat). Clearly, the harmonic Weyl curvature condition is weaker than the locally conformally flat condition (we refer to the reader [12, Remark 1.2]). It is interesting to remember that some classical proofs for the uniqueness of the static Schwarzschild black holes used the conformally flat structure of the static metric to obtain the classification result (cf. [45] and the references therein). Naturally, we can assume weaker integrability conditions on a Riemannian manifold to understand its geometry. Some of our main results were inspired by the idea used by [14] to classify Ricci solitons, where the authors considered that the Weyl tensor is free from divergence as a hypothesis, which is a weaker assumption than harmonic Weyl curvature. These conditions will be discussed ahead.

We remember that an asymptotically flat n-dimensional electrovacuum space is extremal (i.e., $m = |q|$) if, and only if, the magnetic field is zero and

$$f = 1 \pm \sqrt{2(n-2)/(n-1)}\psi,$$

admitting $f = 0$ at ∂M (see Lemma 1 in [36]). Also, in [36, Lemma 3], certain electrovacuum solutions combined with an equation relating ψ and f have implications on the non-existence of magnetic fields. It is worth saying that an extremal RN spacetime contains a unique photon sphere, on which light can be trapped and it has the largest possible ratio of charge to mass (see [15]). The theory of extremal black holes is very important in physics and has very interesting properties. For instance, extremal charged black holes may be quantum mechanically stable, which is consistent with the ideas of cosmic censorship (see [31]). There is also an important type of electrovacuum solution in supergravity theory (see [42]). Moreover, there is evidence that this type of black hole is important to understanding the no-hair theorem (see [8]).

The RN and MP solutions for the electrovacuum system suggest to us that there exists a class of solutions where the electric potential is a smooth function of the lapse function, i.e., $\psi = \psi(f)$. We will prove that an n-dimensional electrovacuum space with null cosmological constant, fourth-order divergence free Weyl tensor, and zero radial Weyl curvature such that the electric potential satisfies the above condition is locally a warped product manifold with an $(n-1)$-dimensional Einstein fibers. Finally, a three-dimensional electrovacuum space with no cosmological constant and third-order divergence-free Cotton tensor is also classified.

The electrovacuum system was constructed in Section 1.2 (Chapter 1). For the sake of completeness and to specify which system will focus on, we will state the definition here once more.

Definition 2.1 *Let (M^n, g) be an n-dimensional smooth Riemannian manifold with $n \geq 3$ and let $f, \psi : M \to \mathbb{R}$ be smooth functions satisfying*

$$\begin{aligned} f\mathrm{Ric} &= \nabla^2 f - \frac{2}{f} d\psi \otimes d\psi + \frac{2}{(n-1)f}|\nabla\psi|^2 g, \\ \Delta f &= 2\left(\frac{n-2}{n-1}\right)\frac{|\nabla\psi|^2}{f} \quad \text{and} \quad 0 = \mathrm{div}\left(\frac{\nabla\psi}{f}\right). \end{aligned}$$

In this chapter, we refer to (M^n, g, f, ψ) as the electrovacuum system (or space).

Moreover, using the electrovacuum equations, we get

$$f^2 R = 2|\nabla\psi|^2. \tag{2-1}$$

2.1 Structural lemmas

In this section, motivated by [6, 12, 37, 44] we will obtain some structural lemmas, which are fundamental to prove the main results of this chapter. To that end, we will first demonstrate Theorem 2.2 which shows us how related the electric potential and the lapse function can be.

In what follows, we will consider that the critical set of the lapse function f, i.e., $crit(f) = \{x \in M,\ \nabla f(x) = 0\}$, is not dense on M. Moreover, $|\nabla f| \neq 0$ at ∂M is known as the non-degeneracy condition.

Theorem 2.2 *Let (M^n, g, f, ψ), $n \geq 3$, be an electrovacuum space such that $\psi = \psi(f)$. Then, the electric potential (locally) is either*

$$\frac{2(n-2)}{n-1}\psi(f)^2 - \frac{4(n-2)}{n-1}\beta\psi(f) + \frac{2(n-2)}{n-1}\beta^2 + \frac{n-1}{n-2}\sigma = f^2 \tag{2-2}$$

or

$$\psi(f) = \beta \pm \sqrt{\frac{(n-1)}{2(n-2)}}f, \tag{2-3}$$

where σ, $\beta \in \mathbb{R}$. Moreover, $\sigma = 0$ if and only if $\psi(f)$ is an affine function of f.

Proof. Since $\psi = \psi(f)$ we obtain

$$\nabla\psi = \dot{\psi}(f)\nabla f. \tag{2-4}$$

Then,

$$\nabla^2\psi = \ddot{\psi}(f)df \otimes df + \dot{\psi}(f)\nabla^2 f,$$

where $\otimes$ is the tensor product. Now, by contracting the above equation, we obtain

$$\Delta\psi = \ddot{\psi}(f)|\nabla f|^2 + \dot{\psi}(f)\Delta f.$$

From the second equation of our system and (2-4), we have

$$\Delta f = \frac{2}{f}\left(\frac{n-2}{n-1}\right)\dot{\psi}(f)^2|\nabla f|^2.$$

Combining the last equations with the divergence of $\frac{\nabla\psi}{f}$ and (2-4), we get

$$\ddot{\psi}(f)|\nabla f|^2 + 2\left(\frac{n-2}{n-1}\right)\frac{\dot{\psi}(f)^3|\nabla f|^2}{f} = \dot{\psi}(f)\frac{|\nabla f|^2}{f}.$$

Notice that $\{\nabla f = 0\}$ is not dense. By a straightforward computation, we arrive at

$$\dot{h} + 2\left(\frac{n-2}{n-1}\right) f h^3 = 0,$$

where

$$h = \frac{\dot{\psi}}{f}.$$

So, by solving this ODE, we get

$$\dot{\psi}(f) = \frac{\pm f}{\sqrt{2\frac{(n-2)}{(n-1)} f^2 - 2\sigma}}; \quad \sigma \in \mathbb{R}. \tag{2-5}$$

By integration, we obtain, either

$$\psi(f) = \beta \pm \frac{(n-1)}{2(n-2)} \sqrt{2\left(\frac{n-2}{n-1}\right) f^2 - 2\sigma}; \quad \sigma \neq 0,\ \beta \in \mathbb{R},$$

or

$$\psi(f) = \beta \pm \sqrt{\frac{(n-1)}{2(n-2)} f}; \quad \sigma = 0\,,\beta \in \mathbb{R}.$$

Moreover, from (2-5) we have the following useful identity

$$2\dot{\psi}(f)^2 = \frac{(n-1)f^2}{(n-2)f^2 - (n-1)\sigma}. \tag{2-6}$$

Finally, we observe that if $\sigma = 0$, then from the above equation $\dot{\psi}(f)$ is a constant, which implies that $\psi(f)$ is an affine function. □

It is interesting to remark how σ and β given by (2-2) are related with the mass m and electric charge q for an RN solution which satisfies (1-34). A straightforward computation shows us that

$$\beta^2 = \frac{(n-1)}{2(n-2)} \frac{m^2}{q^2} \quad \text{and} \quad \sigma = \frac{(n-2)}{(n-1)} \frac{(q^2 - m^2)}{q^2}.$$

So, we can say that a solution satisfying Theorem 2.2 is called subextremal, extremal, or superextremal depending on if $\sigma < 0$, $\sigma = 0$ or $\sigma > 0$, respectively.

It is worth highlighting that the completeness assumption over (M^n, g) is just to ensure that the critical set $\{\nabla f = 0\}$ is not dense on M. So, here, completeness can be replaced by assuming that the critical set is not dense.

The above theorem shows us that an electrovacuum system such that $\psi = \psi(f)$ has two possible solutions, which are closely related to the RN and MP solutions.

Now we will prove the first structural lemma which relates some tensors described in Section 1.1 (Chapter 1), and to prove that we write the electrovacuum equations in a local coordinate, i.e.,

$$fR_{jk} = \nabla_j\nabla_k f - \frac{2}{f}\nabla_j\psi\nabla_k\psi + \frac{1}{n-1}fRg_{jk}; \tag{2-7}$$

$$\Delta f = \frac{n-2}{n-1}fR = 2\left(\frac{n-2}{n-1}\right)\frac{|\nabla\psi|^2}{f}; \tag{2-8}$$

$$0 = \Delta\psi - \frac{1}{f}\langle\nabla f, \nabla\psi\rangle. \tag{2-9}$$

The following lemma relates the Cotton and Weyl tensors with the electrovacuum structure.

Lemma 2.3 *Let (M^n, g, f, ψ), $n \geq 3$, be an electrovacuum system. Then,*

$$\begin{aligned} fC_{ijk} &= W_{ijkl}\nabla^l f + \frac{1}{n-2}(R_{jl}\nabla^l f g_{ik} - R_{il}\nabla^l f g_{jk}) \\ &+ \frac{R}{(n-1)(n-2)}(\nabla_i f g_{jk} - \nabla_j f g_{ik}) \\ &- \frac{2}{f^2}[f(\nabla_j\psi\nabla_i\nabla_k\psi - \nabla_i\psi\nabla_j\nabla_k\psi) - \nabla_i f\nabla_j\psi\nabla_k\psi + \nabla_j f\nabla_i\psi\nabla_k\psi] \\ &+ \frac{n-1}{n-2}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) + \frac{1}{(n-1)f}(\nabla_i|\nabla\psi|^2 g_{jk} - \nabla_j|\nabla\psi|^2 g_{ik}). \end{aligned}$$

Proof. We take the derivative of (2-7) to obtain

$$\begin{aligned} R_{jk}\nabla_i f + f\nabla_i R_{jk} &= -\frac{2}{f^2}[f(\nabla_i\nabla_j\psi\nabla_k\psi + \nabla_j\psi\nabla_i\nabla_k\psi) - \nabla_i f\nabla_j\psi\nabla_k\psi] \\ &+ \nabla_i\nabla_j\nabla_k f + \frac{1}{n-1}\left(\frac{f}{2}\nabla_i R + \frac{1}{f}\nabla_i|\nabla\psi|^2\right)g_{jk} \end{aligned} \tag{2-10}$$

and

$$\begin{aligned} R_{ik}\nabla_j f + f\nabla_j R_{ik} &= -\frac{2}{f^2}[f(\nabla_j\nabla_i\psi\nabla_k\psi + \nabla_i\psi\nabla_j\nabla_k\psi) - \nabla_j f\nabla_i\psi\nabla_k\psi] \\ &+ \nabla_j\nabla_i\nabla_k f + \frac{1}{n-1}\left(\frac{f}{2}\nabla_j R + \frac{1}{f}\nabla_j|\nabla\psi|^2\right)g_{ik}. \end{aligned} \tag{2-11}$$

Subtracting (2-10) from (2-11) and using that the Hessian operator is symmetric, we can deduce that

$$\begin{aligned} R_{jk}\nabla_i f - R_{ik}\nabla_j f &+ f(\nabla_i R_{jk} - \nabla_j R_{ik}) = \nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f + \frac{f}{2(n-1)}(\nabla_i R g_{jk} - \nabla_j R g_{ik}) \\ &- \frac{2}{f^2}[f(\nabla_j\psi\nabla_i\nabla_k\psi - \nabla_i\psi\nabla_j\nabla_k\psi) - \nabla_i f\nabla_j\psi\nabla_k\psi + \nabla_j f\nabla_i\psi\nabla_k\psi] \\ &+ \frac{1}{(n-1)f}(\nabla_i|\nabla\psi|^2 g_{jk} - \nabla_j|\nabla\psi|^2 g_{ik}). \end{aligned}$$

Then, using the Ricci identity (1-1) and the Cotton tensor (1-5), we can infer that

$$\begin{aligned} fC_{ijk} &= R_{ijkl}\nabla^l f + \frac{1}{(n-1)f}(\nabla_i|\nabla\psi|^2 g_{jk} - \nabla_j|\nabla\psi|^2 g_{ik}) - R_{jk}\nabla_i f + R_{ik}\nabla_j f \\ &- \frac{2}{f^2}[f(\nabla_j\psi\nabla_i\nabla_k\psi - \nabla_i\psi\nabla_j\nabla_k\psi) - \nabla_i f\nabla_j\psi\nabla_k\psi + \nabla_j f\nabla_i\psi\nabla_k\psi]. \end{aligned}$$

Now, using the Weyl formula (1-4), we have

$$\begin{aligned} fC_{ijk} &= W_{ijkl}\nabla^l f + \frac{1}{n-2}(R_{jl}\nabla^j f g_{ik} - R_{il}\nabla^l f g_{jk}) - \frac{R}{(n-1)(n-2)}(g_{ik}\nabla^j f - g_{jk}\nabla^i f) \\ &- \frac{2}{f^2}[f(\nabla_j\psi\nabla_i\nabla_k\psi - \nabla_i\psi\nabla_j\nabla_k\psi) - \nabla_i f\nabla_j\psi\nabla_k\psi + \nabla_j f\nabla_i\psi\nabla_k\psi] \\ &+ \frac{n-1}{n-2}(R_{ik}\nabla^j f - R_{jk}\nabla^i f) + \frac{1}{(n-1)f}(\nabla_i|\nabla\psi|^2 g_{jk} - \nabla_j|\nabla\psi|^2 g_{ik}). \end{aligned}$$

So, the proof is finished. □

In the sequel, we define the covariant 3-tensor V_{ijk} by

$$\begin{aligned} V_{ijk} &= \frac{1}{n-2}(R_{jl}\nabla^l f g_{ik} - R_{il}\nabla^l f g_{jk}) + \frac{R}{(n-1)(n-2)}(\nabla_i f g_{jk} - \nabla_j f g_{ik}) \\ &- \frac{2}{f^2}[f(\nabla_j\psi\nabla_i\nabla_k\psi - \nabla_i\psi\nabla_j\nabla_k\psi) - \nabla_i f\nabla_j\psi\nabla_k\psi + \nabla_j f\nabla_i\psi\nabla_k\psi] \qquad (2\text{-}12) \\ &+ \frac{n-1}{n-2}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) + \frac{1}{(n-1)f}(\nabla_i|\nabla\psi|^2 g_{jk} - \nabla_j|\nabla\psi|^2 g_{ik}). \end{aligned}$$

The tensor V_{ijk} was defined similarly to D_{ijk} in [9, Equation 1.2].

Note that from a straightforward computation, we observe that the tensor V has the same symmetries of the Cotton tensor, C, i.e.,

$$V_{ijk} = -V_{jik} \quad \text{and} \quad V_{ijk} + V_{jki} + V_{kij} = 0.$$

Moreover, this tensor is totally trace-free, i.e.,

$$g^{ij}V_{ijk} = 0, \quad g^{ik}V_{ijk} = 0 \quad \text{and} \quad g^{jk}V_{ijk} = 0.$$

The first equality holds trivially, and the last two are similar. In fact, note that

$$\begin{aligned}
g^{ik}V_{ijk} &= \frac{1}{n-2}(nR_{jl}\nabla^l f - R_{jl}\nabla^l f) + \frac{R}{(n-1)(n-2)}(\nabla_j f - n\nabla_j f)\\
&- \frac{2}{f^2}[f(\nabla_j\psi\Delta\psi - \nabla_i\psi\nabla_j\nabla^i\psi) - \nabla_i f\nabla_j\psi\nabla^i\psi + \nabla_j f|\nabla\psi|^2]\\
&+ \frac{n-1}{n-2}(R\nabla_j f - R_{jl}\nabla^l f) + \frac{1}{(n-1)f}(\nabla_j|\nabla\psi|^2 - n\nabla_j|\nabla\psi|^2)\\
&= -\frac{2}{f^2}[f(\nabla_j\psi\Delta\psi - \nabla_i\psi\nabla_j\nabla^i\psi) - \nabla_i f\nabla_j\psi\nabla^i\psi + \nabla_j f|\nabla\psi|^2]\\
&+ R\nabla_j f - \frac{1}{f}\nabla_j|\nabla\psi|^2.
\end{aligned}$$

Now, note that since $|\nabla\psi|^2 = g^{ik}\nabla_i\psi\nabla_k\psi$, then

$$\begin{aligned}
\nabla_j|\nabla\psi|^2 &= g^{ik}\nabla_j\nabla_i\psi\nabla_k\psi + g^{ik}\nabla_i\psi\nabla_j\nabla_k\psi\\
&= 2\nabla_i\psi\nabla_j\nabla_i\psi.
\end{aligned}$$

Therefore, from this equation and (2-9), we obtain

$$\begin{aligned}
g^{ik}V_{ijk} &= -\frac{2}{f^2}[\nabla_j\psi\langle\nabla f, \nabla\psi\rangle - \nabla_j\psi\langle\nabla f, \nabla\psi\rangle + \nabla_j f|\nabla\psi|^2] + R\nabla_j f\\
&= 0,
\end{aligned}$$

where we used (2-1).

The V-tensor has a fundamental importance in what follows. From Lemma 2.3, we have

$$fC_{ijk} = W_{ijkl}\nabla^l f + V_{ijk}. \tag{2-13}$$

In particular, if we suppose that $\psi = \psi(f)$ in the Lemma 2.3, we obtain the following result.

Lemma 2.4 *Let (M^n, g, f, ψ), $n \geq 3$, be an electrovacuum system such that $\psi = \psi(f)$. Then,*

$$V_{ijk} = P(R_{il}\nabla^l f g_{jk} - R_{jl}\nabla^l f g_{ik}) + Q(R_{ik}\nabla_j f - R_{jk}\nabla_i f) + U(\nabla_i f g_{jk} - \nabla_j f g_{ik}), \tag{2-14}$$

where

$$P = \frac{2\dot{\psi}(f)^2}{n-1} - \frac{1}{n-2}, \qquad Q = \frac{n-1}{n-2} - 2\dot{\psi}(f)^2$$

and

$$U = \frac{R}{n-1}\left[\frac{1}{(n-2)} - \frac{2\dot{\psi}(f)^2}{(n-1)} + \frac{f\ddot{\psi}(f)}{\dot{\psi}(f)}\right].$$

Proof. In fact, since $\psi = \psi(f)$, the equation (2-4) is satisfied. Now, using (2-7) we obtain

$$\begin{aligned}\nabla_k\nabla_i\psi &= \ddot{\psi}(f)\nabla_k f\nabla_i f+\dot{\psi}(f)\nabla_k\nabla_i f\\ &= \ddot{\psi}(f)\nabla_k f\nabla_i f+f\dot{\psi}(f)R_{ki}+\frac{2}{f}\dot{\psi}(f)^3\nabla_k f\nabla_i f-\frac{1}{n-1}f\dot{\psi}(f)Rg_{ki}.\end{aligned}$$

Replacing the above equation in (2-12) we can rewrite the V-tensor in the following form:

$$\begin{aligned}V_{ijk} &= \frac{1}{n-2}(R_{jl}\nabla^l fg_{ik}-R_{il}\nabla^l fg_{jk})+\left[\frac{R}{(n-1)(n-2)}-\frac{2}{n-1}\dot{\psi}(f)^2R\right](\nabla_i fg_{jk}-\nabla_j fg_{ik})\\ &+ \left[\frac{n-1}{n-2}-2\dot{\psi}(f)^2\right](R_{ik}\nabla_j f-R_{jk}\nabla_i f)+\frac{1}{(n-1)f}(\nabla_i|\nabla\psi|^2g_{jk}-\nabla_j|\nabla\psi|^2g_{ik}).\end{aligned}\tag{2-15}$$

Now, by taking the derivative of (2-1) and using (2-4) we deduce that

$$4\ddot{\psi}(f)\dot{\psi}(f)\nabla_i f|\nabla f|^2+2\dot{\psi}(f)^2\nabla_i|\nabla f|^2=2fR\nabla_i f+f^2\nabla_i R.$$

Combining (2-7) and (2-4), we obtain

$$4\ddot{\psi}(f)\dot{\psi}(f)\nabla_i f|\nabla f|^2+4\dot{\psi}(f)^2\left(fR_{il}\nabla_l f+\frac{2}{f}\dot{\psi}(f)^2\nabla_i f|\nabla f|^2-\frac{1}{n-1}fR\nabla_i f\right)=2fR\nabla_i f+f^2\nabla_i R,$$

which implies that

$$\begin{aligned}f^2\nabla_i R &= 4\ddot{\psi}(f)\dot{\psi}(f)\nabla_i f|\nabla f|^2+4\dot{\psi}(f)^2\left(fR_{il}\nabla_l f+\frac{2}{f}\dot{\psi}(f)^2\nabla_i f|\nabla f|^2-\frac{1}{n-1}fR\nabla_i f\right)\\ &- 2fR\nabla_i f\\ &= 2fR\left(\frac{f\ddot{\psi}(f)}{\dot{\psi}(f)}+\frac{2(n-2)}{n-1}\dot{\psi}(f)^2-1\right)\nabla_i f+4f\dot{\psi}(f)^2R_{il}\nabla_l f.\end{aligned}\tag{2-16}$$

Then, using (2-1) and (2-16), we get

$$\begin{aligned}\nabla_i|\nabla\psi|^2 &= fR\nabla_i f+\frac{f^2}{2}\nabla_i R\\ &= fR\left(\frac{f\ddot{\psi}(f)}{\dot{\psi}(f)}+\frac{2(n-2)}{n-1}\dot{\psi}(f)^2\right)\nabla_i f+2f\dot{\psi}(f)^2R_{il}\nabla_l f.\end{aligned}$$

Thus, replacing the above equation in (2-15) the result follows. □

On the other hand, by a conformal change of the metric, we get our next lemma.

Lemma 2.5 *Let (M^n, g, f, ψ), $n \geq 3$, be an electrovacuum system such that $\psi = \psi(f)$ is given by (2-3). Then, the Cotton tensor satisfies*

$$(n-2)^2 f C_{ijk} = W_{ijkl}\nabla^l f. \tag{2-17}$$

In particular, when $n = 3$, then (M^3, g) is locally conformally flat, i.e, $C = 0$.

Proof. We consider the conformal change of the metric

$$\widetilde{g} = f^{\frac{2}{n-2}} g.$$

From [12, Appendix] the Cotton tensor for $\widetilde{g}$ is given by

$$(n-2)\widetilde{C}_{ijk} = (n-2)C_{ijk} - \frac{1}{(n-2)f} W_{ijkl}\nabla^l f. \tag{2-18}$$

Moreover, for $\widetilde{g}$ (see [5, page 58]) we obtain

$$\begin{aligned} \widetilde{\mathrm{Ric}} &= \mathrm{Ric} - \frac{1}{f}\nabla^2 f + \frac{(n-1)}{(n-2)f^2} df \otimes df - \frac{\Delta f}{(n-2)f} g \\ &= \mathrm{Ric} - \frac{1}{f}\nabla^2 f + \frac{(n-1)}{(n-2)f^2} df \otimes df - \frac{R}{(n-1)} g, \end{aligned} \tag{2-19}$$

where in the last equation we used (2-8).

Considering $\psi = \psi(f)$, from (1-29), we get

$$\begin{aligned} \widetilde{\mathrm{Ric}} &= \frac{1}{f^2}\frac{(n-1)}{(n-2)} df \otimes df - \frac{2}{f^2} d\psi \otimes d\psi + \frac{1}{(n-2)f}\left[2\frac{(n-2)}{(n-1)}\frac{|\nabla\psi|^2}{f} - \Delta f\right] f^{\frac{-2}{n-2}}\widetilde{g} \\ &= \frac{1}{f^2}\frac{(n-1)}{(n-2)} df \otimes df - \frac{2}{f^2} d\psi \otimes d\psi = \frac{1}{f^2}\left[\frac{(n-1)}{(n-2)} - 2\dot{\psi}^2\right] df \otimes df. \end{aligned} \tag{2-20}$$

Moreover,

$$\widetilde{R} = \frac{1}{f^2}\left[\frac{(n-1)}{(n-2)} - 2\dot{\psi}^2\right] |\widetilde{\nabla} f|^2.$$

By hypothesis, $\psi = \psi(f)$ satisfies (2-3). So,

$$2\dot{\psi}^2 = \frac{(n-1)}{(n-2)}. \tag{2-21}$$

Consequently, from (2-20) and (2-21), we conclude that $(M^n, \widetilde{g})$ is Ricci-flat. In this case, the Schouten tensor (1-13) for $\widetilde{g}$ is given by

$$\begin{aligned}\widetilde{S} &= \frac{1}{n-2}\left(\widetilde{\mathrm{Ric}}-\frac{1}{2(n-1)}\widetilde{R}\tilde{g}\right)\\ &= \frac{\left[\frac{(n-1)}{(n-2)}-2\dot{\psi}^2\right]}{(n-2)f^2}\left(df\otimes df-\frac{|\widetilde{\nabla} f|^2}{2(n-1)}\tilde{g}\right)=0.\end{aligned}$$

This shows that $\widetilde{S}$ is Codazzi, i.e., $(\widetilde{\nabla}_X\widetilde{S})(Y)=(\widetilde{\nabla}_Y\widetilde{S})(X)$ for all $X,\ Y\in TM$. Therefore, the Cotton tensor for the metric $\widetilde{g}$ is identically zero. So, from (2-18) we have

$$(n-2)^2 fC_{ijk}=W_{ijkl}\nabla^l f.$$

Thus, we conclude our proof. □

Now our goal is to obtain a useful formula for the norm of the Cotton tensor involving the divergence of the tensor V.

Lemma 2.6 *Let* (M^n,g,f,ψ), $n\geq 4$, *be an electrovacuum system. Then,*

$$(n-2)B_{ij}=-\nabla^k\left(\frac{V_{ikj}}{f}\right)+\frac{n-3}{n-2}\frac{C_{jki}\nabla^k f}{f}+\frac{1}{f^2}W_{ikjl}(\nabla^k f\nabla^l f-2\nabla^k\psi\nabla^l\psi). \qquad (2\text{-}22)$$

Proof. In fact, from (1-10) and (2-13), we can deduce that

$$\begin{aligned}(n-2)B_{ij} &= -\nabla^k C_{ikj}+R^{kl}W_{ikjl}\\ &= -\nabla^k\left(\frac{V_{ikj}}{f}+\frac{W_{ikjl}\nabla^l f}{f}\right)+R^{kl}W_{ikjl}\\ &= -\nabla^k\left(\frac{V_{ikj}}{f}\right)-\frac{\nabla^k W_{ikjl}\nabla^l f}{f}+\frac{W_{ikjl}\nabla^k f\nabla^l f}{f^2}-\frac{W_{ikjl}\nabla^k\nabla^l f}{f}+R^{kl}W_{ikjl}.\end{aligned}$$

Now, using (2-7), we obtain

$$\begin{aligned}(n-2)B_{ij} &= -\nabla^k\left(\frac{V_{ikj}}{f}\right)-\frac{\nabla^k W_{ikjl}\nabla^l f}{f}+\frac{W_{ikjl}\nabla^k f\nabla^l f}{f^2}\\ &- \frac{W_{ikjl}}{f}\left(fR^{kl}+\frac{2}{f}\nabla^k\psi\nabla^l\psi-\frac{1}{n-1}fRg^{kl}\right)+R^{kl}W_{ikjl}.\end{aligned}$$

Since the Weyl tensor is trace-free we have

$$(n-2)B_{ij}=-\nabla^k\left(\frac{V_{ikj}}{f}\right)-\frac{\nabla^k W_{ikjl}\nabla^l f}{f}+\frac{1}{f^2}W_{ikjl}(\nabla^k f\nabla^l f-2\nabla^k\psi\nabla^l\psi).$$

From (1-8), we get the result. □

Proceeding, we can use the previous lemma to obtain the following result.

Lemma 2.7 *Let* (M^n, g, f, ψ), $n \geq 4$, *be an electrovacuum system. Then,*

$$\begin{aligned} C_{jki}R^{ik} &= (n-2)\nabla^i\nabla^k\left(\frac{V_{ikj}}{f}\right) - (n-2)\frac{1}{f}W_{ikjl}R^{il}\nabla^k f \\ &+ 2(n-2)\frac{W_{ikjl}}{f^2}\nabla^k\psi\nabla^i\nabla^l\psi - 2(n-2)\frac{W_{ikjl}}{f^3}\nabla^i f\nabla^k\psi\nabla^l\psi. \end{aligned} \tag{2-23}$$

Proof. Taking the divergence of (2-22) and using (1-7), we can infer that

$$\begin{aligned} (n-2)\nabla^i B_{ij} &= -\nabla^i\nabla^k\left(\frac{V_{ikj}}{f}\right) + \frac{n-3}{n-2}\frac{C_{jki}}{f^2}(f\nabla^i\nabla^k f - \nabla^k f\nabla^i f) \\ &+ \frac{1}{f^2}W_{ikjl}\left(\nabla^i\nabla^k f\nabla^l f + \nabla^k f\nabla^i\nabla^l f - 2\nabla^i\nabla^k\psi\nabla^l\psi - 2\nabla^k\psi\nabla^i\nabla^l\psi\right) \\ &+ \frac{1}{f^2}\nabla^i W_{ikjl}\left(\nabla^k f\nabla^l f - 2\nabla^k\psi\nabla^l\psi\right) - \frac{2}{f^3}W_{ikjl}\left(\nabla^i f\nabla^k f\nabla^l f - 2\nabla^i f\nabla^k\psi\nabla^l\psi\right). \end{aligned} \tag{2-24}$$

Since the Hessian is symmetric, renaming indices and using the symmetries of the Weyl tensor we get

$$2\nabla^i\nabla^k\psi W_{ikjl} = \nabla^i\nabla^k\psi W_{ikjl} + \nabla^k\nabla^i\psi W_{kijl} = \nabla^i\nabla^k\psi(W_{ikjl} + W_{kijl}) = 0. \tag{2-25}$$

Analogously, we have the same expression for the lapse function f, i.e.,

$$\nabla^i\nabla^k f W_{ikjl} = 0.$$

Combining (2-24) and (2-25), we obtain

$$\begin{aligned} (n-2)\nabla^i B_{ij} &= -\nabla^i\nabla^k\left(\frac{V_{ikj}}{f}\right) + \frac{n-3}{n-2}\frac{C_{jki}}{f^2}(f\nabla^i\nabla^k f - \nabla^k f\nabla^i f) \\ &+ \frac{4}{f^3}W_{ikjl}\nabla^i f\nabla^k\psi\nabla^l\psi - \frac{1}{f^2}\nabla^i W_{jlki}\left(\nabla^k f\nabla^l f - 2\nabla^k\psi\nabla^l\psi\right) \\ &+ \frac{1}{f^2}W_{ikjl}\left(\nabla^k f\nabla^i\nabla^l f - 2\nabla^k\psi\nabla^i\nabla^l\psi\right). \end{aligned}$$

Since the Cotton and Weyl tensors are trace-free, using the symmetries of the Weyl tensor, (1-8) and (2-7) we get

$$\begin{aligned} (n-2)\nabla^i B_{ij} &= -\nabla^i\nabla^k\left(\frac{V_{ikj}}{f}\right) + \frac{n-3}{n-2}C_{jki}R^{ik} + \frac{1}{f}W_{ikjl}R^{il}\nabla^k f \\ &- \frac{2}{f^2}W_{ikjl}\nabla^k\psi\nabla^i\nabla^l\psi + \frac{2}{f^3}W_{ikjl}\nabla^i f\nabla^k\psi\nabla^l\psi. \end{aligned} \tag{2-26}$$

Now, we need to remember some facts. Firstly, $B_{ij}=B_{ji}$, $R^{ij}=R^{ji}$ and the Cotton tensor is skew-symmetric, then an analogous computation as the one made in (2-25) gives us

$$C_{ikj}R^{ik}=0. \tag{2-27}$$

Secondly, using (1-6), we infer can that $C_{jik}=C_{jki}+C_{kij}$, this implies that $C_{jik}R^{ik}=C_{jki}R^{ik}$. Thus, from (1-12) and using these observations, after renaming the indices, we obtain

$$\nabla^i B_{ij}=\frac{n-4}{(n-2)^2}C_{jik}R^{ik}=\frac{n-4}{(n-2)^2}C_{jki}R^{ik}.$$

Finally, using the above equation in (2-26) the result holds. □

Now, we will prove the last structural lemma of this section.

Lemma 2.8 *Let (M^n,g,f,ψ), $n\geq 4$, be an electrovacuum system. Then,*

$$\begin{aligned}\frac{1}{2}|C|^2+R^{ik}\nabla^j C_{jki} &= (n-2)\nabla^j\nabla^i\nabla^k\left(\frac{V_{ikj}}{f}\right)-(n-2)\nabla^j\left[\frac{1}{f}W_{ikjl}R^{il}\nabla^k f\right]\\ &- 2(n-2)\nabla^j\left[\frac{W_{ikjl}}{f^3}\nabla^i f\nabla^k\psi\nabla^l\psi\right]+2(n-2)\nabla^j\left[\frac{W_{ikjl}}{f^2}\nabla^k\psi\nabla^i\nabla^l\psi\right].\end{aligned}$$

Proof. Taking the divergence of (2-23), we have

$$\begin{aligned}C_{jki}\nabla^j R^{ik}+R^{ik}\nabla^j C_{jki} &= (n-2)\nabla^j\nabla^i\nabla^k\left(\frac{V_{ikj}}{f}\right)-(n-2)\nabla^j\left[\frac{1}{f}W_{ikjl}R^{il}\nabla^k f\right]\\ &\quad -2(n-2)\nabla^j\left[\frac{W_{ikjl}}{f^3}\nabla^i f\nabla^k\psi\nabla^l\psi\right]\\ &\quad +2(n-2)\nabla^j\left[\frac{W_{ikjl}}{f^2}\nabla^k\psi\nabla^i\nabla^l\psi\right].\end{aligned} \tag{2-28}$$

Hence, from the symmetries of the Cotton tensor and renaming indices,

$$2C_{jki}\nabla^j R^{ik}=C_{jki}\nabla^j R^{ik}+C_{kji}\nabla^k R^{ij}=C_{jki}(\nabla^j R^{ik}-\nabla^k R^{ij}). \tag{2-29}$$

Then, combining (2-28) and (2-29), we can infer that

$$\begin{aligned}\frac{1}{2}C_{jki}(\nabla^j R^{ik}-\nabla^k R^{ij})+R^{ik}\nabla^j C_{jki} &= (n-2)\nabla^j\nabla^i\nabla^k\left(\frac{V_{ikj}}{f}\right)\\ &\quad -(n-2)\nabla^j\left[\frac{1}{f}W_{ikjl}R^{il}\nabla^k f\right]\\ &\quad -2(n-2)\nabla^j\left[\frac{W_{ikjl}}{f^3}\nabla^i f\nabla^k\psi\nabla^l\psi\right]\\ &\quad +2(n-2)\nabla^j\left[\frac{W_{ikjl}}{f^2}\nabla^k\psi\nabla^i\nabla^l\psi\right].\end{aligned}$$

From (1-5) and using that the Cotton tensor is trace-free, we obtain the result. □

2.2 Classification Results

Now we are ready to present and prove the main results of this chapter. In what follows, we demonstrate that an electrovacuum space, under specific hypotheses, necessarily must be in the Majumdar-Papapetrou class, i.e., $(M^n, \widetilde{g})$ is Ricci-flat (see Lemma 2.5) with respect to the metric $\widetilde{g} = f^{2/(n-2)}g$, the inverse of the electric potential $\frac{1}{\psi(f)}$ given by (2-3) is harmonic with respect to $\widetilde{g}$, [36, Remark 1]. Then, if we consider asymptotic conditions, by the positive mass theorem, $(M^n, \widetilde{g})$ is isometric to the Euclidean space minus a compact set. These facts are important for the classification of electrovacuum solutions. For $n = 3$, the space $(M^n, \widetilde{g})$ is trivially flat, and this is a direct consequence of $(M^n, \widetilde{g})$ being Ricci-flat; however, in higher dimensions, this need not be the case.

As pointed out in [36, Remark 1] and [42], any suitably regular asymptotically flat black hole solution in the Majumdar-Papapetrou class must have a space isometric to Euclidean space (minus a point for each horizon) and a harmonic function of the form (1-35). In this case, the spacetime is in the class of Majumdar–Papapetrou multi-centered black hole solution (see [42]). We need to emphasize that we are not considering any asymptotic condition, so the positive mass theorem is not necessarily valid here.

Theorem 2.9 *Let (M^n, g, f, ψ), $n \geq 3$, be an electrovacuum space satisfying (2-3). Then, the Schouten tensor for the metric g is Codazzi. If (M^n, g) is locally conformally flat, then the space must be in the Majumdar–Papapetrou class, i.e., the static spacetime $(\widehat{M}^{n+1}, \hat{g}) = M^n \times_f \mathbb{R}$ must have (locally) metric tensor given by*

$$\widehat{g}(x,t) = f^{-2/(n-2)}(dx_1^2 + \ldots + dx_n^2) - f^2 dt^2.$$

Moreover, any four or five-dimensional electrovacuum spacetime such that the spatial factor (M, g, f, ψ) satisfies (2-3) must have (locally) the above geometric structure.

Proof. The proof follows from the previous section. In fact, remember that when ψ is an affine function of f, we have equation (2-21). Then, from (2-14) we conclude that $P = Q = U = 0$, so the V-tensor is identically zero. Thus, from (2-13) we obtain $fC_{ijk} = W_{ijkl}\nabla^l f$. Immediately, for $n = 3$ the Cotton tensor is identically zero which means that (M^3, g, f, ψ) is locally conformally flat.

Considering $n>3$, from the proof of Lemma 2.5 we obtain that the Ricci tensor, $\widetilde{\mathrm{Ric}}$, for the conformal change of the metric $\widetilde{g}=f^{2/(n-2)}$ is identically zero, and so the Cotton tensor $\widetilde{C}_{ijk}$. At the same time, using (2-17), we can infer that

$$(n-2)^2 f C_{ijk} = W_{ijkl}\nabla^l f,$$

which combined with (2-13) gives us

$$[(n-2)^2-1]fC_{ijk}=0.$$

Consequently, the Schouten tensor (1-13) is Codazzi, i.e., $(\nabla_X S)(Y)=(\nabla_Y S)(X)$ for all X, $Y\in\mathscr{X}(M)$. Furthermore, since $\widetilde{\mathrm{Ric}}$ is identically zero, we conclude $(M^3,\widetilde{g})$ is isometric to $\mathbb{R}^3$.

Using again the conformal change of the metric $\widetilde{g}=f^{2/(n-2)}g$ (see [5, page 58]), we have

$$\widetilde{R}_{ijkl}=f^{2/(n-2)}\Big[R_{ijkl}-(g_{ik}T_{jl}+g_{jl}T_{ik}-g_{il}T_{jk}-g_{jk}T_{il})\Big], \tag{2-30}$$

where

$$\begin{aligned} T_{ij} &= \frac{1}{n-2}\left(\frac{1}{f}\nabla_i\nabla_j f-\frac{n-1}{(n-2)f^2}\nabla_i f\nabla_j f+\frac{1}{2(n-2)f^2}|\nabla f|^2 g_{ij}\right)\\ &= \frac{1}{(n-2)}\left(\frac{1}{f}\nabla_i\nabla_j f-\frac{(n-1)}{(n-2)f^2}\nabla_i f\nabla_j f+\frac{R}{2(n-1)}g_{ij}\right). \end{aligned}$$

In the last equality, we have used (2-8) and (2-21). Then, from (2-19), we get

$$R_{ij}=\frac{1}{f}\nabla_i\nabla_j f-\frac{(n-1)}{(n-2)f^2}\nabla_i f\nabla_j f+\frac{R}{(n-1)}g_{ij}.$$

Combining these two last identities, we obtain

$$T_{ij}=\frac{1}{n-2}\left(R_{ij}-\frac{R}{2(n-1)}g_{ij}\right).$$

Note that the tensor T coincides with the Schouten tensor S given by (1-13). If the Weyl tensor for g is identically zero, then from (1-4) we have

$$g_{ik}T_{jl}+g_{jl}T_{ik}-g_{il}T_{jk}-g_{jk}T_{il}=R_{ijkl},$$

see [9, Remark 2.1]. Therefore, replacing the above formula in (2-30), we can conclude that

$$\widetilde{R}_{ijkl} = 0.$$

Thus, we can say that (locally) $\widetilde{g} = \delta$, where δ is standard Euclidean metric. Hence, we can infer that $g = f^{-2/(n-2)}\delta$.

We finish the proof considering the four-dimensional case (see [9, Lemma 4.3]). First, remember that in any open set of the level set $\Sigma = \{f = c\}$, where c is any regular value for f, and using the local coordinates system

$$(x_1, x_2, x_3, x_4) = (f, \theta_2, \theta_3, \theta_4),$$

we can always express the metric g in the form

$$g_{ij} = \frac{1}{|\nabla f|^2} df^2 + g_{ab}(f, \theta) d\theta_a d\theta_b,$$

where $g_{ab}(f,\theta)d\theta_a d\theta_b$ is the induced metric and $(\theta_2, \theta_3, \theta_4)$ is any local coordinate system on Σ (see [9, Comment 3.4]). We use a, b, c to represent indices on the level sets that range from 2 to 4. While i, j, k are used to represent indices ranging from 1 to 4. Next, consider that $\nu = \frac{-\nabla f}{|\nabla f|}$ is the normal vector field to Σ. Consider the referential $\{e_1, e_2, e_3, e_4\}$, where e_1 is normal and e_a are tangent to Σ. Since the Schouten tensor is Codazzi and the V-tensor is identically zero, from (2-13) we have $W_{ijk1} = 0$. Hence, we only need to show that

$$W_{abcd} = 0; \quad \forall a, b, c, d \in \{2, 3, 4\}.$$

The Weyl tensor has all the symmetries of the curvature tensor and is trace-free in any two indices. Thus,

$$W_{2121} + W_{2222} + W_{2323} + W_{2424} = 0,$$

this implies that

$$W_{2323} = -W_{2424}.$$

Thus, from

$$W_{2424} = -W_{3434} = W_{2323},$$

we conclude that $W_{2323} = 0$. Moreover,

$$W_{1314} + W_{2324} + W_{3334} + W_{4344} = 0,$$

which implies that $W_{2324} = 0$. This shows that $W_{abcd} = 0$, unless a, b, c, d are all distinct. But there are only three choices for the indices since they range from 2 to 4. Then, the Weyl tensor W_{ijkl} is identically zero. Therefore, (M^4, g) is locally conformally flat. $\square$

As an interesting consequence of the above theorem, we get the following corollary.

Corollary 2.10 *Any five-dimensional electrovacuum spacetime satisfying* (2-3) *must be in the Majumdar–Papapetrou class.*

Remark 3 *Lucietti [42, Theorem 1] proved that an asymptotically flat higher dimensional* $(n > 3)$ *extremal electrovacuum space is in the MP class, by requiring a mild extension of the positive mass theorem to manifolds with conical singularities. Furthermore, the author was able to prove that* f *must be given by* (1-35). *Remembering that we are not assuming any asymptotic condition.*

Next, we prove Theorem 2.11 concerning the classification of an electrovacuum space without any asymptotic condition as a hypothesis. To that end, it is convenient to remember that we say that a Riemannian manifold has *a harmonic Weyl curvature* when

$$\operatorname{div} W = 0$$

and a Riemannian manifold also has radial Weyl curvature if

$$W(\cdot,\cdot,\cdot,\nabla f) = 0.$$

A straightforward computation from (1-8) shows us that the harmonic Weyl tensor condition is equivalent to the Schouten tensor being Codazzi when $n > 3$. For the sake of simplicity of the next results, we will now adopt this new definition of harmonic Weyl tensor as the terminology whenever necessary.

Now, we are ready to announce our next classification result.

Theorem 2.11 *Let* (M^n, g, f, ψ), $n \geq 3$, *be an electrovacuum space with harmonic Weyl curvature and zero radial Weyl curvature such that* ψ *is in the Reissner-Nordström class, i.e., such that* ψ *is given by* (2-2) *and* $\sigma < 0$. *Then, around any regular point of* f, *the manifold is locally a warped product with* $(n-1)$*-dimensional Einstein fibers.*

Proof. We consider an orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ diagonalizing the Ricci tensor Ric at a regular point $p \in \Sigma = f^{-1}(c)$, with associated eigenvalues R_{kk}, $k = 1, \ldots, n$, respectively. That is, $R_{ij}(p) = R_{ii}\delta_{ij}(p)$. From Lemma 2.4, we infer

$$\nabla_j f [P R_{jj} + Q R_{ii} - U] = 0, \quad \forall i \neq j, \tag{2-31}$$

where P, Q and U are given by (2-14). Without lost of generality, consider $\nabla_i f \neq 0$ and $\nabla_j f = 0$ for all $i \neq j$. Observe that $\mathrm{Ric}(\nabla f) = R_{ii}\nabla f$, i.e., ∇f is an eigenvector for Ric. From (2-31), we obtain that $\lambda = R_{ii}$ and $\mu = R_{jj}$, $j \neq i$, have multiplicity 1 and $n-1$, respectively. Moreover, if $\nabla_i f \neq 0$ for at least two distinct directions, then we have that $\mu = R_{11} = \ldots = R_{nn}$ and we also obtain that ∇f is an eigenvector for Ric. It is important to point out that for the above discussion, the solutions satisfy $P \neq 0$, $Q \neq 0$, and $U \neq 0$.

Therefore, in any case, we have that ∇f is an eigenvector for Ric. From the above discussion we can take $\{e_1 = \frac{-\nabla f}{|\nabla f|}, e_2, \ldots, e_n\}$ as an orthonormal frame for Σ diagonalizing the Ricci tensor Ric for the metric g.

Now, from (2-7) we obtain

$$fR_{a\ell}\nabla^{\ell} f = \frac{1}{2}\nabla_a|\nabla f|^2 - \frac{2\dot{\psi}^2}{f}|\nabla f|^2\nabla_a f + \frac{Rf}{(n-1)}\nabla_a f; \quad a \in \{2, \ldots, n\}. \tag{2-32}$$

Hence, equation (2-32) gives us $|\nabla f|$ is a constant in Σ. Thus, we can express the metric g in the form

$$g_{ij} = \frac{1}{|\nabla f|^2}df^2 + g_{ab}(f, \theta)d\theta_a d\theta_b,$$

where $g_{ab}(f, \theta)d\theta_a d\theta_b$ is the induced metric and $(\theta_2, \ldots, \theta_n)$ is any local coordinate system on Σ. We can find a good overview of the level set structure in [9, 37].

Observe that there is no open subset Ω of M^n where $\{\nabla f = 0\}$ is dense. In fact, if f is constant in Ω since M^n is complete, we have that f is analytic, which implies f is constant everywhere. Thus, we consider Σ a connected component of the level set $f^{-1}(c)$ (possibly disconnected) where c is any regular value of the function f. Suppose that I is an open interval containing c such that f has no critical points in the open neighborhood $U_I = f^{-1}(I)$ of Σ. For sake of simplicity, let $U_I \subset M\backslash\{f = 0\}$ be a connected component of $f^{-1}(I)$. Then, we can make a change to the variables

$$r(x) = \int \frac{df}{|\nabla f|}$$

such that the metric g in U_I can be expressed by

$$g_{ij} = dr^2 + g_{ab}(r, \theta)d\theta_a d\theta_b.$$

Let $\nabla r = \frac{\partial}{\partial r}$, then $|\nabla r| = 1$ and $\nabla f = f'(r)\frac{\partial}{\partial r}$ on U_I. Note that $f'(r)$ does not change sign on U_I. Moreover, we have $\nabla_{\partial r}\partial r = 0$.

From (2-7), i.e.,

$$fR_{jk} = \nabla_j\nabla_k f - \frac{2}{f}\nabla_j\psi\nabla_k\psi + \frac{1}{n-1}fRg_{jk}$$

and the fact that ∇f is an eigenvector of Ric, the second fundamental form on Σ is given by

$$\begin{aligned} h_{ab} &= -\langle e_1, \nabla_a e_b\rangle = \frac{\nabla_a\nabla_b f}{|\nabla f|} \\ &= \frac{1}{|\nabla f|}\left(fR_{ab} - \frac{Rf}{n-1}g_{ab}\right) = \frac{f}{|\nabla f|}\left(\mu - \frac{R}{n-1}\right)g_{ab} = \frac{H}{n-1}g_{ab}, \end{aligned}$$

where μ is the eigenfunction associated to Ric at Σ. Moreover, contracting the Codazzi equation

$$R_{1cab} = \nabla_a h_{bc} - \nabla_b h_{ac}$$

over c and b, it gives

$$R_{1a} = \nabla_a(H) - \frac{1}{n-1}\nabla_a(H) = \frac{n-2}{n-1}\nabla_a(H).$$

On the other hand, since $R_{1a} = 0$, we conclude that H is constant in Σ.

In what follows, we fix a local coordinate system

$$(x_1, \ldots, x_n) = (r, \ldots, \theta_n)$$

in U_l, where $(\theta_2, \ldots, \theta_n)$ is any local coordinate system on the level surface Σ. Considering $a, b, c, \dots \in \{2, \ldots, n\}$, we have

$$h_{ab} = -g(\partial_r, \nabla_a\partial_b) = -g(\partial_r, \Gamma^l_{ab}\partial_l) = -\Gamma^1_{ab}.$$

Now, by definition of Christoffel's symbols (1-14) we have

$$\Gamma^1_{ab} = \frac{1}{2}g^{11}\left(-\frac{\partial}{\partial r}g_{ab}\right) = -\frac{1}{2}\frac{\partial}{\partial r}g_{ab}.$$

Then,

$$\frac{2}{n-1}H(r)g_{ab} = \frac{\partial}{\partial r}g_{ab}.$$

Hence, we can infer that

$$g_{ab}(r,\theta) = \varphi(r)^2 g_{ab}(r_0,\theta),$$

where $\varphi(r) = e^{\frac{1}{n-1}\left(\int_{r_0}^{r} H(s)ds\right)}$ and the level set $\{r = r_0\}$ corresponds to the connected component Σ of $f^{-1}(c)$.

Now, we can apply the warped product structure (see [5, Chapter 9] and Section 1.2 of Chapter 1). Hence, considering

$$(M^n, g) = (I, dr^2) \times_\varphi (N^{n-1}, \overline{g}); \quad g = dr^2 + \varphi^2 \overline{g},$$

we deduce that

$$W_{1a1b} = \frac{1}{n-2}\overline{R}_{ab} - \frac{\overline{R}}{(n-2)(n-1)} g_{ab}.$$

Note that if $W(\cdot,\cdot,\cdot,\nabla f) = 0$ we obtain that N is an Einstein manifold.

Since

$$(M^n, g) = (I, dr^2) \times_\varphi (N^{n-1}, \overline{g}),$$

applying the warped product formulas discussed in Section 1.2 of Chapter 1, the Ricci tensor of (M^3, g) is

$$R_{11} = -(n-1)\frac{\varphi''}{\varphi}, \qquad R_{1a} = 0 \tag{2-33}$$

and

$$R_{ab} = \overline{R}_{ab} - \left[(n-2)(\varphi')^2 + \varphi\varphi''\right]\overline{g}_{ab} \qquad (a, b \in \{2, 3\}).$$

On the other hand, since

$$R = \varphi^{-2}\overline{R} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2 - 2(n-1)\frac{\varphi''}{\varphi},$$

we get

$$\overline{R} = \varphi^2 R + (n-1)(n-2)(\varphi')^2 + 2(n-1)\varphi\varphi''.$$

Since $R = 2\frac{\dot{\psi}^2}{f^2}|\nabla f|^2$ and $|\nabla f|$ is constant at Σ we get

$$\overline{R} = 2\varphi^2\frac{\dot{\psi}(f)^2}{f^2}|\nabla f|^2 + (n-1)(n-2)(\varphi')^2 + 2(n-1)\varphi\varphi''.$$

We can conclude that $\overline{R}$ does not depend on θ. Therefore, $\overline{R}$ is a constant. $\square$

2.3 Fourth-order divergence-free Weyl tensor

In this section, we prove some integral theorems in dimension $n \geq 4$ with a fourth-order divergence-free Weyl tensor for an electrovacuum space in the RN class. To that end, we use the lemmas provided in the previous section. We are considering Riemannian manifolds satisfying (2-2) with the zero radial Weyl curvature is considered in our results. Indeed, the fact that electrovacuum space can not satisfies (2-3) appears naturally in the following theorem.

Theorem 2.12 *Let (M^n, g, f, ψ), $n \geq 4$, be an electrovacuum space satisfying (1-29), (2-2) and (1.1). For every $\phi : \mathbb{R} \to \mathbb{R}$, C^2 function with $\phi(f)$ having compact support $K \subseteq M$. Then,*

$$\frac{1}{2(n-1)^2\sigma}\int_M |C|^2\phi(f)\left[(n-1)\sigma-(n-2)f^2\right] = -\frac{n-2}{n-3}\int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j\nabla^l W_{jkil},$$

where σ is a non-null constant.

Remark 4 *It is important to point out that the choice of ϕ in the above theorem should be made in such a way that terms like $\dfrac{\phi(f)}{f^m}$, where $m = 1, 2$ or 3, will be integrable at K.*

Now, let us prove Theorem 2.12.

Proof. From Lemma 2.8, we have

$$\begin{aligned}
\frac{1}{2}|C|^2\phi(f) + \phi(f)R^{ik}\nabla^j C_{jki} &= (n-2)\phi(f)\nabla^j\nabla^i\nabla^k\left(\frac{V_{ikj}}{f}\right)\\
&- (n-2)\phi(f)\nabla^j\left[\frac{1}{f}W_{ikjl}R^{il}\nabla^k f\right]\\
&- 2(n-2)\phi(f)\nabla^j\left[\frac{W_{ikjl}}{f^3}\nabla^i f\nabla^k\psi\nabla^l\psi\right]\\
&+ 2(n-2)\phi(f)\nabla^j\left[\frac{W_{ikjl}}{f^2}\nabla^k\psi\nabla^i\nabla^l\psi\right].
\end{aligned}$$

Now, integration by parts leads us to

$$\begin{aligned}\frac{1}{2}\int_M |C|^2\phi(f)+\int_M \phi(f)R^{ik}\nabla^j C_{jki} = & -\ (n-2)\int_M \dot{\phi}(f)\nabla^j f\nabla^i\nabla^k\left(\frac{V_{ikj}}{f}\right)\\ & +\ (n-2)\int_M \dot{\phi}(f)\nabla^j f\left[\frac{1}{f}W_{ikjl}R^{il}\nabla^k f\right]\\ & -\ 2(n-2)\int_M \dot{\phi}(f)\nabla^j f\left[\frac{W_{ikjl}}{f^2}\nabla^k\psi\nabla^i\nabla^l\psi\right]\\ & +\ 2(n-2)\int_M \dot{\phi}(f)\nabla^j f\left[\frac{W_{ikjl}}{f^3}\nabla^i f\nabla^k\psi\nabla^l\psi\right].\end{aligned}$$

From Lemma 2.7, we obtain

$$\frac{1}{2}\int_M |C|^2\phi(f)+\int_M \phi(f)R^{ik}\nabla^j C_{jki} = -\int_M \dot{\phi}(f)\nabla^j f C_{jki}R^{ik}.$$

Using (2-7), we deduce that

$$\begin{aligned}\frac{1}{2}\int_M |C|^2\phi(f) \ & +\ \int_M \frac{\phi(f)}{f}\left(\nabla^i\nabla^k f-\frac{2}{f}\nabla^i\psi\nabla^k\psi+\frac{1}{n-1}fRg_{ik}\right)\nabla^j C_{jki}\\ =\ & -\ \int_M \frac{\dot{\phi}(f)}{f}\nabla^j f\left(\nabla^k\nabla^i f-\frac{2}{f}\nabla^i\psi\nabla^k\psi+\frac{1}{n-1}fRg_{ik}\right)C_{jki}.\end{aligned}$$

Since the Cotton tensor is totally trace-free, we can infer that

$$\begin{aligned}\frac{1}{2}\int_M |C|^2\phi(f) \ & +\ \int_M \frac{\phi(f)}{f}\nabla^i\nabla^k f\nabla^j C_{jki}-2\int_M \frac{\phi(f)}{f^2}\nabla^i\psi\nabla^k\psi\nabla^j C_{jki}\\ =\ & -\ \int_M \frac{\dot{\phi}(f)}{f}\nabla^j f\nabla^k\nabla^i f C_{jki}+2\int_M \frac{\dot{\phi}(f)}{f^2}\nabla^j f\nabla^i\psi\nabla^k\psi C_{jki}.\end{aligned} \tag{2-34}$$

Analogously to (2-25), we have the following equation

$$2\nabla^j\nabla^k\psi C_{jki} = \nabla^k\nabla^j\psi C_{jki}+\nabla^j\nabla^k\psi C_{kji} = \nabla^k\nabla^j\psi(C_{jki}+C_{kji}) = 0. \tag{2-35}$$

Then, using this, we get

$$\begin{aligned}-2\int_M \frac{\phi(f)}{f^2}\nabla^i\psi\nabla^k\psi\nabla^j C_{jki} \ =\ & 2\int_M\left(\frac{\dot{\phi}(f)}{f^2}-\frac{2\phi(f)}{f^3}\right)\nabla^j f\nabla^i\psi\nabla^k\psi C_{jki}\\ +\ & 2\int_M \frac{\phi(f)}{f^2}\nabla^j\nabla^i\psi\nabla^k\psi C_{jki}.\end{aligned}$$

Replacing this equation in (2-34), since the Cotton tensor is skew-symmetric, and renaming indices we obtain

$$\begin{aligned}
\frac{1}{2}\int_M |C|^2\phi(f) &+ \int_M \frac{\phi(f)}{f}\nabla^i\nabla^k f\nabla^j C_{jki} - 4\int_M \frac{\phi(f)}{f^3}\nabla^j f\nabla^i\psi\nabla^k\psi C_{jki} \\
&+ 2\int_M \frac{\phi(f)}{f^2}\nabla^j\nabla^i\psi\nabla^k\psi C_{jki} \\
&= -\int_M \frac{\dot\phi(f)}{f}\nabla^j f\nabla^k\nabla^i f C_{jki} \\
&= \int_M \frac{\dot\phi(f)}{f}\nabla^j f\nabla^i f\nabla^k C_{jki} + \int_M \left(\frac{\ddot\phi(f)}{f} - \frac{\dot\phi(f)}{f^2}\right) C_{jki}\nabla^i f\nabla^k f\nabla^j f \\
&+ \int_M \frac{\dot\phi(f)}{f}\nabla^k\nabla^j f\nabla^i f C_{jki} \\
&= \int_M \frac{\dot\phi(f)}{f}\nabla^j f\nabla^i f\nabla^k C_{jki} \\
&= -\int_M \frac{\dot\phi(f)}{f}\nabla^i f\nabla^k f\nabla^j C_{jki} = -\int_M \frac{\nabla^i\phi(f)}{f}\nabla^k f\nabla^j C_{jki} \\
&= \int_M \frac{\phi(f)}{f}\nabla^i\nabla^k f\nabla^j C_{jki} - \int_M \frac{\phi(f)}{f^2}\nabla^i f\nabla^k f\nabla^j C_{jki} \\
&+ \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki}.
\end{aligned}$$

Hence, from ([1-7](#)), i.e.,

$$\nabla^i C_{jki} = 0,$$

and the symmetries of the Cotton tensor, by integration we have

$$\begin{aligned}
\frac{1}{2}\int_M |C|^2\phi(f) &+ \int_M \frac{\phi(f)}{f^2}\nabla^i f\nabla^k f\nabla^j C_{jki} \\
&= \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki} + 4\int_M \frac{\phi(f)}{f^3}\nabla^j f\nabla^i\psi\nabla^k\psi C_{jki} \\
&- 2\int_M \frac{\phi(f)}{f^2}\nabla^j\nabla^i\psi\nabla^k\psi C_{jki} = \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki} \qquad (2\text{-}36) \\
&+ 2\int_M \frac{\phi(f)}{f^3}(f\nabla^j\psi\nabla^k\nabla^i\psi + 2\nabla^j f\nabla^k\psi\nabla^i\psi)C_{jki}.
\end{aligned}$$

Now, considering that $\psi = \psi(f)$, we deduce

$$\begin{aligned}
&\int_M \frac{\phi(f)}{f^3}(f\nabla^j\psi\nabla^k\nabla^i\psi + 2\nabla^j f\nabla^k\psi\nabla^i\psi)C_{jki} \\
&\qquad = \int_M \frac{\phi(f)}{f^3}[f\dot\psi\nabla^j f(\dot\psi\nabla^k\nabla^i f + \ddot\psi\nabla^i f\nabla^k f) + 2\dot\psi^2\nabla^j f\nabla^k f\nabla^i f]C_{jki} \\
&\qquad = \int_M \frac{\phi(f)}{f^2}\dot\psi^2\nabla^j f\nabla^k\nabla^i f C_{jki}.
\end{aligned}$$

Again from the symmetries of the Cotton tensor and renaming indices we obtain

$$\begin{aligned}\int_M \frac{\phi(f)}{f^3}(f\nabla^j\psi\nabla^k\nabla^i\psi + 2\nabla^j f\nabla^k\psi\nabla^i\psi)C_{jki} &= \int_M \frac{\phi(f)}{f^2}\dot{\psi}^2\nabla^j f\nabla^k\nabla^i f C_{jki} \\ &= \int_M \frac{\phi(f)}{f^2}\dot{\psi}(f)^2\nabla^k f\nabla^i f\nabla^j C_{jki}.\end{aligned}$$

Thus, replacing the above equation in (2-36), we get

$$\frac{1}{2}\int_M |C|^2\phi(f) + \int_M \frac{\phi(f)}{f^2}(1-2\dot{\psi}(f)^2)\nabla^k f\nabla^i f\nabla^j C_{jki} = \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki}. \quad (2\text{-}37)$$

From now on, we will analyze just one part of the above equation. Since the Cotton tensor is trace-free and skew-symmetric, another integration by parts gives us

$$\begin{aligned}\int_M \frac{\phi(f)}{f^2}(1-2\dot{\psi}(f)^2)\nabla^k f\nabla^i f\nabla^j C_{jki} &= 4\int_M \frac{\phi(f)}{f^2}\dot{\psi}(f)\ddot{\psi}(f)\nabla^k f\nabla^j f\nabla^i f C_{jki} \\ &- \int_M \left(\frac{\dot{\phi(f)}}{f^2} - \frac{2\phi(f)}{f^3}\right)(1-2\dot{\psi}(f)^2)\nabla^k f\nabla^j f\nabla^i f C_{jki} \\ &- \int_M \frac{\phi(f)}{f^2}(1-2\dot{\psi}(f)^2)\nabla^j\nabla^k f\nabla^i f C_{jki} \\ &- \int_M \frac{\phi(f)}{f^2}(1-2\dot{\psi}(f)^2)\nabla^k f\nabla^j\nabla^i f C_{jki} \\ &= \int_M \frac{\phi(f)}{f}(1-2\dot{\psi}(f)^2)R^{ji}\nabla^k f C_{kji}.\end{aligned}$$

In the last equality, we used (2-7) and renamed indices. Now, since M^n has zero radial Weyl curvature and the Cotton tensor is totally trace-free, from (2-13) and (2-14), we can infer that

$$\begin{aligned}R^{ji}\nabla^k f C_{kji} &= \frac{1}{2}C_{kji}(R^{ji}\nabla^k f - R^{ki}\nabla^j f) \\ &= -\frac{1}{2Q}C_{kji}V^{kji} \\ &= -\frac{1}{2Q}f|C|^2,\end{aligned}$$

where Q is the same as given in Lemma 2.4, i.e., $Q = \frac{n-1}{n-2} - 2\dot{\psi}(f)^2$. Therefore, we have

$$\begin{aligned}\int_M \frac{\phi(f)}{f^2}(1-2\dot{\psi}(f)^2)\nabla^k f\nabla^i f\nabla^j C_{jki} &= -\frac{1}{2}\int_M \frac{\phi(f)}{Q}(1-2\dot{\psi}(f)^2)|C|^2 \\ &= -\frac{1}{2}\int_M |C|^2\phi(f)\left[\frac{(n-2)(1-2\dot{\psi}(f)^2)}{n-1-2(n-2)\dot{\psi}(f)^2}\right].\end{aligned}$$

Now, from (2-6) we can conclude that

$$\int_M \frac{\phi(f)}{f^2}(1-2\dot{\psi}(f)^2)\nabla^k f \nabla^i f \nabla^j C_{jki} = -\frac{n-2}{2(n-1)^2\sigma}\int_M \phi(f)\left[f^2+(n-1)\sigma\right]|C|^2.$$

Replacing it in (2-37), we obtain

$$\frac{1}{2(n-1)^2\sigma}\int_M |C|^2\phi(f)\left[(n-1)\sigma-(n-2)f^2\right] = \int_M \frac{\phi(f)}{f}\nabla^k f \nabla^i \nabla^j C_{jki}.$$

Using (1-8), the result holds. □

Next, we will take an appropriate $\phi(f)$ satisfying the conditions of integrability in Theorem 2.12 to obtain an important result concerning the geometric structure of electrovacuum space.

For our next result, remember the following definition [41, Page 372].

Definition 2.13 *[41, Page 372] Let M and N be two topological spaces. A map $f: M \longrightarrow N$ is said to be proper if for each compact subset $K \subset N$, the preimage $f^{-1}(K) \subset M$ is compact.*

Now we can demonstrate the next theorem.

Theorem 2.14 *Let (M^n, g, f, ψ), $n \geq 4$, be an electrovacuum space satisfying (1-29), (2-2) and (1.1) with fourth-order divergence-free Weyl tensor, i.e.,* $\mathrm{div}^4 W = 0$. *If f is a proper function, then the Weyl tensor is harmonic, i.e.,* $\mathrm{div} W = 0$.

Proof. Let $s > 0$ be a real number fixed, so we take $\chi \in C^3(\mathbb{R})$ a real non-negative function defined by $\chi = 1$ in $[0, s]$, $\dot{\chi} \leq 0$ in $[s, 2s]$ and $\chi = 0$ in $[2s, +\infty]$ (see Figure 2.1). Since f is a proper function, we have that $\phi(f) = f^4\chi(f)$ has compact support in

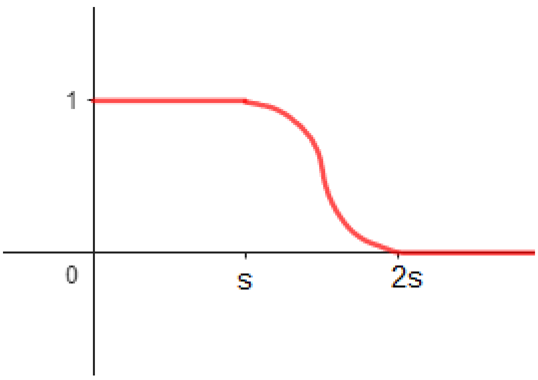

Figure 2.1: An example for χ.

M for $s > 0$. From Theorem 2.12, we get

$$\begin{aligned}
\frac{1}{2(n-1)^2\sigma}\int_M |C|^2 f^4\chi(f)\left[(n-1)\sigma-(n-2)f^2\right] &= -\frac{n-2}{n-3}\int_M f^3\chi(f)\nabla^k f\nabla^i\nabla^j\nabla^l W_{jkil}\\
&= -\frac{n-2}{4(n-3)}\int_M \chi(f)\nabla^k f^4\nabla^i\nabla^j\nabla^l W_{jkil}\\
&= \frac{n-2}{4(n-3)}\int_M \chi(f)f^4\nabla^k\nabla^i\nabla^j\nabla^l W_{jkil}\\
&+ \frac{n-2}{4(n-3)}\int_M \dot{\chi}(f)f^4\nabla^k f\nabla^i\nabla^j\nabla^l W_{jkil}.
\end{aligned}$$

In the last equality, we use integration by parts. Now, we take $\phi(f) = f^5\dot{\chi}(f)$ in the Theorem 2.12 and since $\operatorname{div}^4 W = 0$, we obtain

$$\begin{aligned}
&\frac{1}{2(n-1)^2\sigma}\int_M |C|^2 f^4\chi(f)\left[(n-1)\sigma-(n-2)f^2\right] =\\
&\frac{-1}{8(n-1)^2\sigma}\int_M |C|^2 f^5\dot{\chi}(f)\left[(n-1)\sigma-(n-2)f^2\right].
\end{aligned}$$

Hence,

$$\int_M f^4|C|^2[\chi(f)+\frac{1}{4}f\dot{\chi}(f)]\left[(n-1)\sigma-(n-2)f^2\right] = 0.$$

Define $M_s = \{x\in M; f(x)\le s\}$. We have, by definition of χ that $\chi(f)+\frac{1}{4}f\dot{\chi}(f) = 1$ on the compact set M_s. Thus, on M_s, since $\sigma < 0$,

$$0 \le \int_{M_s} f^4|C|^2\left[(n-2)f^2-(n-1)\sigma\right] = 0,$$

i.e., $C = 0$ in M_s. Taking $s \to +\infty$, we obtain that $C = 0$ on M. □

In conclusion, we have the main result of this section which follows from Theorem 2.11 and Theorem 2.14.

Corollary 2.15 *Let (M^n, g, f, ψ), $n > 3$, be an electrovacuum space with fourth-order divergence free Weyl curvature and zero radial Weyl curvature such that the electric potential ψ is in the Reissner-Nordström class (i.e., satisfying Equation (2-2)). Around any regular point of f, if f is a proper function, then the manifold is locally a warped product with $(n-1)$-dimensional Einstein fibers.*

2.4 Third-order divergence-free Cotton tensor

We will return to the previous results of the last section and study them in dimension $n = 3$. Firstly, it is essential to point out that Lemma 2.6, Lemma 2.7, and

Lemma 2.8 are not valid in the three-dimensional case due to (1-8), which was used in the proofs. However, we can prove another version of those lemmas conveniently when $n = 3$. Another point is that Theorem 2.12 is not valid in dimension $n = 3$, but the main issue here is that the Weyl tensor vanishes in dimension three. Therefore, for $n = 3$ the radial Weyl curvature condition is not necessary anymore. Nonetheless, the computations for the three-dimensional case are very similar to the previous results. We will prove all those results for $n = 3$ in this section for completeness.

After these considerations, we can proceed with our results. To that end, since the Weyl tensor vanishes identically in dimension $n = 3$, from (2-13) we can observe that

$$fC_{ijk} = V_{ijk}. \tag{2-38}$$

Consequently, we have the following lemma.

Lemma 2.16 *Let* (M^3, g, f, ψ) *be an electrovacuum space. Then,*

$$C_{kji}R^{ik} = \nabla^i\nabla^k\left(\frac{V_{kij}}{f}\right).$$

Proof. In fact, from (1-11) and (2-38) we obtain

$$B_{ij} = \nabla^k C_{kij} = \nabla^k\left(\frac{V_{kij}}{f}\right).$$

Taking the derivative over i, we have

$$\nabla^i B_{ij} = \nabla^i\nabla^k\left(\frac{V_{kij}}{f}\right).$$

Since $n = 3$, from (1-12), using (1-6) and (2-27) after renamed the indices, we have

$$\nabla^i B_{ij} = -C_{jik}R^{ik} = -C_{jki}R^{ik} = C_{kji}R^{ik}.$$

Thus, combing these two last relations the result holds. □

Lemma 2.17 *Let* (M^3, g, f, ψ) *be an electrovacuum space. Then,*

$$\frac{1}{2}|C|^2 + R^{ik}\nabla^j C_{jki} = -\nabla^j\nabla^i\nabla^k\left(\frac{V_{kij}}{f}\right).$$

Proof. Taking the divergence in Lemma 2.16, we get

$$C_{kji}\nabla^j R^{ik} + R^{ik}\nabla^j C_{kji} = \nabla^j\nabla^i\nabla^k\left(\frac{V_{kij}}{f}\right).$$

Use (2-29) to obtain

$$\frac{1}{2}C_{kji}(\nabla^j R^{ik} - \nabla^k R^{ij}) + R^{ik}\nabla^j C_{kji} = \nabla^j\nabla^i\nabla^k\left(\frac{V_{kij}}{f}\right).$$

Now, since the Cotton tensor is trace-free, from (1-5) and renaming the indices we obtain

$$-\frac{1}{2}C_{kji}C^{kji} - R^{ik}\nabla^j C_{jki} = \nabla^j\nabla^i\nabla^k\left(\frac{V_{kij}}{f}\right).$$

Therefore, the result holds. □

Theorem 2.18 *Let* (M^3, g, f, ψ) *be an electrovacuum space satisfying* (2-2). *For every* $\phi : \mathbb{R} \to \mathbb{R}$, C^2 *function with* $\phi(f)$ *having compact support* $K \subseteq M$. *Then,*

$$\frac{1}{8\sigma}\int_M |C|^2\phi(f)[2\sigma - f^2] = \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki}.$$

where σ *is a non-null constant.*

Proof. The idea is to proceed as in Theorem 2.12. From Lemma 2.17, we obtain

$$\frac{1}{2}|C|^2\phi(f) + \phi(f)R^{ik}\nabla^j C_{jki} = -\phi(f)\nabla^j\nabla^i\nabla^k\left(\frac{V_{kij}}{f}\right).$$

Hence, upon integrating this expression we get

$$\frac{1}{2}\int_M |C|^2\phi(f) + \int_M \phi(f)R^{ik}\nabla^j C_{jki} = \int_M \dot{\phi}(f)\nabla^j f\nabla^i\nabla^k\left(\frac{V_{kij}}{f}\right).$$

Then, from Lemma 2.16 and the symmetries of C_{ijk}, we have

$$\frac{1}{2}\int_M |C|^2\phi(f) + \int_M \phi(f)R^{ik}\nabla^j C_{jki} = -\int_M \dot{\phi}(f)\nabla^j f C_{jki}R^{ik}.$$

Now, from (2-7) and the fact that C_{ijk} is trace-free and skew-symmetric we obtain the following identity

$$
\begin{aligned}
\frac{1}{2}\int_M |C|^2\phi(f) &+ \int_M \frac{\phi(f)}{f}(\nabla^i\nabla^k f - \frac{2}{f}\dot{\psi}(f)^2\nabla^i f\nabla^k f)\nabla^j C_{jki} \\
&= -\int_M \frac{\dot{\phi}(f)}{f}\nabla^j f(\nabla^k\nabla^i f - \frac{2}{f}\dot{\psi}(f)^2\nabla^i f\nabla^k f)C_{jki} \\
&= -\int_M \frac{\dot{\phi}(f)}{f}\nabla^j f\nabla^k\nabla^i f C_{jki} \\
&= \int_M \left(\frac{\ddot{\phi}(f)}{f} - \frac{\dot{\phi}(f)}{f^2}\right)\nabla^j f\nabla^k f\nabla^i f C_{jki} \\
&\quad + \int_M \frac{\dot{\phi}(f)}{f}\nabla^k\nabla^j f\nabla^i f C_{jki} + \int_M \frac{\dot{\phi}(f)}{f}\nabla^j f\nabla^i f\nabla^k C_{jki} \\
&= \int_M \frac{\dot{\phi}(f)}{f}\nabla^j f\nabla^i f\nabla^k C_{jki}.
\end{aligned}
$$

Note that in the last equality, we have used (2-35). From now, we rename the indices and, integrating by parts again, we infer

$$
\begin{aligned}
\frac{1}{2}\int_M |C|^2\phi(f) &+ \int_M \frac{\phi(f)}{f}\nabla^i\nabla^k f\nabla^j C_{jki} - 2\int_M \frac{\phi(f)}{f^2}\dot{\psi}(f)^2\nabla^i f\nabla^k f\nabla^j C_{jki} \\
&= -\int_M \frac{\dot{\phi}(f)}{f}\nabla^k f\nabla^i f\nabla^j C_{jki} = -\int_M \frac{\nabla^i\phi(f)}{f}\nabla^k f\nabla^j C_{jki} \\
&= \int_M \frac{\phi(f)}{f}\nabla^k\nabla^i f\nabla^j C_{jki} - \int_M \frac{\phi(f)}{f^2}\nabla^k f\nabla^i f\nabla^j C_{jki} \\
&\quad + \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki}.
\end{aligned}
$$

Thus,

$$
\frac{1}{2}\int_M |C|^2\phi(f) + \int_M \frac{\phi(f)}{f^2}(1-2\dot{\psi}(f)^2)\nabla^k f\nabla^i f\nabla^j C_{jki} = \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki}. \quad \text{(2-39)}
$$

Furthermore, from the proof of Theorem 2.12 (Equation 2-38), we get

$$
\int_M \frac{\phi(f)}{f^2}(1-2\dot{\psi}(f)^2)\nabla^k f\nabla^i f\nabla^j C_{jki} \quad = \quad \int_M \frac{\phi(f)}{f}(1-2\dot{\psi}(f)^2)R^{ji}\nabla^k f C_{kji}.
$$

As we did in Theorem 2.12, from (2-14) and (2-38), we have

$$
R^{ji}\nabla^k f C_{kji} = -\frac{1}{2Q}f|C|^2.
$$

Note that for $n=3$, from Lemma 2.4 and (2-6), we obtain, respectively,

$$
Q = 2(1-\dot{\psi}(f)^2)
$$

and

$$\dot{\psi}(f)^2 = \frac{f^2}{f^2 - 2\sigma}; \quad \text{where} \quad \sigma \neq 0.$$

Finally,

$$\int_M \frac{\phi(f)}{f^2}(1 - 2\dot{\psi}(f)^2)\nabla^k f \nabla^i f \nabla^j C_{jki} = -\frac{1}{8\sigma}\int_M |C|^2 \phi(f)\left[f^2 + 2\sigma\right].$$

Therefore, replacing the above equation in (2-39) the result holds. □

Theorem 2.19 *Let* (M^3, g, f, ψ) *be an electrovacuum space satisfying* (2-2) *with* $\sigma < 0$ *and third-order divergence-free Cotton tensor, i.e.,* $\mathrm{div}^3 C = 0$. *If* f *is a proper function, then the Cotton tensor is identically zero, i.e.,* (M^3, g) *is locally conformally flat.*

Proof. Let $s > 0$ be a real number fixed, and so we take $\chi \in C^3$ a real non-negative function defined by $\chi = 1$ in $[0, s]$, $\chi' \leq 0$ in $[s, 2s]$ and $\chi = 0$ in $[2s, +\infty]$ (see Figure 2.1). Since f is a proper function, we have that $\phi(f) = f^4\chi(f)$ has compact support in M for $s > 0$. From Theorem 2.12, we get

$$\begin{aligned}
\frac{1}{8\sigma}\int_M |C|^2 f^4 \chi(f)\left[2\sigma - f^2\right] &= \int_M f^3 \chi(f)\nabla^k f \nabla^i \nabla^j C_{jki} \\
&= \frac{1}{4}\int_M \chi(f)\nabla^i f^4 \nabla^k \nabla^j C_{jki} \\
&= -\frac{1}{4}\int_M \chi(f) f^4 \nabla^i \nabla^k \nabla^j C_{jki} \\
&+ \frac{1}{4}\int_M \dot{\chi}(f) f^4 \nabla^i f \nabla^k \nabla^j C_{jki}.
\end{aligned}$$

In the last equality, we used integration by parts. Now, since $\mathrm{div}^3 C = 0$ we take $\phi(f) = f^5\dot{\chi}(f)$ in Theorem 2.12 one more time to obtain

$$\frac{1}{8\sigma}\int_M |C|^2 f^4 \chi(f)\left[2\sigma - f^2\right] = -\frac{1}{32\sigma}\int_M |C|^2 f^5 \chi(\dot{f})\left[4\sigma - f^2\right],$$

i.e.,

$$\int_M f^4 |C|^2 [\chi(f) + \frac{1}{4} f \dot{\chi}(f)]\left[2\sigma - f^2\right] = 0.$$

Let be M_s defined as in Theorem 2.14, i.e., $M_s = \{x \in M; f(x) \leq s\}$. We have, by definition, $\chi(f) + \frac{1}{4} f \dot{\chi}(f) = 1$ on the compact set M_s. Thus, on M_s, since $\sigma < 0$,

$$0 \leq \int_{M_s} f^4 |C|^2 \left[f^2 - 2\sigma\right] = 0.$$

Therefore, $C = 0$ in M_s. Taking $s \to +\infty$, we obtain that $C = 0$ on M. □

We can finish this chapter by announcing the following result concerning the local geometric structure of three-dimensional electrovacuum spaces.

Corollary 2.20 *Let* (M^3, g, f, ψ) *be an electrovacuum space with third-order divergence free Cotton tensor such that* ψ *satisfies* (2-2) *with* $\sigma < 0$. *Around any regular point of* f*, if* f *is a proper function, then the manifold is locally a warped product with a one-dimensional base and a constant curvature surface fiber.*

Proof. This result is a consequence of Theorem 2.11 and Theorem 2.19. □

CHAPTER 3

The electrostatic system with a non-null cosmological constant

The main goal of this chapter is to show that an electrostatic system with divergence-free Bach tensor, i.e., $\mathrm{div}^2 B = 0$, must be locally conformally flat. It is important to say that $\mathrm{div}^2 B = 0$ is less restrictive (topologically speaking) than asymptotically flat conditions. The focus of this chapter is the electrostatic system with a non-null cosmological constant in dimension three. Even though the idea is to prove similar results as we did in the previous chapter, important differences arise in the present chapter. This will become clear in the discussion that follows.

In the three-dimensional case, the Cotton tensor is associated with the Bach tensor, B, accordingly to $B = \mathrm{div} C$. The Bach tensor was defined in 1921 by Rudolf Bach and it is connected to general relativity and conformal geometry. This tensor appeared naturally from studies of Huyghens's principle and has some psychical significance mainly about wave propagation (see for instance [48] and the references therein). Let us start by remembering the definition provided by (1-31).

Definition 3.1 *Let (M^3, g) be a Riemannian manifold with E a tangent vector field on M and $f \in C^\infty(M)$ satisfying*

$$\nabla^2 f = f(\mathrm{Ric} - \Lambda g + 2E^\flat \otimes E^\flat - |E|^2 g),$$

$$\Delta f = (|E|^2 - \Lambda) f, \quad 0 = \mathrm{div}(E) \quad \textit{and} \quad 0 = \mathrm{curl}(fE).$$

Here, Ric, ∇^2, div *and* Δ *stand for the Ricci tensor, Hessian tensor, divergence, and Laplacian operator concerning the metric g, respectively. Moreover, $E^\flat$ is the one-form metrically dual to E. We refer to the above equations as electrostatic system with cosmological constant Λ for the electrostatic spacetime associated to (M^3, g, f, E).*

Remember the curl stands for an operator that describes the circulation (or rotation) of a vector field (Section 1.2). Thus, we have $\mathrm{curl}(fE)=0$ if, and only if,

$$df\wedge E^{\flat}+fdE^{\flat}=0.$$

Moreover, the smooth function f is called the lapse function, the field E is known as the electric field, and M^3 is the spatial factor for the electrostatic spacetime. Furthermore, $f>0$ on M. If M has boundary ∂M, we assume in addition that $f^{-1}(0)=\partial M$ (cf. [18, 19, 23, 36]).

It is also important to remember that with the contraction of the first equation and combining it with the Laplacian of f given by Definition 3.1, we obtain (1-32), i.e.,

$$R=2(|E|^2+\Lambda).$$

Furthermore, since $\mathrm{curl}(fE)=0$ we have that the electric field and the gradient of the lapse function are linearly dependent on $\partial M=f^{-1}(0)$ (Section 3.3 of Chapter 1). Thus, from the electrostatic solutions presented in Section 1.3, we see that the electric field and the lapse function are related.

There are some well-known classification results of some important geometric structures like static vacuum manifolds and Ricci solitons carrying a metric such that the Bach tensor is free from divergence (cf. [9, 13, 32, 37, 44]). Any three-dimensional Riemannian manifold is locally conformally flat if, and only if, its Cotton tensor C is identically zero. In what follows we will present a classification result for the electrostatic space carrying this geometric condition over the Bach tensor. We must remember Section 1.3, where explicit solutions for the electrostatic system with a non-null cosmological constant were given.

3.1 Structural lemmas

In this section, our aim is to prove some preliminary results. We will discuss briefly some properties of differential forms based on [25, Chapter 1] and [40, Chapter 2].

Definition 3.2 *[25, Definition 3] An exterior k-form in M^n is a map ω that associates to each $p\in M^n$ an element $\omega(p)\in\bigwedge^k(M^n_p)^*$, that is, a k-linear and alternate map at $p\in M^n$.*

Following the notation used in [25, Chapter 1], let

$$\omega(p)=\sum_{i_1<\ldots<i_n}a_{i_1\ldots i_n}(p)(dx_{i_1}\wedge\ldots\wedge dx_{i_n})_p$$

be a k-form in M^n. Here, $a_I : M \to \mathbb{R}$ are smooth functions and I is the k-upla $(i_1, i_2, \cdots, i_k)$ with $i_1 < i_2 < \cdots < i_k$, $i_j \in \{1, 2, \cdots, n\}$. The *exterior differential* $d\omega$ of ω is the $(k+1)$-form defined by

$$d\omega = \sum_I da_I \wedge dx_I,$$

where $\wedge$ performs the *wedge product*. Considering $\{e_i\}_{i=1}^n$ as a base for the tangent space of M, we are considering $dx_i(e_j) = \delta_{ij}$. We know that

$$(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k)(e_1, e_2, \cdots, e_k) = \det(\varphi_i(e_j)),$$

here φ_i are 1-forms. In general, if by $\omega = \sum_I a_I dx_I$ and $\phi = \sum_J b_J dx_J$, then

$$\omega \wedge \phi = \sum_{I,J} a_I b_J dx_I \wedge dx_J,$$

where $I = (i_1, \ldots, i_k)$, $i_1 < \ldots < i_k$, and $J = (i_1, \ldots, i_s)$, $i_1 < \ldots < i_s$. Moreover, if $I = J$ we can define the sum of forms:

$$\omega + \phi = \sum_I (a_I + b_I) dx_I.$$

It is important to highlight some facts about differential forms. For instance, since $dx_i \wedge dx_j = -dx_j \wedge dx_i$, then $dx_i \wedge dx_i = 0$. Other facts are described in the following proposition.

Proposition 3.3 *[25, Chapter 1] Let $\omega \in \bigwedge^k (M^n)^*$ and $\varphi \in \bigwedge^s (M^n)^*$. Then,*

a) $(\omega \wedge \varphi) = (-1)^{ks} (\varphi \wedge \omega)$;
b) $d(\omega \wedge \varphi) = d\omega \wedge \varphi + (-1)^k \omega \wedge d\varphi$;
c) $d(d\omega) = d^2\omega = 0$.

Now we will construct a covariant V-tensor similar to (2-12) defined in Chapter 2. First of all, we note that by combining the first equation of Definition 3.1 with (1-32) we get

$$\nabla^2 f = f\left(\mathrm{Ric} + 2E^\flat \otimes E^\flat - \frac{R}{2} g\right). \tag{3-1}$$

On the other hand, since the Hessian operator is symmetric, taking the covariant derivative of Equation (3-1) over i and j and using the Ricci identity (1-1), we get

$$\begin{aligned} R_{ijkl}\nabla^l f &= \nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f \\ &= f(\nabla_i R_{jk} - \nabla_j R_{ik}) - \frac{f}{2}(\nabla_i R g_{jk} - \nabla_j R g_{ik}) \\ &\quad -\frac{R}{2}(\nabla_i f g_{jk} - \nabla_j f g_{ik}) + (R_{jk}\nabla_i f - R_{ik}\nabla_j f) \\ &\quad +2f(E^{\flat_j}\nabla_i E^{\flat_k} - E^{\flat_i}\nabla_j E^{\flat_k} + \nabla_i E^{\flat_j} E^{\flat_k} - \nabla_j E^{\flat_i} E^{\flat_k}) \\ &\quad +2(\nabla_i f E^{\flat_j} E^{\flat_k} - \nabla_j f E^{\flat_i} E^{\flat_k}). \end{aligned}$$

Here, we are considering $\{e_i\}_{i=1}^3$ as a base for the tangent space of M. Moreover, we define

$$E^{\flat_i} = E^\flat(e_i),$$

i.e.,

$$E^\flat = \sum_{i=1}^3 E^{\flat_i} dx_i.$$

Note that the Cotton tensor (1-5) over a three-dimensional Riemannian manifold is defined by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{4}(\nabla_i R g_{jk} - \nabla_j R g_{ik}). \tag{3-2}$$

Furthermore, since the Weyl tensor (1-4) in dimension 3 is identically null, the Riemann curvature tensor is given by

$$R_{ijkl} = R_{ik}g_{jl} - R_{il}g_{jk} + R_{jl}g_{ik} - R_{jk}g_{il} - \frac{R}{2}(g_{ik}g_{jl} - g_{il}g_{jk}).$$

Therefore, by combining these equations we get

$$\begin{aligned} fC_{ijk} &= (R_{jl}\nabla^l f g_{ik} - R_{il}\nabla^l f g_{jk}) + R(\nabla_i f g_{jk} - \nabla_j f g_{ik}) + 2(R_{ik}\nabla_j f - R_{jk}\nabla_i f) \\ &\quad -2f(E^{\flat_j}\nabla_i E^{\flat_k} - E^{\flat_i}\nabla_j E^{\flat_k} + \nabla_i E^{\flat_j} E^{\flat_k} - \nabla_j E^{\flat_i} E^{\flat_k}) \\ &\quad -2E^{\flat_k}(E^{\flat_j}\nabla_i f - E^{\flat_i}\nabla_j f) + \frac{f}{4}(\nabla_i R g_{jk} - \nabla_j R g_{ik}). \end{aligned} \tag{3-3}$$

Now, using $\mathrm{curl}(fE) = 0$, from the definition of the wedge product of differential forms we can infer that

$$\begin{aligned} f dE^\flat(e_i, e_j) = -(df \wedge E^\flat)(e_i, e_j) &= E^\flat(e_i)df(e_j) - E^\flat(e_j)df(e_i) \\ &= E^{\flat_i}\nabla_j f - E^{\flat_j}\nabla_i f. \end{aligned}$$

Denoting

$$dE^\flat(e_i, e_j) = E^{\flat_{ij}},$$

from the above equation, we have

$$fE^{\flat_{ij}} = E^{\flat_i}\nabla_j f - E^{\flat_j}\nabla_i f.$$

On the other hand, let $E^\flat = \sum_{i=1}^3 a_i dx_i$, where $a_i = E^\flat(e_i) = E^{\flat_i}$. Then,

$$\begin{aligned} E^{\flat_{ij}} &= dE^\flat(e_i, e_j) \\ &= \left(\sum_{\ell=1}^3 da_\ell \wedge dx_\ell\right)(e_i, e_j) \\ &= da_j(e_i) - da_i(e_j) \\ &= (dE^{\flat_j})(e_i) - (dE^{\flat_i})(e_j). \end{aligned}$$

Therefore,

$$E^{\flat_{ij}} = \nabla_i E^{\flat_j} - \nabla_j E^{\flat_i}, \tag{3-4}$$

Further, we can see that

$$f(\nabla_i E^{\flat_j} - \nabla_j E^{\flat_i}) = E^{\flat_i}\nabla_j f - E^{\flat_j}\nabla_i f. \tag{3-5}$$

We can rewrite (3-3) using the above discussion about differential forms and curl$(fE) = 0$. So,

$$\begin{aligned} fC_{ijk} &= (R_{jl}\nabla^l f g_{ik} - R_{il}\nabla^l f g_{jk}) + 2(R_{ik}\nabla_j f - R_{jk}\nabla_i f) + R(\nabla_i f g_{jk} - \nabla_j f g_{ik}) \\ &\quad + \frac{f}{4}(\nabla_i R g_{jk} - \nabla_j R g_{ik}) - 2f(E^{\flat_j}\nabla_i E^{\flat_k} - E^{\flat_i}\nabla_j E^{\flat_k}). \end{aligned} \tag{3-6}$$

Thus, we define the covariant 3-tensor V_{ijk} by

$$\begin{aligned} V_{ijk} &= 2f(E^{\flat_i}\nabla_j E^{\flat_k} - E^{\flat_j}\nabla_i E^{\flat_k}) + \frac{f}{4}(\nabla_i R g_{jk} - \nabla_j R g_{ik}) + R(\nabla_i f g_{jk} - \nabla_j f g_{ik}) \\ &\quad - (R_{il}\nabla^l f g_{jk} - R_{jl}\nabla^l f g_{ik}) - 2(\nabla_i f R_{jk} - \nabla_j f R_{ik}), \end{aligned} \tag{3-7}$$

where $E^{\flat_i} = E^\flat(e_i)$. The V-tensor has the same symmetries as the Cotton tensor C, i.e.,

$$V_{ijk} = -V_{jik} \quad \text{and} \quad V_{ijk} + V_{jki} + V_{kij} = 0.$$

This tensor is totally trace-free, and the proof follows from the same ideas of the previous chapter.

From (3-6) and (3-7) we can conclude our next result.

Lemma 3.4 *Let* (M^3, g, f, E) *be an electrostatic system. Then,*

$$fC_{ijk} = V_{ijk}. \tag{3-8}$$

Consequently, we have the following lemmas concerning the divergence of the V-tensor.

Lemma 3.5 *Let* (M^3, g, f, E) *be an electrostatic system. Then,*

$$C_{kji}R^{ik} = \nabla^i \nabla^k \left(\frac{V_{kij}}{f}\right).$$

Proof. In dimension $n = 3$, the Bach tensor is defined as in (1-11). Thus,

$$B_{ij} = \nabla^k C_{kij} = \nabla^k \left(\frac{V_{kij}}{f}\right).$$

Taking the derivative over i, we have

$$\nabla^i B_{ij} = \nabla^i \nabla^k \left(\frac{V_{kij}}{f}\right).$$

On the other hand, remember (1-12) and the properties of the Cotton tensor, i.e.,

$$\nabla^j B_{ij} = -C_{ijk}R^{jk}, \qquad C_{ijk} = -C_{jik}, \qquad \nabla^k C_{kij} = \nabla^k C_{kji},$$

and

$$C_{ijk} + C_{kij} + C_{jki} = 0.$$

Then, from a straightforward computation, we obtain

$$\nabla^i \nabla^k \left(\frac{V_{kij}}{f}\right) = \nabla^i B_{ij} = -C_{jik}R^{ik} = -C_{jki}R^{ik} = C_{kji}R^{ik},$$

which is the expected result. □

Lemma 3.6 *Let* (M^3, g, f, E) *be an electrostatic system. Then,*

$$\frac{1}{2}|C|^2 + R^{ik}\nabla^j C_{jki} = -\nabla^j \nabla^i \nabla^k \left(\frac{V_{kij}}{f}\right).$$

Proof. Taking the divergence in Lemma 3.5, we get

$$C_{kji}\nabla^j R^{ik} + R^{ik}\nabla^j C_{kji} = \nabla^j \nabla^i \nabla^k \left(\frac{V_{kij}}{f}\right).$$

Note that from the symmetries of the Cotton tensor and renaming indices, we have

$$2C_{jki}\nabla^j R^{ik} = C_{jki}\nabla^j R^{ik} + C_{kji}\nabla^k R^{ij} = C_{jki}(\nabla^j R^{ik} - \nabla^k R^{ij}).$$

Hence,

$$\frac{1}{2}C_{kji}(\nabla^j R^{ik} - \nabla^k R^{ij}) + R^{ik}\nabla^j C_{kji} = \nabla^j\nabla^i\nabla^k\left(\frac{V_{kij}}{f}\right).$$

Now, since the Cotton tensor is trace-free, from (3-2) and renaming the indices, we obtain

$$-\frac{1}{2}C_{kji}C^{kji} - R^{ik}\nabla^j C_{jki} = \nabla^j\nabla^i\nabla^k\left(\frac{V_{kij}}{f}\right).$$

□

It is important to notice that in Chapter 2 we demonstrated Theorem 2.2, which is true for a null cosmological constant and assuming that $\psi = \psi(f)$. Since we are dealing with a more general framework, we cannot apply this result. Nonetheless, we proved a similar result.

Theorem 3.7 *Let (M^3, g, f, E) be an electrostatic system. For every C^2-function $\phi : \mathbb{R} \to \mathbb{R}$, with $\phi(f)$ having compact support $K \subseteq M$ such that $K \cap \partial M = \emptyset$ we have*

$$\frac{1}{4}\int_M \phi(f)|C|^2 = \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki} + \int_M \phi(f)E^{\flat_j}\nabla^k E^{\flat_i} C_{jki}.$$

Proof. From Lemma 3.6, we obtain

$$\frac{1}{2}|C|^2\phi(f) + \phi(f)R^{ik}\nabla^j C_{jki} = -\phi(f)\nabla^j\nabla^i\nabla^k\left(\frac{V_{kij}}{f}\right).$$

Integrating this expression, we get

$$\frac{1}{2}\int_M |C|^2\phi(f) + \int_M \phi(f)R^{ik}\nabla^j C_{jki} = \int_M \dot{\phi}(f)\nabla^j f\nabla^i\nabla^k\left(\frac{V_{kij}}{f}\right).$$

Thus, from Lemma 3.5, we have

$$\frac{1}{2}\int_M |C|^2\phi(f) + \int_M \phi(f)R^{ik}\nabla^j C_{jki} = -\int_M \dot{\phi}(f)R^{ik}\nabla^j f C_{jki}.$$

We will perform integration in some parts of the above equation, separately, using

Definition 3.1 and the fact that C_{ijk} is trace-free and skew-symmetric. First,

$$\begin{aligned}\int_M \phi(f) R^{ik}\nabla^j C_{jki} &= \int_M \frac{\phi(f)}{f}\nabla^i\nabla^k f\nabla^j C_{jki} - 2\int_M \phi(f)E^{\flat_i}E^{\flat_k}\nabla^j C_{jki}\\ &= \int_M \frac{\phi(f)}{f}\nabla^i\nabla^k f\nabla^j C_{jki} + 2\int_M \dot{\phi}(f)\nabla^j f E^{\flat_i}E^{\flat_k} C_{jki}\\ &\quad +2\int_M \phi(f)\nabla^j(E^{\flat_i}E^{\flat_k})C_{jki}.\end{aligned}$$

On the other hand,

$$\int_M \dot{\phi}(f)R^{ik}\nabla^j f C_{jki} = \int_M \frac{\dot{\phi}(f)}{f}\nabla^j f\nabla^i\nabla^k f C_{jki} - 2\int_M \dot{\phi}(f)\nabla^j f E^{\flat_i}E^{\flat_k}C_{jki}.$$

Note that, since the Hessian tensor is symmetric

$$2\nabla^j\nabla^k f C_{jki} = \nabla^k\nabla^j f C_{jki} + \nabla^j\nabla^k f C_{kji} = \nabla^k\nabla^j f(C_{jki}+C_{kji}) = 0.$$

Hence,

$$\begin{aligned}&\frac{1}{2}\int_M |C|^2\phi(f) + \int_M \frac{\phi(f)}{f}\nabla^i\nabla^k f\nabla^j C_{jki} + 2\int_M \phi(f)\nabla^j(E^{\flat_i}E^{\flat_k})C_{jki}\\ &\quad = -\int_M \frac{\dot{\phi}(f)}{f}\nabla^j f\nabla^i\nabla^k f C_{jki} = \int_M \frac{\dot{\phi}(f)}{f}\nabla^j f\nabla^i f\nabla^k C_{jki}\\ &\quad = -\int_M \frac{\dot{\phi}(f)}{f}\nabla^i f\nabla^k f\nabla^j C_{jki} = -\int_M \frac{\nabla^i\phi(f)}{f}\nabla^k f\nabla^j C_{jki}\\ &\quad = -\int_M \frac{\phi(f)}{f^2}\nabla^i f\nabla^k f\nabla^j C_{jki} + \int_M \frac{\phi(f)}{f}\nabla^i\nabla^k f\nabla^j C_{jki}\\ &\qquad + \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki} &= \frac{1}{2}\int_M |C|^2\phi(f) + \int_M \frac{\phi(f)}{f^2}\nabla^i f\nabla^k f\nabla^j C_{jki}\\ &\quad +2\int_M \phi(f)\nabla^j(E^{\flat_i}E^{\flat_k})C_{jki}. \qquad (3\text{-}9)\end{aligned}$$

Then, since the Cotton tensor is trace-free and skew-symmetric, another integration by parts gives us

$$\begin{aligned}\int_M \frac{\phi(f)}{f^2}\nabla^i f\nabla^k f\nabla^j C_{jki} &= \int_M \frac{\phi(f)}{f^2}\nabla^j\nabla^i f\nabla^k f C_{kji}\\ &= \int_M \frac{\phi(f)}{f}(R^{ij} + 2E^{\flat_i}E^{\flat_j})\nabla^k f C_{kji}.\end{aligned}$$

We used Definition 3.1 in the last equality. Thus, (3-9) can be rewritten in the following form:

$$\begin{aligned}\int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki} &= \frac{1}{2}\int_M |C|^2\phi(f) + 2\int_M \phi(f)\nabla^j(E^{\flat_i}E^{\flat_k})C_{jki}\\ &\quad -2\int_M \frac{\phi(f)}{f}E^{\flat_i}E^{\flat_j}\nabla^k f C_{jki} + \int_M \frac{\phi(f)}{f}R^{ij}\nabla^k f C_{kji}.\end{aligned}$$

Now, from (3-7) and (3-8), we have

$$\begin{aligned}R^{ij}\nabla^k f C_{kji} &= \frac{1}{2}C_{kji}(\nabla^k f R^{ji} - \nabla^j f R^{ki})\\ &= -\frac{1}{2}f C_{kji}\left[\frac{1}{2}C^{kji} + (E^{\flat_j}\nabla^k E^{\flat_i} - E^{\flat_k}\nabla^j E^{\flat_i})\right]\\ &= -\frac{1}{4}f|C|^2 - \frac{1}{2}f\left(E^{\flat_j}\nabla^k E^{\flat_i} - E^{\flat_k}\nabla^j E^{\flat_i}\right)C_{kji}.\end{aligned}$$

So,

$$\begin{aligned}&\frac{1}{4}\int_M \phi(f)|C|^2 = \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki}\\ &+2\int_M \frac{\phi(f)}{f}\left[E^{\flat_j}E^{\flat_i}\nabla^k f - f\nabla^j(E^{\flat_i}E^{\flat_k}) + \frac{f}{4}\left(E^{\flat_k}\nabla^j E^{\flat_i} - E^{\flat_j}\nabla^k E^{\flat_i}\right)\right]C_{jki}.\end{aligned}$$

Furthermore, from (3-5) we have

$$E^{\flat_i}\nabla^k f - E^{\flat_k}\nabla^i f = f(\nabla^i E^{\flat_k} - \nabla^k E^{\flat_i}).$$

Combining the last two equations and the fact that the Cotton tensor is skew-symmetric, yields to

$$\begin{aligned}&\frac{1}{4}\int_M \phi(f)|C|^2 = \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki}\\ &+2\int_M \phi(f)\left[E^{\flat_j}(\nabla^i E^{\flat_k} - \nabla^k E^{\flat_i}) - \nabla^j(E^{\flat_i}E^{\flat_k}) + \frac{1}{4}\left(E^{\flat_k}\nabla^j E^{\flat_i} - E^{\flat_j}\nabla^k E^{\flat_i}\right)\right]C_{jki},\end{aligned}$$

i.e.,

$$\begin{aligned}&\frac{1}{4}\int_M \phi(f)|C|^2 = \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki}\\ &+2\int_M \phi(f)\left[E^{\flat_j}\nabla^i E^{\flat_k} - E^{\flat_i}\nabla^j E^{\flat_k} - \frac{3}{4}E^{\flat_k}\nabla^j E^{\flat_i} - \frac{5}{4}E^{\flat_j}\nabla^k E^{\flat_i}\right]C_{jki}.\end{aligned}$$

Note that

$$E^{\flat_j}\nabla^i E^{\flat_k}C_{jki} = -E^{\flat_k}\nabla^i E^{\flat_j}C_{jki}, \qquad E^{\flat_i}\nabla^j E^{\flat_k}C_{jki} = -E^{\flat_i}\nabla^k E^{\flat_j}C_{jki},$$
$$E^{\flat_k}\nabla^j E^{\flat_i}C_{jki} = -E^{\flat_j}\nabla^k E^{\flat_i}C_{jki} \quad \text{and} \quad E^{\flat_j}\nabla^k E^{\flat_i}C_{jki} = -E^{\flat_k}\nabla^j E^{\flat_i}C_{jki}.$$

Then,

$$\begin{aligned}\frac{1}{4}\int_M \phi(f)|C|^2 &= \int_M \phi(f)\left[2E^{\flat_j}\nabla^i E^{\flat_k} - 2E^{\flat_i}\nabla^j E^{\flat_k} + E^{\flat_k}\nabla^j E^{\flat_i}\right]C_{jki} \\ &\quad + \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki}.\end{aligned}$$

From (3-4), since

$$\nabla^j E^{\flat_k}C_{jki} = -\nabla^k E^{\flat_j}C_{jki},$$

hence

$$2E^{\flat_i}\nabla^j E^{\flat_k}C_{jki} = E^{\flat_i}(\nabla^j E^{\flat_k} - \nabla^k E^{\flat_j})C_{jki} = E^{\flat_i}E^{\flat_{jk}}C_{jki},$$

we can infer that

$$\begin{aligned}\frac{1}{4}\int_M \phi(f)|C|^2 &= \int_M \phi(f)\left[-2E^{\flat_k}\nabla^i E^{\flat_j} - E^{\flat_i}E^{\flat_{jk}} + E^{\flat_k}\nabla^j E^{\flat_i}\right]C_{jki} \\ &\quad + \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki} \\ &= \int_M \phi(f)\left[-E^{\flat_k}\nabla^i E^{\flat_j} + E^{\flat_i}E^{\flat_{kj}} + E^{\flat_k}E^{\flat_{ji}}\right]C_{jki} \\ &\quad + \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki} \\ &= \int_M \phi(f)\left[E^{\flat_k}E^{\flat_{ji}} + E^{\flat_j}\nabla^i E^{\flat_k} + E^{\flat_i}E^{\flat_{kj}}\right]C_{jki} \\ &\quad + \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki} \\ &= \int_M \phi(f)\left[E^{\flat_k}E^{\flat_{ji}} + E^{\flat_j}E^{\flat_{ik}} + E^{\flat_i}E^{\flat_{kj}} + E^{\flat_j}\nabla^k E^{\flat_i}\right]C_{jki} \\ &\quad + \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki}.\end{aligned}$$

On the other hand, from (3-5), we get

$$\begin{aligned} f(E^{\flat_k}E^{\flat_{ji}} + E^{\flat_j}E^{\flat_{ik}} + E^{\flat_i}E^{\flat_{kj}}) &= E^{\flat_k}E^{\flat_j}\nabla^i f - E^{\flat_k}E^{\flat_i}\nabla^j f + E^{\flat_j}E^{\flat_i}\nabla^k f \\ &\quad - E^{\flat_j}E^{\flat_k}\nabla^i f + E^{\flat_i}E^{\flat_k}\nabla^j f - E^{\flat_i}E^{\flat_j}\nabla^k f \\ &= 0.\end{aligned}$$

Finally,

$$\frac{1}{4}\int_M \phi(f)|C|^2 = \int_M \frac{\phi(f)}{f}\nabla^k f\nabla^i\nabla^j C_{jki} + \int_M \phi(f)E^{\flat_j}\nabla^k E^{\flat_i} C_{jki}.$$

□

3.2 Divergence-free Bach tensor

It is well-known that in $\partial M = f^{-1}(0)$, the electric field and the gradient of the lapse function are linearly dependent (LD). Motivated by the Reissner-Nordström-de Sitter solution, the charged Nariai solution, and the (ultra)cold black hole system as presented in Section 1.3, we assume that both fields are linearly dependent on M, that is, there exists a smooth function $\rho : M \to \mathbb{R}$ such that $E = \rho\nabla f$. Thus, we can rewrite the V-tensor (3-7) as follows.

Lemma 3.8 *Let (M^3, g, f, E) be an electrostatic system in which $E = \rho\nabla f$. Then, the V-tensor is given by*

$$\begin{aligned} V_{ijk} &= f\rho|\nabla f|^2(\nabla_i\rho g_{jk} - \nabla_j\rho g_{ik}) + \left(R - \frac{1}{2}Rf^2\rho^2 - f^2\rho^2\Lambda\right)(\nabla_i f g_{jk} - \nabla_j f g_{ik}) \\ &\quad +2(f^2\rho^2 - 1)(\nabla_i f R_{jk} - \nabla_j f R_{ik}) + (f^2\rho^2 - 1)(R_{il}\nabla^l f g_{jk} - R_{jl}\nabla^l f g_{ik}). \end{aligned}$$

Proof. Since the electric field and the lapse function are linearly dependent (LD), a smooth function ρ exists such that $E = \rho\nabla f$. Using (3-1) we get

$$\begin{aligned} \nabla_i E^{\flat_j} &= \nabla_i(\rho\nabla_j f) \\ &= \nabla_i\rho\nabla_j f + \rho\nabla_i\nabla_j f \\ &= \nabla_i\rho\nabla_j f + 2f\rho^3\nabla_j f\nabla_i f + f\rho R_{ij} - \frac{f}{2}\rho R g_{ij}. \end{aligned}$$

In opposite side, from (1-32), we get

$$\begin{aligned} \nabla_i R &= 2\nabla_i|E|^2 \\ &= 4\rho|\nabla f|^2\nabla_i\rho + 2\rho^2\nabla_i|\nabla f|^2. \end{aligned}$$

From (3-1), we know that

$$\nabla_i|\nabla f|^2 = 2f\left(R_{il}\nabla^l f + 2\rho^2|\nabla f|^2\nabla_i f - \frac{R}{2}\nabla_i f\right).$$

Combining the last two equations and using Definition 3.1, we have

$$\nabla_i R = 4\rho|\nabla f|^2\nabla_i\rho + 4f\rho^2\left(R_{il}\nabla^l f + 2\rho^2|\nabla f|^2\nabla_i f - \frac{R}{2}\nabla_i f\right).$$

Then, from (3-7) it follows that

$$\begin{aligned} V_{ijk} = {} & 2f\rho(\nabla_i f\nabla_j\rho\nabla_k f - \nabla_j f\nabla_i\rho\nabla_k f) + f\rho|\nabla f|^2(\nabla_i\rho g_{jk} - \nabla_j\rho g_{ik}) \\ & + 2(f^2\rho^2 - 1)(\nabla_i f R_{jk} - \nabla_j f R_{ik}) + (f^2\rho^2 - 1)(R_{il}\nabla^l f g_{jk} - R_{jl}\nabla^l f g_{ik}) \\ & + \left[\left(1 - \frac{3}{2}f^2\rho^2\right)R + 2f^2\rho^4|\nabla f|^2\right](\nabla_i f g_{jk} - \nabla_j f g_{ik}). \end{aligned}$$

Note that $\operatorname{curl}(fE) = 0$ implies that $df \wedge E^\flat + f dE^\flat = 0$, since we are considering E and ∇f linearly dependent, we get that $E^{\flat ij} = 0$, then $\nabla_i E^{\flat j} = \nabla_j E^{\flat i}$ (cf. Equation (3-4) and Equation (3-5)). So,

$$\nabla_i\rho\nabla_j f = \nabla_j\rho\nabla_i f.$$

The above identity plays a vital role in the following results. We recommend the reader's attention to this identity.

Finally, the result follows by combining Definition 3.1 with the last two identities. $\square$

We define the following function that appears in the Lemma 3.8,

$$Q = 1 - f^2\rho^2.$$

Remark 5 *It is important to point out that $Q > 0$ at the boundary ∂M. Moreover, since $\Lambda \neq 0$, there is no open set $\Omega \subseteq M$ such that $Q = 0$, and $E = \rho\nabla f$. Otherwise, taking the derivative of $f^2\rho^2 = 1$ we can see that*

$$f\rho^2\nabla f + f^2\rho\nabla\rho = 0.$$

So, we have $|E|^2 + f\rho\langle\nabla\rho, \nabla f\rangle = \langle(f\rho^2\nabla f + f^2\rho\nabla\rho), \nabla f\rangle = 0$.

On the other hand, $0 = \operatorname{div}(E) = \rho\Delta f + \langle\nabla\rho, \nabla f\rangle = \rho f(|E|^2 - \Lambda) + \langle\nabla\rho, \nabla f\rangle$, i.e., $f\rho\langle\nabla\rho, \nabla f\rangle = f^2\rho^2(\Lambda - |E|^2) = \Lambda - |E|^2$.

Combining these equations we get $\Lambda = 0$ on Ω, which is a contradiction.

Moreover, from Theorem 3.7 we obtain the following corollary.

Corollary 3.9 *Let (M^3, g, f, E) be an electrostatic system where the electric field and gradient of the lapse function are linearly dependent. For every $\phi : \mathbb{R} \to \mathbb{R}$, C^2 function* *with* $\phi(f)$ *having compact support* $K \subseteq M$ *such that* $K \cap \partial M = \emptyset$ *we have*

$$\frac{1}{4}\int_M \frac{1}{Q}|C|^2\phi(f) = \int_M \frac{\phi(f)}{f}\nabla^k f \nabla^i \nabla^j C_{jki},$$

where we are assuming $Q = 1 - f^2\rho^2 \neq 0$.

Proof. Taking into account that $E = \rho\nabla f$ in Theorem 3.7, since the Cotton tensor is skew-symmetric and trace-free we obtain

$$\frac{1}{4}\int_M \phi(f)|C|^2 = \int_M \frac{\phi(f)}{f}\nabla^k f \nabla^i \nabla^j C_{jki} + \int_M \phi(f) f \rho^2 \nabla^j f R^{ki} C_{jki},$$

where we used that $\mathrm{curl}(fE) = 0$, i.e.,

$$\nabla^k \rho \nabla^i f = \nabla^i \rho \nabla^k f.$$

By contrast, from Lemma 3.8 we have

$$\begin{aligned} \nabla^j f R^{ki} C_{jki} &= \frac{1}{2} C_{jki}(\nabla^j f R^{ki} - \nabla^k f R^{ji}) \\ &= -\frac{1}{4Q} C_{jki} V^{jki} \\ &= -\frac{1}{4Q} f|C|^2. \end{aligned}$$

Therefore, replacing this equality in the integral, the result holds. □

Now, from this corollary and considering $\mathrm{div}^3 C = 0$, we can prove our next theorem.

Theorem 3.10 *Let* (M^3, g, f, E) *be a compact electrostatic system such that the electric field and the gradient of the lapse function are linearly dependent. Suppose that the Bach tensor is divergence-free and* $Q > 0$ *(or* $Q < 0$*). Then,* (M^3, g) *is locally conformally flat.*

Proof. Considering that M is compact, $f^{-1}(0) = \partial M$, and $\phi(f) = f$, from Corollary 3.9 and

$$\nabla^k C_{kij} = \nabla^k C_{kji},$$

we obtain

$$\begin{aligned} \frac{1}{4}\int_M \frac{1}{Q}|C|^2 f &= \int_M \nabla^k f \nabla^i \nabla^j C_{jki} \\ &= -\int_M f \nabla^i \nabla^k \nabla^j C_{jki}. \end{aligned}$$

Since $\mathrm{div}^2 B = 0$ (i.e., $\mathrm{div}^3 C = 0$), then the right-hand side is identically zero, i.e.,

$$\int_M \frac{f}{Q}|C|^2 = 0.$$

Since $f > 0$ on M and $Q > 0$ (or $Q < 0$) we notice that the function $\frac{f}{Q}|C|^2$ has a defined sign everywhere on M, then the above integral shows us that the Cotton tensor C must be identically zero, i.e., (M^3, g) is a locally conformally flat manifold.

□

Now we can prove the non-compact case of the previous theorem.

Theorem 3.11 *Let (M^3, g, f, E) be an electrostatic system such that the electric field and the gradient of the lapse function are linearly dependent. Suppose that the Bach tensor is divergence-free and $Q > 0$ (or $Q < 0$). If f is a proper function, then (M^3, g) is locally conformally flat.*

Proof. Let $s > 0$ be a real number fixed, and so we take $\chi \in C^3$ a real non-negative function defined by $\chi(s) = 1$ in $[0, s]$, $\chi'(s) \leq 0$ in $[s, 2s]$ and $\chi(s) = 0$ in $[2s, +\infty]$ (see Figure 2.1). Since f is a proper function, we have that $\phi(f) = f\chi(f)$ has compact support in M for $s > 0$. From Corollary 3.9 and

$$\nabla^k C_{kij} = \nabla^k C_{kji},$$

we get

$$\begin{aligned} \frac{1}{4}\int_M \frac{1}{Q}|C|^2 f\chi(f) &= \int_M \chi(f)\nabla^k f \nabla^i \nabla^j C_{jki} \\ &= -\int_M \chi(f) f \nabla^i \nabla^k \nabla^j C_{jki} \\ &\quad + \int_M \dot{\chi}(f) f \nabla^i f \nabla^k \nabla^j C_{jki}. \end{aligned}$$

In the last equality, we used integration by parts. Now, since $\mathrm{div}^2 B = 0$ and taking $\phi(f) = f^2\dot{\chi}(f)$ in Corollary 3.9 one more time, we obtain

$$\frac{1}{4}\int_M \frac{1}{Q}|C|^2 f\chi(f) = \frac{1}{4}\int_M \frac{1}{Q}|C|^2 f^2\dot{\chi}(f),$$

i.e.,

$$\int_M \frac{1}{Q} f|C|^2[\chi(f) - f\dot{\chi}(f)] = 0.$$

Let be $M_s = \{x \in M; f(x) \leq s\}$. Thus, by the definition of χ, $\chi(f) - f\dot{\chi}(f) = 1$ on M_s. Thus, on M_s,

$$\int_{M_s} \frac{1}{Q} f|C|^2 = 0.$$

Therefore, since $Q > 0$ (or $Q < 0$) and f is positive, $C = 0$ in M_s. Taking $s \to +\infty$, we obtain that $C = 0$ on M. $\square$

3.3 The Warped Product Structure

In this section, we will provide once more for the sake of completeness the warped product structure of a 3−dimensional locally conformally flat electrostatic system following the ideas of [9, 10].

We consider an orthonormal frame $\{e_1, e_2, e_3\}$ diagonalizing the Ricci tensor Ric at a regular point $p \in \Sigma = f^{-1}(c)$, with associated eigenvalues R_{kk}, $k = 1, 2, 3$, respectively. That is, $R_{ij}(p) = R_{ii}\delta_{ij}(p)$. Now, from Theorem 2.19 we can infer that $V_{ijk} = 0$ (since (M, g) is locally conformally flat). Then, from Lemma 3.8, for all $i \neq j$ we get

$$\begin{aligned} 0 = V_{ijj} &= f\rho|\nabla f|^2\nabla_i\rho + (f^2\rho^2 - 1)(2R_{jj} + R_{ii})\nabla_i f \\ &\quad + \left(R - \frac{1}{2}Rf^2\rho^2 - f^2\rho^2\Lambda\right)\nabla_i f. \end{aligned} \tag{3-10}$$

Without loss of generalization, consider $\nabla_1 f \neq 0$ and $\nabla_j f = 0$ for all $1 \neq j$. Observe that $\mathrm{Ric}(\nabla f) = R_{11}\nabla f$, i.e., ∇f is an eigenvector for Ric. From (3-10), we obtain that R_{11} and R_{jj}, $j \neq 1$, have multiplicity 1 and 2, respectively. In fact,

$$-f\rho|\nabla f|^2\frac{\nabla_1\rho}{\nabla_1 f} - \left(R - \frac{1}{2}Rf^2\rho^2 - f^2\rho^2\Lambda\right) - (f^2\rho^2 - 1)R_{11} = 2(f^2\rho^2 - 1)R_{jj},$$

for $j = 2, 3$. The left-hand side of the above identity does not depend on j.

Moreover, suppose that $\nabla_i f \neq 0$ for at least two distinct directions. Assume $\nabla_1 f \neq 0$, $\nabla_2 f \neq 0$ and $\nabla_3 f = 0$. So, for instance, we have

$$-f\rho|\nabla f|^2\frac{\nabla_1\rho}{\nabla_1 f} - \left(R - \frac{1}{2}Rf^2\rho^2 - f^2\rho^2\Lambda\right) - (f^2\rho^2 - 1)R_{11} = 2(f^2\rho^2 - 1)R_{33}$$

and

$$-f\rho|\nabla f|^2\frac{\nabla_2\rho}{\nabla_2 f} - \left(R - \frac{1}{2}Rf^2\rho^2 - f^2\rho^2\Lambda\right) - (f^2\rho^2 - 1)R_{22} = 2(f^2\rho^2 - 1)R_{33}.$$

Then, using that $\mathrm{curl}(fE)=0$ and $E=\rho\nabla f$, we already know that

$$\nabla^k\rho\nabla^i f=\nabla^i\rho\nabla^k f.$$

We can conclude that

$$\frac{\nabla_1\rho}{\nabla_1 f}=\frac{\nabla_2\rho}{\nabla_2 f}.$$

Thus, $R_{11}=R_{22}$. Analogously, if $\nabla_i f\neq 0$ for all $i\in\{1,2,3\}$. Then, $R_{11}=R_{22}=R_{33}$. So, we can conclude that Ric has at most two distinct eigenvalues λ and μ with one of them having multiplicity 2, let us say μ.

Therefore, in any case, we have that ∇f is an eigenvector for Ric. From the above discussion we can take $\{e_1=\frac{\nabla f}{|\nabla f|},e_2,e_3\}$ as an orthonormal frame for Σ diagonalizing the Ricci tensor for the metric g.

Now, we have

$$\nabla_a|\nabla f|^2=2f\left(R_{al}\nabla^l f+2\rho^2|\nabla f|^2\nabla_a f-\frac{R}{2}\nabla_a f\right);\quad a\in\{2,3\}.$$

Hence, $|\nabla f|$ is a constant in Σ. Thus, we can express locally the metric g in the form

$$g_{ij}=\frac{1}{|\nabla f|^2}df^2+g_{ab}(f,\theta)d\theta_a d\theta_b,$$

where $g_{ab}(f,\theta)d\theta_a d\theta_b$ is the induced metric and (θ_2,θ_3) is any local coordinate system on Σ. We can find a good overview of the level set structure in [9, 37].

Observe that there is no open subset Ω of M^n where $\{\nabla f=0\}$ is dense. In fact, if f is constant in Ω and M^n is complete, we have that f is analytic, which implies f is constant everywhere. Thus, we consider Σ a connected component of the level surface $f^{-1}(c)$ (possibly disconnected) where c is any regular value of the function f. Suppose that I is an open interval containing c such that f has no critical points in the open neighborhood $U_I=f^{-1}(I)$ of Σ. For sake of simplicity, let $U_I\subset M\backslash\{f=0\}$ be a connected component of $f^{-1}(I)$. Then, we can make a change to the variables

$$r(x)=\int\frac{df}{|\nabla f|}$$

such that the metric g in U_I can be expressed by

$$g_{ij}=dr^2+g_{ab}(r,\theta)d\theta_a d\theta_b.$$

Let $\nabla r=\frac{\partial}{\partial r}$, then $|\nabla r|=1$ and $\nabla f=f'(r)\frac{\partial}{\partial r}$ on U_I. Note that $f'(r)$ does not change sign on U_I. Thus, we may assume $I=(-\varepsilon,\varepsilon)$ with $f'(r)>0$ for $r\in I$. Moreover, we have $\nabla_{\partial_r}\partial r = 0$.

The second fundamental form on Σ is given by

$$\begin{aligned} h_{ab} &= -\langle e_1, \nabla_a e_b\rangle = \frac{\nabla_a\nabla_b f}{|\nabla f|} \\ &= \frac{1}{|\nabla f|}\left(fR_{ab} - \frac{Rf}{2}g_{ab}\right) = \frac{f}{|\nabla f|}\left(\mu - \frac{R}{2}\right)g_{ab} = \frac{H}{2}g_{ab}, \end{aligned} \tag{3-11}$$

where $H = H(r)$, since H is constant in Σ. So, Σ is totally umbilic. In fact, contracting the Codazzi equation

$$R_{1cab} = \nabla_a h_{bc} - \nabla_b h_{ac}$$

over c and b, it gives

$$R_{1a} = \nabla_a(H) - \frac{1}{2}\nabla_a(H) = \frac{1}{2}\nabla_a(H).$$

On the other hand, since $R_{1a} = 0$, we conclude that H is constant in Σ.

For what follows, we fix a local coordinate system

$$(x_1, x_2, x_3) = (r, \theta_2, \theta_3)$$

in U_l, where (θ_2, θ_3) is any local coordinate system on the level surface Σ_c. Considering that $a, b, c, \dots \in \{2,3\}$, we have

$$h_{ab} = -g(\partial_r, \nabla_a\partial_b) = -g(\partial_r, \Gamma^1_{ab}\partial_r) = -\Gamma^1_{ab}.$$

Now, by definition

$$\Gamma^1_{ab} = \frac{1}{2}g^{11}\left(-\frac{\partial}{\partial r}g_{ab}\right) = \frac{-1}{2}\frac{\partial}{\partial r}g_{ab}.$$

Then,

$$\frac{\partial}{\partial r}g_{ab} = H(r)g_{ab}$$

implies that

$$g_{ab}(r,\theta) = \varphi(r)^2 g_{ab}(r_0,\theta),$$

where $\varphi(r) = e^{\left(\int_{r_0}^r H(s)ds\right)}$ and the level set $\{r = r_0\}$ corresponds to the connected component Σ of $f^{-1}(c)$.

Now, we can apply the warped product structure (see [5] and Section 1.2). Hence, considering

$$(M^3, g) = (I, dr^2) \times_\varphi (N^2, \overline{g}),$$

where $g = dr^2 + \varphi^2 \overline{g}$. The Ricci tensor of (M^3, g) is

$$R_{11} = -2\frac{\varphi''}{\varphi}, \qquad R_{1a} = 0 \tag{3-12}$$

and

$$R_{ab} = \overline{R}_{ab} - \left[(\varphi')^2 + \varphi\varphi''\right] \bar{g}_{ab} \qquad (a, b \in \{2, 3\}).$$

Since $\overline{R}_{ab} = \frac{\overline{R}}{2}\overline{g}_{ab}$,

$$R_{ab} = \left[\frac{\overline{R}}{2} - (\varphi')^2 - \varphi\varphi''\right] \overline{g}_{ab}.$$

On the other hand, since

$$R = \varphi^{-2}\overline{R} - 2\left(\frac{\varphi'}{\varphi}\right)^2 - 4\frac{\varphi''}{\varphi},$$

we get

$$\overline{R} = \varphi^2 R + 2(\varphi')^2 + 4\varphi\varphi''.$$

From $R = 2(\rho^2|\nabla f|^2 + \Lambda)$ we get

$$\overline{R} = 2\varphi^2\rho^2(f')^2 + 2(\varphi')^2 + 4\varphi\varphi'' + 2\varphi^2\Lambda. \tag{3-13}$$

Moreover, from the electrostatic system, we know that

$$\frac{1}{|\nabla f|^2}\langle\nabla|\nabla f|^2, \nabla f\rangle = 2f\left(R_{11} + 2\rho^2|\nabla f|^2 - \frac{R}{2}\right).$$

That is, from (1-30) and (3-12), we get

$$\langle\nabla|\nabla f|^2, \nabla f\rangle = 2f(f')^2\left[\rho^2(f')^2 - 2\frac{\varphi''}{\varphi} - \Lambda\right].$$

Hence, using that $\nabla f = f'\partial_r$, we obtain

$$2(f')^2 f'' = 2f(f')^2\left[\rho^2(f')^2 - 2\frac{\varphi''}{\varphi} - \Lambda\right].$$

So,

$$\rho^2 = \frac{1}{(f')^2}\left[\frac{f''}{f} + 2\frac{\varphi''}{\varphi} + \Lambda\right].$$

Combining the above identity with (3-13) we can conclude that $\overline{R}$ does not depend on θ. Therefore, $\overline{R}$ is a constant. Furthermore, from (3-11) we have

$$\frac{1}{2}|\nabla f|Hg_{ab} = \nabla_a\nabla_b f = f\left(R_{ab} + 2\rho^2\nabla_a f\nabla_b f - \frac{R}{2}g_{ab}\right).$$

Thus,

$$\left(\frac{1}{2}|\nabla f|H + \frac{Rf}{2}\right)\varphi^2\overline{g}_{ab} = fR_{ab}.$$

On the other hand,

$$fR_{ab} = f\left[\frac{\overline{R}}{2} - (\varphi')^2 - \varphi\varphi''\right]\overline{g}_{ab}.$$

Then,

$$f\left[\frac{1}{2}\varphi^2 R + \varphi\varphi''\right] = \left(\frac{1}{2}f'H + \frac{Rf}{2}\right)\varphi^2,$$

i.e.,

$$H = 2\frac{f}{f'}\frac{\varphi''}{\varphi}.$$

Since $\varphi = e^{\int_{r_0}^{r} H(s)ds}$, we have

$$\varphi' - 2\frac{f}{f'}\varphi'' = 0$$

which implies that

$$\varphi'(r) = c_1 f(r)^{1/2},$$

where $c_1 \in \mathbb{R}$.

Therefore, we can conclude the next results of this dissertation.

Theorem 3.12 *Let (M^3, g, f, E) be an electrostatic system such that the electric field and the gradient of the lapse function are linearly dependent. Suppose that the Bach tensor is divergence-free and $Q > 0$ (or $Q < 0$). If f is a proper function, around any regular point of f the manifold is locally a warped product with a one-dimensional* *base and fiber* $(N^2, \overline{g})$ *of constant curvature, i.e.,*

$$(M^3, g) = (I, dr^2) \times_{\varphi} (N^2, \overline{g}),$$

where $I \subset \mathbb{R}$ *and* $\varphi(r) = c_1 \int \sqrt{f(r)}dr + c_2$; c_1 *and* c_2 *are constants.*

It is important to point out that if M^3 is compact in Theorem 3.12, it is not necessary to ask for f to be a proper function.

Theorem 3.13 *Let* (M^3, g, f, E) *be a compact electrostatic system such that the electric field and the gradient of the lapse function are linearly dependent. Suppose that the Bach tensor is divergence-free and* $Q > 0$ *(or* $Q < 0$*). Then, around any regular point of* f *the manifold is locally a warped product with a one-dimensional base and fiber* $(N^2, \overline{g})$ *of constant curvature, i.e.,*

$$(M^3, g) = (I, dr^2) \times_{\varphi} (N^2, \overline{g}),$$

where $I \subset \mathbb{R}$ *and* $\varphi(r) = c_1 \int \sqrt{f(r)}dr + c_2$; c_1 *and* c_2 *are constants.*

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