UNIVERSIDADE FEDERAL DE GOIÁS (UFG)
INSTITUTO DE FÍSICA (IF)
PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA (PPGF)

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On the controversies of the Unruh effect

GOIÂNIA
2021

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JOÃO VÍTOR BARNEZ PIGNATA LEAL DINIZ

On the controversies of the Unruh effect

Dissertação apresentada ao Programa de Pós-Graduação em Física (PPGF), do Instituto de Física (IF), da Universidade Federal de Goiás (UFG), como requisito para obtenção do título de Mestre em Física.

Área de concentração: Física
Linha de pesquisa: Óptica Quântica e Informação Quântica

Orientador: Ardiley Torres Avelar
Coorientador: Emile Raymond Ferreira Taillebois

GOIÂNIA
2021

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Diniz, João Vitor Barnez Pignata Leal
On the controversies of the Unruh effect [manuscrito] / João Vitor Barnez Pignata Leal Diniz. - 2021.
46 f.: il.

Orientador: Prof. Dr. Ardiley Torres Avelar; co-orientador Dr. Emile Raymond Ferreira Taillebois.
Dissertação (Mestrado) - Universidade Federal de Goiás, Instituto de Física (IF), Programa de Pós-Graduação em Física, Goiânia, 2021.
Bibliografia. Apêndice.

1. Efeito Unruh. 2. Teoria quantica de campos. 3. Condições de contorno. I. Avelar, Ardiley Torres, orient. II. Título.

CDU 53

UNIVERSIDADE FEDERAL DE GOIÁS

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Ata nº 196 da sessão de Defesa de Dissertação de João Vítor Barnez Pignata Leal Diniz, que confere o título de Mestre em Física, na área de concentração em Física.

Aos 16 dias do mês de agosto de 2021, a partir das 09h00min, por meio de videoconferência, realizou-se a sessão pública de Defesa de Dissertação intitulada “On the controversies of the Unruh effect”. Os trabalhos foram instalados pelo Orientador, Professor Doutor Ardiley Torres Avelar (IF/UFG), com a participação dos demais membros da Banca Examinadora: Professor Doutor George Emanuel Avraam Matsas (IFT/UNESP), membro titular externo; e Professor Doutor Lucas Chibebe Céleri (IF/UFG), membro titular interno. Durante a arguição, os membros da banca não fizeram sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Dissertação, tendo sido o candidato APROVADO pelos seus membros. Proclamados os resultados pelo Professor Doutor Ardiley Torres Avelar, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos membros da Banca Examinadora, aos 16 dias do mês de agosto de 2021.

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Resumo

Nos 50 anos desde proposto, o efeito Unruh tem sido discutido extensivamente na literatura, com vários resultados teóricos apoiando sua existência e alguns questionando se é de fato observado. O efeito Unruh afirma que observadores com aceleração própria constante no espaço tempo de Minkowski responderão como se interagindo com um banho térmico a uma temperatura proporcional à aceleração. Para acelerações comuns essa temperatura é muito baixa, tornando difícil verificações experimentais. Para entender esse efeito e a discussão que o acompanha, revisamos alguns resultados importantes da teoria quântica de campos e mostramos uma derivação detalhada do efeito Unruh para campos com e sem massa, incluindo o cálculo dos coeficientes de Bogoliubov para esses problemas. Além disso, acompanhamos uma discussão sobre o papel das condições de contorno nesse problema, mostrando que podem levar a uma descrição incompleta do campo no espaço tempo de Minkowski quando escrito em termos dos modos no espaço tempo de Rindler.

PALAVRAS CHAVE: Efeito Unruh, Teoria quântica de campos, Condições de contorno.

Abstract

In the 50 years since it was first proposed, the Unruh effect has been discussed extensively in literature, with many theoretical results supporting its existence and some questioning whether it is actually observed. The Unruh effect states that observers with constant proper acceleration in Minkowski spacetime will respond as if interacting with a thermal bath at a temperature proportional to the acceleration. For common accelerations this temperature is very low, so experimental confirmation has been challenging. To understand the effect and the discussion around it, we review some important results from quantum field theory and work out a detailed derivation of the Unruh effect for massless and massive fields, including the calculation of the Bogoliubov coefficients for these problems. Furthermore, we follow a discussion on the role that the boundary conditions play in this effect, showing that they may lead to an incomplete description of the field in Minkowski spacetime when written in terms of the modes in Rindler spacetime.

KEYWORDS: Unruh effect, Quantum field theory, Boundary conditions.

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Introduction

The Unruh effect was first proposed almost 50 years ago [1] and has been discussed extensively in literature, with many theoretical results supporting its existence [2, 3, 4, 5, 6, 7, 8, 9] and some questioning whether it is actually observed or not [10, 11, 12, 13, 14]. Some of these show that the Unruh effect is very important for the consistency of quantum field theory, providing strong evidence that the effect must exist, at least in some form [15].

The Unruh effect states that observers with constant proper acceleration a in Minkowski spacetime detects the vacuum state as a thermal bath at temperature $T = a/2\pi$ [2]. The absence of additional hypothesis regarding the physics of the detector indicates that its characteristics are not important: the effect is *universal*.

Now, uniformly accelerated observers are particularly interesting once we consider Einstein's equivalence principle [16], indicating that those observers connect with observers under the influence of gravity. As there is still no satisfactory understanding of quantum gravity, accelerated observers may hint at what might be the nature of this fundamental interaction.

In this master's dissertation we show a derivation of the Unruh effect and bring to light some of the points that make the Unruh effect an ongoing discussion. This is by no means an exhaustive review on this long going discussion, but hopefully may serve as a stepping stone in a challenging subject.

We start our discussion with a review of some important aspects of quantum field theory, showing how this theory can lead to surprising results, one of the most curious being the Unruh effect.

Chapter 2 shows a derivation of the Unruh effect for massless fields in two dimensional spacetime. The simplifications brought here allows us to focus on certain crucial details that may be overlooked otherwise. In particular, we discuss the connection between different description of the quantum field as seen by inertial and accelerated observers.

The next chapter is dedicated to a detailed derivation of the Unruh effect in four dimensional spacetime for massive fields, including the quantization in both Minkowski and Rindler spacetimes. We also calculate the Bogoliubov coefficients, connecting the field description in both spacetimes, which eventually leads to the Unruh effect.

The fourth chapter follows a discussion regarding some implicit boundary conditions and what their implications may be. To simplify the calculations, we will use an equivalent two dimensional model, and it is useful to review the quantization schemes and introduce a new one for Minkowski spacetime, with an interesting application to the discussion.

Finally, we discuss our findings and evaluate the Unruh effect for a reasonable value of acceleration, showing that the effect is extremely small, thus very difficult to measure experimentally. Moreover, in the appendix we show a derivation for the density matrix of a boson gas, discuss in details the trajectory of accelerated detectors, and show the connection between black-holes and the Unruh effect.

Chapter 1

A brief review

To satisfy the requirements of both relativity and quantum theory, we need to abandon some comfortable notions that work fine in less demanding contexts, such as a universal time coordinate or a unique zero energy (vacuum) state. In this chapter we review some important background concepts related to the Unruh effect, a significant one being that the vacuum state may have drastically different properties when perceived by different observers.

Before we proceed, it should be made clear that we are using natural units, in which $G = k_B = \hbar = c = 1$. We will also use spacetime metrics with signature $(+, -, -, -)$. Under these considerations, the metric for Minkowski spacetime reads

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{1.1}$$

1.1 Classical field theory

Before heading into quantum field theory and the Unruh effect, we should understand how classical fields are promoted to quantum operators, so it is interesting to review some concepts from classical field theory. In analogy with classical mechanics, we define the system in terms of its Hamiltonian or Lagrangian, which are commonly written as integrals of Hamiltonian or Lagrangian densities [17]:

$$H = \int_{\mathbb{R}^3} d^3x \; \mathcal{H} \qquad \text{and} \qquad L = \int_{\mathbb{R}^3} d^3x \; \mathcal{L}. \tag{1.2}$$

The Lagrangian density is defined as a functional of the fields ϕ and their conjugate momenta π, and the Hamiltonian density is then defined as the Legendre transform of the Lagrangian density.

As in classical mechanics, in most cases it is possible to identify the Hamiltonian as the sum of potential and kinetic energy, or the total system energy, and the Lagrangian as their difference. The Hamiltonian is then a constant of motion, and for that reason it is commonly used in non-relativistic systems. The Lagrangian, however, is not a constant of motion, but it is Lorentz invariant, and therefore much more convenient when working with relativistic systems.

To obtain the equations of motion, we define the action, S, as the integral of the Lagrangian over time, or of the Lagrangian density over the entire spacetime:

$$S = \int_{\mathbb{R}} dt\ L = \int_{\mathbb{R}^4} d^4x\ \mathcal{L}. \tag{1.3}$$

Minimizing the action we obtain the Euler-Lagrange equations, which describes the dynamics of the field [17]:

$$\frac{\partial}{\partial \phi}\mathcal{L} - \nabla_\mu \frac{\partial}{\partial(\nabla_\mu \phi)}\mathcal{L} = 0, \tag{1.4}$$

where we have used the Einstein summation convention, which will be used throughout this text. In particular, we are interested in Lagrangian densities that may be written as

$$\mathcal{L} = \sqrt{-g}\ (\nabla_\mu \phi\ \nabla^\mu \phi + m^2\phi^2)/2, \tag{1.5}$$

with g being the metric determinant and ∇_μ indicating the covariant derivative. Lagrangian densities like this leads to equations of motion given by the minimally coupled Klein-Gordon equation

$$\left(\nabla_\mu \nabla^\mu + m^2\right)\phi = 0. \tag{1.6}$$

Given two solutions to this last equation, f_A and f_B, we define the Klein-Gordon current as

$$J^\mu_{(f_A, f_B)}(x) = f_A^*(x)\ \nabla^\mu f_B(x) - f_B(x)\ \nabla^\mu f_A^*(x), \tag{1.7}$$

and it is simple to show that $\nabla_\mu J^\mu_{(f_A,f_B)} = 0$:

$$\begin{aligned}
\nabla_\mu J^\mu_{(f_A,f_B)} &= \nabla_\mu f_A^*(x)\ \nabla^\mu f_B(x) + f_A^*(x)\ \nabla_\mu \nabla^\mu f_B(x) - \nabla_\mu f_B(x)\ \nabla^\mu f_A^*(x) \\
&\qquad - f_B(x)\ \nabla_\mu \nabla^\mu f_A^*(x) \\
&= f_A^*(x)\ \nabla_\mu \nabla^\mu f_B(x) - f_B(x)\ \nabla_\mu \nabla^\mu f_A^*(x) \\
&= -m^2 f_A^*(x)\ f_B(x) + m^2 f_B(x)\ f_A^*(x) = 0.
\end{aligned} \tag{1.8}$$

On the first line we simply expand the divergence using the usual product rule. From the first to the second line we write $\nabla_\mu f_A^*(x) = g_{\mu\nu}\nabla^\nu f_A^*(x)$ and similarly for $\nabla_\mu f_B(x)$, then, with a simple change of labels on the indices and using the metric symmetry $g_{\mu\nu} = g_{\nu\mu}$, we cancel the first and third terms. On the second line we use the Klein-Gordon equation

(1.6) on both terms, that are finally canceled out in the third line.

The fact that $\nabla_\mu J^\mu_{(f_A,f_B)} = 0$ implies the existence of a conserved quantity, namely, the Klein-Gordon inner product:

$$(f_A,\ f_B)_{KG} = i\int_\Sigma d^3X\sqrt{G}\ \eta_\mu J^\mu_{(f_A,f_B)}, \tag{1.9}$$

where d^3X is the integration element for the spatial coordinates X, G is the metric associated with these coordinates, Σ is a hypersurface with constant time coordinate, and η_μ is the future-oriented unit vector normal to the hypersurface Σ. Using Stokes's theorem it can be shown that this inner product is time independent (see Appendix E from reference [18]).

We will be working with metrics of the form

$$ds^2 = N(x)^2 dt^2 - G_{ab}(x)\ dx^a dx^b. \tag{1.10}$$

It should be noted that the indices a and b run only through the spatial coordinates, and that every spacetime discussed in this text has metric of this form. These metrics describe globally hyperbolic spacetimes, which can be understood as spacetimes with well defined causality. This is particularly important because it allows us to obtain the behavior of a field for all time once we know the equations of motion and the field's state at some initial time.

For metrics like the one in equation (1.10), the Klein-Gordon inner product is written as

$$(f_A,\ f_B)_{KG} = i\int_\Sigma d^3X\sqrt{G}N^{-1}(f_A^*\ \partial_t f_B - f_B\ \partial_t f_A^*), \tag{1.11}$$

and the conjugate momentum, defined as $\pi = \partial\mathcal{L}/\partial\dot{\phi}$, is given by

$$\pi(x) = N^{-1}\sqrt{G}\ \partial_t\phi(x). \tag{1.12}$$

We now define the momenta p_A and p_B, corresponding to the fields f_A and f_B, respectively, as $p_j(x) = N^{-1}\sqrt{G}\ \partial_t f_j(x)$, and rewrite equation (1.11) as

$$(f_A,\ f_B)_{KG} = i\int_\Sigma d^DX\{f_A^*(x)\ p_B(x) - f_B(x)\ p_A^*(x)\}. \tag{1.13}$$

1.2 Quantum field theory

As classical field theory provides a framework to work with classical fields, quantum field theory does the same for quantum fields. The canonical field quantization proceeds as follows.

We denote by $\hat{\phi}$ and $\hat{\pi}$ the operators corresponding to the fields ϕ and π, respectively, and impose the canonical equal time commutation relations

$$[\hat{\phi}(t,X),\ \hat{\pi}(t,X')] = i\delta(X-X'), \tag{1.14}$$

$$[\hat{\phi}(t,X),\ \hat{\phi}(t,X')] = [\hat{\pi}(t,X),\ \hat{\pi}(t,X')] = 0. \tag{1.15}$$

For two arbitrary solutions to the Klein-Gordon equation, f_A and f_B, we have

$$\begin{aligned}[(f_A,\ \hat{\phi})_{KG},\ (\hat{\phi},\ f_B)_{KG}] = -\int_\Sigma d^3X \int_\Sigma d^3X' \\ \times [f_A^*(t,X)\ \hat{\pi}(t,X) - \hat{\phi}(t,X)\ p_A^*(t,X),\ \hat{\phi}^\dagger(t,X')\ p_B(t,X') - f_B(t,X')\ \hat{\pi}^\dagger(t,X')],\end{aligned} \tag{1.16}$$

where the integrand may be simplified using (1.14), being reduced to

$$[...] = -i\delta(X-X')\ \{f_A^*(t,X)\ p_B(t,X') - f_B(t,X')\ p_A^*(t,X)\}. \tag{1.17}$$

Thus equation (1.16) becomes

$$\begin{aligned}[(f_A,\ \hat{\phi})_{KG},\ (\hat{\phi},\ f_B)_{KG}] &= \int_\Sigma d^3X \int_\Sigma d^3X' \\ &\quad \times i\delta(X-X')\ \{f_A^*(t,X)\ p_B(t,X') - p_A^*(t,X)\ f_B(t,X')\} \\ &= i\int_\Sigma d^3X\{f_A^*(t,X)\ p_B(t,X) - f_B(t,X)\ p_A^*(t,X)\} \\ &= (f_A,\ f_B)_{KG}.\end{aligned} \tag{1.18}$$

We now assume that there exists a complete set of solutions to the Klein-Gordon equation (1.6), $\{f_i, f_i^*\}$, satisfying the orthonormalization conditions

$$(f_i,\ f_j)_{KG} = -(f_i^*,\ f_j^*)_{KG} = \delta_{ij} \qquad (f_i^*,\ f_j)_{KG} = (f_i,\ f_j^*)_{KG} = 0 \tag{1.19}$$

where, for simplicity, it was assumed a discrete index. Expressions for the general case may be obtained by replacing the following sums by the appropriate integrals, and the inner products of equation (1.19) by the appropriate delta function. In Minkowski spacetime we choose f_i as the positive frequency modes, however, in more general spacetimes there may not always be a natural choice [15].

Since the modes $\{f_i, f_i^*\}$ form a complete set of solutions to the Klein-Gordon equation, we may expand the field in this base:

$$\hat{\phi} = \sum_i \left[(f_i,\ \hat{\phi})_{KG}\ f_i + (\hat{\phi},\ f_i)_{KG}\ f_i^* \right] = \sum_i \left[\hat{a}_i f_i + \hat{a}_i^\dagger f_i^* \right], \tag{1.20}$$

where we implicitly defined the operators $\hat{a}_i = (f_i,\ \hat{\phi})_{KG}$ and $\hat{a}_i^\dagger = (\hat{\phi},\ f_i)_{KG}$. Notice that these operators are time independent and can be obtained by the inner product between the modes f_i and f_i^* and the field $\hat{\phi}$ at some starting time t_0, where the field operator $\hat{\phi}$ at $t = t_0$ is known from the initial conditions.

With the help of equations (1.18) and (1.19) we obtain the commutation relations for the operators $\hat{a}$ and $\hat{a}^\dagger$:

$$[\hat{a}_i,\ \hat{a}_j] = [\hat{a}_i^\dagger,\ \hat{a}_j^\dagger] = 0, \qquad [\hat{a}_i,\ \hat{a}_j^\dagger] = \delta_{ij}, \tag{1.21}$$

and we identify them with the *annihilation* and *creation* operators, respectively. The vacuum state is now implicitly defined by $\hat{a}_i |vac\rangle = 0$, for every operator $\hat{a}_i$, and the Fock space may be obtained by successive applications of the creation operators $\hat{a}_i^\dagger$ to the vacuum state.

1.2.1 Bogoliubov transformation

We now assume that there is a second complete set of solutions to the Klein-Gordon equation, $\{f_I, f_I^*\}$, satisfying relations similar to those in equation (1.19). The completeness of this set allows us to expand the field $\hat{\phi}$ in terms of either set:

$$\hat{\phi} = \sum_i \left[\hat{a}_i f_i + \hat{a}_i^\dagger f_i^* \right] = \sum_I \left[\hat{a}_I f_I + \hat{a}_I^\dagger f_I^* \right], \tag{1.22}$$

where the operators $\hat{a}_I$ and $\hat{a}_I^\dagger$ are defined in a similar manner to the operators $\hat{a}_i$ and $\hat{a}_i^\dagger$, and thus satisfy equivalent commutation relations.

Since both sets are complete, we may write the elements from one as a linear combination of the elements from the other:

$$f_I = \sum_i \alpha_{Ii} f_i + \beta_{Ii} f_i^*, \qquad f_I^* = \sum_i \alpha_{Ii}^* f_i^* + \beta_{Ii}^* f_i, \tag{1.23}$$

and using the relations in equation (1.19) we obtain the coefficients α_{Ii} and β_{Ii}:

$$\alpha_{Ii} = (f_i,\ f_I)_{KG} = (f_I,\ f_i)^*_{KG}, \qquad \beta_{Ii} = -(f_i^*,\ f_I)_{KG} = (f_I^*,\ f_i)^*_{KG}. \tag{1.24}$$

After substituting the expressions for f_I and f_I^* in terms of the functions f_i and f_i^* in the field expansion, we compare the coefficients of each mode f_i and f_i^*, and obtain a relation between the annihilation and creation operators associated to the modes in different sets:

$$\hat{a}_i = \sum_I \alpha_{Ii}\hat{a}_I + \beta^*_{Ii}\hat{a}^\dagger_I \qquad \hat{a}_I = \sum_i \alpha^*_{Ii}\hat{a}_i - \beta^*_{Ii}\hat{a}^\dagger_i. \tag{1.25}$$

These are known as Bogoliubov transformations, and the coefficients α_{Ii} and β_{Ii} are the Bogoliubov coefficients.

1.2.2 Vacuum states

To see how the Bogoliubov relates to the Unruh effect, we consider the vacuum states associated the annihilation and creation operators of modes in the different sets, which are defined by

$$\hat{a}_i \left|vac^i\right\rangle = 0, \qquad \hat{a}_I \left|vac^I\right\rangle = 0, \tag{1.26}$$

and look at the expectation value of the number operator, $\hat{N}_i = \hat{a}^\dagger_i\hat{a}_i$, on the state $\left|vac^I\right\rangle$:

$$\begin{aligned}
\left\langle vac^I\right| \hat{a}^\dagger_i\hat{a}_i \left|vac^I\right\rangle &= \sum_{I,J} \left\langle vac^I\right| (\alpha^*_{Ii}\hat{a}^\dagger_I + \beta_{Ii}\hat{a}_I)(\alpha_{Ji}\hat{a}_J + \beta^*_{Ji}\hat{a}^\dagger_J) \left|vac^I\right\rangle \\
&= \sum_{I,J} \left\langle vac^I\right| (\beta_{Ii}\hat{a}_I)(\beta^*_{Ji}\hat{a}^\dagger_J) \left|vac^I\right\rangle \\
&= \sum_{I,J} \beta_{Ii}\beta^*_{Ji} \left\langle vac^I\right| \hat{a}_I\hat{a}^\dagger_J \left|vac^I\right\rangle .
\end{aligned} \tag{1.27}$$

Where we have used the vacuum state definition to eliminate the terms associated with the α_{Ii} coefficients.

Using the commutation relation for the operators $\hat{a}_I$ and $\hat{a}^\dagger_J$, it is easy to see that the expectation value inside the sum in the previous equation reduces to δ_{IJ}, from where it follows

$$\left\langle vac^I\right| \hat{N}_i \left|vac^I\right\rangle = \sum_I |\beta_{Ii}|^2, \tag{1.28}$$

and with a similar derivation for the expectation value of $\hat{N}_i$ in respect to the vacuum associated with the set I we have:

$$\left\langle vac^i \right| \hat{N}_I \left| vac^i \right\rangle = \sum_i |\beta_{Ii}|^2. \tag{1.29}$$

From these calculations it is clear that the states $\left| vac^i \right\rangle$ and $\left| vac^I \right\rangle$ are distinct if the coefficients β_{Ii} are non zero.

Although generally there is not a unique choice of the vacuum state, for spacetimes with metric in the form of equation (1.10) with the functions $N(x)$ and $G_{ab}(x)$ independent of time, the Klein-Gordon equation becomes

$$\partial_t^2 f_i = N G^{-\frac{1}{2}} \partial_a \left(N G^{\frac{1}{2}} G^{ab} \partial_b f_i \right) - N^2 m^2 f_i, \tag{1.30}$$

and is natural to choose the positive frequency modes, f_i, to depend on time with the form $exp(-i\omega_i t)$, where ω_i can be understood as the energy of the particle with respect to the Killing vector ∂_t ‡. This choice leads to a well defined vacuum state, which satisfies time translation symmetry [15]. Such state is referred to as *static vacuum.*

‡Killing vectors are vector fields associated with metric symmetries [18], which relate to conserved quantities. They may be understood as translation generators. For instance, in Minkowski spacetime, the vector ∂_t is the time translation generator, with the energy being the related conserved quantity. Another example is the vector $z\partial_t + t\partial_z$, the *boost* generator in the z direction, related to some notion of mass center conservation.

Chapter 2

The massless scalar field

In this chapter we derive the Unruh effect for a massless scalar field in two $(1+1)$ dimensional spacetime. Although we will do the same calculations for the massive scalar field in four dimensional spacetime, the simpler calculations here may give a clearer idea on the fundamental concepts of the Unruh effect. In particular, it should be easier to understand how to extend the modes associated with the Rindler spacetimes into the whole Minkowski spacetime, a critical step in the calculation of the Bogoliubov coefficients and the Unruh effect.

2.1 Massless scalar field in Minkowski spacetime

Roughly following the procedure laid out in the previous chapter, we start by evaluating the Klein-Gordon equation in Minkowski spacetime, which reads

$$\left(\partial_t^2 - \partial_z^2\right)\,\hat{\Phi} = 0. \tag{2.1}$$

The solutions to this equation are plane waves, which we use to write the most general solution in integral form:

$$\hat{\Phi} = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi k}} \left[\hat{b}_k\; e^{-i|k|t+ikz} + \hat{b}_k^\dagger\; e^{i|k|t-ikz}\right]. \tag{2.2}$$

We now split the integral at $k = 0$ and change the integration variable signal in the negative integral, which leads to

$$\hat{\Phi} = \int_{0}^{\infty} \frac{dk}{\sqrt{4\pi k}} \left[\hat{b}_{-k}\; e^{-ik(t-z)} + \hat{b}_{+k}\; e^{-ik(t+z)} + H.c.\right]. \tag{2.3}$$

With the variables change $U = t - z$ and $V = t + z$, the field can be rewritten as

$$\hat{\Phi}(t,z) = \hat{\Phi}_-(U) + \hat{\Phi}_+(V), \tag{2.4}$$

that is, it can be factored into two non interacting sectors, one moving to the left, the other to the right. For that reason, we may focus on the left moving sector, which reads

$$\hat{\Phi}_+(V) = \int_0^\infty dk \left[\hat{b}_{+k}\ f_{+k}(V) + \hat{b}^\dagger_{+k}\ f^*_{+k}(V)\right], \tag{2.5}$$

with the modes $f_{+k}(V)$ satisfying

$$f_{+k}(V) = (4\pi k)^{-\frac{1}{2}}\ e^{-ikV}. \tag{2.6}$$

2.2 Rindler spacetime

Minkowski spacetime is invariant under *boost* transformations, motivating the following coordinates change:

$$t = \rho \sinh \eta, \qquad\qquad z = \rho \cosh \eta, \tag{2.7}$$

where ρ is a positive real number and η any real number. The metric is then written as

$$ds^2 = \rho^2 d\eta^2 - d\rho^2 - dx^2 - dy^2. \tag{2.8}$$

It should be noted that the coordinates ρ and η map only the region $|t| < z$ of Minkowski spacetime, also known as the *right Rindler wedge*. With appropriate signal changes to the right side of equation (2.7) we obtain a map to the region $|t| < -z$, or the *left Rindler wedge*, with metric identical to the one in equation (2.8).

Maps for the remaining regions of Minkowski spacetime, named *expanding* and *contracting degenerate Kasner universes*, may be obtained with a similar coordinates change:

$$t' = \pm\rho \cosh \eta, \qquad\qquad z' = \rho \sinh \eta. \tag{2.9}$$

However, this leads to a time dependent metric, describing a non static spacetime, where there is not a natural vacuum [15]. Therefore, the fields in the *degenerate Kasner universes* do not play an important role in the following discussion.

With an appropriate choice for the starting conditions, the trajectory of a point particle with constant proper acceleration a along the z axis may be obtained from special relativity calculations (check appendix B), and is given by

$$t(\tau) = \frac{1}{a}\sinh(a\tau), \qquad\qquad z(\tau) = \frac{1}{a}\cosh(a\tau). \tag{2.10}$$

Comparing with the transformation given by equation (2.7), it follows that an observer at rest in Rindler spacetime, with trajectory (η, ρ_0, x_0, y_0), corresponds to an observer in Minkowski spacetime with constant proper acceleration ρ_0^{-1} in the z direction, and the relation between Rindler spacetime and accelerated observers becomes clear.

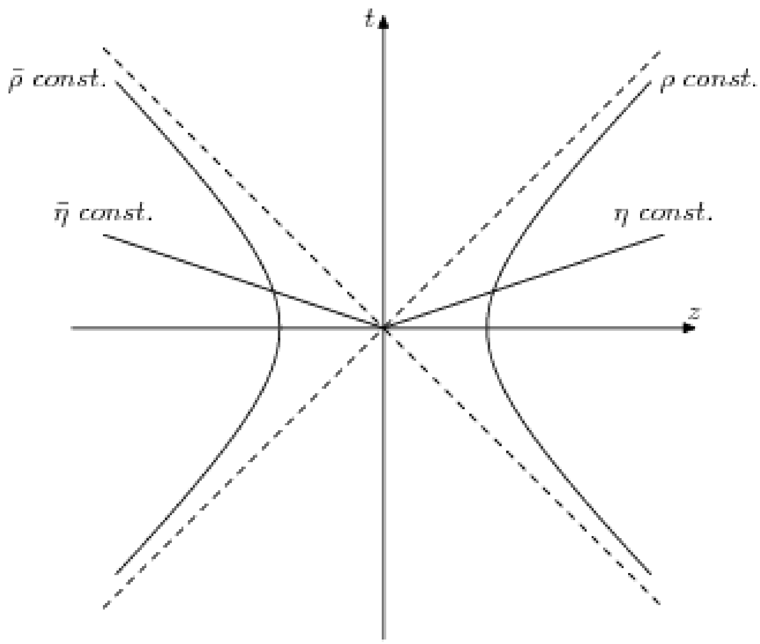

Figure 2.1: Rindler coordinates system for both Rindler wedges.

The map for the left Rindler wedge is obtained by taking $z \to -z$ and $(\eta, \rho) \to (\bar{\eta}, \bar{\rho})$ in equation (2.7). Figure 2.2 shows the lines with constant ρ, η, $\bar{\rho}$ and $\bar{\eta}$ in that equation, describing the Rindler wedges in terms of these variables. Notice that the dashed lines indicate the "light cone", expressed mathematically as $|t| = |z|$ or $\rho = \bar{\rho} = 0$.

2.3 Massless scalar field in Rindler spacetime

To obtain a description of the field in Rindler spacetime we introduce the following coordinates change, with ξ and τ accepting any real value:

$$t = a^{-1} e^{a\xi} \sinh a\tau, \qquad z = a^{-1} e^{a\xi} \cosh a\tau. \tag{2.11}$$

This transformation is similar to the one used to map Rindler spacetime, equation (2.7), which may be obtained by setting

$$\rho = a^{-1} e^{a\xi} \quad \text{and} \quad \eta = a\tau. \tag{2.12}$$

Notice that this transformation maps the entire right Rindler wedge and only that, and that taking $\xi = 0$ gives the trajectory of an object in Minkowski spacetime with constant proper acceleration a.

The metric then takes the form

$$ds^2 = e^{2a\xi}\left(d\tau^2 - d\xi^2\right) - dx^2 - dy^2 \tag{2.13}$$

and the Klein-Gordon equation for the massless scalar field reads

$$\left(\partial_\tau^2 - \partial_\xi^2\right)\hat{\Phi} = 0, \tag{2.14}$$

which is the same equation as in Minkowski spacetime, with a simple variable label change. As was the case in Minkowski spacetime, we can separate the field into two non interacting sectors, one depending on $u = \tau - \xi$ and the other on $v = \tau + \xi$:

$$\hat{\Phi}(\tau, \xi) = \hat{\Phi}_-(u) + \hat{\Phi}_+(v), \tag{2.15}$$

with the fields $\hat{\Phi}_\pm(v|u)$ satisfying

$$\hat{\Phi}_\pm(v|u) = \int_0^\infty d\omega\left[\hat{a}_{\pm\omega}\; f_{\pm\omega}(v|u) + \hat{a}^\dagger_{\pm\omega}\; f^*_{\pm\omega}(v|u)\right]. \tag{2.16}$$

The variables U, V, u and v are related by the following expressions

$$U = -a^{-1}e^{-au}, \qquad\qquad V = a^{-1}e^{av}, \tag{2.17}$$

which can be used to write v in terms of V as $v(V) = a^{-1}\ln(aV)$. We now write the modes $f_{+\omega}$ as functions of V, obtaining

$$\begin{aligned} f_\omega(v(V)) &= (4\pi\omega)^{-\frac{1}{2}} e^{-i\omega\ln(aV)/a} \\ &= (4\pi\omega)^{-\frac{1}{2}}\,(aV)^{-i\omega/a}. \end{aligned} \tag{2.18}$$

It's important to remember that this expression is only valid in the right Rindler wedge, that is, for $V > 0$ and $U < 0$. However, this is an analytic function for $V > 0$, so there is no reason not to take this as the definition of the modes $f_{+\omega}$ for all $V > 0$, or both the right Rindler wedge and the expanding Kasner universe ($t > |z|$). As such, we can write the field expansion for this region of Minkowski spacetime

$$\hat{\Phi}_+(V > 0) = \int_0^\infty d\omega\left[\hat{a}_{+\omega}\; f_{+\omega}(v) + \hat{a}^\dagger_{+\omega}\; f^*_{+\omega}(v)\right]. \tag{2.19}$$

Similarly, the modes associated with the field $\hat{\Phi}_-$, also originally defined only on the right Rindler wedge, can be extended to the whole $U < 0$ region, that is, the right Rindler wedge and the contracting Kasner universe ($t < -|z|$).

For the left Rindler wedge, we use the map

$$t = a^{-1} e^{a\bar{\xi}} \sinh a\bar{\tau} \qquad\qquad z = -a^{-1} e^{a\bar{\xi}} \cosh a\bar{\tau}, \tag{2.20}$$

,

where the variables $\bar{v} = \bar{\tau} + \bar{\xi}$ and V are related by $V = -a^{-1} e^{-a\bar{v}}$, and similarly for $\bar{u}$ and U. Following the same procedure from the right wedge, we find two non interacting sectors, depending on the variables $\bar{u}$ or $\bar{v}$. Once again we extend the modes of each sector to an entire half of the Minkowski spacetime, and thus may write

$$\hat{\Phi}_+(V < 0) = \int_0^\infty d\omega \left[\hat{a}_{+\omega}\ f_{+\omega}(\bar{v}) + \hat{a}^\dagger_{+\omega}\ f^*_{+\omega}(\bar{v})\right]. \tag{2.21}$$

We now have two sets of modes, $f_{+\omega}(v)$ and $f_{+\omega}(\bar{v})$, each covering half the Minkowski spacetime, and may write an expression for the field that is valid over the whole Minkowski spacetime [1]:

$$\hat{\Phi}_+(V) = \int_0^\infty d\omega \left[\hat{a}_{+\omega}\ \theta(V)\ f_{+\omega}(v) + \hat{a}_{+\omega}\ \theta(-V)\ f_{+\omega}(\bar{v}) + H.c.\right], \tag{2.22}$$

where we have used the Heaviside function $\theta(x)$ to select the appropriate contribution when evaluating the field modes in each half of the Minkowski spacetime.

2.4 The Bogoliubov coefficients

Knowing both field descriptions, we are in a position to calculate the Bogoliubov coefficients. We do that by writing the modes from one space as a linear combination of the modes from another space, as indicated by equation (1.23), and find

$$\theta(V)\ f_\omega(v) = \int_0^\infty \frac{dk}{\sqrt{4\pi k}} \left(\alpha^R_{\omega k}\ e^{-ikV} + \beta^R_{\omega k}\ e^{ikV}\right), \tag{2.23}$$

$$\theta(-V)\ f_\omega(\bar{v}) = \int_0^\infty \frac{dk}{\sqrt{4\pi k}} \left(\alpha^L_{\omega k}\ e^{-ikV} + \beta^L_{\omega k}\ e^{ikV}\right). \tag{2.24}$$

These last equations are just Fourier integrals, so the Bogoliubov coefficients may be obtained by multiplying the equations by $e^{\pm ikV}$ and then integrating over V. For $\alpha^R_{\omega k}$ we find

$$
\begin{aligned}
\alpha^R_{\omega k} &= \frac{\sqrt{4\pi k}}{2\pi}\int_0^\infty f_\omega(v(V))\ e^{ikV} dV \\
&= \frac{\sqrt{k/\omega}}{2\pi}\int_0^\infty (aV)^{-i\omega/a}\ e^{ikV} dV \\
&= \frac{1}{2\pi\sqrt{k\omega}}\left(\frac{a}{k}\right)^{-i\omega/a}\int_0^\infty x^{-i\omega/a}\ e^{ix}\ dx.
\end{aligned} \tag{2.25}
$$

For this integral to converge we must introduce a small imaginary part to x by setting $x \to x + i\epsilon$, with $\epsilon \to 0^+$. The integral then gives

$$
\begin{aligned}
\int_0^\infty x^{-i\omega/a}\ e^{ix}\ dx &= i\ (-i)^{i\omega/a}\ \Gamma(1 - i\omega/a) \\
&= i\ e^{\pi\omega/2a}\ \Gamma(1 - i\omega/a),
\end{aligned} \tag{2.26}
$$

and we obtain the first set of Bogoliubov coefficients. An identical procedure may be performed to obtain the remaining coefficients. In explicit form, they read

$$
\alpha^R_{\omega k} = i\beta^L_{-\omega,k} = \frac{i\ e^{\pi\omega/2a}}{2\pi\sqrt{k\omega}}\ \Gamma(1 - i\omega/a), \tag{2.27}
$$

$$
\beta^R_{\omega k} = i\alpha^L_{-\omega,k} = -\frac{i\ e^{-\pi\omega/2a}}{2\pi\sqrt{k\omega}}\ \Gamma(1 - i\omega/a). \tag{2.28}
$$

2.5 The Unruh Effect for massless fields

An inspection of equations (2.27) and (2.28) shows us that the Bogoliubov coefficients satisfy the relations

$$
\beta^L_{\omega k} = -e^{-\pi\omega/a}\alpha^{R*}_{\omega k}, \qquad \beta^R_{\omega k} = -e^{-\pi\omega/a}\alpha^{L*}_{\omega k}. \tag{2.29}
$$

These relations are actually more important than the explicit form of the coefficients, as they will also appear in the derivation for the massive scalar field, with equivalent consequences.

The relations in equation (2.29) are now used to write the Rindler spacetime field expansion in terms of the modes in Minkowski spacetime, where we notice that the following expressions must be satisfied

$$
(\hat{a}^R_{+\omega} - e^{-\pi\omega/a}\ \hat{a}^{L\dagger}_{+\omega})\left|vac^M\right\rangle = 0, \qquad (\hat{a}^L_{+\omega} - e^{-\pi\omega/a}\ \hat{a}^{R\dagger}_{+\omega})\left|vac^M\right\rangle = 0. \tag{2.30}
$$

Together with the canonical commutation relations for the creation and annihilation operators, these last equations lead to a very important result:

$$(\hat{a}^{R\dagger}_{+\omega}\hat{a}^{R}_{+\omega} - \hat{a}^{L\dagger}_{+\omega}\hat{a}^{L}_{+\omega}) \left|vac^M\right\rangle = 0, \tag{2.31}$$

that is, there must be the same number of excitations on both Rindler wedges, which means the Minkowski vacuum state can be written in terms of the Rindler vacuum state and the creation operators as

$$\left|vac^M\right\rangle \propto \prod_{\omega} \sum_{n_\omega \geq 0} \frac{K_{n_\omega}}{n_\omega!} (\hat{a}^{R\dagger}_{\omega}\hat{a}^{L\dagger}_{\omega})^{n_\omega} \left|vac^R\right\rangle, \tag{2.32}$$

where we have used an approximation for discrete ω. For a more general verification of the thermal state, one should check whether the state satisfies the Kubo-Martin-Schwinger equilibrium condition or not [19]. However, for the discussion here it suffices to introduce an approximation for discrete ω, explicitly calculate the state's density operator, and compare it with the density operator for a relativistic boson gas (check appendix A).

With further manipulation of these results, we find the coefficients K_{n_ω} and construct the density operator corresponding to the Minkowski vacuum seen by an observer with constant proper acceleration:

$$\hat{\rho}_M = \left|vac^M\right\rangle\!\left\langle vac^M\right| = \prod_{\omega} \left(K_0^2(\omega) \sum_{n_\omega \geq 0} e^{-2\pi\omega n_\omega/a} \left|n_\omega\right\rangle\!\left\langle n_\omega\right| \right), \tag{2.33}$$

with $K_0(\omega) = \sqrt{1 - \exp(-2\pi\omega/a)}$. This is, of course, the density operator for a relativistic gas with temperature $a/2\pi$, i.e., the Unruh effect: $T = a/2\pi$.

Chapter 3

The massive scalar field

We now tackle the complete problem of a massive scalar field in four $(3+1)$ dimensional spacetime. The overall procedure is the same as for the massless scalar field, but the calculations are more complicated, requiring some interesting insights. One of the results showed here is that the problem can be reduced to two dimensions without loss of generality, which will come in handy for the discussion that follows.

3.1 Massive scalar field in Minkowski spacetime

We start with a description of the quantum field in Minkowski spacetime. The massive Klein-Gordon equation (1.6) takes the form

$$(\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 + m^2)\ \phi = 0, \tag{3.1}$$

and has plane waves as solutions:

$$f_P(x) = [(2\pi)^3\ 2P_0]^{-\frac{1}{2}} \exp(-iP_0 t + iP \cdot X), \qquad P_0 = \sqrt{P^2 + m^2}, \tag{3.2}$$

which form a complete set of solutions to the Klein-Gordon equation (3.1), and are orthonormal with respect to the Klein-Gordon inner product (1.9). Notice that X and P are three dimensional vectors.

The quantization procedure described in the first chapter leads to the quantum field operator

$$\hat{\phi}(x) = \int d^3P\ \{\hat{a}_P^M f_P(x) + \hat{a}_P^{M\dagger} f_P^*(x)\}, \tag{3.3}$$

where the operators $\hat{a}_P^M$ and $\hat{a}_P^{M\dagger}$ are the annihilation and creation operators, satisfying the canonical commutation relations.

3.2 Massive scalar field in Rindler spacetime

The description of the massive scalar field in Rindler spacetime can be obtained by using the same coordinates changes as in equation (2.11). In this coordinates system, the Klein-Gordon equation is

$$\partial_\tau^2 \phi = \left[\partial_\xi^2 + e^{2a\xi}\left(\partial_x^2 + \partial_y^2\right) - m^2 e^{2a\xi}\right]\phi. \tag{3.4}$$

As discussed in the previous chapter, we choose the positive frequency solutions to have time dependency in the form $\exp(-i\omega\tau)$. It is not hard to see that the general solution to this equation may be written as

$$v^R_{\omega K_\perp} = \left(2\pi\sqrt{2\omega}\right)^{-\frac{1}{2}} e^{-i\omega\tau + iK_\perp \cdot X_\perp} g_{\omega K_\perp}(\xi), \tag{3.5}$$

with $X_\perp = (x, y)$, $K_\perp = (K_x, K_y)$, and $g_{\omega K_\perp}(\xi)$ satisfying the following equation:

$$\left[\frac{d^2}{d\xi^2} + e^{2a\xi}\left(K_\perp^2 + m^2\right)\right] g_{\omega K_\perp}(\xi) = \omega^2 \; g_{\omega K_\perp}(\xi). \tag{3.6}$$

This is a time independent Schrödinger equation with an exponential potential, and it is known that solutions of physical interest go to zero when $\xi \to \infty$ and behave as $\exp(\pm i\omega\xi)$ for $\xi \to -\infty$. We also impose the orthonormalization of the modes $v^R_{\omega K_\perp}$ with respect to the Klein-Gordon inner product (1.9), and it can be shown that they should satisfy [15]

$$v^R_{\omega K_\perp}(\tau, \xi, X_\perp) = \sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}} \; e^{iK_\perp \cdot X_\perp - i\omega\tau} \; K_{i\omega/a}\left(\frac{\kappa}{a} e^{a\xi}\right), \quad \kappa = \sqrt{K_\perp^2 + m^2}, \tag{3.7}$$

where $K_\nu(x)$ is the modified Bessel function of second kind.

After performing the canonical quantization procedure we obtain an expansion for the quantum field in Rindler spacetime in terms of the modes $v^R_{\omega K_\perp}(x')$:

$$\hat{\phi}^R(x') = \int_0^\infty d\omega \int d^2K_\perp \; \{\hat{a}^R_{\omega K_\perp} v^R_{\omega K_\perp}(x') + \hat{a}^{R\dagger}_{\omega K_\perp} v^{R*}_{\omega K_\perp}(x')\}, \tag{3.8}$$

with the Rindler spacetime annihilation and creation operators, $a^R_{\omega K_\perp}$ and $a^{R\dagger}_{\omega K_\perp}$, also satisfying the canonical commutation relations.

Performing a coordinates change similar to the one in equation (2.11), namely

$$t = a^{-1} e^{a\bar{\xi}} \sinh a\bar{\tau}, \qquad z = -a^{-1} e^{a\bar{\xi}} \cosh a\bar{\tau}, \tag{3.9}$$

we obtain the quantum field operator for the left Rindler wedge:

$$\hat{\phi}^L(\bar{x'}) = \int_0^\infty d\omega \int d^2K_\perp \{\hat{a}^L_{\omega K_\perp} v^L_{\omega K_\perp}(\bar{x'}) + \hat{a}^{L\dagger}_{\omega K_\perp} v^{L*}_{\omega K_\perp}(\bar{x'})\}, \tag{3.10}$$

with the operators $\hat{a}^L_{\omega K_\perp}$ and $\hat{a}^{L\dagger}_{\omega K_\perp}$ satisfying the canonical commutation relations. The modes $v^L_{\omega K_\perp}$ are defined as

$$v^L_{\omega K_\perp}(\bar{\tau}, \bar{\xi}, X_\perp) = \sqrt{\frac{\sinh(\pi\omega/a)}{4\pi^4 a}}\ e^{iK_\perp \cdot X_\perp - i\omega\bar{\tau}}\ K_{i\omega/a}\left(\frac{\kappa}{a} e^{a\bar{\xi}}\right), \tag{3.11}$$

which is just the expression for $v^R_{\omega K_\perp}$ with a label change.

Finally we write the quantum field operator for the union of both Rindler wedges:

$$\hat{\phi}^{RL} = \int_0^\infty d\omega \int d^2K_\perp\ \{\hat{a}^R_{\omega K_\perp} v^R_{\omega K_\perp} + \hat{a}^{R\dagger}_{\omega K_\perp} v^{R*}_{\omega K_\perp} + \hat{a}^L_{\omega K_\perp} v^L_{\omega K_\perp} + \hat{a}^{L\dagger}_{\omega K_\perp} v^{L*}_{\omega K_\perp}\}, \tag{3.12}$$

and define the vacuum state for this region of the spacetime as:

$$\hat{a}^R_{\omega K_\perp} \left|vac^{RL}\right\rangle = 0, \qquad\qquad \hat{a}^L_{\omega K_\perp} \left|vac^{RL}\right\rangle = 0. \tag{3.13}$$

3.3 The Bogoliubov coefficients

We are now able to calculate the Bogoliubov coefficients. Before we proceed, it should be noted that the equations (3.7) and (3.11) only define the modes $v^R_{\omega K_\perp}$ and $v^L_{\omega K_\perp}$ in the Rindler wedges, and they should be extended to the appropriate region in Minkowski spacetime.

Equation (1.23) shows that we may use the Bogoliubov coefficients to write the modes in Rindler spacetime as a linear combination of the plane wave modes (3.2) in Minkowski spacetime:

$$v^R_{\omega K_\perp} = \int \frac{d^3P}{\sqrt{2P_0\ (2\pi)^3}} \{\alpha^R_{\omega K_\perp P}\ e^{-iP_0 t + iP\cdot X} + \beta^R_{\omega K_\perp P}\ e^{iP_0 t - iP\cdot X}\}, \tag{3.14}$$

$$v^L_{\omega P_\perp} = \int \frac{d^3P}{\sqrt{2P_0\ (2\pi)^3}} \{\alpha^L_{\omega K_\perp P}\ e^{-iP_0 t + iP\cdot X} + \beta^L_{\omega K_\perp P}\ e^{iP_0 t - iP\cdot X}\}. \tag{3.15}$$

Notice that by replacing $P_\perp$ with $-P_\perp$ in the rightmost term of the previous two equations we can rewrite them as

$$v^R_{\omega K_\perp} = \int \frac{d^3P}{\sqrt{4\pi P_0}} \{\alpha^R_{\omega K_\perp P}\ e^{-iP_0 t + iP_z z} + \beta^R_{\omega K_\perp P}\ e^{iP_0 t - iP_z z}\} \frac{e^{iP_\perp \cdot X_\perp}}{2\pi}, \tag{3.16}$$

$$v^L_{\omega K_\perp} = \int \frac{d^3P}{\sqrt{4\pi P_0}} \{\alpha^L_{\omega K_\perp P}\ e^{-iP_0t+iP_zz} + \beta^L_{\omega K_\perp P}\ e^{iP_0t-iP_zz}\} \frac{e^{iP_\perp \cdot X_\perp}}{2\pi}, \tag{3.17}$$

and comparing these with equation (3.7) it becomes clear that the Bogoliubov coefficients with $P_\perp$ different from $K_\perp$ are zero, i.e.

$$\alpha^R_{\omega K_\perp P} = \alpha^R_{\omega P_z}\ \delta(P_\perp - K_\perp), \qquad \beta^R_{\omega K_\perp P} = \beta^R_{\omega P_z}\ \delta(P_\perp - K_\perp), \tag{3.18}$$

and similarly for the coefficients associated with the modes on the left Rindler wedge.

Equations (3.16) and (3.17) are then further simplified to

$$v^R_{\omega K_\perp} = \frac{e^{iK_\perp \cdot X_\perp}}{2\pi} \int_{-\infty}^{\infty} \frac{dP_z}{\sqrt{4\pi P_0}} \{\alpha^R_{\omega P_z}\ e^{-iP_0t+iP_zz} + \beta^R_{\omega P_z}\ e^{iP_0t-iP_zz}\}, \tag{3.19}$$

$$v^L_{\omega K_\perp} = \frac{e^{iK_\perp \cdot X_\perp}}{2\pi} \int_{-\infty}^{\infty} \frac{dP_z}{\sqrt{4\pi P_0}} \{\alpha^L_{\omega P_z}\ e^{-iP_0t+iP_zz} + \beta^L_{\omega P_z}\ e^{iP_0t-iP_zz}\}. \tag{3.20}$$

This is very interesting as it shows that the components x and y are not important for the Unruh effect and we may work with models in lower dimensions, which significantly simplifies the calculations.

The modes $v^L_{\omega K_\perp}$ were defined as the modes $v^R_{\omega K_\perp}$ with a change of labels $\tau \to \bar{\tau}$ and $\xi \to \bar{\xi}$, which corresponds to the change $z \to -z$. Comparing equations (3.19) and (3.20) under these changes we obtain a simple expression for the Bogoliubov coefficients associated with the left Rindler wedge in terms of the coefficients from the right wedge:

$$\alpha^L_{\omega,P_z} = \alpha^R_{\omega,-P_z} \qquad \text{and} \qquad \beta^L_{\omega,P_z} = \beta^R_{\omega,-P_z}.$$

Therefore, we may now focus on equation (3.19).

It is interesting to look at what happens in the right future Killing horizon, $t = z$ for $t > 0$, where we have

$$v^R_{\omega K_\perp} \to \frac{e^{iK_\perp \cdot X_\perp}}{2\pi} \int_{-\infty}^{\infty} \frac{dP_z}{\sqrt{4\pi P_0}} \{\alpha^R_{\omega P_z}\ e^{-i(P_0-P_z)V/2} + \beta^R_{\omega P_z}\ e^{i(P_0-P_z)V/2}\}, \tag{3.21}$$

with $V = t + z$, and it's convenient to define $U = t - z$. Equation (3.7) is valid in the right Rindler wedge, with the map in equation (2.11), where the same Killing horizon is located at $\xi \to -\infty$, so we should look at that equation under this condition.

We start with an approximation for the modified Bessel functions, $K_\nu(x)$, under the condition $|x| \to 0$. The modified Bessel function is defined as [20]

$$K_\nu(x) = \frac{\pi}{2} \frac{i^\nu J_{-\nu}(ix) - i^{-\nu} J_\nu(ix)}{\sin \nu\pi}, \tag{3.22}$$

where $J_\nu(x)$ are the Bessel function of the first kind, which satisfies the small arguments approximation [21]

$$J_\nu(x) \approx \frac{(x/2)^\nu}{\Gamma(1+\nu)}, \tag{3.23}$$

and $K_\nu(x)$ takes the form

$$K_\nu(x) \approx \frac{\pi}{2 \sin \nu\pi} \left[\frac{(x/2)^{-\nu}}{\Gamma(1-\nu)} - \frac{(x/2)^\nu}{\Gamma(1+\nu)} \right]. \tag{3.24}$$

We now write the approximation for the modes $v^R_{\omega K_\perp}$ in equation (3.7) near the future Killing horizon $\xi \to -\infty$:

$$\begin{aligned} v^R_{\omega K_\perp} &\to \frac{i\ e^{iK_\perp \cdot X_\perp - i\omega\tau}}{4\pi\sqrt{a \sinh(\pi\omega/a)}} \left[\frac{(\frac{\kappa}{2a})^{i\omega/a} e^{i\omega\xi}}{\Gamma(1+i\omega/a)} - \frac{(\frac{\kappa}{2a})^{-i\omega/a} e^{-i\omega\xi}}{\Gamma(1-i\omega/a)} \right] \\ &\to \frac{i\ e^{iK_\perp \cdot X_\perp}}{4\pi\sqrt{a \sinh(\pi\omega/a)}} \left[\frac{(\frac{\kappa}{2a})^{i\omega/a} e^{-i\omega u}}{\Gamma(1+i\omega/a)} - \frac{(\frac{\kappa}{2a})^{-i\omega/a} e^{-i\omega v}}{\Gamma(1-i\omega/a)} \right], \end{aligned} \tag{3.25}$$

where we implicitly defined $u = \tau - \xi$ and $v = \tau + \xi$. Inverting equation (2.11) we obtain expressions for τ and ξ as functions of t and z:

$$\tau = a^{-1} \tanh^{-1}(t/z) = a^{-1} \ln \sqrt{\frac{z+t}{z-t}} \qquad \xi = a^{-1} \ln\left(a\sqrt{z^2 - t^2}\right), \tag{3.26}$$

which gives us functions of u and v in terms of U and V, respectively:

$$u = -a^{-1} \ln(-aU) \qquad \text{and} \qquad v = a^{-1} \ln(aV). \tag{3.27}$$

Equation (3.25) may then be written as functions of U and V:

$$\begin{aligned} v^R_{\omega K_\perp} &\to \frac{i\ e^{iK_\perp \cdot X_\perp}}{4\pi\sqrt{a \sinh(\pi\omega/a)}} \left[\frac{(\frac{\kappa}{2a})^{i\omega/a} (-aU)^{i\omega/a}}{\Gamma(1+i\omega/a)} - \frac{(\frac{\kappa}{2a})^{-i\omega/a} (aV)^{-i\omega/a}}{\Gamma(1-i\omega/a)} \right] \\ &\to \frac{i\ e^{iK_\perp \cdot X_\perp}}{4\pi\sqrt{a \sinh(\pi\omega/a)}} \left[\frac{(-U\kappa/2)^{i\omega/a}}{\Gamma(1+i\omega/a)} - \frac{(V\kappa/2)^{-i\omega/a}}{\Gamma(1-i\omega/a)} \right]. \end{aligned} \tag{3.28}$$

We approach the right future Rindler horizon from $t < z$, so $U = t - z$ goes to zero from the left, i.e. $U \to 0^-$, and it follows, ignoring any delta contributions for $\omega \to 0^+$, that the left term inside the square brackets is zero.

Notice that both equations (3.21) and (3.28) have the modes $v^R_{\omega K_\perp}$ written as functions of V and $X_\perp$. Comparing them gives:

$$\int_{-\infty}^{\infty} \frac{dP_z}{\sqrt{4\pi P_0}} \{\alpha^R_{\omega P_z}\ e^{-i(P_0-P_z)V/2} + \beta^R_{\omega P_z}\ e^{i(P_0-P_z)V/2}\} = \\ -\frac{i}{\sqrt{4a\sinh(\pi\omega/a)}} \frac{(V\kappa/2)^{-i\omega/a}}{\Gamma(1-i\omega/a)}, \quad (3.29)$$

and, with the variable change $\gamma = (P_0 - P_z)/2$, the left side of equation (3.29) can be written as

$$\int_0^{\infty} d\gamma\ \frac{1+\kappa^2/4\gamma^2}{\sqrt{4\pi P_0(\gamma)}} \{\alpha^R_{\omega P_z}\ e^{-i\gamma V} + \beta^R_{\omega P_z}\ e^{i\gamma V}\}, \quad (3.30)$$

which is clearly a Fourier expansion with coefficients

$$\frac{1+\kappa^2/4\gamma^2}{\sqrt{4\pi P_0(\gamma)}}\ \alpha^R_{\omega P_z} \qquad \text{and} \qquad \frac{1+\kappa^2/4\gamma^2}{\sqrt{4\pi P_0(\gamma)}}\ \beta^R_{\omega P_z}, \quad (3.31)$$

obtainable by multiplying the expansion (3.30) by $\exp(\pm i\gamma' V)/2\pi$ and integrating over V. Doing this with equation (3.29) for the Bogoliubov coefficient $\alpha^R_{\omega P_z}$ yields

$$\alpha^R_{\omega P_z} = -\frac{i\ \kappa^{-i\omega/a}}{\Gamma(1-i\omega/a)} \frac{P_0 - P_z}{\sqrt{16\pi P_0 a \sinh(\pi\omega/a)}} \int_0^{\infty} dV\ (V/2)^{-i\omega/a}\ e^{i(P_0-P_z)V/2}. \quad (3.32)$$

Working out the integral and applying the same procedure to equation (3.29) multiplied by $\exp(-i\gamma' V)/2\pi$ in order to obtain $\beta^R_{\omega P_z}$ we find an explicit expression for the Bogoliubov coefficients:

$$\alpha^R_{\omega P_z} = \alpha^L_{\omega,-P_z} = \frac{e^{\pi\omega/2a}}{\sqrt{4\pi P_0 a\sinh(\pi\omega/a)}} \left(\frac{P_0+P_z}{P_0-P_z}\right)^{-i\omega/2a}, \quad (3.33)$$

$$\beta^R_{\omega P_z} = \beta^L_{\omega,-P_z} = -\frac{e^{-\pi\omega/2a}}{\sqrt{4\pi P_0 a\sinh(\pi\omega/a)}} \left(\frac{P_0+P_z}{P_0-P_z}\right)^{-i\omega/2a}. \quad (3.34)$$

We finish this section with a discussion on the extension of the Rindler modes to the whole Minkowski spacetime. The Bogoliubov coefficients in equations (3.33) and (3.34) were obtained with calculations using only the right future Rindler horizon, which should not really be a surprise considering that the field dynamics are entirely determined from its state in a ‘slice’ of the spacetime, i.e., the initial conditions. Due to symmetries in the definitions of the modes in the left and right Rindler wedges, we have a further reduction in the information required to obtain the coefficients, and may focus on only the future half of the 'slice' $t = z$. For that reason we can write the explicit form of the Bogoliubov coefficients in equations (3.14) and (3.15) and take these as the extensions of the Rindler modes to the whole Minkowski spacetime.

3.4 The Unruh effect

We now use the Rindler modes defined in equations (3.14) and (3.15), with the Bogoliubov coefficients found in equations (3.33) and (3.34), to write the quantum field expansion in both Rindler wedges, as shown in equation (3.12), in terms of the modes in Minkowski spacetime:

$$\hat{\phi}^{RL} = \int_0^\infty d\omega \int d^2K_\perp \int \frac{d^3P}{\sqrt{2P_0\,(2\pi)^3}}\{ \\ \left(\hat{a}^R_{\omega K_\perp}\ \alpha^R_{\omega K_\perp P} + \hat{a}^{R\dagger}_{\omega K_\perp}\ \beta^{R*}_{\omega K_\perp P} + \hat{a}^L_{\omega K_\perp}\ \alpha^L_{\omega K_\perp P} + \hat{a}^{L\dagger}_{\omega K_\perp}\ \beta^{L*}_{\omega K_\perp P}\right)\ e^{-iP_0 t + iP\cdot X} \\ + H.c.\}. \quad (3.35)$$

Now, equation (3.3) indicates that the operators that appear as coefficients to the positive frequency modes are the annihilation operators of the field in Minkowski spacetime, which means that we may write the operator inside the big parenthesis in equation (3.35) as a linear combination of the Minkowski annihilation operators $\hat{a}^M_P$. All of these operators annihilate the Minkowski vacuum, so we may write

$$\int d^3P\ f(P)\ \hat{a}^M_P \left|vac^M\right\rangle = 0, \quad (3.36)$$

for every distribution $f(P)$.

For that reason we may write, for some function $f(P)$,

$$(\hat{a}^R_{\omega K_\perp}\ \alpha^R_{\omega K_\perp P} + \hat{a}^{R\dagger}_{\omega K_\perp}\ \beta^{R*}_{\omega K_\perp P} + \hat{a}^L_{\omega K_\perp}\ \alpha^L_{\omega K_\perp P} + \hat{a}^{L\dagger}_{\omega K_\perp}\ \beta^{L*}_{\omega K_\perp P})\left|vac^M\right\rangle = \\ \int d^3P\ f(P)\ \hat{a}^M_P \left|vac^M\right\rangle = 0. \quad (3.37)$$

An inspection of equations (3.33) and (3.34) shows that the Bogoliubov coefficients satisfy the relations

$$\beta^R_{\omega K_\perp P} = -e^{-\pi\omega/a}\ \alpha^{L*}_{\omega K_\perp P}, \qquad \beta^L_{\omega K_\perp P} = -e^{-\pi\omega/a}\ \alpha^{R*}_{\omega K_\perp P},$$
$$\text{and} \qquad \alpha^L_{\omega K_\perp P} = e^{2i\omega Q(P)/a}\ \alpha^R_{\omega K_\perp P}, \tag{3.38}$$

in which we introduced the *rapidity* $Q(P) = \ln\sqrt{\frac{P_0+P_z}{P_0-P_z}} = \tanh^{-1}(P_z/P_0)$. Equation (3.37) may then be rewritten as

$$\begin{aligned} \Big[\alpha^R_{\omega K_\perp P}\ (\hat{a}^R_{\omega K_\perp} - e^{-\pi\omega/a}\ \hat{a}^{L\dagger}_{\omega K_\perp}) + \alpha^L_{\omega K_\perp P}\ (\hat{a}^L_{\omega K_\perp} - e^{-\pi\omega/a}\ \hat{a}^{R\dagger}_{\omega K_\perp})\Big] \left|vac^M\right\rangle = 0 \\ \Big[(\hat{a}^R_{\omega K_\perp} - e^{-\pi\omega/a}\ \hat{a}^{L\dagger}_{\omega K_\perp}) + e^{2i\omega Q(P)/a}\ (\hat{a}^L_{\omega K_\perp} - e^{-\pi\omega/a}\ \hat{a}^{R\dagger}_{\omega K_\perp})\Big] \left|vac^M\right\rangle = 0. \end{aligned} \tag{3.39}$$

This last equation must be satisfied for all P_z, implying that the operators in parenthesis must annihilate the Minkowski vacuum state:

$$(\hat{a}^R_\omega - e^{-\pi\omega/a}\ \hat{a}^{L\dagger}_\omega) \left|vac^M\right\rangle = 0, \qquad (\hat{a}^L_\omega - e^{-\pi\omega/a}\ \hat{a}^{R\dagger}_\omega) \left|vac^M\right\rangle = 0, \tag{3.40}$$

where we have omitted the $K_\perp$ index on the annihilation and creation operator in order to simplify the notation.

We now take the expectation value of $(\hat{a}^{R\dagger}_\omega + e^{-\pi\omega/a}\ \hat{a}^L_\omega)(\hat{a}^R_\omega - e^{-\pi\omega/a}\ \hat{a}^{L\dagger}_\omega)$ with respect the the vacuum state $\left|vac^M\right\rangle$, which must be zero according to the previous equation, and obtain:

$$\left\langle vac^M\right| [\hat{a}^{R\dagger}_\omega\ \hat{a}^R_\omega - e^{-2\pi\omega/a}\ \hat{a}^L_\omega\ \hat{a}^{L\dagger}_\omega - e^{-\pi\omega/a}\ (\hat{a}^{R\dagger}_\omega\ \hat{a}^{L\dagger}_\omega - \hat{a}^L_\omega\ \hat{a}^R_\omega)] \left|vac^M\right\rangle = 0. \tag{3.41}$$

The operator in parenthesis is anti-Hermitian while the remaining terms form a Hermitian operator, so their expectation values must go to zero independently, and we may write

$$\left\langle vac^M\right| (\hat{a}^{R\dagger}_\omega\ \hat{a}^R_\omega - e^{-2\pi\omega/a}\ \hat{a}^L_\omega\ \hat{a}^{L\dagger}_\omega) \left|vac^M\right\rangle = 0, \tag{3.42}$$

where we use the commutation relations $[\hat{a}^L_\omega,\ \hat{a}^{L\dagger}_\omega] = 1$ to find

$$\left\langle vac^M\right| \hat{a}^{R\dagger}_\omega\ \hat{a}^R_\omega \left|vac^M\right\rangle = e^{-2\pi\omega/a} \left\langle vac^M\right| \hat{a}^{L\dagger}_\omega\ \hat{a}^L_\omega \left|vac^M\right\rangle + e^{-2\pi\omega/a}. \tag{3.43}$$

Similarly, the expectation value of $(\hat{a}^{L\dagger}_\omega + e^{-\pi\omega/a}\ \hat{a}^R_\omega)(\hat{a}^L_\omega - e^{-\pi\omega/a}\ \hat{a}^{R\dagger}_\omega)$, which is also zero according to the right expression of equation (3.40), leads to

$$\left\langle vac^M\right| \hat{a}^{L\dagger}_\omega\ \hat{a}^L_\omega \left|vac^M\right\rangle = e^{-2\pi\omega/a} \left\langle vac^M\right| \hat{a}^{R\dagger}_\omega\ \hat{a}^R_\omega \left|vac^M\right\rangle + e^{-2\pi\omega/a}. \tag{3.44}$$

We now simultaneously solve the two previous equations to obtain the expectation value of the number operators associated with both Rindler wedges:

$$\langle vac^M | \, \hat{a}_\omega^{L\dagger} \, \hat{a}_\omega^L \, | vac^M \rangle = \langle vac^M | \, \hat{a}_\omega^{R\dagger} \, \hat{a}_\omega^R \, | vac^M \rangle = (e^{2\pi\omega/a} - 1)^{-1}, \tag{3.45}$$

which is the Bose-Einstein statistics, exactly what we would expect for a boson gas at temperature $T = a/2\pi$ [22]. Of course, this is not enough to establish the Unruh effect, as it does not imply that the other matrix elements characterize a thermal bath.

Applying the creation operators $\hat{a}_\omega^{R\dagger}$ and $\hat{a}_\omega^{L\dagger}$ to the left and right expressions in equation (3.40), respectively, and subtracting the equations we find

$$[\hat{a}_\omega^{R\dagger} \, \hat{a}_\omega^R - \hat{a}_\omega^{L\dagger} \, \hat{a}_\omega^L - e^{-\pi\omega/a} \, (\hat{a}_\omega^{R\dagger} \, \hat{a}_\omega^{L\dagger} - \hat{a}_\omega^{L\dagger} \, \hat{a}_\omega^{R\dagger})] \, | vac^M \rangle = 0, \tag{3.46}$$

where using the commutation relation $[\hat{a}_\omega^{R\dagger}, \ \hat{a}_\omega^{L\dagger}] = 0$ we obtain

$$(\hat{a}_\omega^{R\dagger} \, \hat{a}_\omega^R - \hat{a}_\omega^{L\dagger} \, \hat{a}_\omega^L) \, | vac^M \rangle = 0. \tag{3.47}$$

This means that, for each mode ω, the Minkowski vacuum has the same number of excitations on both Rindler wedges. We now use the same approximation for discrete ω discussed in the previous chapter, and may write

$$| vac^M \rangle \propto \prod_\omega \sum_{n_\omega \geq 0} \frac{K_{n_\omega}}{n_\omega!} (\hat{a}_\omega^{R\dagger} \hat{a}_\omega^{L\dagger})^{n_\omega} \, | vac^R \rangle \, . \tag{3.48}$$

Multiplying the last equation by $(\hat{a}_\omega^R - e^{-\pi\omega/a} \ \hat{a}_\omega^{L\dagger})$ we find the recursion relation satisfied by the coefficients K_{n_ω}:

$$K_{n_\omega} = K_{n_\omega - 1} \ e^{-\pi\omega/a} = K_0 \ e^{-\pi\omega n_\omega/a}. \tag{3.49}$$

Minkowski vacuum may then be written as

$$| vac^M \rangle = \prod_\omega \left(K_0(\omega) \sum_{n_\omega \geq 0} e^{-\pi\omega n_\omega/a} \, | n_\omega^R \rangle \otimes | n_\omega^L \rangle \right), \tag{3.50}$$

where the states $| n_\omega^R \rangle$ and $| n_\omega^L \rangle$ are the eigenstates of the Hamiltonian operator written in terms of Rindler spacetime coordinates. The product symbol indicates the tensor product of the states in parenthesis for every mode ω, and the constant $K_0(\omega)$ are obtained by the normalization condition of the vacuum state, which reads $K_0(\omega) = \sqrt{1 - \exp(-2\pi\omega/a)}$. We now write the density operator associated with the Minkowski spacetime vacuum

$$\hat{\rho}_M = | vac^M \rangle\langle vac^M | = \prod_\omega \left[K_0^2(\omega) \sum_{n_\omega \geq 0} e^{-2\pi\omega n_\omega/a} \left(| n_\omega^R \rangle\langle n_\omega^R | \otimes | n_\omega^L \rangle\langle n_\omega^L | \right) \right], \tag{3.51}$$

Due to the causal structure of Minkowski spacetime, there must be no communications between observers located on the left Rindler wedge and observers on the right wedge, therefore we trace out the components from either wedges from the tensor product in equation (3.50). We finally obtain the Minkowski vacuum state operator, accessible to observers in either wedges of Rindler spacetime, written in terms of latter spacetime's energy eigenstates:

$$\hat{\rho}_R = \prod_{\omega} \left(K_0^2(\omega) \sum_{n_\omega \geq 0} e^{-2\pi\omega n_\omega / a} \, |n_\omega\rangle\langle n_\omega| \right). \tag{3.52}$$

Comparing this last equation with the density operator for a relativistic boson gas (equation (A.4)), we find the same expressions if we set $2\pi/a = \beta$, and we have the Unruh effect:

$$T = \frac{a}{2\pi}. \tag{3.53}$$

In words, an observer with constant proper acceleration a in Minkowski spacetime cannot distinguish the vacuum state (described by the Minkowski vacuum density operator $\hat{\rho}_M$ in its reference frame) from a thermal bath with temperature $T = a/2\pi$ (described by the density operator in equation (A.4)).

Chapter 4

Implicit boundary conditions

We now focus on certain conditions that may have been overlooked in our derivation of the Unruh effect. The discussion here is based on calculations published in reference [23].

As discussed in the previous chapter, the perpendicular components $X_\perp$ and $P_\perp$ are not important for the calculations of the Unruh effect. Therefore, in order to simplify the calculations, we will consider the problem of a massive scalar field in 1+1 dimensions spacetime.

4.1 Quantum fields in two dimensional spacetime

For future reference, in this section we rewrite some of the results from the previous chapter for two-dimensional spacetimes. The overall similarities between the equations should make clear that there is no generality loss if using this simpler model.

In Minkowski spacetime the massive Klein-Gordon equation is written as

$$\left(\partial_t^2 - \partial_z^2 + m^2\right) \phi_M(x) = 0, \tag{4.1}$$

and its solutions are the plane waves indexed by the momentum $p \in \mathbb{R}$:

$$\Theta_p(x) = \frac{1}{\sqrt{4\pi p_0}} e^{ipz - ip_0 t}, \qquad p_0 = \sqrt{p^2 + m^2}. \tag{4.2}$$

These functions form a complete set of solutions to the Klein-Gordon equation, and are orthonormal under the Klein-Gordon inner product for Minkowski spacetime:

$$(f, g)_M = i \int_{-\infty}^{\infty} f^*(x) \overleftrightarrow{\partial_t} g(x)\, dz. \tag{4.3}$$

The quantization procedure laid out in the first chapter leads to the following field description:

$$\hat{\phi}_M(x) = \int_{-\infty}^{\infty} dp \left[\hat{a}_p \ \Theta_p(x) + \hat{a}_p^\dagger \ \Theta_p^*(x)\right], \tag{4.4}$$

where the operators $\hat{a}_p$ and $\hat{a}_p^\dagger$ are the annihilation and creation operators in Minkowski spacetime. As we have seen, these operators satisfy the canonical commutation relations and define the Minkowski vacuum state $\hat{a}_p \left|vac^M\right\rangle = 0$.

In Rindler spacetime, described by the coordinates change in equation (2.7) (and similarly for the left Rindler wedge), the Klein-Gordon equation reads

$$\left(\partial_\eta^2 - \rho\partial_\rho \ \rho\partial_\rho + m^2\rho^2\right) \phi_R(\xi) = 0, \tag{4.5}$$

and has the following functions as solutions:

$$\Phi_\mu(x') = \frac{e^{-i\mu\eta}}{\sqrt{2\mu}} \ \psi_\mu(\rho), \qquad \text{with} \qquad \psi_\mu(\rho) = \frac{1}{\pi}\sqrt{2\mu \sinh \pi\mu} \ K_{i\mu}(m\rho), \tag{4.6}$$

where $K_{i\mu}(x)$ are the modified Bessel functions and μ is a non negative real number. Notice the similarities between equations (3.7) and (4.6) under the transformation described by equation (2.12).

The functions described by equation (4.6) also form a complete set of solutions to the Klein-Gordon equation and are orthonormal when used with the inner product (1.11) in Rindler spacetime:

$$(f, g)_R = i \int_0^\infty \frac{d\rho}{\rho} \ f^*(x')\partial_\eta g(x'). \tag{4.7}$$

Therefore the quantization procedure leads to the expansion of field in Rindler spacetime the as a linear combination of the modes $\Phi_\mu(x')$, and the annihilation and creation operators $\hat{c}_\mu$ and $\hat{c}_\mu^\dagger$, respectively:

$$\hat{\phi}_R(x') = \int_0^\infty d\mu \ \{\hat{c}_\mu \ \Phi_\mu(x') + \hat{c}_\mu^\dagger \ \Phi_\mu^*(x')\} \tag{4.8}$$

with the operators $\hat{c}_\mu$ and $\hat{c}_\mu^\dagger$ satisfying the canonical commutation relations. It follows that the vacuum state in Rindler spacetime is defined by $\hat{c}_\mu \left|vac^R\right\rangle = 0$.

4.2 Boundary Conditions

In this section we'll consider the one-particle amplitude $\phi_f(x)$, defined as

$$\phi_f(x) = \langle vac| \, \hat{\phi}_M(x) \, |f\rangle \,, \tag{4.9}$$

which determines all matrix elements of the field operator $\hat{\phi}_M$ [23]. In Minkowski spacetime, the one-particle state $|f\rangle$ is defined as

$$|f\rangle = \int_{-\infty}^{\infty} dp \; f(p) \; a_p^\dagger \left|vac^M\right\rangle, \qquad \langle f|f\rangle = \int_{-\infty}^{\infty} dp \; |f(p)|^2 = 1. \tag{4.10}$$

From the field Lagrangian we obtain the Hamiltonian density and may evaluate the field energy, which reads

$$\langle f| \, \hat{H} \, |f\rangle = \frac{1}{2}\int_{-\infty}^{\infty} dz \; \{|\partial_t \phi_f|^2 + |\partial_z \phi_f|^2 + m^2|\phi_f|^2\}. \tag{4.11}$$

The finiteness of the one-particle field energy requires that each one of the three independent integrals be finite, from which it follows the finiteness of

$$\int_{-\infty}^{\infty} |\partial_z \phi_f|^2 dz \qquad \text{and} \qquad \int_{-\infty}^{\infty} |\phi_f|^2 dz, \tag{4.12}$$

implying, respectively, the field continuity and the boundary conditions

$$\phi_f(t, z \to \pm\infty) = 0. \tag{4.13}$$

The solution to the Klein-Gordon equation in Rindler spacetime (4.5), can also be written as

$$\phi_R(x') = e^{-i\eta \mathcal{K}_R^{1/2}} \psi(\rho) + e^{i\eta \mathcal{K}_R^{1/2}} \psi^*(\rho), \tag{4.14}$$

with the functions $\psi(\rho)$ obtained from the field's initial conditions:

$$\psi(\rho) = \frac{1}{2}\phi_R(0,\rho) + \frac{i}{2}\mathcal{K}_R^{1/2}\dot{\phi}_R(0,\rho). \tag{4.15}$$

It should be noted that $\mathcal{K}_R = -\rho\partial_\rho \, \rho\partial_\rho + m^2\rho^2$ in these last equations acts as an operator on the fields.

We once again consider the one-particle amplitude, which in Rindler spacetime reads

$$\phi_g(x') = \left\langle vac^R \right| \hat{\phi}_R(x') \, |g\rangle = \exp\Big(-i\eta\mathcal{K}_R^{1/2}\Big)\psi_g(x'), \tag{4.16}$$

where we used equation (4.14) and the fact that the term on the left is associated with the creation operators and are eliminated once applied to the vacuum state $\left\langle vac^R \right|$. The one-particle state is written as

$$|g\rangle = \int_0^\infty \frac{d\mu}{\mu^{1/2}}\, g(\mu)\, c_\mu^\dagger \left|vac^R\right\rangle \qquad \text{with} \qquad \langle g|g\rangle = \int_0^\infty \frac{d\mu}{\mu}|g(\mu)|^2 = 1, \tag{4.17}$$

and it follows that the spatial component $\psi_g(\rho)$ satisfies

$$\psi_g(\rho) = \int_0^\infty d\mu\, \frac{g(\mu)}{\pi} \left(\frac{\sinh \pi\mu}{\mu}\right)^{1/2} K_{i\mu}(m\rho). \tag{4.18}$$

Performing the Langer transformation $u = \ln(m\rho)$, mapping $\rho = 0$ to $u = -\infty$, the operator $\mathcal{K}_R$ is rewritten as

$$\mathcal{K}_R = -\partial_u^2 + V(u) \qquad \text{with} \qquad V(u) = e^{2u}. \tag{4.19}$$

The form of the potential $V(u)$ immediately tells us that physically relevant solutions satisfy the boundary condition $\phi_g(\eta, +\infty) = 0$. In order to obtain the condition for $u \to -\infty$, we look at the one-particle field energy, which reads:

$$\langle g|\, H\, |g\rangle = \frac{1}{2}\int_0^\infty \frac{d\rho}{\rho}\{|\partial_\eta\phi_g|^2 + \rho^2\,|\partial_\rho\phi_g|^2 + m^2\rho^2\,|\phi_g|^2\} = \int_0^\infty d\mu\, |g(\mu)|^2, \tag{4.20}$$

implying the finiteness of each of the following integrals

$$\int_0^\infty \frac{d\rho}{\rho}|\rho\partial_\rho\phi_g|^2, \qquad \int_0^\infty \frac{d\rho}{\rho}|\rho\phi_g|^2 \qquad \text{and} \qquad \int_0^\infty d\mu\, |g(\mu)|^2. \tag{4.21}$$

The equivalent to these last expressions for Minkowski spacetime led us to the boundary conditions (4.13), but here there is no obvious limitation for the behaviour of $\phi_g(x')$ when $\rho \to 0$ from these alone. However, there *is* an important condition that must be satisfied, namely:

$$\phi_g(\eta, 0) = 0. \tag{4.22}$$

In order to prove this we split the integral in equation (4.18) in three parts:

$$I_1 = \int_0^{\mu_1} G_\mu(\rho)\, d\mu, \qquad I_2 = \int_{\mu_1}^{\mu_2} G_\mu(\rho)\, d\mu \qquad \text{and} \qquad I_3 = \int_{\mu_2}^\infty G_\mu(\rho)\, d\mu \tag{4.23}$$

where and μ_1 and μ_2 are arbitrarily small and large positive numbers, respectively, and $G(\mu, \rho)$ is the integrand in that equation, given by:

$$G_\mu(\rho) = \frac{g(\mu)}{\pi}\left(\frac{\sinh \pi\mu}{\mu}\right)^{1/2} K_{i\mu}(m\rho). \tag{4.24}$$

The small arguments approximation for the modified Bessel equation (3.24) is once again useful, and we write

$$K_{i\mu}(e^u) \approx \frac{\pi}{2i\sinh \mu\pi}\left[\frac{e^{-i\mu u+i\mu\ln 2}}{\Gamma(1-i\mu)} - \frac{e^{i\mu u-i\mu\ln 2}}{\Gamma(1+i\mu)}\right], \qquad \text{for} \quad u \to -\infty. \tag{4.25}$$

For small μ we may set $\Gamma(1-i\mu) \approx \Gamma(1+i\mu) \approx \Gamma(1)$ and $\sinh \mu\pi \approx \mu\pi$ and obtain an asymptotic expression for $G_\mu(\rho)$:

$$G_\mu(\rho) \approx \frac{g(\mu)}{\mu\sqrt{\pi}} - \sin(\mu u - \mu\ln 2), \qquad \text{for} \quad \mu \ll 1 \quad \text{and} \quad u \to -\infty. \tag{4.26}$$

Analysis on the behaviour of the $\Gamma(1+i\mu)$ and $\sinh(\pi\mu)$ functions for the conditions $\mu \approx 1$ and $\mu \gg 1$ gives us the approximation

$$G_\mu(\rho) \approx \frac{g(\mu)}{\mu\sqrt{\pi}}\begin{cases} -\sin(\mu u - \mu\ln 2), & \mu \ll 1 \\ \cos(\mu u - \mu\ln 2 - \arg\Gamma(i\mu)), & \mu \approx 1 \\ \sin(\mu\ln\mu - \mu u + \mu(\ln 2 - 1) + \pi/4), & \mu \gg 1 \end{cases}. \tag{4.27}$$

The asymptotic expression in equation (4.27) shows that the functions $G_\mu(\rho)$ oscillate everywhere, therefore, from the normalization condition of $|g\rangle$ and the inequality $|g| \leq \frac{1}{2}(1+|g|^2)$, it is clear that the integral $\int_{\mu_1}^{\mu_2} |g|/\mu \, d\mu$ converges. The Riemann-Lebesgue lemma then tells us that I_2 must go to zero as $\rho \to 0$. Applying the Schwartz inequality to I_3, we obtain

$$|I_3(\rho)|^2 \leq \frac{1}{\pi\mu_2}\int_{\mu_2}^{\infty} |g(\mu)|^2 d\mu \tag{4.28}$$

and it follows that I_3 yields arbitrarily small values by the appropriate choice of μ_2. It should be clear that only I_1 contributes to the field as $\rho \to 0$.

Looking at the normalization condition for $\langle g|g\rangle$, it is obvious that continuous functions $g(\mu)$ must vanish for $\mu = 0$. We now consider functions approaching zero as a power of μ for $\mu \to 0$: $g(\mu) = a\mu^\alpha$, $\alpha > 0$. Using the approximation from equation (4.27), the indefinite integral $I_1(\mu)$ is given by

$$\begin{aligned} I_1(\mu) &\approx -\frac{a}{\sqrt{\pi}}\int_0^{\mu} \bar{\mu}^{\ \alpha-1} sin(\bar{\mu}u - \bar{\mu}\ln 2)\ d\bar{\mu} \\ &\approx \frac{ia}{2\sqrt{\pi}}\bar{u}^{-\alpha}\{i^{\alpha}\ \Gamma(\alpha, -i\mu\bar{u}) - i^{-\alpha}\ \Gamma(\alpha, i\mu\bar{u}) - 2i\sin(\pi\alpha/2)\ \Gamma(\alpha)\}, \end{aligned} \tag{4.29}$$

where we implicitly defined $\bar{u} = u - \ln 2 = \ln(m\rho/2)$ and the incomplete gamma function $\Gamma(\alpha, z)$ satisfies, for imaginary z and $|z| \to \infty$ [24]:

$$\Gamma(\alpha, z) \approx z^{\alpha-1}e^{-z}\sum_{k\geq 0}\frac{\Gamma(\alpha)}{\Gamma(\alpha-k)}z^{-k}. \tag{4.30}$$

Substituting this asymptotic expansion's leading term into equation (4.29) we have

$$\begin{aligned} I_1(\mu) &\approx -\frac{ia}{\bar{u}\sqrt{\pi}}\mu^{\alpha-1}\sin(\mu\bar{u}) + \frac{a}{\bar{u}^{\alpha}\sqrt{\pi}}\sin(\pi\alpha/2)\ \Gamma(\alpha) \\ &\approx \frac{a}{\bar{u}^{\alpha}\sqrt{\pi}}\sin(\pi\alpha/2)\ \Gamma(\alpha) \quad \text{for} \quad \mu \ll 1, \end{aligned} \tag{4.31}$$

which clearly goes to zero as $\bar{u} \to -\infty$. It is quite complicated to evaluate this integral for arbitrary functions, but it can be shown that this integral must vanish for all physically realizable states $|g\rangle$ [23], implying (4.22). This means that, when deriving the Unruh effect in different spacetimes, we could be obtaining solutions to effectively different problems, and comparing the fields is pointless.

4.3 Boost modes quantization

There is an interesting derivation of the expression for the Rindler modes in terms of the plane waves modes, which will be very useful for the following discussion. We start by defining the operators $\hat{\alpha}_q$, labeled by the rapidity $q = \ln\sqrt{\frac{p_0-p}{p_0+p}} = \tanh^{-1}\left(\frac{p}{p_0}\right)$, as

$$\hat{\alpha}_q = \sqrt{m\cosh q}\ \hat{a}_p, \tag{4.32}$$

in which $p = m\sinh q$ and it follows $p_0 = m\cosh q$. It is straightforward to show that these operators, along with their Hermitian conjugate, satisfy the canonical commutation relations. We now define the operator $\hat{b}_\kappa$ as the Fourier transform of the operators $\hat{\alpha}_q$, therefore they satisfy

$$\hat{b}_k = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dq\ e^{iqk}\ \hat{\alpha}_q, \qquad \text{and} \qquad \hat{\alpha}_q = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk\ e^{-iqk}\ \hat{b}_k. \tag{4.33}$$

From these definitions it follows that the operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ satisfy $[\hat{b}'_k, \hat{b}_{k'}^\dagger] = [\hat{\alpha}_k, \hat{\alpha}_{k'}^\dagger]$, that is, the canonical commutation relations. Moreover, since the Fourier transform is known to be unitary, the operators $\hat{b}_k$ define the same vacuum as the operators $\hat{\alpha}_q$, which is clearly the same state defined by $\hat{a}_p$: the Minkowski spacetime static vacuum.

The field expansion (4.4) is now rewritten in terms of the operators $\hat{b}_k$ and $\hat{b}_k^\dagger$:

$$
\begin{aligned}
\hat{\phi}_M(x) &= \int_{-\infty}^{\infty} dp \int_{-\infty}^{\infty} dk\ (2^{\frac{3}{2}}\pi\ p_0)^{-1}\{e^{-iqk}\ \hat{b}_k\ e^{ipz-ip_0t} + H.c.\} \\
&= \int_{-\infty}^{\infty} dk\ \{\hat{b}_k \int_{-\infty}^{\infty} dp\ (2^{\frac{3}{2}}\pi\ p_0)^{-1} e^{-iqk+ipz-ip_0t} + H.c.\} \\
&= \int_{-\infty}^{\infty} dk\ \{\hat{b}_k\ \Psi_k(x) + \hat{b}_k^\dagger\ \Psi_k^*(x)\},
\end{aligned} \tag{4.34}
$$

with the modes $\Psi_k(x)$ defined by the integral form

$$
\begin{aligned}
\Psi_k(x) &= \int_{-\infty}^{\infty} dp\ (2^{\frac{3}{2}}\pi\ p_0)^{-1} \exp[-iqk + ipz - ip_0t] \\
&= (2^{\frac{3}{2}}\pi)^{-1} \int_{-\infty}^{\infty} dq \exp[-iqk + im(z\sinh q - t\cosh q)].
\end{aligned} \tag{4.35}
$$

These modes are also orthonormal with respect to the inner product in Minkowski spacetime and, along with $\Psi_k^*(x)$, form a complete set of solutions to the Klein-Gordon equation. From this and the commutation relations for the operators $\hat{b}_k$ and $\hat{b}_k^\dagger$ we can say that the quatization scheme in equation (4.34) is appropriate.

Acting the boost generator $\mathcal{B} = i(z\ \partial_t + t\ \partial_z)$ on the modes $\Psi_k(x)$ we find

$$
\mathcal{B}\ \Psi_k(x) = \ (2^{\frac{3}{2}}\pi)^{-1} \int_{-\infty}^{\infty} dq\ m\ (z\cosh q - t\sinh q)\ \exp[...], \tag{4.36}
$$

where we notice that the integrand may be written as $(k - i\partial_q)\exp[...]$, and the last equation is reduced to

$$
\begin{aligned}
\mathcal{B}\ \Psi_k(x) &= \ (2^{\frac{3}{2}}\pi)^{-1} \int_{-\infty}^{\infty} dq\ (k - i\partial_q)\exp[...] \\
&= k\ \Psi_k(x) - i\ (2^{\frac{3}{2}}\pi)^{-1} \exp[...]\ \Big|_{-\infty}^{+\infty} = k\ \Psi_k(x).
\end{aligned} \tag{4.37}
$$

Therefore the modes $\Psi_k(x)$ are eigenfunctions of the boost generator with eigenvalue k. Using the coordinates defined in equation (2.7) the boost generator reads $\mathcal{B} = i\partial_\eta$, and has as eigenfunctions the modes $\Phi_\mu(x')$ from equation (4.6), with μ as eigenvalues. It should be clear now that the modes $\Psi_k(x)$ may be identified as $\Phi_k(x')$ and $\hat{b}_k$ as the operators $\hat{c}_k$ from equation (4.8).

4.4 The zero boost mode

The Unruh effect is derived from equation (4.34) by splitting the integral at $k = 0$, and changing the integration variable on one of the terms to obtain a single integral with different limits, that is

$$\hat{\phi}_M(x) = \int_0^\infty dk \; \{\hat{b}_k \; \Psi_k(x) + \hat{b}_k^\dagger \; \Psi_k^*(x) + \hat{b}_{-k} \; \Psi_{-k}(x) + \hat{b}_{-k}^\dagger \; \Psi_{-k}^*(x)\}. \tag{4.38}$$

This last equation may then be rewritten as a linear combination of the Unruh modes defined, with $C_\mu = (2 \; \sinh \; \pi\mu)^{-1/2}$, as

$$R_\mu(x) = C_\mu\{e^{\pi\mu/2}\Psi_\mu(x) - e^{-\pi\mu/2}\Psi_{-\mu}^*(x)\} \quad \text{and} \quad L_\mu(x) = C_\mu\{e^{\pi\mu/2}\Psi_{-\mu}^*(x) - e^{-\pi\mu/2}\Psi_\mu(x)\}, \tag{4.39}$$

and the associated annihilation (and creation) operators given by

$$r_\mu = C_\mu\{e^{\pi\mu/2} \; b_\mu + e^{-\pi\mu/2} \; b_{-\mu}^\dagger\} \quad \text{and} \qquad l_\mu = C_\mu\{e^{\pi\mu/2} \; b_{-\mu} + e^{-\pi\mu/2} \; b_\mu^\dagger\}, \tag{4.40}$$

respectively.

These operators naturally satisfy the canonical commutation relations and we may identify the following field expansion

$$\hat{\phi}_M(x) = \int_0^\infty d\mu \; \{\hat{r}_\mu R_\mu(x) + \hat{r}_\mu^\dagger R_\mu^*(x) + \hat{l}_\mu L_\mu^*(x) + \hat{l}_\mu^\dagger L_\mu(x)\}, \tag{4.41}$$

with equation (3.35), from where it follows the Unruh effect.

The procedure above was actually the one used by Unruh on his original derivation in reference [1]. Although it may seem harmless, the splitting of the integral would imply the change of the integral by its Cauchy principal value at the point $\kappa = 0$:

$$\int_{-\infty}^\infty dk\{...\} \to p.v. \int_{-\infty}^\infty dk\{...\}, \tag{4.42}$$

and the exclusion of the zero boost mode from the complete set Ψ_μ. This would be fine if the modes $\Psi_k(x)$ where regular functions at $k = 0$, however, these modes have δ-like singularities at all points in the light cone, which obviously includes the origin.

What this means is that the exclusion of the zero boost mode implies the loss of some degrees of freedom of the field, which makes the modes Ψ_k incomplete when working in the whole Minkowski spacetime. As we have seen, this loss is equivalent to setting the boundary condition (4.22), or removing the entire light cone. This can be easily seen once we consider the field initially described in the 'slice' $\eta = t = 0$, where at $\rho = z = 0$ equation (4.35) reads

$$\Psi_k(0) = (2^{\frac{3}{2}}\pi)^{-1} \int_{-\infty}^{\infty} dq\ e^{-iqk} = \frac{1}{\sqrt{2}}\ \delta(k), \tag{4.43}$$

that is, the field at the origin of Minkowski spacetime is entirely described by the mode $\Psi_0(x)$.

To illustrate this point, we evaluate equation (4.34) for $z = t = 0$, and obtain

$$\hat{\phi}_M(0) = \frac{\hat{b}_0 + \hat{b}_0^\dagger}{2} = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{4\pi p_0}} (\hat{a}_p + \hat{a}_p^\dagger), \tag{4.44}$$

from where, for the one-particle state defined in equation (4.10), it follows

$$\phi_f(0) = \left\langle vac^M \right| \hat{\phi}_M(0) \left| f \right\rangle = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{4\pi p_0}} f(p). \tag{4.45}$$

Notice that this expression is not generally zero, while on the other hand, the exclusion of the zero boost mode implies $\phi_f(0) = 0$. This means that we can construct states in Minkowski spacetime that are not possible when considering only the Unruh modes defined in equation (4.39). More details on this discussion can be found in references [25, 26].

Conclusions

In this dissertation we have reviewed the derivation of the Unruh effect. This important phenomenon states that an observer in Minkowski spacetime with constant proper acceleration detects the quantum vacuum as a thermal bath with temperature proportional to the acceleration [2]. This statement does not depend on the physical characteristics of the detector in question, and is therefore understood as a universal effect. Under Einstein's equivalent principle, accelerated observers are particularly interesting due to their connection with observers under the influence of gravity (this relationship is well illustrated by the Hawking radiation, as can be seen in appendix C). As far we know there is no fully satisfactory understanding of gravity under quantum mechanics or vice versa, these results are important in giving hints on what should be the nature of such theories.

It is worth to mention that using the International System of Units, the Unruh effect reads

$$T_U = \frac{a\hbar}{2\pi c k_B}, \tag{4.46}$$

and evaluates to roughly $T \approx 4 \times 10^{-20} K$ for accelerations near $1g$. We may compare this with the lowest temperature ever seen on Earth (and maybe even in the entire universe), which is about 10^5 times higher than that, and cosmic background radiation corresponds to a temperature of around $2.7K$. Due to the incredibly low temperature, the Unruh effect has not been experimentally observed, but there are some proposals to do so, as in reference [27].

In sequence we have examined the controversies of the existence of the Unruh effect due to the problem of the boundary conditions [23]. Unfortunately, to the present time we have not obtained a decisive conclusion on the importance of the boundary conditions in the thermal response indicated by the Unruh effect. It is known that several theoretical results support that the thermal response is indeed observed [2, 3, 4, 5, 6, 7, 8], so these conditions do not seem to be important, at least in some cases. On the other hand, references [10, 11, 12, 13, 14] question whether the Unruh effect is actually detected, which could illustrate the effects of the boundary conditions showed here.

Finally, theoretical results indeed show accelerated systems reacting as if interacting with a thermal bath at the temperature predicted by the Unruh effect [10, 11, 12, 13, 14], some even indicating that this effect plays an important role in the consistency of quantum field theory in general. This should make clear that the Unruh effect must exist, at least is some sense.

Appendix A

Density matrix for a boson gas

The density operator of a thermodynamic system, which can be used to describe the expectation value of observables in statistical quantum systems, may be expressed as [22]

$$\hat{\rho} = \frac{e^{-\beta \hat{H}}}{Tr(e^{-\beta \hat{H}})} \tag{A.1}$$

where β is the inverse of the temperature and $\hat{H}$ is the systems Hamiltonian operator. Notice that this operator is diagonal when expressed in the energy eigenstates basis, i.e. $|n\rangle$ with $\hat{H}\,|n\rangle = \epsilon_n\,|n\rangle$, so we may write

$$\rho_n = \langle n|\,\hat{\rho}\,|n\rangle = Tr(e^{-\beta \hat{H}})^{-1}\,\langle n|\,e^{-\beta \hat{H}}\,|n\rangle = Tr(e^{-\beta \hat{H}})^{-1}\;e^{-\beta \epsilon_n}. \tag{A.2}$$

For a free relativistic boson gas we have $\epsilon_n = \omega n$, with positive real ω and integer $n \geq 0$, so

$$Tr(e^{-\beta \epsilon_n}) = Tr(e^{-\beta \omega n}) = \sum_{n \geq 0} e^{-\beta \omega n} = (1 - e^{-\beta \omega})^{-1}, \tag{A.3}$$

and the matrix elements are $\rho_n = e^{-\beta \omega n}/(1 - e^{-\beta \omega})$, leading to the density operator

$$\hat{\rho} = \prod_{w} \left((1 - e^{-\beta \omega})^{-1} \sum_{n_\omega \geq 0} e^{-\beta \omega n_\omega}\; |n_\omega\rangle\langle n_\omega| \right). \tag{A.4}$$

Appendix B

Trajectory of an accelerated observer

In this appendix we derive the trajectory of an observer with constant proper acceleration in Minkowski spacetime. With an appropriate choice of starting conditions, the set of trajectories with proper acceleration $\alpha > 0$ form a map (in terms of the acceleration and the observer's proper time) to the region $z > t$ in Minkowski spacetime: the right Rindler wedge.

We start with the position 4-vector, $x = (t, \vec{X})$. The 4-velocity is defined as the total derivative of the position 4-vector in respect with the observer's proper time, τ:

$$v = \frac{dx}{d\tau} = \left(\frac{dt}{d\tau}, \frac{d\vec{X}}{d\tau} \right). \tag{B.1}$$

Now, $dt/d\tau$ is just the instantaneous time dilation factor, so we may write

$$\frac{dt}{d\tau} = \gamma = \frac{1}{\sqrt{1 - \vec{V}^2}} \qquad \text{and} \qquad \frac{d\vec{X}}{d\tau} = \frac{d\vec{X}}{dt}\frac{dt}{d\tau} = \gamma\vec{V}, \tag{B.2}$$

which leads to

$$a = \frac{dv}{d\tau} = \frac{d}{d\tau}\left(\gamma, \gamma\vec{V}\right). \tag{B.3}$$

Working the derivatives give

$$\frac{d\gamma}{d\tau} = \gamma\frac{d\gamma}{dt} = \gamma\ (1 - \vec{V}^2)^{-\frac{3}{2}}\ \vec{V} \cdot \vec{A} = \gamma^4\ \vec{V} \cdot \vec{A} \tag{B.4}$$

$$\frac{d}{d\tau}(\gamma\vec{V}) = \frac{d\gamma}{d\tau}\vec{V} + \gamma\frac{d\vec{V}}{d\tau} = \frac{d\gamma}{d\tau}\vec{V} + \gamma^2\vec{A} \tag{B.5}$$

and we have

$$a = \left(\gamma^4\ \vec{V} \cdot \vec{A}, \gamma^4\ (\vec{V} \cdot \vec{A})\ \vec{V} + \gamma^2\vec{A}\right) \tag{B.6}$$

Let a be a constant with norm α and $\vec{V}$ point in the same direction as $\vec{A}$, it follows

$$\begin{aligned}
\alpha^2 &= -g_{\mu\nu}\ a^\mu\ a^\nu \\
&= -\gamma^8\ (\vec{V}\cdot\vec{A})^2 + \gamma^8\vec{V}^2(\vec{V}\cdot\vec{A})^2 + \gamma^4\vec{A}^2 + 2\gamma^6(\vec{V}\cdot\vec{A})^2 \\
&= \gamma^6(VA)^2 + \gamma^4\vec{A}^2 = \gamma^4(\gamma^2V^2+1)A^2 \\
&= \gamma^6A^2.
\end{aligned} \tag{B.7}$$

These considerations reduce the problem to a single spatial dimension, so we can drop the vector notation. Equation (B.7) gives us the equation of motion in terms of V and t:

$$(1-V^2)^{-\frac{3}{2}}\,\frac{dV}{dt} = \alpha, \tag{B.8}$$

which is solved by

$$V = \frac{\alpha t}{\sqrt{1+\alpha^2t^2}} \quad \rightarrow \quad X = \frac{1}{\alpha}\sqrt{\alpha^2t^2+1}, \tag{B.9}$$

where we have set the initial conditions $V(t=0)=0$ and $X(t=0)=\alpha^{-1}$, for reasons that should become clear by the end of this section.

Assuming that the observers' clocks are synchronized as $t=\tau=0$, the accelerated observer's proper time in terms of t is

$$\frac{dt}{d\tau} = \gamma = \sqrt{1+\alpha^2t^2} \quad \rightarrow \quad \tau = \frac{1}{\alpha}\sinh^{-1}\alpha t. \tag{B.10}$$

This last expression can be inverted into

$$t = \frac{1}{\alpha}\sinh\alpha\tau \qquad \text{and} \qquad X = \frac{1}{\alpha}\cosh\alpha\tau. \tag{B.11}$$

If we set $\eta = \alpha t$ and $\rho = \alpha^{-1}$ we can write

$$t = \rho\sinh\eta \qquad \text{and} \qquad X = \rho\cosh\eta, \tag{B.12}$$

which can be inverted to yield

$$\eta = \tanh^{-1}(t/X) \qquad \text{and} \qquad \rho = \sqrt{X^2-t^2}. \tag{B.13}$$

Notice that equations (B.12) and (B.13) form a smooth invertible map between the variables (t, X) and (η, ρ) in the region $X > |t|$ (or $\rho > 0$) of Minkowski spacetime. This region is known as the right Rindler wedge, and can be understood as a spacetime on it's own.

The right Rindler wedge is often used to study accelerated observers, as a trajectory with fixed ρ describes the same trajectory as would an observer with constant proper acceleration $\alpha = \rho^{-1}$ in Minkowski spacetime (considering the proper choice of initial conditions).

Appendix C

Black holes and the Unruh effect

There is a remarkable similarity between the Hawking temperature[28] and the Unruh effect. This is not a coincidence, as the spacetime describing a black hole and Rindler spacetime has some properties in common. Here we will show a simple calculation of how the Hawking radiation can be obtained using the Unruh effect.

A black hole with no charge nor angular momentum and mass M (in a universe with the cosmological constant set to zero) is well described by the Schwarzschild metric

$$g = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2\ (d\theta^2 + sin^2\theta\ d\phi^2), \tag{C.1}$$

where $2M$ is the Schwarzschild radius of this black hole. Introducing the coordinate change

$$r = 2M + \frac{\rho^2}{8M}, \qquad \text{for} \quad \rho > 0 \tag{C.2}$$

we rewrite the metric and pick the terms of lower order on ρ, indicating an observer close to the event horizon, which gives

$$g = \left(\frac{\rho}{4M}\right)^2\ dt^2 - d\rho^2 - dx^2 - dy^2. \tag{C.3}$$

Notice that we transformed the θ and ϕ coordinates in two locally flat coordinates x and y. Setting $\eta = t/4M$ we find the Rindler spacetime metric.

The transformation in equation (C.2) only maps the universe outside of the black hole. Of course, we have very good reason to believe that the field outside of the black hole should not interact in any way whatsoever with the field inside the black hole. The transformation

$$r = 2M - \frac{\rho^2}{8M}, \qquad \text{with} \quad 0 < \rho < 4M, \tag{C.4}$$

maps the interior of the black hole. What is interesting here is that this transformation leads to a non-static globally hyperbolic metric, similar to the one used the describe the degenerate Kasner universes. As we have seen, spacetimes like these do not have a well defined vacuum state.

The spacetime near the black-hole horizon is described as two separate spacetimes: one static and one non-static, described by the a metric identical to the ones for Rindler spacetime and the Kasner degenerate universes, respectively. It follows that there is a very good analogy between the problem of an observer near the outside of the event horizon and an observer with constant proper acceleration. This should become obvious if we consider that the metric in equation (C.3) describes an observer accelerating with proper acceleration ρ^{-1} to avoid falling into the black hole. This is perfectly compatible with Einstein's equivalence principle.

The problem here is actually simpler than the Unruh effect, as our only concern is some equivalent to a right Rindler wedge, and the calculations are pretty much exactly as was done before. This leads to the result

$$T = \frac{1}{2\pi\rho} = \frac{1}{4\pi\sqrt{2Mr\,(1 - 2M/r)}}. \tag{C.5}$$

An observer at r is under the gravitational acceleration $\sqrt{1 - 2M/r}$. Applying the proper adjustment due to gravitational redshift, an observer at r' detects the temperature

$$T = \frac{1}{4\pi\sqrt{2Mr\,(1 - 2M/r')}}, \tag{C.6}$$

where we take the limit for $r \to 2M$ and $r' \to \infty$, which corresponds to an observer very far away from the black hole (like us), and obtain the Hawking temperature

$$T_H = \frac{1}{8\pi M}. \tag{C.7}$$

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