Universidade Federal de Goiás
Instituto de Matemática e Estatística
Programa de Pós-Graduação em Matemática

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Piecewise Smooth Vector Fields: Conley Index and Asymptotic Analysis

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Angie Tatiana Suárez Romero

Piecewise Smooth Vector Fields: Conley Index and Asymptotic Analysis

Tese apresentada ao Programa de Pós-Graduacão em Matemática do Instituto de Matemática e Estatística da Universidade Federal de Goiás, como requisito parcial para obtenção do título de Doutora em Matemática.

Área de concentração: Sistemas dinâmicos.

Orientador: Prof. Dr. Rodrigo Donizete Euzébio

Co–Orientador: Prof. Dr. Ewerton Rocha Vieira

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ATA DE DEFESA DE TESE

Ata nº 04 da sessão de Defesa de Tese de **Angie Tatiana Suárez Romero,** que confere o título de Doutora em Matemática, **na área de concentração de Sistemas Dinâmicos.**

Ao trigésimo dia do mês de setembro do ano de dois mil e vinte e dois, a partir das quinze horas, via Web videoconferência, realizou-se a sessão pública de Defesa de Tese intitulada **"Piecewise smooth vector fields: Conley index and asymptotic analysis."** Os trabalhos foram instalados pelo Orientador e presidente da banca, Professor Doutor **Rodrigo Donizete Euzébio - IME/UFG** com a participação dos demais membros da Banca Examinadora: Professor Doutor **Ewerton Rocha Vieira - IME/UFG** - coorientador, Professor Doutor **Ronaldo Alves Garcia - IME/UFG** membro titular interno, Professora Doutora **Ketty Abaroa de Rezende - IMECC/Unicamp** membro titular externa, Professora Doutora **Dahisy Valadão de Souza Lima - CMCC/Ufabc,** membro titular externa e Professora Doutora **Mariana Rodrigues da Silveira - CMCC/Ufabc** membro titular externa. Durante a arguição os membros da banca **não fizeram** sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Tese, tendo sido a candidata **aprovada** pelos seus membros. Proclamados os resultados pelo Professor Doutor **Rodrigo Donizete Euzébio - IME/UFG**, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos Membros da Banca Examinadora, ao trigésimo dia do mês de setembro do ano de dois mil e vinte e dois.

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Piecewise smooth vector fields: Conley index and asymptotic analysis

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Angie Tatiana Suárez Romero

Bacharela em matemática pela Universidade Distrital Francisco José de Caldas - UD de Bogotá-Colômbia, e obteve o grau de mestre em matemática pela Universidade Federal de Goiás - UFG em Goiás-Brasil.

A minha mãe Mariela Romero no ceú;
A minha irmã Valentina Suárez;
A meu companheiro de vida Martín Barajas;
A meu pai Enrique e minha tia Alicia.

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Agradeço à minha família pela confiança e apoio que sempre tive deles. Em especial à minha irmã Valentina e meu companheiro de vida Martín Barajas que em todo momento estiveram comigo.

Também agradeço à minha tia Alicia Romero pela motivação nos meus estudos profissionais e pela força que sempre me deu para não desistir ao longo do caminho, também ao meu tio Manuel Romero e ao meu pai Enrique Suárez.

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La soledad le había seleccionado los recuerdos, y había incinerado los entorpecedores montones de basura nostálgica que la vida había acumulado en su corazón, y había purificado, magnificado y eternizado los otros, los más amargos.

Gabriel García Márquez.
Cien años de soledad

Resumo

Angie Tatiana Suárez Romero. **Piecewise Smooth Vector Fields: Conley Index and Asymptotic Analysis**. Goiânia, 2022. 100p. Tese de Doutorado. Instituto de Matemática e Estatística, Universidade Federal de Goiás.

Nesta tese propomos usar a teoria de Conley para investigar uma classe de campos vetoriais suaves por partes (PSVF) onde é possível construir um sistema semi-dinâmico usando a convenção de Fillipov para PSVF. Inicialmente, construímos um semifluxo gerado pelas trajetórias positivas de um PSVF em uma variedade fechada tridimensional com uma variedade de comutação sem regiões de escape. Além disso, o resultado é estendido para qualquer variedade de dimensão finita onde a variedade de comutação admite apenas regiões de cruzamento. Assim, construímos um sistema semi-dinâmico que nos permite aplicar a teoria clássica de Conley para garantir a existência de órbitas periódicas para uma classe de PSVF. Posteriormente, fazemos algumas aplicações deste resultado em dimensão dois e aplicações em sistemas biológicos de dimensão 3. Além disso, estudamos o comportamento de trajetórias positivas próximas à origem na classe de sistemas suaves por partes com origem do tipo cúspide-dobra, resultado na linha do teorema de Poincaré-Bendixson para esta classe de sistemas em dimensão 3.

Palavras–chave

Índice de Conley, campos de vetores suaves por partes, cúspide-dobra, Poincaré Bendixson.

Abstract

Angie Tatiana Suárez Romero. **Piecewise Smooth Vector Fields: Conley Index and Asymptotic Analysis**. Goiânia, 2022. 100p. PhD. Thesis. Instituto de Matemática e Estatística, Universidade Federal de Goiás.

In this thesis, we propose to use Conley theory to investigate a class of piecewise smooth vector fields (PSVF) where it is possible to construct a semi-dynamical system by using Fillipov's convention for PSVF. Initially, we build a semi-flow generated by the forward trajectories of a PSVF on a closed three-dimensional manifold with a switching manifold without escaping regions. Furthermore, the result is extended to any finite-dimensional manifold where the switching manifold admits only crossing regions. Thus, we build a semi-dynamical system that allows us to apply the classical Conley theory to guarantee the existence of periodic orbits for the PSVF class. Subsequently, we make some applications of this result in dimension two, as well as applications in systemic biology with systems of dimension 3. On the other hand, we study the behavior of positive trajectories near the origin in the class of piecewise smooth systems having the origin of cusp-fold type. A result along the lines of the Poincaré-Bendixson theorem for this class of systems in dimension 3 is obtained.

Keywords

Conley index, piecewise smooth vector field, cusp-fold singularity, Poincaré Bendixson.

Introduction

A dynamical system describes the evolution of a phenomenon throughout time, where time could be considered discrete or continuous. At the beginning of the 19th century Henri Poincaré, while investigating the movement of the planets and interested in the study of periodic orbits, gave rise to what today is known as modern dynamical systems; his work combined topology and geometry, thus conducting a qualitative study of dynamical systems. Later, David Hilbert, at the Second International Congress on Mathematics in 1900, proposed a list of 23 relevant problems, now called Hilbert problems. One of the most famous and still unsolved problems being of the 16th, which refers to finding the maximum number of limit cycles of a polynomial vector field of degree greater than or equal to 2. This great interest in the existence of periodic orbits and limit cycles is due to the fact that, in general, it is not easy to explicitly obtain solutions of a dynamical system.

In dynamical systems, a modern research topic concerns the piecewise smooth vector fields (PSVF shortened). PSVFs are vector fields that are not completely differentiable but are differentiable in parts, where the trajectory of a vector field is suddenly interrupted and changed by another different one. These systems are widely used to model problems associated with control theory, economics and biology, see [2]. We are interested in guaranteeing the existence of periodic orbits in PSVF. Moreover, in 1978 Charles Conley introduced a new topological index called the Conley Index as a generalization of the Morse Index. The Conley index guarantees the existence of invariant sets within a particular compact set; we are interested in invariant sets corresponding to periodic orbits. One of the main goals of this work is to use Conley's theory to obtain periodic orbits in PSVF.

Conley theory has a significant number of applications in the study of dynamical and semidynamical systems. In [10], Charles Conley began to develop this theory for two-sided flows in compact or locally compact spaces and this was continued by Dietmar Salamon, see [43], and extended to semiflows by Rybakowski, see [41]. In the paper [31], the authors presented a useful result for finding periodic orbits in semidynamical systems; this result is the main tool in Chapter 2. A modern application in the search for periodic orbits in neuroscience is found in the thesis [42], where the author uses numerical techniques to obtain periodic orbits in dynamics given by Competitive Threshold-Linear Networks and Wilson-Cowan networks.

The Conley theory has been developed for continuous and discrete dynamical systems, multiflows, and semiflows. With respect to flows, one of the first works related to Conley Theory and discontinuous systems corresponds to [8]. The authors use a regularization of a discontinuous vector field, see [46], to adapt Conley index for continuous flows. They define the notion of D-Conley index and show its invariant under homotopy. Most recently, Cameron Thieme has been extending the Conley theory for multiflows. The main objective being to generalize Conley index theory to differential inclusions having the Filippov systems as motivation. Firstly, in [52], it was introduced introduces differential inclusions and Filippov systems, and it was showed the existence of a multiflow for this class of system. Subsequently, in the preprints [51] and [53], the author exposes a definition of perturbation (see Definition 3.1 in [51]) and shows that both the isolating neighborhoods and the attractor-repeller decomposition are stable, which are essential objects in Conley theory.

Our approach is to construct a semiflow for Filippov systems in order to apply the well-established results of Conley theory. Recently, related work has been done by Mrozek and Wanner in [38]. Their engaging paper shows the construction of a continuous semiflow on a finite topological space X for a combinatorial vector field (a discretization of a piecewise smooth vector field).

Another important aspect of the modern dynamical system theory studied by Poincaré is description the asymptotic behavior of solutions and the structure of their limit sets. The Poincaré-Bendixson theorem guarantees that, under suitable hypotheses, a limit set can be an equilibrium point, a periodic solution, or closed curves consisting of a finite number of equilibrium points connected by regular orbits. Some advantages obtained when working with smooth systems in dimension two is that the Poincaré-Bendixson theorem is valid, besides being valid also on some surfaces such as the sphere. Recently, a significant and important topic in research is to answer whether, in the case of PSVF, these results are true or not. A first result in this context can be found in [3].

Another important object concerning PSVF are the tangency points. Particularly in dimension three, there are two important types of generic tangential singularities. One of them corresponds to points where the contact with a co-dimension one smooth manifold is quadratic, and the second where it presents a cubic contact. These are called fold and cusp singularities, respectively. Some work addressing three-dimensional PSVFs are [54], [7] and [13]. In Chapter 4, we study the behavior of forward orbits near the origin, in the spirit of the Poincaré-Bendixson theorem but in dimension three, for a class of piecewise smooth systems having the origin as a cusp-fold singularity.

The overall description of the main results and, in general, the structure of this thesis is as follows:

- In the first chapter, we present the definitions, notations, and the main results on Conley index and discontinuous systems used in this thesis. With respect to the Conley index, we provide a sketch of the proof of the theorem of [31].
- In Chapter 2, we prove that the positive trajectories generated by a piecewise-smooth vector field defined on a closed 3-manifold without escape region produce a semidynamical system. With this result, and using the main theorem of the paper [31] we guarantee the existence of periodic orbits in an isolating neighborhood. Finally, we present some applications.
- In Chapter 3, we present an application of the main result of the previous chapter in biological systems. We present a method that guarantees the existence of periodic orbits for a particular network called Repressilator.
- In Chapter 4, we analyze the behavior of positive trajectories near the origin in a PSVF having the origin of cusp-fold type. We state three results along the lines of the Poincaré-Bendixson theorem for this class of vector fields.
- Finally, in Chapter 5, besides concluding remarks, we indicate some future and in-progress works that arise from this thesis.

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CHAPTER 1

Preliminaries

This preliminary chapter recalls some basic concepts and results from Conley theory and the Filippov convention for piecewise-smooth vector fields defined in dimension 3 which is the main object of study in this work. In Chapter 2, we use topological tools and in Chapter 4 analytics tools.

Throughout this thesis, we denote the intervals $(-\infty,0]$ and $[0,\infty)$ by $\mathbb{R}^-$ and $\mathbb{R}^+$, respectively. Let X be a topological space and f a real-valued function on X. We say that f is *upper semi-continuous* at x^* if and only if $\limsup_{x\to x^*} f(x) \leq f(x^*)$. We adopt the following definition of semidynamical systems, [44].

Definition 1.1 *The pair* (X,ϕ) *is called a continuous semiflow or continuous semidynamical system if* X *is a topological space satisfying the Hausdorff property and* ϕ *is a map* $\phi:\mathsf{X}\times\mathbb{R}^+\longrightarrow\mathsf{X}$*, satisfying the following properties:*

i. $\phi(x,0)=x$*, for every* $x\in\mathsf{X}$ *(initial value property);*

ii. $\phi(\phi(x,t),s)=\phi(x,t+s)$*, for each* $x\in\mathsf{X}$ *and* $t,s\in\mathbb{R}^+$ *(semigroup property);*

iii. ϕ *is continuous on the product space* $\mathsf{X}\times\mathbb{R}^+$ *(continuity property).*

In order to simplify the notation, in some cases, we denote $\phi(x,t)$ by $x\cdot t$.

1.1 Conley Index

In this section, we review Conley index theory, for more details see [9], [10], [31], [32], [33], [35] and [43]. Throughout this section assume that X is a metric space and $\phi:\mathsf{X}\times\mathbb{R}^+\longrightarrow\mathsf{X}$ a semiflow. A subset $S\subset\mathsf{X}$ is said to be an *invariant set* with respect to the semiflow ϕ if, for all $p\in S$, one has $p\cdot t\in S$ for all $t\in\mathbb{R}^+$. In other words, $\phi(S,\mathbb{R}^+)=S$. Let $N\subset\mathsf{X}$ be a subset of X. The *maximal invariant set of* N is defined by: $\mathrm{inv}(N)=\{x\in\mathsf{X}|\ x\cdot t\in N,\text{ for all } t\in\mathbb{R}^+\}$. A subset $S\subset\mathsf{X}$ is called an *isolated invariant set* if there exists a compact neighborhood N of S in

X such that $S \subset \text{int}(N)$ and $S = \text{inv}(N)$. In this case, N is said to be an *isolating neighborhood* for S in X. The set N is called an *isolating neighborhood* for ϕ if it is closed, contained in the domain of ϕ, and $\text{inv}(N) \subset \text{int}(N)$.

Definition 1.2 *Let $S \subset \mathsf{X}$ be an isolated invariant set. A pair (N, L) of compact sets in X is said to be an index pair for S in X if $L \subset N$ and*

1. *$\overline{N \setminus L}$ is an isolating neighborhood for S in X;*
2. *L is positively invariant in N, that is, if $x \in L$ and $x \cdot [0, T] \subset N$ then $x \cdot [0, T] \subset L$;*
3. *L is the exit set of the semiflow, that is, if $x \in N$ and $x \cdot \mathbb{R}^+ \subsetneq N$ then there exists $T > 0$ such that $x \cdot [0, T] \subset N$ and $x \cdot T \in L$.*

Conley defines isolating blocks in [9], as follows.

Definition 1.3 *Let V be a smooth vector field on a smooth manifold M and let $\phi_V : M \times \mathbb{R} \longrightarrow M$. Let $N \subset M$ be a smooth compact submanifold (with boundary) of M with $\dim\ N = \dim\ M$, and let $\partial N = n$. Define*

$$
\begin{aligned}
n^+ &:= \{p \in n |\ \exists \epsilon > 0 \ \textit{with}\ \ p \cdot (-\epsilon, 0) \cap N = \emptyset\}, \\
n^- &:= \{p \in n |\ \exists \epsilon > 0 \ \textit{with}\ \ p \cdot (0, \epsilon) \cap N = \emptyset\}, \\
\tau &:= \{p \in n | X \ \ \textit{is tangent to}\ \ n \ \ \textit{at}\ \ p\}.
\end{aligned}
$$

Definition 1.4 *Using the notation of Definition 1.3, N is an isolating block for ϕ_V if $n^+ \cap n^- = \tau$, if τ is a smooth submanifold of n with codimension one and (as a consequence) n^+ and n^- are submanifolds with common boundary τ.*

It follows that an isolating block is an isolating neighborhood.

Given a pair (N, L) of topological spaces with $L \subset N$ and $L \neq \emptyset$, define:

$$x \sim y \Leftrightarrow x = y \text{ or } x, y \in L. \tag{1-1}$$

Denote by N/L the pointed space $(N/\!\sim, [L])$. Figures 1.1 and 1.2 show the index pair (N, L) for various hyperbolic invariant sets.

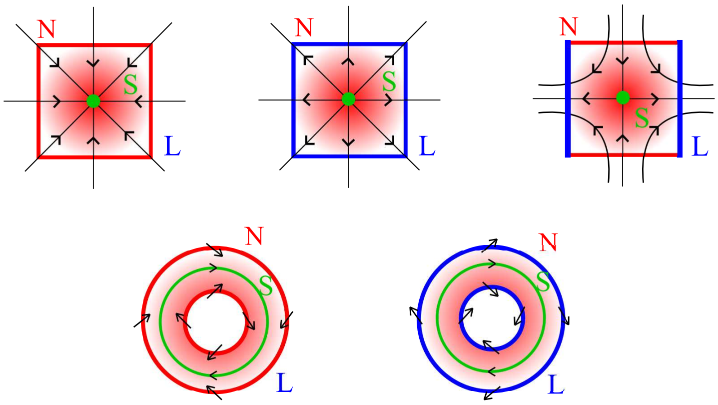

Figure 1.1: The index pair (N, L) for five hyperbolic invariant sets in dimension two.

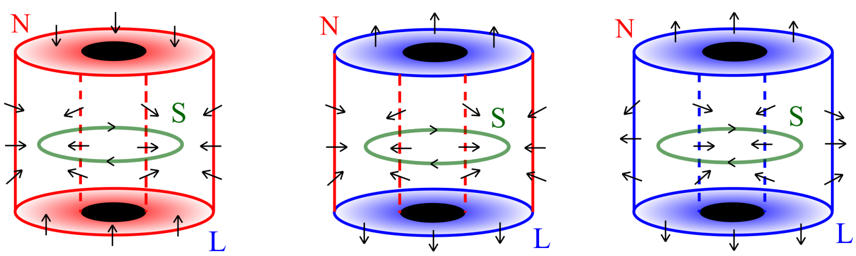

Figure 1.2: The index pair (N, L) for three hyperbolic invariant sets in dimension three.

Next, we present several definitions that we use throughout this work.

Definition 1.5 *The Homotopy Conley Index of S is defined as the homotopy type of the pointed space N/L, where (N, L) is an index pair for S.*

Note that, by definition, the homotopy Conley index is the homotopy type of a topological space. The homotopy Conley index is well defined, that is, it does not depend on the isolating neighborhood that isolates the isolating set, see [32]. Unfortunately, operating with homotopy classes of spaces is difficult. It is helpful to consider the cohomology Conley index to evade this problem.

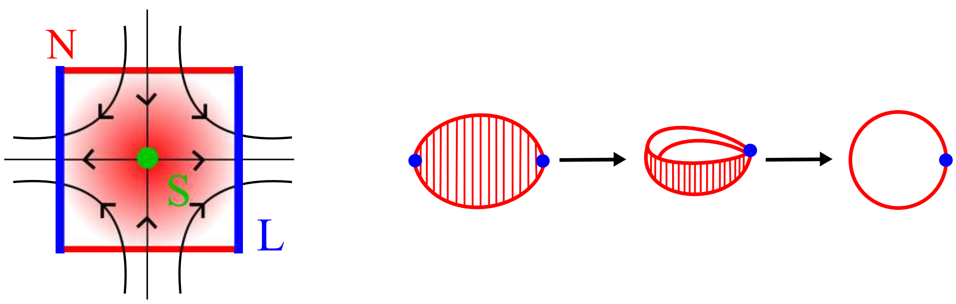

Figure 1.3: Homotopy type of the pointed space N/L for inv N correspond to a saddle.

Definition 1.6 *Let S be an isolated invariant set with respect to the semiflow ϕ and let (N,L) be an index pair for S. The cohomology Conley index of S is defined as*

$$CH^*(S) = CH^*(S,\phi) := H^*(N/L) \approx H^*(N,L) \tag{1-2}$$

where H^* denotes the Alexander-Spanier cohomology with integer coefficients.

The robustness under perturbation of an isolating neighborhood is one of its most important property. To state formally this property, called the continuation theorem, we need the following definition.

Definition 1.7 *Let $N \subset \mathsf{X}$ be a compact set. Let ϕ_λ be a 1-parameter family of flow and $S_\lambda = \mathrm{inv}(N,\phi_\lambda)$. Two isolated invariant sets S_{λ_0} and S_{λ_1} are related by continuation or S_{λ_0} continues to S_{λ_1} if N is an isolating neighborhood for all $S_\lambda, \lambda \in [\lambda_0,\lambda_1]$.*

Now, we state this important theorem for the Conley index, whose proof is nontrivial and it can be found in [43].

Theorem 1.8 (Continuation Property) *Let S_{λ_0} and S_{λ_1} be isolated invariant sets that are related by continuation. Then,*

$$CH^*(S_{\lambda_0}) \approx CH^*(S_{\lambda_1}).$$

Let us now consider, as an example, the Conley index of a stable periodic orbit in two dimensions.

Example 1.9 *The homotopy type of the pointed space N/L of a stable periodic orbit in two dimensions is equal to the homotopy type of $S^1 \vee S^0$ (see Figure 1.4), hence*

$$CH^k(S) \approx \begin{cases} \mathbb{Z}, & k = 0,1, \\ 0, & \textit{otherwise}. \end{cases}$$

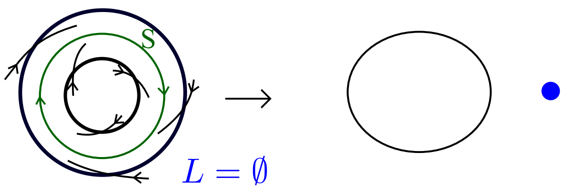

Figure 1.4: The homotopy type of the pointed space N/L of a stable periodic orbit in dimension two.

The following result generalizes Example 1.9.

Proposition 1.10 (Mischaikow, [32]) *Let S be a hyperbolic periodic orbit with an oriented unstable manifold of dimension $n+1$. Then*

$$CH^k(S) \approx \begin{cases} \mathbb{Z}, & k = n, n+1, \\ 0, & \text{otherwise.} \end{cases}$$

1.1.1 Discrete Conley index

In this subsection, we present some definitions and results on discrete Conley index. Further details can be found in [31] and [36]. Let $f : \mathsf{U} \to \mathsf{X}$ be a continuous map defined on a subset U of X. Denote it by $f : \mathsf{X} \nrightarrow \mathsf{X}$ this map. We say that K is an *invariant set* if $f(K) = K$. A function $\gamma_x : \mathbb{Z} \longrightarrow \mathsf{X}$ is called *a (full) solution* to f through $x \in \mathsf{X}$ if $f(\gamma_x(n)) = \gamma_x(n+1)$ for all $n \in \mathbb{Z}$ and $\gamma_x(0) = x$. A full solution γ_x is in $N \subset \mathsf{X}$ if $\gamma_x(\mathbb{Z}) \subset \mathbb{N}$. If $N \subset \mathsf{X}$ then we define *the maximal invariant set* in N by $\text{inv}(N) := \{x \in N \mid \exists \text{ a full solution } \gamma_x \text{ in } N\}$.

Let N be a compact subset of X. We say that N is an *isolating neighborhood* for f if N is closed, contained in the domain of f and $\text{inv}(N) \subset \text{int } N$. A subset K of X is called an *isolated invariant set* if there exits an isolating neighborhood N of K such that $\text{inv}(N) = K$, and $A \subset N$ is *positively invariant* with respect to N if, and only if, $A \cap f^{-1}(N) \subseteq f^{-1}(A)$. If N is an isolating neighborhood for K, we define the sets

- $\text{inv}^+(N) := \{x \in \mathsf{X} \mid \forall i \in \mathbb{Z}^+,\ f^i(x) \in \mathbb{N}\}$,
- $\text{inv}^-(N) := \{x \in \mathsf{X} \mid \forall i \in \mathbb{Z}^-,\ f^i(x) \in \mathbb{N}\}$,
- $\text{inv}(N) := \text{inv}^+(N) \cap \text{inv}^-(N)$.

Definition 1.11 *A pair (P_1, P_2) of compact subsets of N is called an index pair of K in N (with respect to f) if, and only if, the following three conditions are satisfied:*

1. *P_1 and P_2 are positively invariant with respect to N;*

2. *$\text{inv}^-(N) \subseteq \text{int}_N P_1$ and $\text{inv}^+(N) \subseteq N \setminus P_2$;*

3. *$P_1 \setminus P_2 \subseteq \text{int } N \cap f^{-1}(\text{int } N)$.*

The family of all index pairs in N is denoted by $IP(N)$. In [34], the author shows that for every neighborhood W in K there exists an index pair $P = (P_1, P_2)$ in N such that $P_1 \setminus P_2 \subseteq W$.

Example 1.12 *Let $\mathsf{X} = S^2 = \mathbb{R}^2 \cup \{\infty\}$ and $f : \mathsf{X} \longrightarrow \mathsf{X}$ be a continuous map such that $f(R_0) = S_0$ and $f(R_1) = S_1$, see Figure 1.5. This map is called a G-horseshoe,*

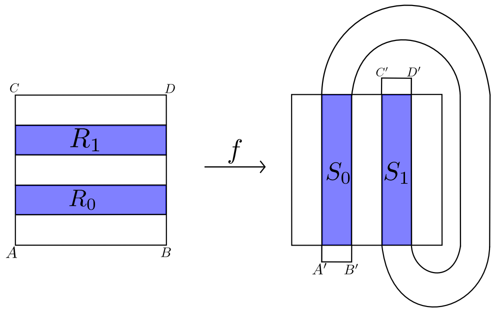

Figure 1.5: G-horseshoe map.

see [36].

We are interested in finding an index pair for $\mathrm{inv}(N)$*, where* N *is the square* $ABCD$*. The description of the dynamics of* f *in* N *is completely analogous with Smale's horseshoe dynamics that can be found in [40]. We take* $N = [0,5] \times [0,5]$ *an isolating neighborhood and let us show that* $P = (P_1, P_2)$ *is an index pair of* N *where* $P_1 = [1,4] \times [0,5]$ *e* $P_2 = [1,4] \times ([0,1] \cup [2,3] \cup [4,5])$.

1. *Note that* $P_1 \cap f^{-1}(N) \subseteq f^{-1}(P_1)$ *and* $P_2 \cap f^{-1}(N) \subseteq f^{-1}(P_2)$*, that is,* P_1 *and* P_2 *are positively invariant with respect to* N*, see Figure 1.6.*

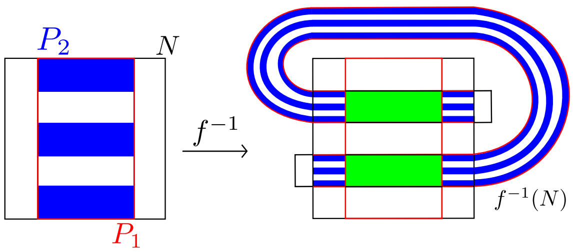

Figure 1.6: The sets P_1 and P_2 are positively invariant.

2. *Observe that* $\mathrm{inv}^-(N) \subseteq \mathrm{int}_N P_1$*, in fact* $\mathrm{inv}^-(N) = \bigcap_{n\geq 0} f^n(N) \subseteq N \cap f(N) \cap f^2(N)$*, see Figure 1.7. Similarly,* $\mathrm{inv}^+(N) \subseteq N \setminus P_2$*, in fact* $\mathrm{inv}^+(N) = \bigcap_{n\leq 0} f^n(N) \subseteq N \cap f^{-1}(N) \cap f^{-2}(N)$.

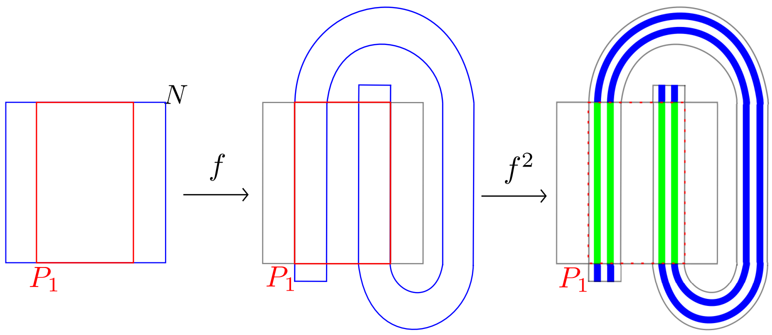

Figure 1.7: $\text{inv}^- N \subseteq \text{int}_N P_1$.

3. *Finally, we have that* $P_1 \setminus P_2 \subseteq \text{int } N \cap f^{-1}(\text{int } N)$, *see Figure 1.8.*

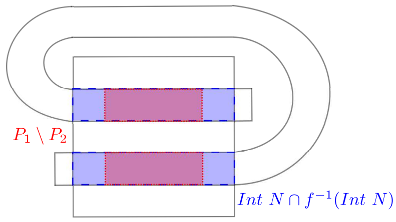

Figure 1.8: $P_1 \setminus P_2 \subseteq \text{int } N \cap f^{-1}(\text{int } N)$.

Leray functor

This subsection is based in [36]. Let ϵ be the category of graded modules over a ring Ξ and homomorphisms of degree zero. We denote by $\epsilon(E,F)$ the set of all morphisms of E in F with $E,F \in \epsilon$. Let ϵE be the category of graded modules equipped with an endomorphism. The objects of this category are pairs (F,f), where $F \in \epsilon$ and $f \in \epsilon(F,F)$ is an endomorphism.

Fix (F,f) and (E,e) in ϵE. A morphism between (F,f) and (E,e) is a map $\varphi : F \longrightarrow E$ such that the following diagram commutes

$$\begin{array}{ccc} F & \xrightarrow{f} & F \\ \varphi \downarrow & & \downarrow \varphi \\ E & \xrightarrow{e} & E \end{array}$$

We define the covariant functor LM which transforms a pair $(F,f) \in \epsilon E$ in a pair in ϵM. We fix $(F,f) \in \epsilon E$ and we define the generalized kernel $gker(f)$ of f as

$$gker(f) := \bigcup \{f^{-n}(0) | n \in \mathbb{N}\} = \{x \in F | f^n(x) = 0 \text{ for some } n \in \mathbb{N}\}.$$

As $f(gker(f)) \subseteq gker(f)$, we have the following induced monomorphism

$$\begin{aligned} f' : F/gker(f) &\longrightarrow F/gker(f) \\ [x] &\longmapsto [f(x)]. \end{aligned}$$

And so, we define $LM(F,f) := (F/gker(f), f') \in \epsilon$.

We take $\varphi : (E,e) \longrightarrow (F,f)$ a morphism, then $\varphi(gker(e)) \subseteq gker(f)$, and so, we have an induced map

$$\varphi' : \ E/gker(e) \ni [x] \longrightarrow [\varphi(x)] \in F/gker(f).$$

Therefore the diagram below is commutative:

$$\begin{array}{ccc} E/gker(e) & \xrightarrow{e'} & E/gker(e) \\ \varphi' \downarrow & & \downarrow \varphi' \\ F/gker(f) & \xrightarrow{f'} & F/gker(f) \end{array}$$

We define $LM(\varphi) := \varphi'$, then LM is a covariant functor $LM : \epsilon E \longrightarrow \epsilon M$.

Now, we define another covariant functor LI which transforms a pair $(F,f) \in \epsilon M$ in a pair belonging to LI. Assume that $(F,f) \in \epsilon E$. We define the generalized image of f as

$$gim(f) := \bigcap \{f^n(F) | n \in \mathbb{N}\}.$$

As $f(gim(f)) \subseteq gim(f)$, we can consider the contraction

$$f'' : \ gim(f) \ni x \longrightarrow f(x) \in gim(f),$$

and we obtain that f'' is a monomorphism that is also an epimorphism, then f'' is an isomorphism. We define $LI(F,f) := (gim(f), f'') \in \epsilon I$.

Now, we take $(E,e),(F,f) \in \epsilon M$ and $\varphi : (E,e) \longrightarrow (F,f)$, we have that $\varphi(gim(e)) \subseteq gim(f)$. Let φ'' be the contraction $\varphi'' : \ gim(e) \ni x \longrightarrow \varphi(x) \in gim(f)$, causing the diagram below to be commutative

$$\begin{array}{ccc} gim(e) & \xrightarrow{e''} & gim(e) \\ \varphi'' \downarrow & & \downarrow \varphi'' \\ gim(f) & \xrightarrow{f''} & gim(f) \end{array}$$

We define $LI(\varphi) := \varphi''$, so LI is a covariant functor $LI:\ \epsilon M \longrightarrow \epsilon I$.

Definition 1.13 *The functor $L:\ \epsilon E \longrightarrow \epsilon I$ defined by the composition $LI \circ LM$ is called Leray functor.*

Let $P = (P_1, P_2)$, $Q = (Q_1, Q_2)$, $R = (R_1, R_2)$ and $S = (S_1, S_2)$ be are index pairs such that $R \subseteq P$ and $S \subseteq Q$, and $f : P_1 \longrightarrow Q_1$ with $f(R) \subseteq S$. Consider the map $f_{R,S} : R_1 \longrightarrow S_1$, hence it $R_1 \cap R_2$ into $S_1 \cap S_2$, $f_{R,S}$ is be called the contraction of f to the pair of pairs (R, S). Moreover, we denote by $i_{P,S(P)}$ the contraction of the identity map $id : S(P) \longrightarrow S(P)$ to the pair of pairs (R, S).

Let $f : \mathsf{X} \longrightarrow \mathsf{X}$ be a homeomorphism and N an isolating neighborhood with respect to f. We consider an index pair $P = (P_1, P_2) \in IP(N)$, and we define the index pairs

$$S(P) := (P_1 \cup f(P_2), P_2 \cup f(P_2)),$$
$$T(P) := (P_1 \cup (X \setminus \text{int } N), P_2 \cup (X \setminus \text{int } N)).$$

Proposition 1.14 (Mrozek-1990, [36]) *Let $P \in IP(N)$. Then $f(P) \subseteq S(P) \subseteq T(P)$ and the inclusions $i_{P,S(P)}$, $i_{S(P),T(P)}$, and $i_{P,T(P)}$ induce isomorphisms in the Alexander-Spanier cohomology.*

We define $f_P := f_{P,T(P)}$ and $i_P := i_{P,T(P)}$. Then by Proposition 1.14, we have that $H^*(i_P)$ is an isomorphism.

Definition 1.15 *The endomorphism $H^*(f_P) \circ H^*(i_P)^{-1}$ of $H^*(P)$ is called the index map associated to P and denoted by I_P.*

Theorem 1.16 *Let K be an isolated invariant set with respect to f. Then for all isolating neighborhoods N, M of K, and for all $P \in IP(N)$, $Q \in IP(M)$, the reductions of Leray of the Alexander-Spanier cohomologies $L(H^*(P), I_P)$ and $L(H^*(Q), I_Q)$ are isomorphic.*

Corollary 1.17 *The Leray reduction of the Alexander-Spanier cohomology of an index pair of an isolated invariant set K depends only on K.*

Definition 1.18 *The cohomological Conley index of K is $L(H^*(P), I_P)$, where P is an index pair of K in N.*

Therefore, the cohomological Conley index of K consists of a graded module over a ring Ξ equipped with a zero-degree endomorphism. This module is represented by $CH^*(\text{inv}(N), f)$ and the isomorphism by $\chi^*(\text{inv}(N), f)$. So,

$$Con^*(\text{inv}(N), f) = (CH^*(\text{inv}(N), f), \chi^*(\text{inv}(N), f)).$$

Remark 1.1 *The Conley index satisfies the Wazewski property of additivity and the homotopy (continuation) property.*

Direct group limit

We define a *directed set* as a non-empty set $\mathcal{I}$ together with a reflexive and transitive binary relation $\leq$, with the property that for any $a \leq b$ in $\mathcal{I}$ there exists c in $\mathcal{I}$ such that $a \leq c$ and $b \leq c$. Let $(\mathcal{I}, \leq)$ be a directed set, $\{A_i | i \in \mathcal{I}\}$ be a family of groups indexed by $\mathcal{I}$ and $f_{ij} : A_i \longrightarrow A_j$ homomorphisms such that for every $i \leq j$, f_{ii} corresponds to the identity map in A_i and $f_{ik} = f_{jk} \circ f_{ij}$, for all $i \leq j \leq k$. $\langle A_i, f_{ij} \rangle$ is called a *direct system* in $\mathcal{I}$.

Definition 1.19 *The direct limit of the system* $\langle A_i, f_{ij} \rangle$ *is* $\varinjlim A_i := \sqcup_i A_i / \sim$ *where if* $x_i \in A_i$ *and* $x_j \in A_j$ *then* $x_i \sim x_j$ *if, and only if, there exist* $k \in \mathcal{I}$ *with* $i \leq k$ *and* $j \leq k$ *such that* $f_{ik}(x_i) = f_{jk}(x_j)$.

We denote by $End(R-GMod)$ the category of endomorphisms of graded R-modules. Similarly, $Auto(R-GMod)$ denotes the category of automorphisms of graded R-modules. The objects of these categories are pairs (F, f), where $F \in R-GMod$ and $f : F \longrightarrow F$ is an endomorphism (automorphism).

Fix (F, f) and (E, e) in $End(R-GMod)$. A morphism between (F, f) and (E, e) is a map $\varphi : F \longrightarrow E$ such that the following diagram commutes

$$\begin{array}{ccc} F & \xrightarrow{f} & F \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ E & \xrightarrow{e} & E. \end{array}$$

Note that, $\varinjlim$ is a functor from $End(R-GMod)$ in $Auto(R-GMod)$. This functor assigns to each endomorphism $f : E \longrightarrow E$ in $End(R-GMod)$ the automorphism induced by the direct limit of the sequence:

$$E \xrightarrow{f} E \xrightarrow{f} E \xrightarrow{f} \cdots.$$

In this case, we have the direct system $\langle E_i, e_{ij} \rangle$ such that $E_i = E$ and $e_{ij} = f$ for all $i, j \in \mathcal{I} = \mathbb{N}$. Then, $\varinjlim E_n = \sqcup_n E_n / \sim$ where $g_n \sim g_m$ if and only if $g_m = f^{m-n}(g_n)$. Each element of $\sqcup_n E_n$ can be written as (n, g), and so, the map $(n, g) \to (n, f(g))$ induces an automorphism in $\varinjlim E$.

Definition 1.20 *Let K be an isolated invariant set for a discrete dynamical system $f : \mathsf{X} \longrightarrow \mathsf{X}$. Let P be an index pair for K. The Conley index for K is given by*

$$Con^*(K) := (CH^*(K,f), \chi^*(K,f)) = \varinjlim\{(H^*(P), \mathcal{I}^*{}_P)\}.$$

By properties of direct limit, we have that $\varinjlim\{(H^*(P), \mathcal{I}^*{}_P)\} \cong \varinjlim\{(H^*(P'), \mathcal{I}^*{}_{P'})\}$ for $P = (P_1, P_2)$ and $P' = (P'_1, P'_2)$ index pairs for an isolated invariant set K. The Conley index satisfies the Wazewski property of additivity and homotopy (continuation) property.

Poincaré section

The content of this subsection can be found in [31]. Suppose N is an isolating neighborhood for a semiflow φ with $S = \text{inv}(N)$. Assume that for a pair of sequences $\{x_n\}_{n=1}^{\infty} \subset N$ and $\{k_n\}_{n=1}^{\infty} \subset \mathbb{N}$ the map $f^i(x_n)$ is defined and contained in N for every $n \in \mathbb{N}$ and $i = 0, 1, 2, \ldots, k_n$. We called N an *admissible isolating neighborhood* if for every such pair of sequences, the set $\{f^{k_n}(x_n)\}_{n=1}^{\infty}$ has compact closure. Particularly, every compact isolating neighborhood is admissible.

We said that $\Xi \subset X$ is a *local section* for φ and N if there exists $\xi > 0$ such that

$$C_\xi^N(\Xi) := \{x \in N | x \cdot (0, \xi) \cap \Xi \neq \emptyset\}$$

is open in N and for every $x \in C_\xi^N(Cl(\Xi))$ there exists an unique element of $x \cdot (0, \Xi) \cap \Xi$.

Definition 1.21 *$\Xi \subset X$ is a **Poincaré section** for φ in N if:*

1. *Ξ is a local section,*
2. *$\Xi_N := \{\Xi \cap N\}$ is closed and*
3. *for every $x \in N$, $x \cdot (0, \infty) \cap \Xi \neq \emptyset$.*

Observe that it is not necessary to know S to find a Poincaré section. Also it is not required to be a subset of N. Certainly, if N has an exit set, then for any subset of N, there will be points in N whose orbits exit N before they cross the section again. So no subset of N can be a Poincaré section.

Suppose (Λ, ρ) is a metric space that parameterizes a continuous family of semiflows $\phi^\lambda : \mathsf{X} \times \mathbb{R}^+ \longrightarrow \mathsf{X}$. Furthermore, suppose that for some $\lambda_0 \in \Lambda$, N is an isolating neighborhood for ϕ^{λ_0} and Ξ is a local section for N and λ_0.

Proposition 1.22 (Mccord, Mrozek and Mischaikow,[31]) *Let W be an admissible set for ϕ^λ for all $\lambda \in \Lambda$. Assume that $N \subset W$ is an admissible isolating neighborhood for ϕ^λ for all $\lambda \in \Delta \subset \Lambda$, a neighborhood of λ_0. Furthermore, assume that N has a Poincaré section for ϕ^{λ_0} and that $\text{inv}(N, \phi^{\lambda_0}) \neq \emptyset$. Then, for Δ sufficiently small, given $\lambda \in \Delta$ there exists an isolating neighborhood $N_\lambda \subset W$ for ϕ^λ such that*

$$\text{inv}(N_\lambda, \phi^\lambda) = \text{inv}(N, \phi^\lambda)$$

and N_λ admits a Poincaré section.

The previous proposition is the main result that guarantees the existence of periodic orbits that persist under perturbations of the system. In other words, for nearby semiflows ϕ^{λ_0}, there is an isolating neighborhood N_λ that admits a Poincaré section and has the same maximal invariant set for $\lambda \in \Delta$.

Given a Poincaré section Ξ for N, there exists a subset Ξ_0 of Ξ, opened in Ξ, such that $\Xi_S = \Xi \cap S \subset \Xi_0$ and such that, for every $x \in \Xi_0$, there exist a unique strictly positive minimal time $\pi_\Xi(x)$ with $x \cdot [0, \pi_\Xi(x)] \subset N$ and $x \cdot \pi_\Xi(x) \in \Xi$.

The *Poincaré map Π_Ξ* associated with the Poincaré section Ξ is defined as

$$\begin{aligned} \Pi_\Xi : &\Xi_0 \longrightarrow \Xi \\ &x \longmapsto x \cdot \pi_\Xi(x). \end{aligned} \tag{1-3}$$

Assume again that N is an isolating neighborhood under φ. We adopt the following notation:

- $\tilde{D} = \tilde{D}_\Xi := \{x \in \mathsf{X} | x \cdot (0, \infty) \cap \Xi \neq \emptyset\}$,
- The first crossing time for a point in $\tilde{D}_\Xi$ is given by $\pi(x) = \pi_\Xi(x) := inf\{t > 0 | x \cdot t \in \Xi\}$,
- $D = D_\Xi := \{x \in \tilde{D}_\Xi | x \cdot [0, \pi_\Xi(x)] \subset N\}$.

Then, $\pi_\Xi : \tilde{D}_\Xi \longrightarrow \mathbb{R}$ has an obvious discontinuity at Ξ itself. However, aside from this necessary discontinuity, π_Ξ is continuous.

Theorem 1.23 (Mccord, Mrozek and Mischaikow,[31]) *Assume N is an isolating neighborhood for the semiflow φ defined on a metric space X which admits a Poincaré section Ξ. Let Π denote the corresponding Poincaré map, $S = \text{inv } N$, and $K = \Xi \cap S$. Then there is the following exact sequence of cohomology Conley indices:*

$$\cdots \longrightarrow CH^n(S, \varphi) \longrightarrow CH^n(K, \Pi) \xrightarrow{id - \chi^n(K,\Pi)} CH^n(K, \Pi) \longrightarrow$$

$$CH^{n+1}(S,\varphi) \longrightarrow \cdots$$

Corollary 1.24 (Mccord, Mrozek and Mischaikow,[31]) *In the setting of Theorem 1.23, the Conley index for S under the semiflow φ is determined from $Con^*(K,\Pi)$ by the exact sequence*

$$0 \to Coker(id - \chi^{n-1}(K,\Pi)) \to CH^n(S,\varphi) \to ker(id - \chi^n(K,\Pi)) \to 0.$$

If field coefficients are used for the cohomology, then

$$Coker(id - \chi^n(K,\Pi)) \cong ker(id - \chi^n(K,\Pi))$$

and

$$CH^n(S,\varphi) \cong Ker(id - \chi^{n-1}(K,\Pi)) \oplus Ker(id - \chi^n(K,\Pi)).$$

Compact attraction maps

The content of this subsection can be found in [31]. A map $f : X \longrightarrow Y$ is *locally compact* if every point $x \in X$ admits a neighborhood U such that the closure of $f(U)$ is compact. If, in addition, there exists a compact set A such that for every $x \in X$, $cl\{f^n(x)|n \in \mathbb{N}\} \cap A \neq \emptyset$, then f is called a map of *compact attraction*. A semiflow φ on X is *locally compact* if there exists $t > 0$ such that φ_t is locally compact and it is of compact attraction if φ_t is a map of compact attraction for some $t > 0$.

Theorem 1.25 (Mrozek, [35]) *Assume $f : \mathsf{X} \longrightarrow \mathsf{X}$ is a map of compact attraction and K is an isolated invariant set for f. Then the Conley index of K under f is of finite type.*

Zeta function

Let $V \subset \mathbb{R}^n$ be an open set, and $g : V \longrightarrow \mathbb{R}^n$ a map. We say that $(\mathbb{R}^{\ltimes}, g, V)$ is an admissible triple if the set $Fix(g) := \{x \in V | g(x) = x\}$ is compact. The content of this subsection can be found in [31] and [15].

Definition 1.26 *Consider an admissible triple $(\mathbb{R}^n, g, V)$, $F = Fix(g)$ and*

$$(i - g)_* : H_n(V, V \setminus F) \to H_n(\mathbb{R}^n, \mathbb{R}^n \setminus 0) \cong \mathbb{Z}.$$

Where $(i - g)(x) = x - g(x)$. The fixed point index $I_g \in \mathbb{Z}$ of g is given by

$$(i - g)_*(0_F) = I_g 0_o,$$

where 0_o *is the generator of* $H_n(\mathbb{R}^n, \mathbb{R}^n)$ *and* 0_F *is the fundamental class of* $H_n(V, V \setminus F)$ *around* F*.*

Recall our assumptions: X is ANR and K an isolated invariant set of a locally compact map $f: \mathsf{X} \multimap \mathsf{X}$. Then $K \cap Fix(f^n)$ is an isolated set of fixed points of f^n. Take the fixed point index $i(K, f^n) = I_{f^n}$. The (cohomological) zeta function of f in K is defined as a formal power series

$$\zeta_{K,f}(t) := exp\Big(\sum_{n=1}^{\infty} \frac{i(K, f^n)}{n} t^n\Big).$$

Observe that, by properties of the fixed point index, if $\zeta_{K,f} \neq 1$, then f has a periodic point in K.

Theorem 1.27 (Mrozek [37]) *Assume* $f: X \to X$ *is a map of compact attraction and* K *is an isolated invariant set for* f*. Then*

$$\zeta_{K,f} = \prod_{n=0}^{\infty} det(Id - t\chi^n(K, f))^{(-1)^{n+1}}.$$

Define the Euler characteristic of the Conley index of K by $EC(K, f) := \sum_{n=0}^{\infty}(-1)^n dim\ CH^n(K, f)$. Then it can be shown that if $EC(K, f) \neq 0$, then f has a periodic trajectory in K.

Lemma 1.28 (Mccord, Mrozek and Mischaikow,[31]) *Assume* A *is an endomorphism of a finite dimensional vector space. If* 1 *is an eigenvalue of* A*, then* 1 *is an eigenvalue of* A^n *for all* n*. If* 1 *is not an eigenvalue of* A*, then for infinitely many* $n \in \mathbb{Z}^+$*,* 1 *is not an eigenvalue of* A^n*.*

Periodic orbits and Conley index

In this subsection, we state the main result of the Conley theory used in this work.

Theorem 1.29 (Mccord, Mrozek and Mischaikow,[31]) *Assume* X *is an absolute neighborhood retract and* $\varphi: X \times [0, \infty) \to X$ *is a semiflow with compact attraction. If* N *is an isolating neighborhood for* φ *which admits a Poincaré section* Ξ *and either*

$$dim\ CH^{2n}(N, \varphi) = dim\ CH^{2n+1}(N, \varphi) \qquad for \quad n \in \mathbb{Z}^+ \tag{1-4}$$

or

$$dim\ CH^{2n}(N, \varphi) = dim\ CH^{2n-1}(N, \varphi) \qquad for \quad n \in \mathbb{Z}^+, \tag{1-5}$$

where not all the above dimensions are zero, then φ *has a periodic trajectory in* N*.*

Proof.[Sketch]

- Fix $t > 0$ such that φ_t is a compact attraction map. By Theorem 1.25 we have that the Conley index of K under φ_t is of finite type, that is, $CH^*(N,\varphi) = CH^*(N,\varphi_t)$ is finite.
- If $\Pi : \Xi_0 \to \Xi$, then $CH^*(\Xi_N, \Pi)$ is finite.
- We define $s_n := dim\ CH^n(S,\varphi)$ and $k_n := dim\ ker(id - \chi^n(K,\Pi))$. Corollary 1.24 implies that $s_n = k_n + k_{n-1}$ for $n \in \mathbb{Z}$.
- The hypothesis 1-4 implies that $k_{2n+1} = 0$ and $k_{2n} = s_{2n}$ for $n \in \mathbb{N}$. Then, 1 is not eigenvalue of $\chi^{odd}(K,\Pi) := \oplus_{n=0}^{\infty} \chi^{2n+1}(K,\Pi)$ but instead it is an eigenvalue of $\chi^{even}(K,\Pi) := \oplus_{n=0}^{\infty} \chi^{2n}(K,\Pi)$ because not all s_{2n} are zeros.
- Using Lemma 1.28, we have the same result as before for $\chi^{odd}(K,\overline{\Pi}^m)$ $\chi^{even}(K,\overline{\Pi}^m)$ for some m sufficiently large.
- Finally, as $\zeta_{K,\overline{\Pi}^m}$ is well defined, by Theorem 1.27, $\zeta_{K,\overline{\Pi}^m} \neq 1$.

□

Corollary 1.30 *Under the hypotheses of Theorem 1.29, if N has the Conley index of a hyperbolic periodic orbit, then* $\mathrm{inv}(N)$ *contains a periodic orbit.*

1.2 Piecewise smooth vector fields and Filippov systems

In this section, we introduce Filippov's convention for piecewise smooth vector fields (PSVF), see [19]. Let M be a closed n-dimensional C^r manifold. Denote by $\mathfrak{X}(M)$ the space of C^r vector fields on M endowed with the C^r-topology where r is finite and sufficiently large. Let Σ be a codimension one compact submanifold of M that divides M in two regions, i.e. $M = \Sigma^+ \cup \Sigma^-$, where Σ^+ and Σ^- are manifolds with common boundary $\partial\Sigma^+ = \partial\Sigma^- = \Sigma$. Let $h : M \to \mathbb{R}$ be a smooth function such that $h^{-1}(0) = \Sigma$, $h^{-1}([0,\infty)) = \Sigma^+$, $h^{-1}((-\infty,0]) = \Sigma^-$ and 0 is a regular value of h. Define a PSVF as follows

$$Z(p) = \begin{cases} X(p), & \text{if } p \in \Sigma^+, \\ Y(p), & \text{if } p \in \Sigma^-. \end{cases} \tag{1-6}$$

For the points belonging to Σ, we consider the Filippov convention which will be described later. Let $Z = (X,Y) \in \mathfrak{X}(M,h) = \mathfrak{X}(M) \times \mathfrak{X}(M)$ equipped with the product topology. We denote $\Sigma^{\pm} \setminus \Sigma$ by $\text{int}(\Sigma^{\pm})$.

For a vector field, $X \in \mathfrak{X}(M)$, the Lie derivative is defined as an operator $\mathcal{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$, such that $\mathcal{L}_X(Y)$ the Lie derivative of a vector field Y in the direction of the vector field X is the vector field defined by

$$\mathcal{L}_X(Y) := \frac{d}{dt}\Big|_{t=0} (D\varphi(x,t))^{-1} Y(\varphi(x,t)),$$

where $(t,x) \mapsto \varphi(x,t)$ is a local flow of X. Generally $\mathcal{L}_X(Y)$ is defined by $[X,Y]$, where $[\cdot,\cdot]$ corresponds to the Lie bracket. The derivative for a differentiable function h in M is $\mathcal{L}_X(h) = (Xh)(p) = \sum_i m_i(p)\frac{\partial h}{\partial x_i}(p)$, where $\left\{\frac{\partial}{\partial x_i}\right\}$ is a basis associated with a parametrization $\mathbf{x} : \mathcal{U} \subset \mathbb{R}^n \to M$ and each $m_i : \mathcal{U} \to \mathbb{R}$ is a function in $\mathcal{U}$. Then $\mathcal{L}_X(h) = \langle X(p), \nabla h(p)\rangle$, where h indicates the expression of h in the parametrization $\mathbf{x}$. Hence Lie derivatives are

$$Xh(p) = \langle X(p), \nabla h(p)\rangle \text{ and } X^{k+1}h(p) = XX^kh(p) = \langle X(p), \nabla X^k h(p)\rangle,$$

where $\langle \cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$.

Following Filippov's convention in [19], we distinguish the following regions in the switching manifold Σ in $\mathbb{R}^n$:

- ◇ crossing region: $\Sigma^c = \{p \in \Sigma; Xh(p)Yh(p) > 0\}$, $\Sigma^{c+} = \{p \in \Sigma; Xh(p) > 0 \text{ and } Yh(p) > 0\}$ and $\Sigma^{c-} = \{p \in \Sigma; Xh(p) < 0 \text{ and } Yh(p) < 0\}$,
- ◇ escaping region: $\Sigma^e = \{p \in \Sigma; Xh(p) > 0 \text{ and } Yh(p) < 0\}$,
- ◇ sliding region: $\Sigma^s = \{p \in \Sigma; Xh(p) < 0 \text{ and } Yh(p) > 0\}$.

Definition 1.31 *The sliding vector field associated to Z is the vector field tangent to Σ^s in $\mathbb{R}^n$ and defined by*

$$Z^s(p) = \frac{1}{Yh(p) - Xh(p)}(Yh(p)X(p) - Xh(p)Y(p)).$$

If $p \in \Sigma^s$ then $p \in \Sigma^e$ for $-Z$ and then we can define the escaping vector field on Σ^e associated to Z by $Z^e = -(-Z)^s$.

Remark 1.32 *The critical points in Σ^s are considered pseudo-equilibria of Z and the sliding vector field Z^s can be C^r extended beyond the boundary of Σ^s, in fact, if $p \in \partial\Sigma^s$ then:*

1. *if $Xh(p) = 0$ and $Yh(p) \neq 0$ then $Z^s(p) = X(p)$;*

2. *if* $Yh(p) = 0$ *and* $Xh(p) \neq 0$ *then* $Z^s(p) = Y(p)$*;*

3. *if* $Xh(p) = 0$ *and* $Yh(p) = 0$ *then* p *is a pseudo-equilibrium for* Z^s *that is* $Z^s(p) = 0$.

For the following definitions, we consider as references the work of S. M. Vishik [55], J. Sotomayor [49], and J. Sotomayor and M. A. Teixeira [46]. These papers define tangential singularities for vector fields on a manifold with boundary using Thom-Boardman singularities. These definitions for PSVF defined on a compact manifold divided into two pieces by a submanifold of codimension 1 can be found in [20], [21] and [23].

Definition 1.33 *A point* $p \in \partial\Sigma^+$ *is a tangential singularity of* $X \in \mathfrak{X}(\Sigma^+)$ *if the orbit passing through* p *is tangent to the boundary of* Σ^+ *in* p*, i.e.,* $Xh(p) = 0$ *and* $X(p) \neq 0$*. If* $dim(\Sigma^+) = m$*, then* p *is a tangential singularity of order* m *of* X *if* $Xh(p) = X^2h(p) = \cdots = X^{m-1}h(p) = 0$*,* $X^m h(p) \neq 0$ *and the set* $\{Dh(p), DXh(p), DX^2h(p), \cdots, DX^{m-1}h(p)\}$ *is linearly independent.*

When $m = 2$ and $m = 3$, the tangential singularities are called fold and cusp tangential singularities, respectively.

Definition 1.34 *Let* $Z = (X,Y) \in \mathfrak{X}(M,h)$*. A point* $p \in \Sigma$ *is said to be a tangential singularity of* Z *if* $Xh(p)Yh(p) = 0$ *and* $X(p) \neq 0$*,* $Y(p) \neq 0$*.*

Denote by S_X and S_Y the tangency sets of X and Y, respectively. If $Z = (X,Y) \in \mathfrak{X}(M,h)$, then the tangency set of Z is given by $S_Z = S_X \cup S_Y$. Also, denote by S_X^i the set of tangential singularities of order i of X, and by S_X^j the set of tangential singularities of order j of X, then S_X^i and S_Y^j are submanifolds of Σ with codimensions $i-1$ and $j-1$, respectively.

If $p \in \Sigma^c$, then the orbit of $Z = (X,Y) \in \mathfrak{X}(M,h)$ at p is defined as the concatenation of the orbits of X and Y at p. Nevertheless, if $p \in \Sigma \setminus \Sigma^c$ then it a lack of uniqueness of solutions may occur. In this case, the flow Z is multivalued and any trajectory passing through p originating in orbits of X, Y, Z^s and Z^e is considered a solution of Z. More details can be found in [19].

Definition 1.35 *Let* $Z = (X,Y) \in \mathfrak{X}(M,h)$*. A point* $p \in \Sigma$ *is said to be a* Σ*-singularity of* Z *if* p *is either a critical point of* X *or* Y*, or a tangential singularity, or a pseudo-equilibrium of* Z*. Otherwise, it is said to be a regular-regular point of* Z*. Hence we distinguish two more regions in the discontinuity manifold* Σ*:*

- *singular points:* $S_Z = \{p \in \Sigma; Xh(p)Yh(p) = 0, X(p) \neq 0$ *and* $Y(p) \neq 0\}$ *and*

- *equilibrium points:* $S_Z^e = \{p \in \Sigma; X(p)Y(p) = 0\}$.

Definition 1.36 *A point $p \in \Sigma$ is said to be a fold point or of even contact for $X \in \mathfrak{X}(\Sigma^+)$ if for $k > 0$, $Xh(p) = \cdots = X^{2k-1}h(p) = 0$ and $X^{2k}h(p) \neq 0$ and $\{Dh(p), DXh(p), DX^2h(p), \cdots, DX^{2k-1}h(p)\}$ is a linearly independent set. If $X^{2k}h(p) > 0$ (resp. $X^{2k}h(p) < 0$), then p is a visible fold or of visible contact (resp. invisible fold or of invisible contact). If $X \in \mathfrak{X}(\Sigma^-)$, the visibility condition is switched.*

Definition 1.37 *A point $p \in \Sigma$ is said to be a cusp or of odd contact of $X \in \mathfrak{X}(\Sigma^+)$ if for $k > 0$, $Xh(0) = \cdots = X^{2k}h(p) = 0$, $X^{2k+1}h(p) \neq 0$ and $\{Dh(p), DXh(p), DX^2h(p), \cdots, DX^{2k}h(p)\}$ is a linearly independent set.*

Remark 1.38 *Generically, a fold point of X belongs to a local curve of fold points of X with the same visibility, and cusp points occur as isolated points located at the extreme of curves of fold points.*

Resuming the previous results we have the following definition for the local trajectory of a point $p \in M$.

Definition 1.39 *The local trajectory $\varphi_Z(p,t)$ of a PSVF of the form (1-6) through a point $p \in M$ is defined as follows where every interval I contains zero:*

1) *For $p \in int(\Sigma^+)$ and $p \in int(\Sigma^-)$ the trajectory is given by $\varphi_Z(p,t) = \varphi_X(p,t)$ and $\varphi_Z(p,t) = \varphi_Y(p,t)$ respectively, where $t \in I \subset \mathbb{R}$.*

2) *For $p \in \Sigma^{c+}$, the trajectory is defined as $\varphi_Z(p,t) = \varphi_Y(p,t)$ for $t \in I \cap (-\infty, 0])$ and $\varphi_Z(p,t) = \varphi_X(p,t)$ for $t \in I \cap [0, \infty)$. For the case $p \in \Sigma^{c-}$ the definition is the same reversing.*

3) *For $p \in \Sigma^s$, the trajectory is defined as $\varphi_Z(p,t) = \varphi_{Z^s}(t,p)$ for $t \in I \cap [0, \infty)$ and $\varphi_Z(p,t) = \varphi_1(t,p)$ and $\varphi_{Z_1}(p,t)$ is either $\varphi_X(p,t)$, $\varphi_Y(p,t)$ or $\varphi_{Z^s}(p,t)$ for $t \in I \cap (-\infty, 0]$.*

4) *For p a tangential singularity either a visible or an invisible tangency for at least one of the fields X or Y, the trajectory is defined as $\varphi_Z(p,t) = \varphi_1(p,t)$ for $t \in I \cap (-\infty, 0]$ and $\varphi_Z(p,t) = \varphi_2(p,t)$ for $t \in I \cap [0, \infty)$ where each φ_1, φ_2 is either φ_X or φ_Y or φ_{Z^s}.*

5) *For p an equilibrium point, i.e., the equilibrium points of X, Y and Z^s, the trajectory is defined as $\varphi_Z(p,t) = p$ for all $t \in \mathbb{R}$.*

CHAPTER 2

Existence of periodic orbits for piecewise-smooth vector fields with sliding region via Conley theory

Conley theory has a tool to guarantee the existence of periodic trajectories in isolating neighborhoods of semidynamical systems. We prove that the forward trajectories generated by a piecewise smooth vector field $Z = (X, Y)$ defined on a closed three dimensional manifold without the escape region produces a semi dynamical system. Thus, we built a semiflow that allows us to apply the classical Conley theory. Furthermore, we use it to guarantee the existence of periodic orbits in this class of piecewise smooth vector fields.

Since periodic orbits do not correspond to local objects, our study of these systems is global. In this chapter, we consider M as a closed manifold. A PSVF defined on M is a tangent vector field on M, which is not differentiable only on a submanifold N of codimension one, where N is, called switching manifold. There exist two approaches in the literature for formulating the equations for PSVF; namely the control method of Utkin and the convex method of Filippov. In Filippov's book, see [19], the author establishes conversions used in this chapter to define solution orbit for a PSVF. The problem of guaranteeing the existence of periodic orbits in piecewise smooth vector fields is of colossal importance. In the literature, one of the tools studied for this purpose is the first recurrence map or Poincaré map; a fixed point for this map corresponds to a periodic orbit. Some of the work done in this context are: [4], [16] and [28]. In the papers [29], and [30], the authors use the first integral to studies the existence of crossing periodic orbits. Finally, the well-known theory of averaging was also used, for example, in [27]. Moreover, in [18], [25] and [54], the authors also study the existence of periodic orbits in piecewise smooth systems.

This chapter is structured as follows. In Section 2.1 we provide the construction of a semiflow using the forward trajectories of a PSVF, where the switching N admits only crossing and sliding regions, in other words, there exists a unique solution for forward trajectories of the PSVF. In Section 2.1, the main result is presented. In Section 2.2, we use Conley theory applied to the semidynamical system constructed in Section 2.1 to find periodic trajectories of PSVF.

2.1 Main Results

The main goal of this section is to generate a semidynamical system that arises from the forward trajectories of a PSVF, where the convex method of Filippov is used. Furthermore, for this semiflow one can use Theorem 1.29 to guarantee the existence of periodic orbits of PSVF without escaping region.

Assume X is a topological space which is a finite union of closed subsets X_i, i.e. $\mathsf{X} = \bigcup_{i=1}^{n} \mathsf{X}_i$. If for some topological space Y, there are continuous maps $f_i : \mathsf{X}_i \longrightarrow \mathsf{Y}$ that agree on overlaps (i.e., $f_i\big|_{\mathsf{X}_i \cap \mathsf{X}_j} = f_j\big|_{\mathsf{X}_i \cap \mathsf{X}_j}$), then the generalized gluing lemma says that there exists a unique continuous function $f : \mathsf{X} \longrightarrow \mathsf{Y}$ with $f\big|_{\mathsf{X}_i} = f_i$, for all i. We apply the gluing lemma for the semiflows generated by forward trajectories of the vector fields X, Y, and Z^s.

Throughout this chapter, we restrict the dimension of the manifold M to $n \leq 3$. This assumption is needed to simplify the construction of a semiflow for the cases where the switching manifold Σ has tangencies. For $n \geq 4$, the ideas will be similar; however, the challenge consists in analyzing more tangency cases and defining Filippov's convention for submanifolds with tangencies. Observe that, for Filippov systems with only sliding and crossing regions, the assumption that M be a three dimensional manifold can be dropped, see Theorem C.

Assume that $M = \Sigma^+ \cup \Sigma^-$ is a closed 3-dimensional C^1 manifold, where Σ^+ and Σ^- are submanifolds with common boundary given by the switching manifold Σ with crossing and sliding regions. We begin by introducing useful notations for same subsets of Σ.

Definition 2.1 *For $X \in \mathfrak{X}(\Sigma^+)$, denote $\Sigma_X^+ = \{p \in \Sigma;\ Xh(p) > 0\}$ and $\Sigma_X^- = \{p \in \Sigma;\ Xh(p) < 0\}$ subsets of Σ_X. Furthermore,*

1. $S_X^v = \{p \in S_X;\ p \text{ is a visible fold of } X\}$,
2. $S_X^i = \{p \in S_X;\ p \text{ is an invisible fold of } X\}$,

3. $S_X^{ic} = \{p \in S_X;\ p$ *is a cusp of* X *and* $X^3h(p) < 0\}$ *and*

4. $S_X^{vc} = \{p \in S_X;\ p$ *is a cusp of* X *and* $X^3h(p) > 0\}$.

Analogously to $Y \in \mathfrak{X}(\Sigma^-)$ *and* $Z^s \in \mathfrak{X}(\Sigma^S)$.

In the following definition, $t_X^+(p)$ is the minimal time necessary for the forward trajectory of p to leave Σ^+. This is essential when defining the domain of a semiflow.

Definition 2.2 *Let* $Z = (X, Y)$ *be a* PSVF *given by* (1-6) *defined on a closed manifold* M *with switching manifold* Σ*. For* $X \in \mathfrak{X}(M)$*, let* $\Lambda_X^+ = \{p \in \Sigma^+;\ \varphi_X(p,[0,\infty)) \nsubseteq \Sigma^+\}$ *and* $t_X^+ : \Sigma^+ \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ *such that*

$$t_X^+(p) = \begin{cases} \infty & \text{if } p \notin \Lambda_X^+, \\ \inf\{t > 0;\ \varphi_X(p,t) \in S_X^{ic} \cup \Sigma_X^-\} & \text{if } \varphi_X(p,[0,t]) \subset \Sigma^+ \text{ for } t \in I \cap \mathbb{R}^+ \text{ with } t \neq 0, \\ 0 & \text{otherwise .} \end{cases}$$

Analogously for $Y \in \mathfrak{X}(M)$.

To better understand the previous definition, we present the following example with different cases for t_X^+.

Example 2.3 *For* $X \in \mathfrak{X}(\Sigma^+)$ *consider points as in Figure 2.1. Note that* $p_i \in \Lambda_X^+$ *for* $i = 1,...,6$*. Observe that only for* p_1, p_2 *and* p_3*,* $\varphi_X(p,[0,t]) \subset \Sigma^+$ *then for* $t \in I \cap \mathbb{R}^+$ *with* $t \neq 0$ *we have that* $t_X^+(p_i) > 0$ *for* $i = 1,2,3$ *with* $t_X^+(p_2) > t_X^+(p_3)$ *and* $t_X^+(p_i) = 0$ *for* $i = 4,5,6$*. Now, for* p_i *for* $i = 7,...,9$ *we have that* $\varphi_X(p,[0,\infty)) \subset \Sigma^+$ *then, these points are not in* Λ_X^+ *and so* $t_X^+(p_i) = \infty$*, for* $i = 7,...,9$.

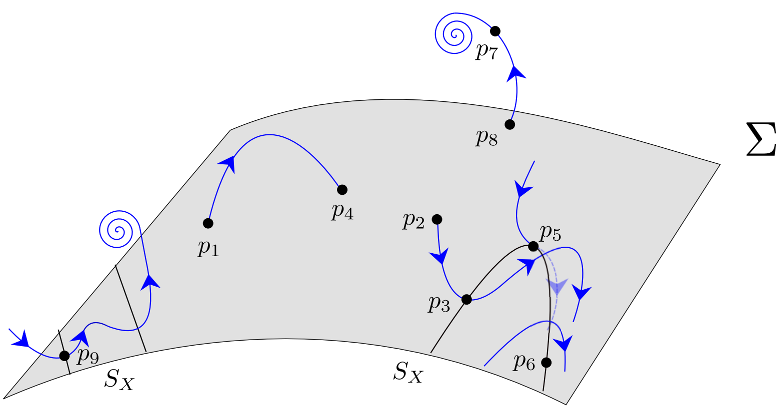

Figure 2.1: $p_i \in \Lambda_X^+$ for $i = 1,...,6$ but $p_j \notin \Lambda_X^+$ for $j = 7,...,9$.

Remark 2.4 *In order to extend Definition 2.2 to the vector field* $Z^s \in \mathfrak{X}(\Sigma^s)$*, note that:*

1. *Locally Σ^s is a submanifold with boundary $\partial\Sigma^s = Xh^{-1}(0)$ or $\partial\Sigma^s = Yh^{-1}(0)$. Without loss of generality, we take $p \in S_X$ such that $Yh(p) > 0$ then locally the set S_X, which is a submanifold of codimension 1, separates the discontinuity manifold into two pieces Σ^S and Σ^{c^+} such that $\Sigma^S = Xh^{-1}((-\infty, 0])$ and $\Sigma^{c+} = Xh^{-1}((0,\infty))$.*

2. *If $p \in S_X$ and $Yh(p) > 0$. If $p \in S_X^i$ then $(Z^s)Xh(p) < 0$ but if $p \in S_X^v$ then $(Z^s)Xh(p) > 0$. If $p \in S_X^{ic}$ then $p \in S_{Z^s}^v$ and if $p \in S_X^{vc}$ then $p \in S_{Z^s}^i$. This assertion is supported by the following Lemma, 6.3 and 6.4 in [50].*

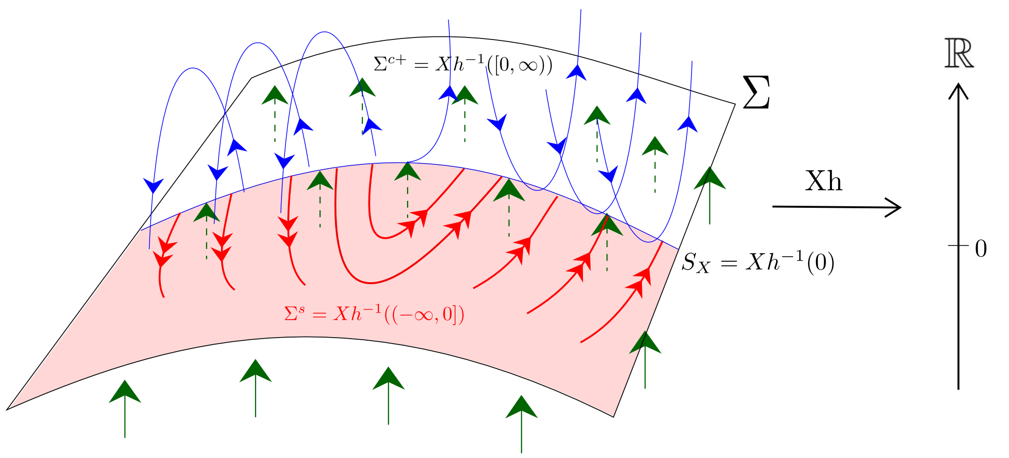

Figure 2.2: Local behavior of cusp-regular singularities in a PSVF.

Next result shows that the vector field Z^s can be extended beyond the boundary of Σ^s as in [50]. The same extension is applicable to Definition 2.2.

Lemma 2.5 *Let $Z = (X,Y)$ with Filippov vector field Z^s, then:*

(i) The vector field Z^s can be smoothly extended beyond the boundary of Σ^s.

(ii) If a point p in $\partial\Sigma^s$ is a fold point (resp. cusp point) of X and a regular point of Y then Z^s is transverse to $\partial\Sigma^s$ at p (resp. Z^s has a quadratic contact with $\partial\Sigma^s$ at p).

In the following proposition, we show that the function t_X^+ is upper semi-continuous. Consider the notation, if U is open in M then $\gamma_X(p) = \{\varphi_X(p,t);\ t \in I\}$ denotes the local orbit of a point $p \in U$; $\gamma_X^+(p)$ and $\gamma_X^-(p)$ the local positive and negative orbits of p, respectively.

Proposition 2.6 *Let $Z = (X,Y)$ be a* PSVF *given by (1-6) defined on a closed manifold M with switching manifold Σ. The map $t_X^+ : \Sigma^+ \to \mathbb{R}^+ \cup \{\infty\}$ such that $p \mapsto t_X^+(p)$ is upper semicontinuous. Analogously for $t_{Z^s}^+$ and t_Y^+.*

Proof. We divided the proof into cases depending on whether p is in or not in Λ_X^+. For $p \notin \Lambda_X^+$ then $t_X^+(p) = \infty$ and $\limsup_{q \to p} t_X^+(q) \le t_X^+(p)$, thus t_X^+ is upper semicontinuous at p.

Assume that $p \in \Lambda_X^+$, then either $p \in \Sigma$ or $p \notin \Sigma$. Without loss of generality, we can assume $p \in \Sigma$. In fact, for $p \notin \Sigma$ it is enough to use the first tangency point of γ on Σ, denoted by $p_1 \in \Sigma$, to be in the same case as $p \in \Sigma$ when proving $\limsup_{q \to p} t_X^+(q) \le t_X^+(p)$. Hence, it remains to prove the upper semi-continuity for $p \in \Lambda_X^+ \cap \Sigma$.

Let $p \in \Lambda_X^+ \cap \Sigma$. Denote $q = \varphi_X(p, t_X^+(p))$ and $\gamma = \varphi_X(p, [0, t_X^+(p)])$. For $p \neq q$, assume that $q \in \Sigma_X^-$. In fact, the other case is when $q \in S_X^{vc}$ and the proof is similar. Choose local coordinates (x_1, x_2, x_3) of the vector field X at $p \in \Sigma$ such that locally $X = (1,0,0)$. Let $x_3 = g(x_1, x_2)$ be a C^∞ solution of $h(x_1, x_2, x_3) = 0$ with $g(0,0) = 0$. Let $N = \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ x_1 = 0\}$ be the transverse section to X at p. Define the projection $\sigma_{X_p} : (\Sigma, p) \longrightarrow (N, p)$ of Σ along the orbits of X onto N given by

$$\sigma_{X_p}(x_1, x_2, g(x_1, x_2)) = (0, x_2, g(x_1, x_2)).$$

Since $\Sigma = \Sigma_X \cup S_X^2 \cup S_X^3$ is the disjoint union of submanifolds of decreasing dimension, then the map σ_{X_p} is an immersion at p. This implies that every orbit meets the boundary of Σ^+ in a discrete set of points. Let $\mathcal{U}$ be an open neighborhood of p in M and $\mathcal{U}_{\Sigma^+} = \mathcal{U} \cap \Sigma^+$. Since $\Sigma = \Sigma_X \cup S_X^2 \cup S_X^3$, we split the proof in three cases: (1) $p \in \Sigma_X$; (2) $p \in S_X^2$; and (3) $p \in S_X^3$. By Definition 2.2, we have that $q \in \Sigma_X^- \cup S_X^{ic}$, where $q \in \Sigma_X^-$ for cases (1) and (2), and $p = q \in S_X^{ic}$ for case (3).

(1) Assume that $p \in \Sigma_X$ and the trajectory γ does not have internal tangencies.

(1.1) If $p \in \Sigma_X^-$ then $t_X^+(p) = 0$ and so for all $\tilde{p} \in \mathcal{U}_{\Sigma^+}$ (case (a) of Figure 2.4), we have that $t_X^+(\tilde{p}) \longrightarrow t_X^+(p)$ whenever $\tilde{p} \longrightarrow p$.

(1.2) If $p \in \Sigma_X^+$ and the trajectory γ does not have internal tangencies in $\mathcal{U}$ other than p, then for all $\tilde{p} \in \mathcal{U}_{\Sigma^+}$ we have that $t_X^+(\tilde{p}) \longrightarrow t_X^+(p)$ whenever $\tilde{p} \longrightarrow p$. In fact, by the long tubular flow theorem (see [39]), there exists a tubular flow (F, f) of X such that $F \supset \gamma$. Take F small enough such that the vector field in the box induced by f and X is the constant field $f_* X = (1,0,0)$ (see Figure 2.3), thus if $\tilde{p} \longrightarrow p$ then $t_X^+(\tilde{p}) \longrightarrow t_X^+(p)$.

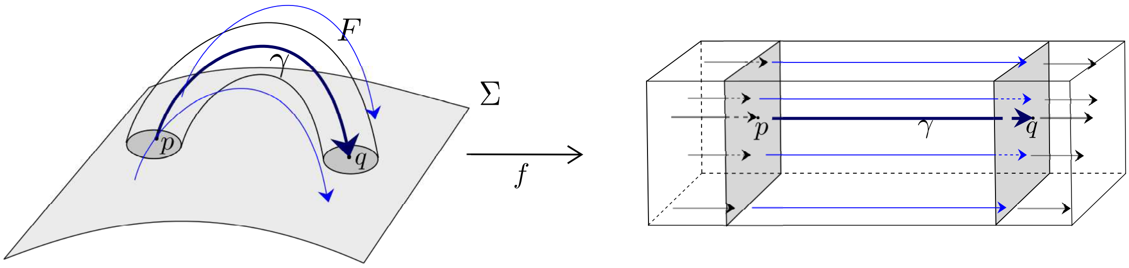

Figure 2.3: The tubular flow (F, f) of X such that $\gamma \subset F$.

(2) Assume that p is a tangential singularity of order two and the trajectory γ does not have internal tangencies other than p. Observe that p is a fold of σ_X $(X^2h(p) = \frac{\partial^2 g}{\partial {x_1}^2} \neq 0)$.

(2.1) If $p \in S_X^i$ then $g(x_1, x_2)$ is conjugate to $(x_1, x_2) \longmapsto x_1^2$. If $\tilde{p} = (\tilde{x_1}, \tilde{x_2}, \tilde{x_3}) \in \Sigma^+$, and only if, $\tilde{x}_3 \geq \tilde{x}_1^2$ (case (b) of Figure 2.4). Hence, for all $\tilde{p} \longrightarrow p$ with $\tilde{p} \in \Sigma^+$ there exists a unique $t(\tilde{p}) \geq 0$ such that the orbit solution $t \longrightarrow \phi_X(\tilde{p}, t)$ of X through $\tilde{p}$ meets Σ at a point $\tilde{q} = \phi_X(\tilde{p}, t(\tilde{p}))$, so

$$\lim_{\tilde{p}\to p} t_X^+(\tilde{p}) = \lim_{\tilde{p}\to p} t(\tilde{p}) = t(p) = 0 = t_X^+(p).$$

(2.2) Now, assume that $p \in S_X^v$ then $g(x_1, x_2)$ is conjugate to $(x_1, x_2) \longmapsto -{x_1}^2$. If $\tilde{p} = (\tilde{x_1}, \tilde{x_2}, \tilde{x_3}) \in \Sigma^+$, and only if, $\tilde{x}_3 \geq -\tilde{x}_1^2$ (case (c) of Figure 2.4). Reduce the neighborhood $\mathcal{U}$ if necessary to guarantee that γ does not have internal tangencies in $\mathcal{U}$ other than p. Thus, $\mathcal{U}_{\Sigma^+} = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3$ with $\mathcal{U}_1 = \{\tilde{p} \in \mathcal{U}_{\Sigma^+}; \gamma_X^+(\tilde{p}) \cap \Sigma_X^- \neq \emptyset\}$, $\mathcal{U}_2 = \{\tilde{p} \in \mathcal{U}_{\Sigma^+}; \gamma_X^+(\tilde{p}) \cap S_X \neq \emptyset$ or $\gamma_X^+(\tilde{p}) \cap \Sigma = \emptyset\}$ and $\mathcal{U}_3 = \{\tilde{p} \in \mathcal{U}_{\Sigma^+}; \gamma_X^-(\tilde{p}) \cap \Sigma_X^+ \neq \emptyset\}$. Hence, for points $\tilde{p} \in \mathcal{U}_1$, $\gamma_X^+(\tilde{p}) \cap \Sigma_X^- \neq \emptyset$ and $\lim_{\tilde{p}\to p} t_X^+(\tilde{p}) = 0$.

Let $\tilde{p} \in \mathcal{U}_2 \cup \mathcal{U}_3$. Using the long tubular flow theorem (reduce F, if necessary, such that there is no internal tangency in F) and $Xh(q) < 0$ then for all $\tilde{p} \in \mathcal{U}_2 \cup \mathcal{U}_3$ there exists a unique $t(\tilde{p}) \geq 0$ such that the orbit-solution $t \longrightarrow \phi_X(\tilde{p}, t)$ of X through $\tilde{p}$ meets Σ at q. Hence, $t(\tilde{p}) = t_X^+(\tilde{p})$. Thus, $\limsup_{\tilde{p}\to p} t_X^+(\tilde{p}) = \inf\{\sup\{t_X^+(\tilde{p});\ \tilde{p} \in (\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3) \setminus \{p\})\}\} \leq t_X^+(p)$.

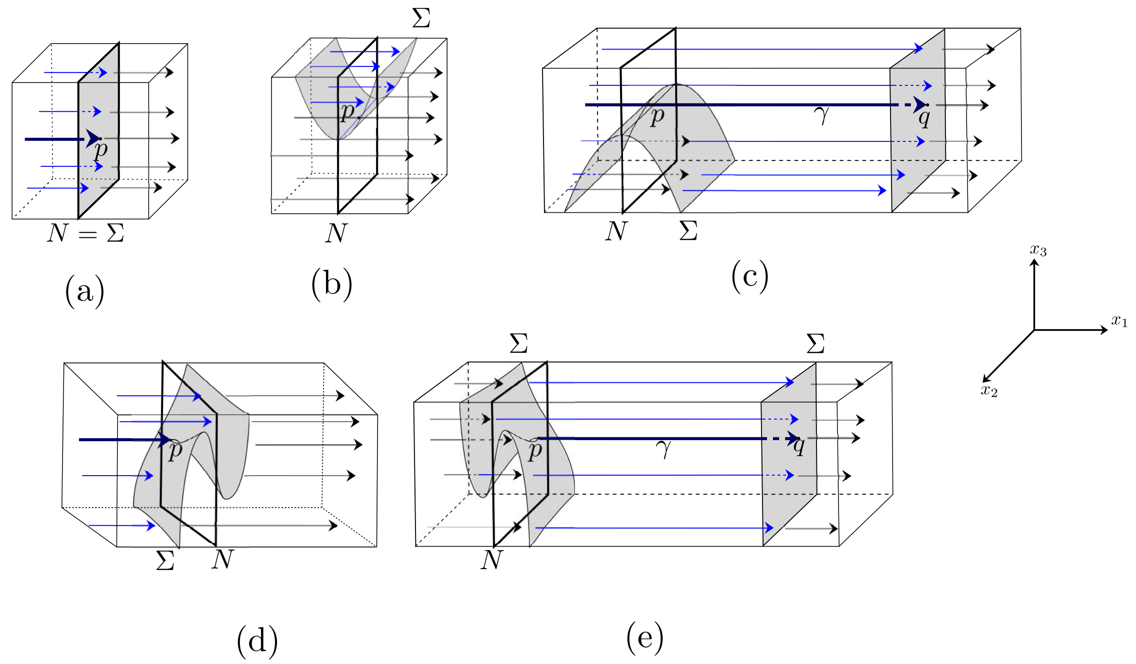

Figure 2.4: Local coordinates where X is conjugate to a constant vector field and Σ represents the switching manifold in the local coordinates.

(3) Assume that p is a tangential singularity of order three and the trajectory γ does not have internal tangencies other than p, then p is a cusp of σ_X.

(3.1) If $p \in S_X^{ic}$ then $g(x_1,x_2)$ is conjugate to $(x_1,x_2) \longmapsto {x_1}^3 + x_1x_2$. Then, $\tilde{p} = (\tilde{x_1},\tilde{x_2},\tilde{x_3}) \in \Sigma^+$ if, and only if, $\tilde{x}_3 \geq \tilde{x}_1^3 + \tilde{x}_1\tilde{x}_2$ (case (d) of Figure 2.4). Thus $\gamma^+(\tilde{p}) \cap \text{int}(U \cap \Sigma^-) \neq \emptyset$ and $\lim_{\tilde{p}\to p} t_X^+(\tilde{p}) = t_X^+(p)$.

(3.2) If $p \in S_X^{vc}$ then $g(x_1,x_2)$ is conjugate to $(x_1,x_2) \longmapsto -{x_1}^3 - x_1x_2$ (case (e) of Figure 2.4). Hence, $\mathcal{U}_{\Sigma^+} = \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{V}_3$ with $\mathcal{V}_1 = \{\tilde{p} \in \mathcal{U}_{\Sigma^+};\ \tilde{p} \in S_X^i \text{ or } \gamma_X^+(\tilde{p}) \cap \Sigma_X^- \neq \emptyset\}$, $\mathcal{V}_2 = \{\tilde{p} \in \mathcal{U}_{\Sigma^+};\ \gamma_X^+(\tilde{p}) \in S_X^v\}$ and $\mathcal{V}_3 = \{\tilde{p} \in \mathcal{U}_{\Sigma^+};\ \tilde{p} \notin \mathcal{V}_1 \cup \mathcal{V}_2\}$.

For the points in $\mathcal{V}_1$, by definition of t_X^+, we have that $\lim_{\tilde{p}\to p} t_X^+(\tilde{p}) = 0$. For points in $\mathcal{V}_2 \cup \mathcal{V}_3$ it follows that $\lim_{\tilde{p}\to p} t_X^+(\tilde{p}) = t_X^+(\tilde{p})$, since it is precisely case 2.2. Therefore t_X^+ is upper semicontinuous at p.

Now assume that γ has a finite number of internal tangencies. If $p \in \Sigma_X^- \cup S_X^i \cup S_X^{ic}$ then $p = q$ and there is nothing to prove. Assume that $p \in \Sigma_X^+$ and $q \in \Sigma_X^-$ (for the cases $p \in S_X^v$ and $p \in S_X^{vc}$ the proof is similar). Let $p_1 \in \gamma$ be the next tangency in S_X^v after p. Consider $t_1 > 0$ such that $\varphi_X(p,t_1) = p_1$ then $t_1 \ll t_X^+(p)$ and $\lim_{\tilde{p}\to p} t_X^+(\tilde{p}) = t_1$. Thus t_X^+ is upper semicontinuous at p. □

The following corollary is a noteworthy consequence from the previous proposition.

Corollary 2.7 *Let $Z=(X,Y)$ be a* PSVF *given by* (1-6) *defined on a closed manifold M with switching manifold Σ. For $X \in \mathfrak{X}(M)$, if $p \in \Sigma^+$ and $\tilde{p} \longrightarrow p$ then $\limsup_{\tilde{p}\to p} t_X^+(\tilde{p})$ is equal to 0 or $t_X^+(p)$ or the time $t>0$ such that $\varphi_X(p,t)$ is the first point of internal tangency of γ.*

The upper semicontinuity of t_X^+, t_Y^+ and $t_{Z^s}^+$ proved in Proposition 2.6 is crucial to guarantee that the forward trajectories of the vector fields X, Y and Z^s over Σ^+, Σ^- and Σ are continuous functions and therefore the gluing lemma is applicable. In particular, when t_X^+, t_Y^+ and $t_{Z^s}^+$ are continuous functions, the forward trajectories are continuous and then we can apply the gluing lemma without using Proposition 2.6. In other words, this is the particular case where the hypothesis on the dimension of M and the types of tangency points on Σ are not required. Hence, the natural question to be answered is: What are the conditions for the function t_X^+ to be continuous? We address this question by observing that Proposition 2.6 together with Corollary 2.7 imply the following corollary.

Corollary 2.8 *Let $Z=(X,Y)$ be a* PSVF *given by* (1-6) *defined on a closed manifold M^3 with switching manifold, given by $p \mapsto t_X^+(p)$, is continuous whenever $\Sigma = \Sigma_X \cup S_X^i$. Analogously for $t_{Z^s}^+$ and t_Y^+.*

Proof. Following the proof of Proposition 2.6, the map t_X^+ is not lower semi-continuous when $\Sigma_X^v \neq \emptyset$, in other words, this happens when there exists tangential singularities of order two. Hence, when $\Sigma = \Sigma_X \cup S_X^i$, t_X^+ is also lower semi-continuous, therefore it is also continuous. □

Now, we describe the regions where it is possible to create a semidynamical system by using the gluing lemma. Recall that M is a closed 3-dimensional C^1 manifold and the switching manifold only has crossing regions and sliding regions. The boundary of these regions is composed by tangency points. We classify the points in Σ in types A and B, as follows:

Type A. The Σ-singularity points of singular-regular type;

- A1. $p \in S_X^v$ and $Yh(p)>0$ (or $p \in S_Y^v$ and $Xh(p)<0$);
- A2. $p \in S_X^i$ and $Yh(p)>0$ (or $p \in S_Y^i$ and $Xh(p)<0$);
- A3. $p \in S_X^{ic}$ and $Yh(p)>0$ (or $p \in S_Y^{ic}$ and $Xh(p)<0$);
- A4. $p \in S_X^{vc}$ and $Yh(p)>0$ (or $p \in S_Y^{vc}$ and $Xh(p)<0$).

Type B. The Σ-singularity points of singular-singular type;

- B1. $p \in S_X^v \cap S_Y^i$ (or $p \in S_X^i \cap S_Y^v$) with $p \in \partial\Sigma^{c+} \cap \partial\Sigma^{c-}$;

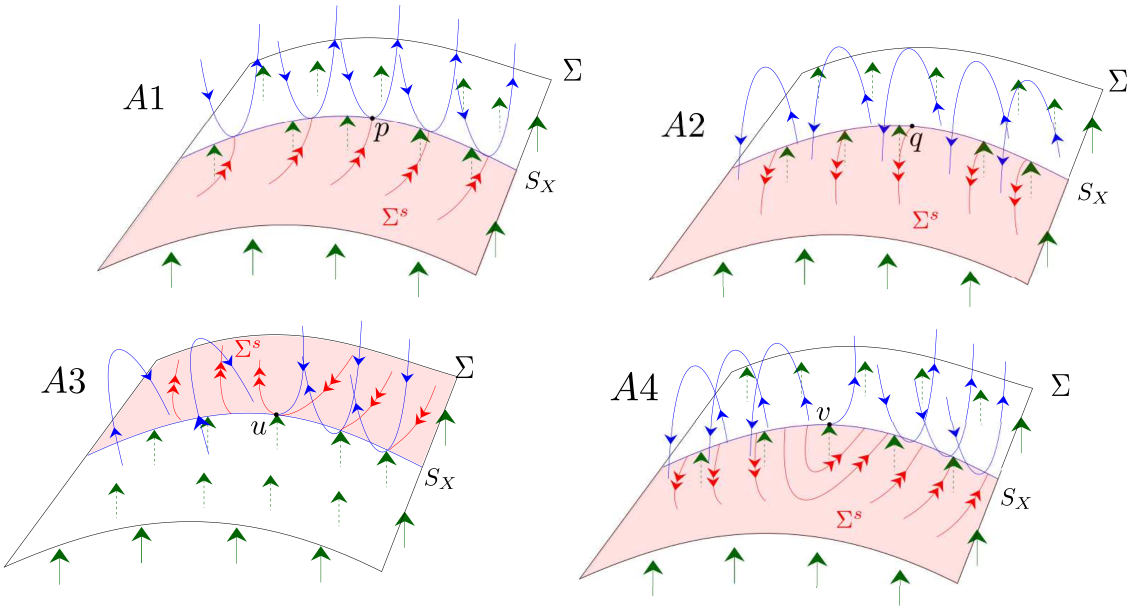

Figure 2.5: Local behavior of the points of the type A.

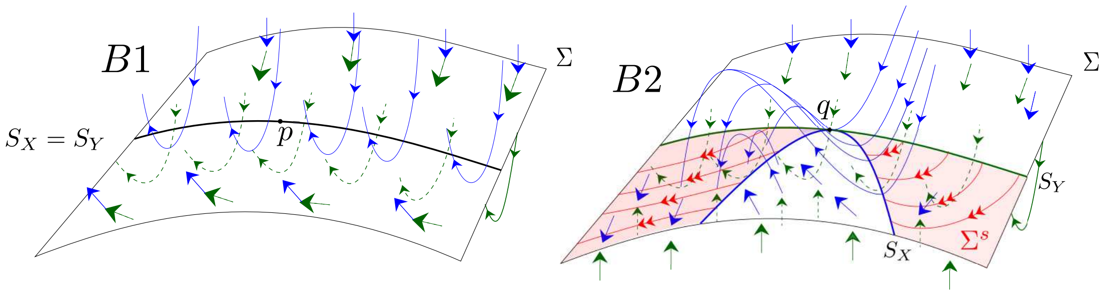

Figure 2.6: Local behavior of the points of the type B.

B2. $p \in S_X^{ic}$ and $p \in S_Y^{i}$ (or $p \in S_Y^{ic}$ and $p \in S_X^{i}$) with S_X and S_Y tangent of p.

We denote "$A1$ to $B2$" the sets of points of the type $A1$, $A2$, $A3$, $A4$, $B1$ and $B2$. Note that, by Definition 1.31, if p is a point of type B2 then $Xh(p) = Yh(p) = 0$ and so $Z^s h(p) = 0$. With the following lemma, we guarantee that the forward local trajectory for tangency points of types $A1$ to $B2$ is unique.

Lemma 2.9 *The forward local trajectory $\varphi_Z(p,t)$ of a* PSVF *of the form (1-6) through a point $p \in S_Z$ of some type* A1 *to* B2 *is unique.*

Proof. In fact, by Definition 1.39 and Remark 2.4:

1) for p of type A1, A4 and B1, the trajectory is given by $\varphi_Z(p,t) = \varphi_X(p,t)$ ($\varphi_Z(p,t) = \varphi_Y(p,t)$);

2) for p of type A2 and A3, the trajectory is given by $\varphi_Z(p,t) = \varphi_{Z^s}(p,t)$;

3) for p of type B2, the trajectory is given by $\varphi_Z(p,t) = \varphi_{Z^s}(p,t) = p$

for $t \in I \cap \mathbb{R}^+$ and $0 \in I$. □

In the next definition, for each $p \in M$, we build a sequence of points on the switching manifold using the function t_X^+. We will use these sequences to define intervals in which the solutions of the systems of the vector fields X, Y, and Z^s are defined. The objective of these intervals is to use them to restrict the domains of the flows generated by the associated vector fields.

Definition 2.10 *Let $Z = (X,Y)$ be a* PSVF *given by* (1-6) *defined on a closed manifold M with switching manifold Σ. Assume $\Sigma = \Sigma^c \cup \Sigma^s \cup S_Z$ and $p \in S_Z$ is some type* A1 *to* B2. *If $p \in M$ then it is denoted by $p_0 = p$ and $p_i = \varphi_{Z_{i-1(p)}}(p_{i-1}, t^+_{Z_{i-1(p)}}(p_{i-1}))$ for $i \in \mathbb{Z}^+$, where*

$$Z_{0(p)} = \begin{cases} X & \text{if } p \in \operatorname{int}(\Sigma^+), \\ Y & \text{if } p \in \operatorname{int}(\Sigma^-); \end{cases}$$

and

$$Z_{i(p)} = \begin{cases} X & \text{if } p_i \in \Sigma^{c+} \text{ or } p_i \in (\mathrm{A1} \cup \mathrm{A4} \cup \mathrm{B1}) \cap (S_X^v \cup S_X^{vc}), \\ Z^s & \text{if } p_i \in \Sigma^s \text{ or } p_i \in \mathrm{A2} \cup \mathrm{A3} \cup \mathrm{B2}, \\ Y & \text{if } p_i \in \Sigma^{c-} \text{ or } p_i \in (\mathrm{A1} \cup \mathrm{A4} \cup \mathrm{B1}) \cap (S_Y^v \cup S_Y^{vc}). \end{cases}$$

Fix $\Delta_X^+(p) = \{i \in \mathbb{Z}^+;\ Z_{i(p)} = X\}$ and $\Omega_X = \{(p,t) \in M \times \mathbb{R}^+;\ t \in I_X(p)\}$ with

$$I_X(p) = \bigcup_{i \in \Delta_X^+(p)} I_X^i(p),$$

where $I_X^0(p) = \left[0, a_{0(p)}\right]$, $I_X^i(p) = \left[a_{i-1(p)}, a_{i(p)}\right]$ for $i > 0$ such that $a_{i(p)} = \sum_{j=0}^{i} t^+_{Z_{j(p)}}(p_j)$. Analogously, we define Ω_Y and Ω_{Z^s} for the vector fields Y and Z^s, respectively.

Now, we define the functions glued together by the generalized gluing lemma, hence obtaining the desired semiflow.

Definition 2.11 *Let φ_X, φ_Y and φ_{Z^s} be the flows of the vector fields $X \in M$, $Y \in M$ and $Z^s \in \mathfrak{X}(\Sigma^S)$, respectively. Denote*

$$\phi_X : \Omega_X \longrightarrow \Sigma^+,$$

$$\phi_Y : \Omega_Y \longrightarrow \Sigma^- \ \textit{and}$$

$$\phi_{Z^s} : \Omega_{Z^s} \longrightarrow \Sigma^S,$$

by the restriction of φ_X, φ_Y and φ_{Z^s} to the domains Ω_X, Ω_Y and Ω_{Z^s}, respectively. For $(p,t) \in \Omega_X$ then $t \in I^i_X(p)$ with $i \in \Delta^+_X(p)$ and $\phi_X(p,t) = \varphi_X(p_i, t - a_{i-1(p)})$. Analogously, for ϕ_Y and ϕ_{Z^s}.

Remark 2.12 *The function $\phi_X : \Omega_X \longrightarrow M$ introduced in 2.11 is continuous, in fact $\phi_X = \mathfrak{i} \circ \varphi_X|_{\Omega_X}$ where $\mathfrak{i}$ is the inclusion function of Σ^+ in M. Analogously, for ϕ_Y and ϕ_{Z^s}.*

The following lemma guarantees that the limit points of the domain of ϕ_X, ϕ_Y, and ϕ_{Z^s} are correctly glued when the gluing lemma is applied.

Lemma 2.13 *If $(p,s) \in \overline{\Omega_X}$ then $\phi_{Z_{i(p)}}(p,s) \in \Sigma^+$, where $s \in I^i_{Z_{i(p)}}(p)$.*

Proof. Assume by contradiction that $q = \phi_{Z_{i(p)}}(p,s) \notin \Sigma^+$ then $q \in \operatorname{int}(\Sigma^-)$. Let V be an open neighborhood of $\operatorname{int}(\Sigma^-)$ such that $q \in V$. Then by Remark 2.12, $\phi_Y^{-1}(V)$ is an open neighborhood of $M \times \mathbb{R}^+$ such that $(p,s) \in \phi_Y^{-1}(V)$ and for all $(\tilde{p},\tilde{s}) \in \phi_Y^{-1}(V)$, $\tilde{s} \in I_Y(\tilde{p}) \setminus I_X(\tilde{p})$ and so $\phi_Y^{-1}(V) \cap \Omega_X = \emptyset$ that is $(p,s) \notin \overline{\Omega_X}$. □

For $(p,s) \in M \times \mathbb{R}^+$, fix $\gamma_i(p) = \varphi_{Z_{i(p)}}(p_i, [0, t^+_{Z_{i(p)}}(p_i)])$, for each $i \in \Delta^+_Z(p)$, and let Γ be the concatenation of all $\gamma_i(p)$. Fix $\Delta^+_Z(p) = \Delta^+_X(p) \cup \Delta^+_Y(p) \cup \Delta^+_{Z^s}(p)$.

Proposition 2.14 *Let $(p,s) \in \overline{\Omega_X}$ with $s \in I^i_{Z_{i(p)}}(p)$ and $i \in \Delta^+_Z(p)$ then*

$$\lim_{\substack{\tilde{p} \to p \\ \tilde{s} \to s}} \phi_X(\tilde{p},\tilde{s}) = \phi_{Z_{i(p)}}(p,s)$$

whenever $(\tilde{p},\tilde{s}) \in (V \times I) \cap \Omega_X$, where $V \times I$ is a neighborhood of (p,s) in $M \times \mathbb{R}^+$. Analogously for (p,s) in $\overline{\Omega_Y}$ and $\overline{\Omega_{Z^s}}$.

Proof. Since ϕ_X is continuous, then $\lim_{\substack{\tilde{p} \to p \\ \tilde{s} \to s}} \phi_X(\tilde{p},\tilde{s}) = \phi_{Z_{i(p)}}(p,s)$, for all $(p,s) \in \Omega_X$. If $(p,s) \notin \Omega_X$ then $s \in (I_Y(p) \cup I_{Z^s}(p)) \setminus I_X(p)$, that is, there exists $i \in \Delta_Z(p)^+$ such that $s \in I^i_{Z_{i(p)}}(p)$ with $Z_{i(p)} \neq X$. Fixing $p \in M$, we proceed by induction on i.

Induction basis: Assume that $i = 0$ then $(p,s) \in M \times \mathbb{R}^+$ is such that $s \in I^0_{Z_{0(p)}}(p)$, hence $p \in \Sigma^-$. Firstly, assume that $p \in \operatorname{int}(\Sigma^-)$ then $Z_{0(p)} = Y$, and by Lemma 2.13 we have that $s \neq 0$. Thus $\gamma_0(p) = \varphi_Y(p, [0, a_{0(p)}]) = \varphi_Y(p, [0, t^+_Y(p)])$ and so we have the two cases with respect to s in $[0, a_{0(p)}]$: either (1) $s < a_{0(p)}$ or (2) $s = a_{0(p)}$.

1. For $s < a_{0(p)}$, that is $\gamma_0(p)$ has internal tangencies. An internal tangency is an adherent point of Ω_X if, and only if, it is a point of type B1. Let $q = \phi_Y(p,s) = \varphi_Y(p,s)$ be the first internal tangency of $\gamma_0(p)$, so $q \in S^i_X \cap S^v_Y$ and also $q \in \partial\Sigma^{c+} \cap \partial\Sigma^{c-}$. We are going to show that $\phi_X(\tilde{p},\tilde{s}) \longrightarrow q$ when $\tilde{p} \longrightarrow p$ and $\tilde{s} \longrightarrow s$.

   Note that $\tilde{s} \in I^1_X(\tilde{p})$ (see Figure 2.7). As $p \in \text{int}(\Sigma^-)$ and $\gamma_0(p)$ has internal tangencies, by Corollary 2.7, we have that $\lim\limits_{\tilde{p}\to p} t^+_Y(\tilde{p}) = s$ and $\lim_{\tilde{p}\to p}\tilde{p}_1 = \lim_{\tilde{p}\to p}\varphi_Y(\tilde{p}, t^+_Y(\tilde{p})) = \varphi_Y(\lim_{\tilde{p}\to p}\tilde{p}, \lim_{\tilde{p}\to p} t^+_Y(\tilde{p})) = \varphi_Y(p,s) = q$, and so

$$\begin{aligned}
\lim_{\substack{\tilde{p}\to p\\ \tilde{s}\to s}} \phi_X(\tilde{p},\tilde{s}) &= \lim_{\substack{\tilde{p}\to p\\ \tilde{s}\to s}} \varphi_X(\tilde{p}_1, \tilde{s} - a_{0(\tilde{p})})\\
&= \lim_{\substack{\tilde{p}\to p\\ \tilde{s}\to s}} \varphi_X(\tilde{p}_1, \tilde{s} - t^+_Y(\tilde{p}))\\
&= \varphi_X\left(\lim_{\tilde{p}\to p}\tilde{p}_1, \lim_{\tilde{s}\to s}\tilde{s} - \lim_{\tilde{p}\to p} t^+_Y(\tilde{p})\right)\\
&= \varphi_X(q, s-s) = q.
\end{aligned}$$

   Now, if $\varphi_Y(p,s)$ is not the first internal tangency, the proof follows by induction in k, where k is the position of internal tangency, in fact $\gamma_0(p)$ has a finite number of tangencies with Σ.

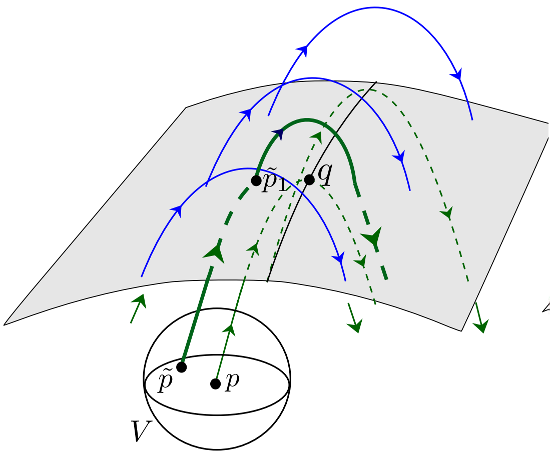

Figure 2.7: $q = \phi_Y(p,s)$ is a point of type B1 and $\tilde{p}_1 \in \Sigma^{c+}$.

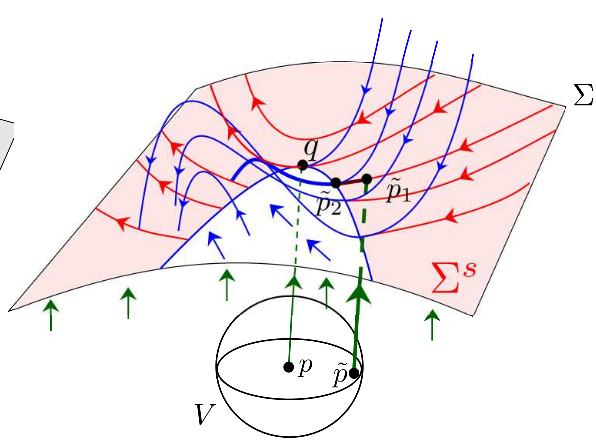

Figure 2.8: $q = \phi_Y(p,s)$ is a point of type A3, $\tilde{p}_1 \in \Sigma^s$ and $\tilde{p}_2 \in \partial\Sigma^s$.

2. For $s = a_{0(p)}$, we have two cases to analyze: (2.1) assuming that $\gamma_{0(p)}$ has no internal tangencies or (2.2) assuming that $\gamma_{0(p)}$ has internal tangencies.

(2.1) Assume that $s = a_{0(p)}$ and $\gamma_{0(p)}$ have no internal tangencies. Then $q = \phi_Y(p,s) = \phi_Y(p,a_{0(p)}) = \phi_Y(p,t_Y^+(p)) = p_1$ is a point of type either A1 or A3. For $q \in$ A1, the proof is analogous to case (1). It remains to prove for $q \in$ A3. As (p,s) is an adherent point of Ω_X then either $\tilde{s} \in I_X^1(p)$ if $\tilde{p}_1 \in \Sigma^{c+} \cup \partial\Sigma^s$ (analogous to case (1)) or $\tilde{s} \in I_X^2(p)$ if $\tilde{p}_1 \in \Sigma^s$ (see Figure 2.8).

For $\tilde{s} \in I_X^2(p)$ then $\tilde{p}_1 \longrightarrow q$ when $\tilde{p} \longrightarrow p$ and, by using part 2.2 of the proof of Proposition 2.6 with the map $t_{Z^s}^+ : \Sigma^S \longrightarrow \mathbb{R}^+$, we have that $\lim\limits_{\tilde{p}_1 \to q} t_{Z^s}^+(\tilde{p}_1) = 0$. In fact, $\gamma_1(\tilde{p}) = \varphi_{Z^s}(\tilde{p}_1,[0,t_{Z^s}^+(\tilde{p}_1)])$ has no internal tangencies with $\partial\Sigma^s$ and so

$$\lim_{\tilde{p}\to p} \tilde{p}_2 = \lim_{\substack{\tilde{p}\to p \\ \tilde{p}_1\to q}} \varphi_{Z^s}(\tilde{p}_1,t_{Z^s}^+(\tilde{p}_1)) = \varphi_{Z^s}(\lim_{\tilde{p}\to p}\tilde{p}_1, \lim_{\tilde{p}_1\to q} t_{Z^s}^+(\tilde{p}_1)) = \varphi_{Z^s}(q,0) = q.$$

Thus,

$$\begin{aligned}
\lim_{\substack{\tilde{p}\to p \\ \tilde{s}\to s}} \phi_X(\tilde{p},\tilde{s}) &= \lim_{\substack{\tilde{p}\to p \\ \tilde{s}\to s}} \varphi_X(\tilde{p}_2, \tilde{s} - a_1(\tilde{p})) \\
&= \lim_{\substack{\tilde{p}\to p \\ \tilde{s}\to s}} \varphi_X(\tilde{p}_2, \tilde{s} - (t_Y^+(\tilde{p}) + t_{Z^s}^+(\tilde{p}_1))) \\
&= \varphi_X(\lim_{\tilde{p}\to p}\tilde{p}_1, \lim_{\tilde{s}\to s}\tilde{s} - (\lim_{\tilde{p}\to p} t_Y^+(\tilde{p}) + \lim_{\tilde{p}_1\to q} t_{Z^s}^+(\tilde{p}_1))) \\
&= \varphi_X(q, s-(s+0)) = q.
\end{aligned}$$

(2.2) Assume that $s = a_{0(p)}$ and $\gamma_{0(p)}$ has internal tangencies. Then each internal tangency point is a point of type B1. Thus, we use item (1) and item (2.1) replacing $\tilde{p}$ by $\tilde{p}_j$ with $j \in \Delta_Z^+(\tilde{p})$ such that $\tilde{p}_j \to \tilde{q}$ where q^* is the last of the internal tangencies of $\gamma_{0(p)}$ such that $(p,s^*) \in \overline{\Omega_X}$ with $\phi_{Z_{0(p)}}(p,s^*) = \phi_Y(p,s^*) = q^*$.

Now, assume that $p \in \Sigma$. Since $s \in I^0_{Z_{0(p)}}(p)$ then either $s = 0$ or $0 < s \leq t^+_{Z_{0(p)}}(p)$. If $s = 0$ then p is a point of type $A2$, $A3$ or $B1$ and by similar analysis as in items (1) and (2) we have that $\lim\limits_{\substack{\tilde{p}\to p \\ \tilde{s}\to 0}} \phi_X(\tilde{p},\tilde{s}) = p$. If $0 < s \leq t^+_{Z_{0(p)}}(p)$ then $\lim\limits_{\substack{\tilde{p}\to p \\ \tilde{s}\to s}} \phi_X(\tilde{p},\tilde{s})$ is either an internal tangency of $\gamma_0(p)$ (tangency contact with Σ when $Z_{0(p)} = Y$ or with $\partial\Sigma^s$ when $Z_{0(p)} = Z^s$) of p_1, and the proof is analogous to the previous items (1) and (2). Analogously for (p,s) in $\overline{\Omega_Y}$ and $\overline{\Omega_{Z^s}}$ with $s \in I^0_{Z_{0(p)}}(p)$.

Inductive step: We assume that the proposition is true for all $(p,s) \in \overline{\Omega_X}$ ($(p,s) \in \overline{\Omega_Y}$ or $(p,s) \in \overline{\Omega_{Z^s}}$) with $s \in I^i_{Z_{i(p)}}(p) = [a_{i-1(p)}, a_{i(p)}]$. For $s = a_{i(p)}$, if $(p, a_{i(p)}) \in \overline{\Omega_{Z^*}}$ with $Z^* = X$, Y or Z^s

$$\lim_{\substack{\tilde{p}\to p \\ \tilde{s}\to a_{i(p)}}} \phi_{Z^*}(\tilde{p},\tilde{s}) = \phi_{Z_{i(p)}}(p,s) = p_{i+1}.$$

Then there exists $j \in \Delta^+_Z(\tilde{p})$ such that $\tilde{p}_j \longrightarrow p_{i+1}$ and by continuity of $\varphi_{Z_{j-1(\tilde{p})}}$ we have that $a_{j-1(\tilde{p})} \longrightarrow a_{i(p)}$ whenever $\tilde{p} \longrightarrow p$. Now, assume that $a_{i(p)} < s < a_{i+1(p)}$, that is, $\gamma_{i+1}(p) \subset \Sigma$ has internal tangencies. We analyze $Z_{i+1(p)} = Z^s$, $Z_{i+1(p)} = Y$ and $Z_{i+1(p)} = X$.

(a) If $Z_{i+1(p)} = Z^s$ then $\gamma_{i+1}(p) \subset \Sigma$ and $q = \phi_{Z^s}(p,s) \in$ A3 with q visible cusp points for X, when $p_{i+1} \in \Sigma^s$ (See Figure 2.9). Since V is a neighborhood of M containing p by the induction hypothesis $\tilde{s} \in I^{j+1}_{Z_{j+1(\tilde{p})}}(\tilde{p})$ with $Z_{j+1(\tilde{p})} = X$ for each $(\tilde{p},\tilde{s}) \in (V \times I) \cap \Omega_X$. Then

$$\lim_{\tilde{p}_j \to p_{i+1}} t^+_{Z^s}(\tilde{p}_j) = s - a_{i(p)},$$

$$\lim_{\tilde{p}\to p} \tilde{p}_{j+1} = \lim_{\tilde{p}_j \to p_{i+1}} \varphi_{Z^s}(\tilde{p}_j, t^+_{Z^s}(\tilde{p}_j)) = \varphi_{Z^s}(p_{i+1}, s - a_{i(p)}) = \phi_{Z^s}(p,s) = q, \text{ and}$$

$$\lim_{\tilde{p}\to p} a_{j(\tilde{p})} = \lim_{\tilde{p}\to p} a_{j-1(\tilde{p})} + \lim_{\tilde{p}_{j-1}\to p_{i+1}} t^+_{Z^s}(\tilde{p}_j) = a_{i(p)} + (s - a_{i(p)}) = s.$$

Thus,

$$\begin{aligned}
\lim_{\substack{\tilde{p}\to p \\ \tilde{s}\to s}} \phi_X(\tilde{p},\tilde{s}) &= \lim_{\substack{\tilde{p}\to p \\ \tilde{s}\to s}} \varphi_X(\tilde{p}_{j+1}, \tilde{s} - a_{j(\tilde{p})}) \\
&= \varphi_X(\lim_{\tilde{p}\to p} \tilde{p}_{j+1}, \lim_{\tilde{s}\to s} \tilde{s} - \lim_{\tilde{p}\to p} a_{j(\tilde{p})}) \\
&= \varphi_X(q, 0) = q.
\end{aligned}$$

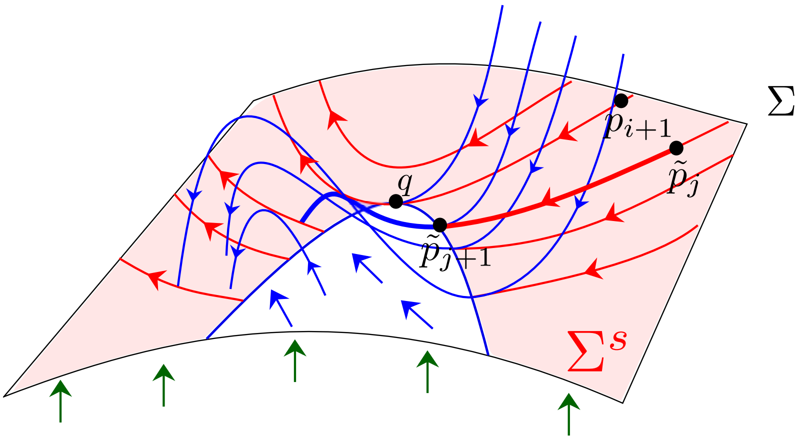

Figure 2.9: $q = \phi_Y(p,s)$ with $a_{i(p)} < s < a_{i+1(p)}$ is a point of type A3, $\tilde{p}_j \in \Sigma^s$ and $\tilde{p}_{j+1} \in \partial\Sigma^s$.

(b) If $Z_{i+1(p)} = Y$ or $Z_{i+1(p)} = X$ then the proof is analogous to item (1) of the induction basis using the induction hypothesis as in item (a).

For $(p,s) \in \overline{\Omega_X}$ with $s = a_{i+1(p)}$ the proof is similar to item (*b*) in the induction basis. In this case we use the induction hypothesis as in item (2) of the induction step. Analogously for (p,s) in $\overline{\Omega_Y}$ and $\overline{\Omega_{Z^s}}$.

□

A direct consequence of the previous proposition is the corollary below.

Corollary 2.15 *The map* $\phi_X : \overline{\Omega_X} \longrightarrow \Sigma^+$ *is continuous and if* $(p,s) \in \overline{\Omega_X} \setminus \Omega_X$ *with* $s \in I^i_{Z_{i(p)}}(p)$ *then* $\phi_X(p,s) = \phi_{Z_{i(p)}}(p,s)$. *Analogously for the maps* $\phi_Y : \overline{\Omega_Y} \longrightarrow \Sigma^-$ *and* $\phi_{Z^s} : \overline{\Omega_{Z^s}} \longrightarrow \Sigma^S$.

Now, we are ready to construct the semidynamical system for a PSVF.

Theorem A *Assume* M *is a closed 3-dimensional* C^1 *manifold and* $Z \in \mathfrak{X}(M,h)$. *If* $\Sigma = \Sigma^c \cup \Sigma^s \cup S_Z$ *with* $p \in S_Z$ *being of type* Ai, $i = 1,\ldots,4$ *or* Bj, $j = 1,2$ *then the trajectories of* $Z = (X,Y) \in \mathfrak{X}(M,h)$ *generate a semidynamical system* (M,ϕ_Z).

In order to prove Theorem A we need the following lemmas.

Lemma 2.16 *Under the hypothesis of Theorem A:*

(a) For all $(p,t) \in \overline{\Omega_X} \cap \overline{\Omega_{Z^s}}$ *we have that* $\phi_X(p,t) = \phi_{Z^s}(p,t)$;

(b) For all $(p,t) \in \overline{\Omega_Y} \cap \overline{\Omega_{Z^s}}$ *we have that* $\phi_Y(p,t) = \phi_{Z^s}(p,t)$ *and*

(c) For all $(p,t) \in \overline{\Omega_X} \cap \overline{\Omega_Y}$ *we have that* $\phi_X(p,t) = \phi_Y(p,t)$.

Proof. Fix $(p,t) \in \overline{\Omega_X} \cap \overline{\Omega_{Z^s}}$ then:

(1) If $(p,t) \in \Omega_X \cap \Omega_{Z^s}$ then there exists $i \in \Delta_Z^+(p)$ such that $t \in I_X^i(p) \cap I_{Z^s}^{i+1}(p)$ (similarly for $t \in I_{Z^s}^i(p) \cap I_X^{i+1}(p)$) so $t = a_{i(p)}$ and

$$\phi_X(p,t) = \varphi_X(p_{i+1}, t - a_{i(p)}) = \varphi_X(p_{i+1}, 0) = p_{i+1} = \varphi_{Z^s}(p_{i+1}, 0) = \phi_{Z^s}(p,t).$$

(2) If $(p,t) \in (\overline{\Omega_X} \cap \overline{\Omega_{Z^s}}) \setminus \Omega_X$ (similarly for $(p,t) \in (\overline{\Omega_X} \cap \overline{\Omega_{Z^s}}) \setminus \Omega_{Z^s}$) then there exists $i \in \Delta_{Z^s}(p)$ such that $t \in I_{Z^s}^i(p)$ and by Corollary 2.15 we obtain that $\phi_X(p,t) = \phi_{Z^s}(p,t)$.

(3) If $(p,t) \in (\overline{\Omega_X} \cap \overline{\Omega_{Z^s}}) \setminus (\Omega_X \cap \Omega_{Z^s})$ then $t = 0$ and p is a point of the type A1. Thus $\phi_X(p,t) = \varphi_X(p,0) = p = \varphi_{Z^s}(p,0) = \phi_{Z^s}(p,t)$.

The proves of items *b*) and *c*) are analogous.

□

Lemma 2.17 *Under the hypothesis of Theorem A, the map $\phi_Z : M \times \mathbb{R}^+ \longrightarrow M$ such that*

$$\phi_Z(p,t) = \begin{cases} \phi_X(p,t), & \text{if } (p,t) \in \overline{\Omega_X}, \\ \phi_{Z^s}(p,t), & \text{if } (p,t) \in \overline{\Omega_{Z^s}}, \\ \phi_Y(p,t), & \text{if } (p,t) \in \overline{\Omega_Y} \end{cases}$$

is well defined for all $(p,t) \in M \times \mathbb{R}^+$.

Proof. For all $p \in M$, applying Definition 1.39 and Lemma 2.9 we have that:

(a) if $p \in \mathrm{int}(\Sigma^+)$ then $\phi_Z(p,t) = \phi_X(p,t)$ for all $t \in [0, a_{0(p)}]$;

(b) if $p \in \mathrm{int}(\Sigma^-)$ then $\phi_Z(p,t) = \phi_X(p,t)$ for all $t \in [0, a_{0(p)}]$;

(c) if $p \in \Sigma^{c+}$ then $\phi_Z(p,t) = \phi_X(p,t)$ for all $t \in [0, a_{0(p)}]$;

(d) if $p \in \Sigma^{c-}$ then $\phi_Z(p,t) = \phi_Y(p,t)$ for all $t \in [0, a_{0(p)}]$;

(e) if $p \in \Sigma^s$ then $\phi_Z(p,t) = \phi_{Z^s}(p,t)$ for all $t \in [0, a_{0(p)}]$;

(f) if p is of the type A1, A4 or B1 for X then $\phi_Z(p,t) = \varphi_X(p,t)$ for all $t \in [0, a_{0(p)}]$,

(g) if p is of the type A2, A3 or B2 then $\phi_Z(p,t) = \phi_{Z^s}(p,t)$ for all $t \in [0, a_{0(p)}]$,

(h) if p is the type A1, or A4, or B1 for Y then $\phi_Z(p,t) = \varphi_Y(p,t)$ for all $t \in [0, a_{0(p)}]$.

If $a_{0(p)}$ is a real number then $\phi_{Z_1(p)}(p, a_{0(p)}) \in \Sigma$ hence apply item c) to h) above for all $t \in [a_{0(p)}, a_{1(p)}]$ and do it again if $a_1(p) = t^+_{Z_{0(p)}}(p_0) + t^+_{Z_{1(p)}}(p_1) < \infty$ and so forth. □

Proof.[Theorem A]

Applying the gluing lemma for $\phi_X : \overline{\Omega_X} \longrightarrow M$, $\phi_Y : \overline{\Omega_Y} \longrightarrow M$ and $\phi_{Z^s} : \overline{\Omega_{Z^s}} \longrightarrow M$, and by Remark 2.12, Proposition 2.14, Lemmas 2.16 and 2.17 we have that the map $\phi_Z : M \times \mathbb{R}^+ \longrightarrow M$ given by

$$\phi_Z(p,t) = \begin{cases} \phi_X(p,t) & \text{if } (p,t) \in \overline{\Omega_X}, \\ \phi_{Z^s}(p,t) & \text{if } (p,t) \in \overline{\Omega_{Z^s}}, \\ \phi_Y(p,t) & \text{if } (p,t) \in \overline{\Omega_Y} \end{cases}$$

is continuous.

Now, we prove the conditions in items (1) and (2) of Definition 1.1 for $\phi_Z(p,t)$. The initial value property of $\phi_Z(p,t)$ is satisfied since φ_X, φ_{Z^s} and φ_Y are flows. Now, we prove the semigroup property for $\phi_Z(p,t)$. In fact, let $p \in M$ and $t, s \in \mathbb{R}^+$. Let $j \in \mathbb{N} \cup \{0\}$ such that $t \in \mathrm{I}^j_{Z_{j(p)}}(p)$ and

$$q = \phi_Z(p,t) = \phi_{Z_{j(p)}}(p,t) = \varphi_{Z_{j(p)}}(p_j, t - a_{j-1(p)}). \tag{2-1}$$

Let $i \in \mathbb{N} \cup \{0\}$ be such that $s \in I^i_{Z_{i(p)}}(p)$ then $t + s \in I^{j+i}_{Z_{j+i(p)}}(p)$ and $Z_{i(q)} = Z_{j+i(p)}$. Hence, we use induction on $i \in \Delta^+_Z(p)$ to prove that:

1. $q_i = p_{j+i}$ for $i > 0$;
2. $a_{i-1(q)} = a_{j+i-1(p)} - t$ for $i > 0$;
3. and $\phi_Z(\phi_Z(p,t),s) = \phi_Z(p,t+s)$.

Induction basis: Assume that $i = 0$ then $s \in I^0_{Z_{0(p)}}(p)$, $t + s \in I^j_{Z_{j(p)}}(p)$ and $Z_{0(q)} = Z_{j(p)}$. Thus,

$$\begin{aligned} \phi_Z(\phi_Z(p,t),s) &= \phi_Z(q,s) = \phi_{Z_{0(q)}}(q,s) \\ &= \varphi_{Z_{0(q)}}(\varphi_{Z_{j(p)}}(p_j, t - a_{j(p)}), s) \\ &= \varphi_{Z_{j(p)}}(\varphi_{Z_{j(p)}}(p_j, t - a_{j(p)}), s) \\ &= \varphi_{Z_{j(p)}}(p_j, t - a_{j(p)} + s) \\ &= \varphi_{Z_{j(p)}}(p_j, (t+s) - a_{j(p)}) \\ &= \varphi_{Z_{j(p)}}(p, t+s) \\ &= \phi_Z(p, t+s). \end{aligned}$$

Inductive step: Assume that $i \in \Delta_Z(p)$ such that $s \in I^i_{Z_{i(p)}}(p)$, then $t + s \in I^{j+i}_{Z_{j+i(p)}}(p)$ and (1), (2) and (3) are satisfied, that is, $q_i = p_{j+i}$, $a_{i-1}(q) = a_{j+i-1}(p) - t$, and $\phi_Z(\phi_Z(p,t),s) = \phi_Z(p,t+s)$. We need to prove that, for $i+1 \in \Delta_Z(p)$ and $s' \in I^{i+1}_{Z_{i+1(p)}}(p)$ then properties (1), (2) and (3) are satisfied. In other words, $q_{i+1} = p_{j+i+1}$, $a_i(q) = a_{j+i}(p) - t$, and $\phi_Z(\phi_Z(p,t),s') = \phi_Z(p,t+s')$. In fact,

1. $q_{i+1} = \varphi_{Z_{i(q)}}(q_i, t^+_{Z_{i(q)}}(q_i)) = \varphi_{Z_{j+i(p)}}(p_{j+i}, t^+_{Z_{j+i(p)}}(p_{j+i})) = p_{j+i+1}$,
2. $a_{i(q)} = a_{i-1(q)} + t^+_{Z_{i(q)}}(q_i) = (a_{j+i-1(p)} - t) + t^+_{Z_{j+i(p)}}(p_{j+i}) = a_{j+i(p)} - t$, and
3.

$$\begin{aligned}
\phi_Z(\phi_Z(p,t),s') = \phi_Z(q,s') &= \phi_{Z_{i+1(q)}}(q_{i+1}, s' - a_{i(q)}) \\
&= \varphi_{Z_{j+i+1(p)}}(p_{j+i+1}, s' - (a_{j+i(p)} - t)) \\
&= \varphi_{Z_{j+i+1(p)}}(p_{j+i+1}, (t+s') - a_{j+i(p)}) \\
&= \phi_{Z_{j+i+1(p)}}(p, t+s') \\
&= \phi_Z(p, t+s').
\end{aligned}$$

Therefore (M,ϕ_Z) and $\Sigma = \Sigma^{c+} \cup \Sigma^{c-} \cup \Sigma^s \cup S_Z$ and with $p \in S_Z$ being of type A1 to B2 is a semidynamical system. □

As consequences of Theorem A, we obtain two significant corollaries.

Corollary 2.18 *Assume M is a closed 2-dimensional C^1 manifold and $Z = (X,Y) \in \mathfrak{X}(M,h)$. If $\Sigma = \Sigma^c \cup \Sigma^s \cup S_Z$ and $p \in S_Z$ has type* A1, A2 *or* B1 *then the trajectories of Z generate a semidynamical system (M,ϕ_Z).*

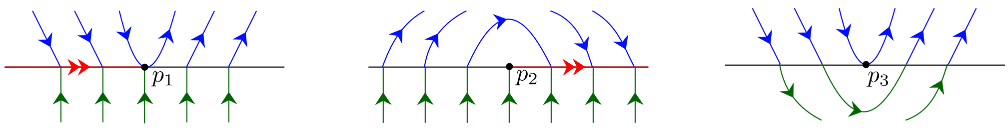

Figure 2.10: Example of points of type A and B in 2-dimension.

Corollary 2.19 *Assume a closed n-dimensional C^1 manifold M and $Z \in \mathfrak{X}(M,h)$. If $\Sigma = \Sigma^c \cup \Sigma^s$ then the trajectories of $Z = (X,Y) \in \mathfrak{X}(M,h)$ generate a semi-dynamical system (M,ϕ_Z).*

Proof. Note that, in the proof of Theorem A, if Σ^s empty then we can work with φ_X and φ_Y instead of ϕ_X and ϕ_Y. In this case, Definitions 2.2 and 2.10 are valid for $\mathbb{R}^-$. For $X \in \mathfrak{X}(M)$, let $\Lambda^-_X = \{p \in \Sigma^+;\ \varphi_X(p,(-\infty,0]) \not\subseteq \Sigma^+\}$ and $t^+_X : \Sigma^+ \longrightarrow \mathbb{R}^- \cup \{\infty\}$ such that

$$t^-_X(p) = \begin{cases} -\infty & \text{if } p \notin \Lambda^-_X, \\ \sup\{t<0;\ \varphi_X(p,t) \in S^{ic}_X \cup \Sigma^-_X\} & \text{if } \varphi_X(p,[t,0]) \subset \Sigma^+ \text{ for } t \in I \cap \mathbb{R}^- \text{ with } t \neq 0, \\ 0 & \text{otherwise .} \end{cases}$$

Analogously, for $Y \in \mathfrak{X}(M)$. Let $p \in M$, denote $p_0 = p$ and $p_i = \varphi_{Z^-_{(i-1)(p)}}(p_{i-1}, t^-_{Z_{i-1(p)}}(p_{i-1}))$ for $i \in \mathbb{Z}^-$, where

$$Z_{0(p)} = \begin{cases} X & \text{if } p \in \text{int}(\Sigma^+), \\ Y & \text{if } p \in \text{int}(\Sigma^-); \end{cases}$$

and

$$Z_{i(p)} = \begin{cases} X & \text{if } p_i \in \Sigma^{c+} \text{or } p_i \in \text{B1} \cap S^v_X, \\ Y & \text{if } p_i \in \Sigma^{c-} \text{or } p_i \in \text{B1} \cap S^v_Y. \end{cases}$$

Fix $\Delta^-_X(p) = \{i \in \mathbb{Z}^-;\ Z_{i(p)} = X\}$, $\Omega_X = \{(p,t) \in M \times \mathbb{R};\ t \in I_X(p)\}$ and

$$I_X(p) = \bigcup_{i \in \Delta^+_X \cup \Delta^-_X(p)} I^i_X(p),$$

where $I^0_X(p) = \left[a^-_{0(p)}, a_{0(p)}\right]$, $I^i_X(p) = \left[a^-_{i-1(p)}, a^-_{i(p)}\right]$ for $i < 0$ such that $a^-_{i(p)} = \sum_{j=0}^{i} t^-_{Z_{j(p)}}(p_j)$. Analogously, define Ω_Y for the vector field Y.

□

Finally, the following theorem is the main result of this chapter. It provides very general conditions under which an isolated invariant set must contain a periodic orbit of a PSVF.

Theorem B *Let $Z = (X,Y) \in \mathfrak{X}(M,h)$ be a* PSVF *given by (1-6) defined on a 3-dimensional C^1 manifold M with switching manifold Σ. Assume that $\Sigma = \Sigma^c \cup \Sigma^s \cup S_Z$ and $p \in S_Z$ being of type* Ai, *$i = 1, \dots, 4$ or* Bj, *$j = 1,2$, and $\phi_Z : M \times [0,\infty) \to M$ is the semiflow generated by the trajectories of Z. If N is an isolating neighborhood for φ which admits a Poincaré section Ξ and either*

$$dim\ CH^{2n}(N,\varphi) = dim\ CH^{2n+1}(N,\varphi) \qquad for \quad n \in \mathbb{Z}^+ \tag{2-2}$$

or

$$dim\ CH^{2n}(N,\varphi) = dim\ CH^{2n-1}(N,\varphi) \qquad for \quad n \in \mathbb{Z}^+, \tag{2-3}$$

where not all the above dimensions are zero, then φ has a periodic trajectory in N.

Corollary 2.20 *Under the hypotheses of Theorem B, if N has the Conley index of a hyperbolic periodic orbit, then* inv(N) *contains a periodic orbit.*

The following theorem is an n-dimensional version of Theorem B, where we assume that the system has only crossing and sliding regions without tangency points.

Theorem C *Let $Z=(X,Y)\in\mathfrak{X}(M,h)$ be a* PSVF *given by* (1-6) *defined on a closed n-dimensional C^1 manifold M with switching manifold Σ. Assume that $\Sigma=\Sigma^c\cup\Sigma^s$ and $\phi_Z : M\times[0,\infty)\to M$ is the semiflow generated by the trajectories of Z. If N is an isolating neighborhood for φ which admits a Poincaré section Ξ and either*

$$dim\ CH^{2n}(N,\varphi) = dim\ CH^{2n+1}(N,\varphi) \qquad for \quad n\in\mathbb{Z}^+ \tag{2-4}$$

or

$$dim\ CH^{2n}(N,\varphi) = dim\ CH^{2n-1}(N,\varphi) \qquad for \quad n\in\mathbb{Z}^+, \tag{2-5}$$

where not all the above dimensions are zero, then φ has a periodic trajectory in N.

2.2 Some applications

2.2.1 Regularization

In this subsection, we use Proposition 1.22, the regularization of discontinuous vector fields in [46], and the robustness of the Conley index under perturbation to provide an immediate consequence of Corollary 2.18 and Theorem B. Denote by *transition function* a C^∞ function $\varphi:\mathbb{R}\longrightarrow\mathbb{R}$ such that: $\varphi(t)=0$ if $t\le -1$, $\varphi(t)=1$ if $t\ge 1$ and $\varphi'>0$ if $t\in(-1,1)$.

Definition 2.21 *A $\varphi_\epsilon-$regularization of $Z=(X,Y)\in\mathfrak{X}(M,h)$ defined on a closed 2-dimensional manifold is a one parameter family of vector fields Z_ϵ in $\mathfrak{X}(M)$ given by*

$$Z_\epsilon(q) = (1-\varphi_\epsilon(h(q)))Y(q)+\varphi_\epsilon(h(q))X(q),$$

where $\varphi_\epsilon(t)=\varphi(\frac{t}{\epsilon})$.

Using the homotopy invariance of the Conley index and Proposition 1.22 we have the following proposition.

Proposition 2.22 *Let γ be a periodic orbit in an isolating neighborhood N of the semiflow generated by the trajectories of $Z=(X,Y)\in\mathfrak{X}(M,h)$ defined on a closed 2-dimensional manifold. Then there is ϵ_0 such that for every $\epsilon\le\epsilon_0$, Z_ϵ contains a periodic orbit in N.*

2.2.2 Closed poly-trajectories

For a PSVF defined on a region of the plane, in [47], the authors present a concatenation of arcs of trajectories called closed poly-trajectories. They show that if the regularization of a PSVF contains a poly-trajectory, then there exists one periodic orbit. Applying our results, we can further provide necessary conditions to guarantee the existence of closed poly-trajectories solutions in a disk of M when a PSVF has sliding motion.

Definition 2.23 *Let M be a closed 2-dimensional C^1 manifold and $Z=(X,Y)\in\mathfrak{X}(M,h)$.*

1. *A curve Γ is a closed poly-trajectory if Γ is closed and*
   - *Γ contains regular arcs of at least two of the vector fields $X|\Sigma^+$, $Y|\Sigma^-$, Z^e and Z^s or is composed by a single regular arc of either Z^s or Z^e;*
   - *the transition between arcs of X and arcs of Y happens in sewing points (and vice versa);*
   - *the transition between arcs of X (or Y) and arcs of Z^s or Z^e happens through fold points or regular points in the escape or sliding arc, respecting the orientation. Moreover if $\Gamma\neq\Sigma$ then there exists at least one visible fold point on each connected component of $\Gamma\cap\Sigma$.*
2. *Let Γ be a canard cycle of Z. We say that Γ is a closed poly-trajectory*
   - *of type I if Γ meets Σ just in sewing points;*
   - *of type II if $\Gamma=\Sigma$;*
   - *of type III if Γ contains at least one visible fold point of Z.*

   In Figures 2.11, 2.12 and 2.13 poly-trajectories of type I, II and III are respectively illustrated.
3. *Let Γ be a closed poly-trajectory. We say that Γ is hyperbolic if it*
   - *is of type I and $\eta\prime(p)\neq 1$ where η is the first return map defined on a segment T with $p\in T\cap\Gamma$;*
   - *is of type II;*
   - *is of type III and either $\Gamma\cap\Sigma\subseteq\Sigma^c\cup\Sigma^s$ or $\Gamma\cap\Sigma\subseteq\Sigma^c\cup\Sigma^e$.*

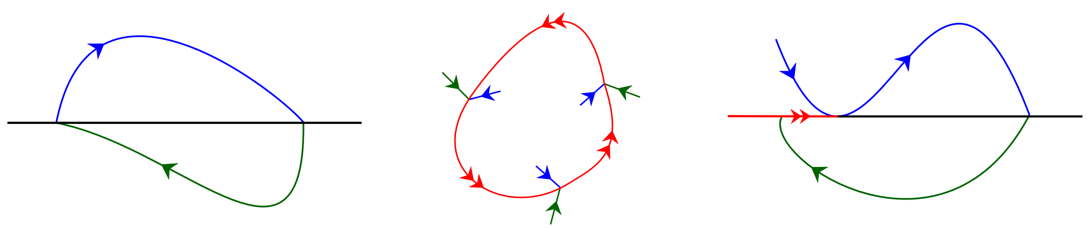

Figure 2.11: Closed poly-trajectories of type I (left), II (center) and III (right).

Let $(pq)_{Z^*}$ be the arc $\phi_{Z^*}(p,[0,t_q])$, where $t_q > 0$ is the time such that $\phi_{Z^*}(p,t_q) = q$, and Z^* is either X or Y. We say that $(pq)_{Z^*}$ has *focal type* if there are no fold points between p and q (see Figure 2.12) and we say that $(pq)_{Z^*}$ has *graphic type* if it has only one fold point between p and q (see Figure 2.13).

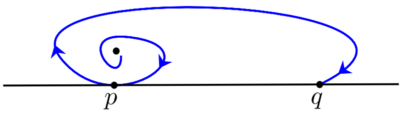

Figure 2.12: Focal type arc.

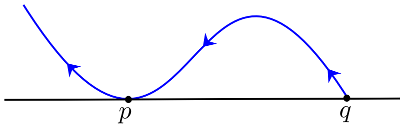

Figure 2.13: Graphic type arc.

Applying Theorem C, we obtain existence results for poly-trajectories.

Proposition 2.24 *Let* $\mathcal{U} = \{(x,y,z) \in \mathbb{R}^2;\ x^2+y^2 < \delta\}$ *be a disk of* $\mathbb{R}^2$ *where* $\delta > 0$ *and* $Z = (X,Y) \in \mathfrak{X}(\mathcal{U},h)$ *such that* $\Sigma = \Sigma^c \cup \Sigma^s \cup S_Z$*, and for all* $p \in S_Z$*,* p *is of type* A1*,* A2 *or* B1*. Moreover, assume that* X *is a linear vector field, and*

(1) $int(\mathcal{U}) \cap \Sigma \neq \emptyset$,

(2) $\mathcal{U} \cap int(\Sigma^+)$ *contains only one equilibrium point* $\tilde{x}$*, which is an unstable focus and the unstable manifold of* $\tilde{x}$ *intercepts* Σ *in an arc,*

(3) there are no pseudo-equilibrium points in $\mathcal{U} \cap \Sigma$,

(4) $Yh(x) > 0$ *for all* $x \in \mathcal{U} \cap \Sigma$.

Then, there exists a hyperbolic poly-trajectory of type III *in* $\mathcal{U}$.

Proof. A closed poly-trajectory of $Z = (X,Y) \in \mathfrak{X}(\mathcal{U},h)$ is a periodic orbit for semiflow ϕ_Z. Thus under the hypothesis of Theorem B we obtain the existence of a poly-trajectory. Therefore, it remains to show that the hypothesis of Theorem B is satisfied.

Let $\tilde{x}$ be the unstable focus in $\mathcal{U} \cap \text{int}(\Sigma^+)$ and p be the visible fold point for X such that $\gamma_0(p) = p \cdot [0, t_X^+(p)]$ is the focal type arc. Let $\tilde{p}$ and $\tilde{q}$ be the intersection of $\partial\mathcal{U}$ with Σ such that $Xh(x) > 0$ for all $x \in (\tilde{p}p)_\Sigma \setminus \{p\}$, where $(\tilde{p}p)_\Sigma$ is the arc in Σ from the point $\tilde{p}$ to p, and $Xh(x) < 0$ for all $x \in (p\tilde{q})_\Sigma \setminus \{p\}$. Take $q \in \text{int}(\tilde{p}p)_\Sigma$ such that $q_1 \in \text{int}(p\tilde{q})_\Sigma$ and $\gamma_0(q) \subset \mathcal{U}$. Consider $\mu < 0$ such that $\Sigma_\mu = h^{-1}(\mu)$ is parallel to Σ and $\Sigma_\mu \cap \text{int}(\mathcal{U}) \neq \emptyset$. Consider the points $v, \tilde{v} \in \Sigma_\mu$ satisfying that $q = \tilde{v}_1$ and $q_1 = v_1$. Let $\tilde{N}$ be the region bounded by the curve $(v\tilde{v})_{\Sigma_\mu} \cup \gamma_0(\tilde{v}) \cup \gamma_0(q) \cup \gamma_0(v)$, see Figure 2.14.

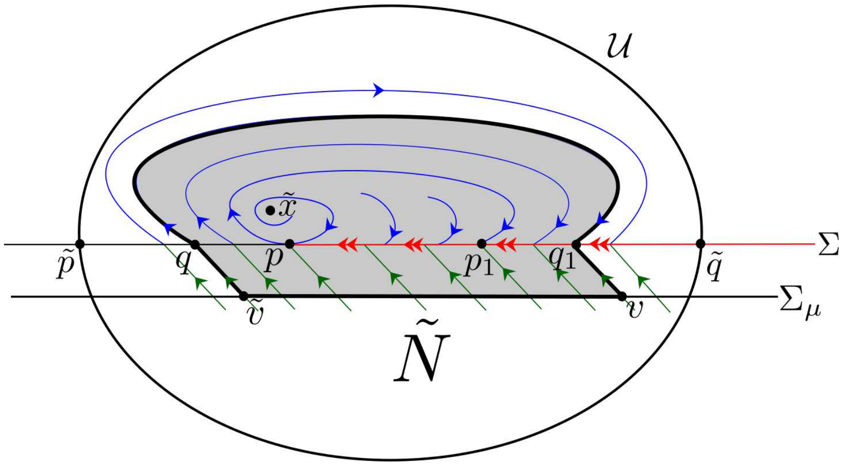

Figure 2.14: Construction of the isolating neighborhood.

Let $r = (x_r, y_r) \in \mathcal{U}$ be a point such that $x_r = x_{x_0}$ and $y_r > y_{x_0}$, where $x_0 = (x_{x_0}, y_{x_0})$. Let Ξ be a local section for φ_X at r such that $\varphi_X \cap \Sigma = \emptyset$, that is, Ξ is transverse to the flow φ_X, as in Figure 2.15. Let $a \pm bi$ be the eigenvalues of X associated to $\tilde{x}$. Then fix $r \in \Xi$ such that $\theta(r) < 90°$ when $b < 0$ or $90° < \theta(r) < 180°$ when $b > 0$, where $\theta(r)$ is the angle between the vector $X(r)$ and the tangent space of Σ at r. Since $\tilde{x}$ is an unstable focus, using the Poncaré map there exists $\tilde{r} \in \Xi$ such that $\tilde{r} = \pi_\Xi(r)$. Let $\tilde{\gamma}_r$ be the arc from r to $\tilde{r}$ by flow φ_X.

By Jordan curve theorem, the curve $\tilde{\gamma}_r \cup (r\tilde{r})_\Xi$ delimits a topological disk D. Therefore, $N = \tilde{N} \setminus \text{int}(D)$ is a compact region.

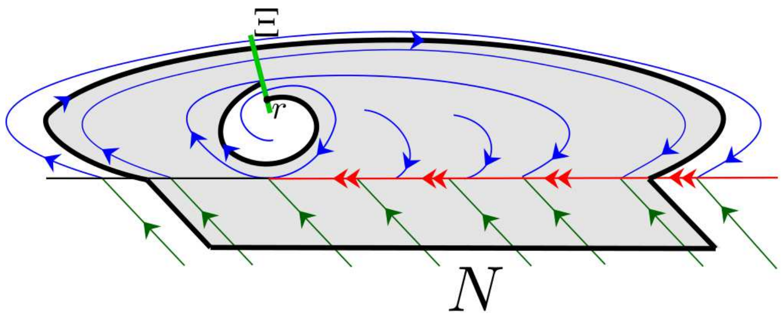

Figure 2.15: Poincaré Section.

Note that, $\text{inv}(N) \subset \text{int}(N)$; in fact, in what follows, we will show that the exit set of N, denoted by L is empty. Firstly, for all $x \in \Sigma^s \cap \mathcal{U}$ there is $s_x \geq 0$ such that $x \cdot s_x = p$. To show this we can consider that $X = (x,y) = (-x+y-\varepsilon, -x-y+\varepsilon)$ and $Y = (-1,1)$, so $\tilde{x} = (0,\varepsilon)$, $p = (\varepsilon, 0)$, and the forward orbits of the points in Σ^s using $Z^s = (-2x, 0)$ tends towards p. Then, if $x' \in N \cap \Sigma^+$ we have that by hypotheses (3) either $x' \in \Sigma^s$ or there exists $t_X^+(x') \in (0, \infty)$ such that $x' \cdot t_X^+(x') \in \Sigma^s$; then, there exists $s_{x'} \geq 0$ such that $x' \cdot (t_X^+(x') + s_{x'}) = p$. Now, if $x' \in N \cap \Sigma^-$ then there exists $t_Y^+(x') > 0$ such that $x' \cdot t_Y^+(x') \in \Sigma \subset \Sigma^+$. Therefore $L = \emptyset$. Note that N is an annular region since it is a set homomorphic to a disk with a hole, see Figure 2.16. Therefore, the homotopy type of N is the same as that of a stable periodic orbit, thus

$$CH^k(N) \approx \begin{cases} \mathbb{Z}, & k = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

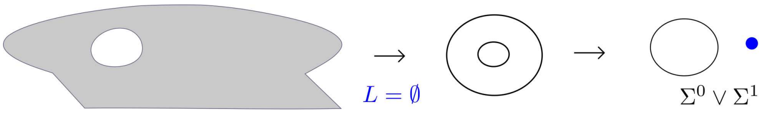

Figure 2.16: The homotopy type of N/L.

Poincaré Section. Let Ξ be a local section for φ_X crossing the stable manifold of $\tilde{x}$ as in Figure 2.15. We claim that Ξ is the required Poincaré section for N. It is a closed set and transverse to the semiflow generated by $Z = (X, Y)$. Finally, we must show that the forward orbit of every point in N intersects Ξ. But this was already shown when it was proved that $L = \emptyset$. Thus Ξ is a Poincaré section for the semiflow ϕ_Z in N.

□

Let $Z = (X, Y) \in \mathfrak{X}(\mathcal{U}, h)$ be a PSVF. We assume X under the same hypotheses of the vector field X in Proposition 2.24. Assume that $\Sigma \cap \mathcal{U}$ contains a point q that is an invisible fold of Y on the left side of Figure 2.17. Then using the ideas of Proposition 2.24 it is possible to show that in the disk $\mathcal{U}$, there exists a hyperbolic poly-trajectory of type III. We have the same result assuming that $\Sigma \cap \mathcal{U}$ contains a point q that is an invisible fold and a point $\tilde{q}$ that is a visible fold of Y, as on the right side of Figure 2.17.

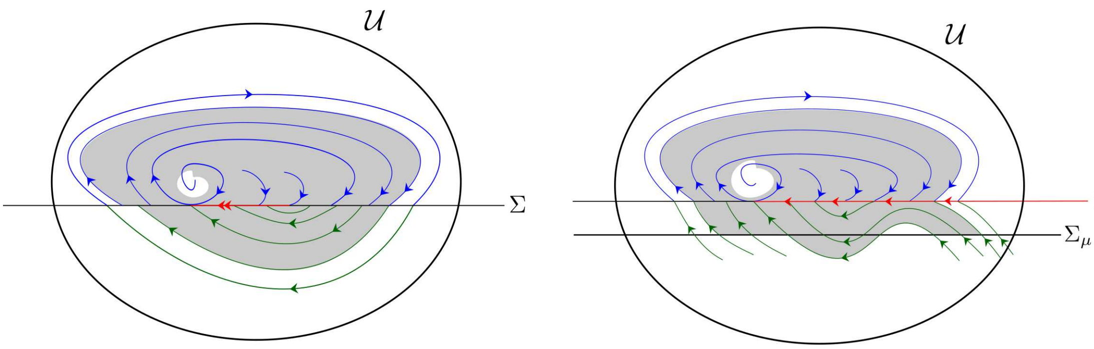

Figure 2.17: In both cases, there exists a hyperbolic poly-trajectory of type III inside in $\mathcal{U}$.

CHAPTER 3

Existence of periodic orbits for switching networks via Conley theory.

In this chapter, we apply the previous results obtained to biological systems. The method presented in this chapter guarantees the existence of periodic orbits for the gene regulatory network called Repressilator.

3.1 Introduction

In the human body, we can find a wide range of different cells; nevertheless, the genetic code inside each cell is identical. The genetic regulation networks have the function of modulating the genes present in each cell and thus can distinguish between them. In particular, the identity of a cell depends on how these genes are turned on and off and how they interact among themselves. To understand the underlying interaction between genes, we present a model for the gene regulatory based on Dynamic Signatures Generated by Regulatory Networks (DSGRN) in [11], where given a regulatory network **RN**, the associated database of dynamic signatures encoding the global dynamics can be built in the entire parameter space.

The Repressilator (See Elowitz and Leibler, 2000) was the first synthetic genetic network built in the bacterium Escherichia coli. It consists of a system of three proteins, called $LacI$, $tetR$, and λcI that, when combined with a green fluorescent protein (a protein produced by the jellyfish Aequorea victoria) makes E. coli cells flash periodically. The protein expressed by $LacI$ acts as a repressor of $tetR$. In turn, the protein expressed by tetR acts as a repressor of $\lambda\ cI$. Finally, the protein expressed by $\lambda\ cI$ acts as a repressor of $LacI$. The mathematical conceptualization of a regulatory network RN, introduced in [11], is a directed graph where the nodes (vertices) $V = \{1, \cdots, \mathrm{M}\}$ indicate the genes, and the edges E indicate the activation or repression of a production of protein from one gene to another. Although the method presented in this chapter is applied to a particular network, our main objective is to guarantee the existence of periodic orbits in a wide variety of gene regulatory networks.

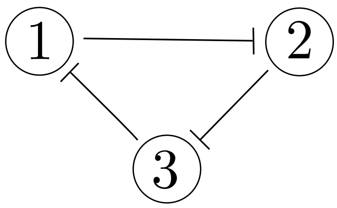

Figure 3.1: The repressilator.

The phase space dimension of the model equals to the number of nodes, and the associated parameter space consists of one parameter for each node and three parameters for each edge. The parameter of each node is called a *decay rate*, and the parameters for each edge are called the *low level* of expression, the *high level* of expression, and the *threshold* at which expression levels change. The associated switching system of a regulatory network **RN** with M nodes is an M-dimensional system of ODE of the form $\dot{x} = -\Gamma x + \Lambda(x)$, where Γ is a diagonal matrix with positive entries and Λ is a piecewise constant function. Each of the terms in the diagonal of Γ (a decay rate) is denoted by γ_i and corresponds to each variable x_i. For simplicity, the decay rates are fixed and equal to 1 for all the nodes. The goal is to show that the repressilator admits a periodic orbit in a region of the parameter space of dimension 9. Moreover, the same idea can be extended to show the existence of periodic orbits in the networks shown in Figure 3.2.

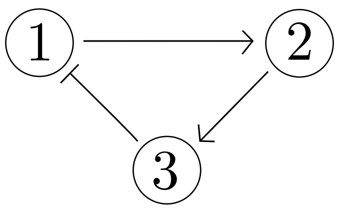

Figure 3.2: Other network which admits periodic orbit.

3.2 Switching networks

The definitions and results presented in this section are based on paper [11]. A switching network is one M-dimensional system of ordinary differential equations generated by a regulatory network **RN** with M nodes. Mathematically, a regulatory network is a directed graph where the nodes (vertices) $V = \{1, \ldots, \mathrm{M}\}$ indicate the gene, and the edges E indicate the activation or repression of the production of proteins produced by the gene represented by the incoming edge.

Definition 3.1 *A* regulatory network $\mathbf{RN} = (V, E)$ *is an annotated directed graph with vertices* $V = \{1, \ldots, \mathrm{M}\}$ *called network nodes and annotated directed edges* $E \subset V \times V \times \{\to, \dashv\}$ *called interactions. An* $\to$ *annotated edge is referred to as an activation and an* $\dashv$ *annotated edge is called a repression. We indicate either* $i \to j$ *or* $i \dashv j$*, without specifying which one by writing* $(i,j) \in E$*. We allow for self-edges but admit at most one edge between any two nodes, e.g., we cannot have both* $i \to j$ *and* $i \dashv j$ *simultaneously. The set of sources and targets of a node* n *are denoted by*

$$\mathrm{S}(n) := \{i|\ (i,n) \in E\} \quad \textit{and} \quad \mathrm{T}(n) := \{j|\ (n,j) \in E\}.$$

The cardinality of $\mathrm{S}(n)$ *and* $\mathrm{T}(n)$ *are denoted by* $\#\mathrm{S}(n)$ *and* $\#\mathrm{T}(n)$*. Each node is equipped with a multilinear function* $M_i : \mathbb{R}^{\#\mathrm{S}(i)} \to \mathbb{R}$*, called the logic of node* i*.*

An essential aspect of the regulatory network is understanding associate parameters to a **RN** that describes a biological system.

Definition 3.2 *Given a regulatory network* $\mathbf{RN} = (V, E)$*, for each edge* $(i,j) \in E$ *(i.e.,* $i \to j$ *or* $i \dashv j$*) we associate three parameters:* $l_{j,i}$*,* $u_{j,i}$*, and* $\theta_{j,i}$*. (Note the matrix-style subscript order convention.) Additionally, to each node* $i \in V$ *we associate a decay rate* γ_i*. Each of these parameters is a real number, so we may regard the collection of all these parameters as a tuple* $(l, u, \theta, \gamma) \in \mathbb{R}^D$*, where* $D = \mathrm{M} + 3\#(E)$*. We call this collection of numbers* a parameter *for* **RN**.

An **RN** with M nodes, can be modeled by a switching system given by M-dimensional piecewise system of ordinary differential equations defined as follows.

Definition 3.3 *Given a regulatory network* **RN** *the associated switching system at parameter* $(l, u, \theta, \gamma) \in \mathbb{R}^D$ *is given by*

$$\dot{x_j} = -\gamma_j x_j + \Delta_j(x), \quad j = 1, \ldots, M,$$

where

$$\Delta_j(x) := M_j \circ \sigma_j.$$

Here $\sigma_j : \mathbb{R}^{\mathrm{M}} \to \mathbb{R}^{\#S(j)}$ *is a multidimensional step function defined componentwise (i.e. by its coordinate projections* $\pi_i(\sigma_j)$*) for each* $i \in S(j)$ *as*

$$\sigma_{i,j} = \pi_i(\sigma_j(x)) := \begin{cases} l_{j,i} & \textit{if } i \to j \textit{ and } x_i < \theta_{j,i}, \textit{ or if } i \dashv j \textit{ and } x_i > \theta_{j,i}. \\ u_{j,i} & \textit{if } i \to j \textit{ and } x_i > \theta_{j,i}, \textit{ or if } i \dashv j \textit{ and } x_i < \theta_{j,i}. \\ \textit{undefined} & \textit{otherwise.} \end{cases} \tag{3-1}$$

The nonlinear functions M_j are logical expressions involving truth variables v_i, logical conjunctives $\wedge$ (i.e. ANDs), and logical disjunctives $\vee$ (i.e., ORs) leads to an analogous arithmetic expression by replacing $\wedge$ with $\cdot$ and $\vee$ with $+$. Note that given truth variables v_i, a logical expression $L(v_1, v_2, \ldots, v_n)$ (without negations) leads unambiguously to the multilinear expression $M(x_1, x_2, \ldots, x_n)$ given by

$$M(x_1, x_2, \ldots, x_n) := \sum_{L(v_1, v_2, \ldots, v_n)=T} \prod_{v_i=F} (1-x_i) \prod_{v_i=T} x_i,$$

where $x_i \in \mathbb{R}$.

Definition 3.4 *A regulatory network* **RN** *and a choice of parameter values $z \in Z$ leads to a uniquely defined switching 3-4 and set of cells $\mathcal{K}(z)$. The k-cell vector field for a cell $k \in \mathcal{K}(z)$ is given by*

$$f^k(x) := -\Gamma x + \Delta(k). \tag{3-2}$$

3.3 Analysis and Method

In this section we show that the repressilator model admits a periodic orbit for the following combination of parameters

$$\begin{cases} 0 < l_{1,3} < \theta_{2,1} < u_{1,3}, \\ 0 < l_{2,1} < \theta_{3,2} < u_{2,1}, \\ 0 < l_{3,2} < \theta_{1,3} < u_{3,2}. \end{cases} \tag{3-3}$$

We begin by constructing an isolating block N in R^3 such that the Conley index corresponds to an unstable hyperbolic periodic orbit. After, we show that N admits a Poincaré section then inv(N) contains one periodic orbit. We say that a compact set N is an *attracting block* if it is an isolating block such that the forward image of N is entirely contained in N. Using Definitions 1.3 and 1.4, we say that a compact set N is an attracting block if $n^- = \emptyset$, where $n = \partial \mathrm{N}$ and $n^- = \{p \in n | \; \exists \epsilon > 0 \text{ with } \; p \cdot (0, \epsilon) \cap \mathrm{N} = \emptyset\}$.

Applying Definition 3.3, the switching equations for the repressilator take the following form

$$\begin{aligned} \dot{x_1} &= -\gamma_1 x_1 + \Delta_1(x) \\ \dot{x_2} &= -\gamma_2 x_2 + \Delta_2(x) \\ \dot{x_3} &= -\gamma_3 x_3 + \Delta_3(x) \end{aligned}$$

Without loss of generality, we consider $\gamma_i = 1$ for $i = 1,2,3$. We use the logic expression $L(v_1,v_2,v_3) = v_1 \wedge v_2 \wedge v_3$ and therefore $M_1 \circ \sigma_1(x_1,x_2,x_3) = \sigma_{1,3}(x_1,x_2,x_3)$, $M_2 \circ \sigma_2(x_1,x_2,x_3) = \sigma_{2,1}(x_1,x_2,x_3)$ and $M_3 \circ \sigma_3(x_1,x_2,x_3) = \sigma_{3,2}(x_1,x_2,x_3)$ is trivial for each node. Then we obtain the following smooth piecewise system:

$$\dot{x_1} = -x_1 + \begin{cases} l_{1,3} & \text{if } x_3 \geq \theta_{1,3}, \\ u_{1,3} & \text{if } x_3 \leq \theta_{1,3}, \end{cases}$$
$$\dot{x_2} = -x_2 + \begin{cases} l_{2,1} & \text{if } x_1 \geq \theta_{2,1}, \\ u_{2,1} & \text{if } x_1 \leq \theta_{2,1}, \end{cases} \tag{3-4}$$
$$\dot{x_3} = -x_3 + \begin{cases} l_{3,2} & \text{if } x_2 \geq \theta_{3,2}, \\ u_{3,2} & \text{if } x_2 \leq \theta_{3,2}. \end{cases}$$

Note that, the system (3-4) is a PSVF, also it has a single regulatory threshold for each variable that divides the phase space into 8 regions,

$$\begin{aligned}
\mathcal{C}_1 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ x_1 \geq \theta_{2,1}, 0 \leq x_2 \leq \theta_{3,2}, 0 \leq x_3 \leq \theta_{1,3}\};\\
\mathcal{C}_2 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ x_1 \geq \theta_{2,1}, 0 \leq x_2 \leq \theta_{3,2}, x_3 \geq \theta_{1,3}\};\\
\mathcal{C}_3 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ 0 \leq x_1 \leq \theta_{2,1}, 0 \leq x_2 \leq \theta_{3,2}, x_3 \geq \theta_{1,3}\};\\
\mathcal{C}_4 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ 0 \leq x_1 \leq \theta_{2,1}, 0 \leq x_2 \leq \theta_{3,2}, 0 \leq x_3 \leq \theta_{1,3}\};\\
\mathcal{C}_5 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ 0 \leq x_1 \leq \theta_{2,1}, x_2 \geq \theta_{3,2}, x_3 \geq \theta_{1,3}\};\\
\mathcal{C}_6 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ 0 \leq x_1 \leq \theta_{2,1}, x_2 \geq \theta_{3,2}, 0 \leq x_3 \leq \theta_{1,3}\};\\
\mathcal{C}_7 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ x_1 \geq \theta_{2,1}, x_2 \geq \theta_{3,2}\, 0 \leq x_3 \leq \theta_{1,3}\};\\
\mathcal{C}_8 &= \{(x_1,x_2,x_3) \in \mathbb{R}^3;\ x_1 \geq \theta_{2,1}, x_2 \geq \theta_{3,2}, x_3 \geq \theta_{1,3}\}.
\end{aligned}$$

Using the system (3-4), in each region $\mathcal{C}_i$, a vector field X_i is defined as

$$\begin{aligned}
X_1 &= (-x_1 + u_{1,3}, -x_2 + l_{2,1}, -x_3 + u_{3,2});\\
X_2 &= (-x_1 + l_{1,3}, -x_2 + l_{2,1}, -x_3 + u_{3,2});\\
X_3 &= (-x_1 + l_{1,3}, -x_2 + u_{2,1}, -x_3 + u_{3,2});\\
X_4 &= (-x_1 + u_{1,3}, -x_2 + u_{2,1}, -x_3 + u_{3,2});\\
X_5 &= (-x_1 + l_{1,3}, -x_2 + u_{2,1}, -x_3 + l_{3,2});\\
X_6 &= (-x_1 + u_{1,3}, -x_2 + u_{2,1}, -x_3 + l_{3,2});\\
X_7 &= (-x_1 + u_{1,3}, -x_2 + l_{2,1}, -x_3 + l_{3,2});\\
X_8 &= (-x_1 + l_{1,3}, -x_2 + l_{2,1}, -x_3 + l_{3,2}).
\end{aligned}$$

The remaining part of the current section is organized as follows: subsection 3.3.1 presents a construction of an isolating block, and subsection 3.3.2 contains the proof of the existence of a Poincaré section.

3.3.1 Isolating block for the Repressilator

In this section, we use the semi-flow obtained from the system 3-4 to construct a neighborhood N that will isolate the desired periodic orbit. We build the set N starting from a piece **N** and then by removing some open sets. Initially, we construct one rectangular set of the form

$$\mathbf{N} = \prod_{i=1}^{3} [a_i, b_i],$$

with $a_i < b_i$. We use **N** to construct a set such that the forward orbits generated by (3-4) stay inside. Note that, the system (3-4) is a linear system that admits a unique hyperbolic attractor for each vector field X_i, and this attractor belongs in some $\mathcal{C}_j$. This information is reflected in the following lemma.

Lemma 3.5 *For each vector field X_i there is a unique hyperbolic attractor p_i, such that $p_1 \in \mathcal{C}_2$, $p_2 \in \mathcal{C}_3$, $p_3 \in \mathcal{C}_5$, $p_4 \in \mathcal{C}_8$, $p_5 \in \mathcal{C}_6$, $p_6 \in \mathcal{C}_7$, $p_7 \in \mathcal{C}_1$ and $p_8 \in \mathcal{C}_4$.*

Observe that $p_i \notin \mathcal{C}_i$, this is fundamental for the existence of a Poincaré section. Note that all $\mathcal{C}_i' s$ are in the first orthant, then by Lemma 3.5 we can assume a rectangular region **N** such that $p_i \in \mathbf{N}$. More specifically, consider η_1, η_2 and η_3 such that:

$$\mathbf{N} = [0, \theta_{2,1} + \eta_1] \times [0, \theta_{3,2} + \eta_2] \times [0, \theta_{1,3} + \eta_3],$$

where $\eta_1 > u_{1,3} - \theta_{2,1}$, $\eta_2 > u_{2,1} - \theta_{3,2}$, and $\eta_3 > u_{3,2} - \theta_{1,3}$. Using the notation of Definition 1.3, note that, $p \in \mathbf{n}^+$ for all $p \in \partial\mathbf{N} = \mathbf{n}$ for the vector field X_i, $i = 1, \ldots, 8$. Topologically, it is represented by a rectangular region in Figure 3.3.

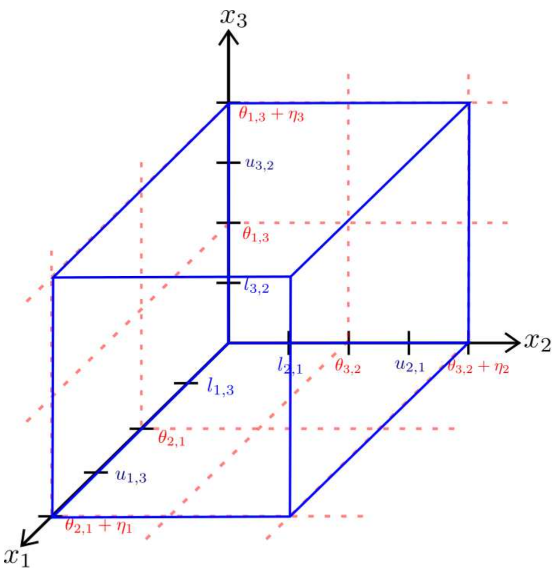

Figure 3.3: The rectangular set **N**.

We delimitate $\mathcal{C}_i$ by the compact region $C_i = \mathbf{N} \cap \mathcal{C}_i$ that are described as follows

$$\begin{aligned}
C_1 &= [\theta_{2,1}, \theta_{2,1} + \eta_1] \times [0, \theta_{3,2}] \times [0, \theta_{1,3}];\\
C_2 &= [\theta_{2,1}, \theta_{2,1} + \eta_1] \times [0, \theta_{3,2}] \times [\theta_{1,3}, \theta_{1,3} + \eta_3];\\
C_3 &= [0, \theta_{2,1}] \times [0, \theta_{3,2}] \times [\theta_{1,3}, \theta_{1,3} + \eta_3];\\
C_4 &= [0, \theta_{2,1}] \times [0, \theta_{3,2}] \times [0, \theta_{1,3}];\\
C_5 &= [0, \theta_{2,1}] \times [\theta_{3,2}, \theta_{3,2} + \eta_2] \times [\theta_{1,3}, \theta_{1,3} + \eta_3];\\
C_6 &= [0, \theta_{2,1}] \times [\theta_{3,2}, \theta_{3,2} + \eta_2] \times [0, \theta_{1,3}];\\
C_7 &= [\theta_{2,1}, \theta_{2,1} + \eta_1] \times [\theta_{3,2}, \theta_{3,2} + \eta_2] \times [0, \theta_{1,3}];\\
C_8 &= [\theta_{2,1}, \theta_{2,1} + \eta_1] \times [\theta_{3,2}, \theta_{3,2} + \eta_2] \times [\theta_{1,3}, \theta_{1,3} + \eta_3].
\end{aligned}$$

Proceeding in the construction, we want to obtain an attracting isolating block. For this, we need to remove a small region containing the intersection of hyperplanes: $x_1 = \theta_{2,1}$, $x_2 = \theta_{3,2}$ and $x_3 = \theta_{1,3}$.

In order to construct the isolating block, we consider the following hyperplanes, which are used to remove pieces from the rectangular set $\mathbf{N}$. Let $f_i : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $f_1 = x_1 - \theta_{2,1}$, $f_2 = x_2 - \theta_{3,2}$ and, $f_3 = x_3 - \theta_{1,3}$. Note that, the points in the intersection between C_i and C_j belong to some hyperplane $f_k^{-1}(0)$. We defined $\widetilde{\Sigma}_i = f_i^{-1}(0)$.

The first step is extract from $\mathbf{N}$ some rectangular cubes C_i, that correspond to open sets in the topology of $\mathbf{N}$. Note that, by Lemma 3.5, $p_1 \in C_2$, $p_2 \in C_3$, $p_3 \in C_5$, $p_5 \in C_6$, $p_6 \in C_7$, and $p_7 \in C_1$, and so, the cubes C_4 and C_8 do not belong to this sequence. Then, the cubes to be removed are C_4 and C_8. As the vector fields X_4 and X_8 are such that $p_4 \in \text{int } C_8$ and $p_8 \in \text{int } C_4$, then $X_8 f_i(p) < 0$ for all $p \in f_i^{-1}(0) \cap (C_2 \cup C_5 \cup C_7)$ and $X_4 f_i(p) > 0$ for all $p \in f_i^{-1}(0) \cap (C_1 \cup C_3 \cup C_6)$. Denote by $\mathbf{N}_1$ the set $\mathbf{N} \setminus (C_4 \cup C_8)$. A direct consequence is that the flow is transverse to the points in $\partial \mathbf{N}_1$, that is, $p \in \mathbf{n}_1^+$ for all $p \in \partial \mathbf{N}_1 = \mathbf{n}_1$.

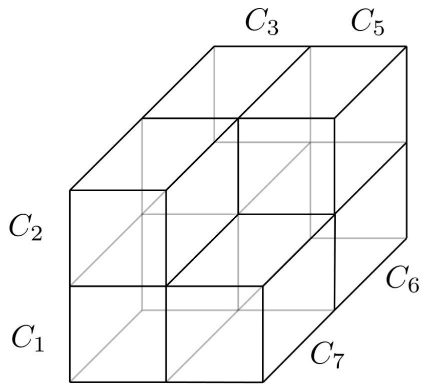

Figure 3.4: The set $\mathbf{N}_1$.

Bear in mind that the points in $\mathbf{N}_1$ that belong to the hyperplanes $\widetilde{\Sigma}_i's$ have two or more vector fields defined on it. However, the Filippov convention is not defined for more than two vector fields, and the next step is to solve this problem. Let us make some cuts in the rectangular cubes C_i to ensure that only two vector fields are defined at the intersections of the rectangular subsets and so to be able to use Filippov's convention for the system (3-4).

Consider the values $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}^+$ sufficiently small such that

$$\begin{aligned} \theta_{2,1}+\varepsilon_1 < u_{1,3} &\text{ and } -\theta_{2,1}+\varepsilon_1 < -l_{1,3}; \\ \theta_{3,2}+\varepsilon_2 < u_{2,1} &\text{ and } -\theta_{3,2}+\varepsilon_2 < -l_{2,1}; \\ \theta_{1,3}+\varepsilon_3 < u_{3,2} &\text{ and } -\theta_{1,3}+\varepsilon_3 < -l_{3,2}; \end{aligned} \tag{3-5}$$

We remove from each C_i the open rectangular sets c_i such that

$$\begin{aligned} c_1 &= [\theta_{2,1}, \theta_{2,1}+\varepsilon_1) \times [0, \theta_{3,2}] \times [0, \theta_{1,3}]; \\ c_2 &= [\theta_{2,1}, \theta_{2,1}+\eta_1] \times (\theta_{3,2}-\varepsilon_2, \theta_{3,2}] \times [\theta_{1,3}, \theta_{1,3}+\eta_3]; \\ c_3 &= [0, \theta_{2,1}] \times [0, \theta_{3,2}] \times [\theta_{1,3}, \theta_{1,3}+\varepsilon_3); \\ c_5 &= (\theta_{2,1}-\varepsilon_1, \theta_{2,1}] \times [\theta_{3,2}, \theta_{3,2}+\eta_2] \times [\theta_{1,3}, \theta_{1,3}+\eta_3]; \\ c_6 &= [0, \theta_{2,1}] \times [\theta_{3,2}, \theta_{3,2}+\varepsilon_2) \times [0, \theta_{1,3}]; \\ c_7 &= [\theta_{2,1}, \theta_{2,1}+\eta_1] \times [\theta_{3,2}, \theta_{3,2}+\eta_2] \times (\theta_{1,3}-\varepsilon_3, \theta_{1,3}]. \end{aligned}$$

The conditions (3-5) are necessary conditions to guarantee that the forward image of the new walls of the cubes are entirely contained in the corresponding cube. In fact, if we consider the functions $f_{\varepsilon_i} : \mathbb{R}^3 \longrightarrow \mathbb{R}$ such that $f_{\varepsilon_1,-} = x_1 - (\theta_{2,1} - \varepsilon_1)$, $f_{\varepsilon_1,+} = x_1 - (\theta_{2,1} + \varepsilon_1)$, $f_{\varepsilon_2,-} = x_2 - (\theta_{3,2} - \varepsilon_2)$, $f_{\varepsilon_2,+} = x_2 - (\theta_{3,2} + \varepsilon_2)$, $f_{\varepsilon_3,-} = x_1 - (\theta_{1,3} - \varepsilon_3)$, $f_{\varepsilon_3,+} = x_1 - (\theta_{1,3} + \varepsilon_3)$, then,

$$X_2 f_{\varepsilon_2,-}(p) < 0 \text{ for all } p \in f^{-1}_{\varepsilon_2,-}(0) \cap \widetilde{C}_2;$$
$$X_1 f_{\varepsilon_1,+}(p) < 0 \text{ for all } p \in f^{-1}_{\varepsilon_1,+}(0) \cap \widetilde{C}_1;$$
$$X_5 f_{\varepsilon_1,-}(p) < 0 \text{ for all } p \in f^{-1}_{\varepsilon_1,-}(0) \cap \widetilde{C}_5;$$
$$X_3 f_{\varepsilon_3,+}(p) < 0 \text{ for all } p \in f^{-1}_{\varepsilon_3,+}(0) \cap \widetilde{C}_3;$$
$$X_7 f_{\varepsilon_3,-}(p) < 0 \text{ for all } p \in f^{-1}_{\varepsilon_3,-}(0) \cap \widetilde{C}_7;$$
$$X_6 f_{\varepsilon_2,+}(p) < 0 \text{ for all } p \in f^{-1}_{\varepsilon_2,+}(0) \cap \widetilde{C}_6.$$

We define $\widetilde{C}_i := C_i \setminus c_i$ and $\mathbf{N}_2 := \widetilde{C}_1 \cup \widetilde{C}_2 \cup \widetilde{C}_3 \cup \widetilde{C}_5 \cup \widetilde{C}_6 \cup \widetilde{C}_7$, see Figure 3.5.

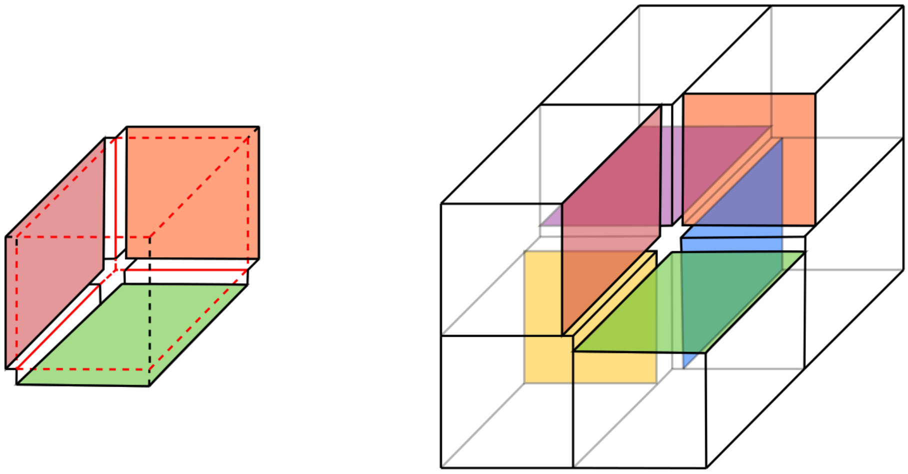

Figure 3.5: The set on the right is $\mathbf{N}_2$, and the rectangular set on the left is one of the pieces that was removed.

Note that we construe the set $\mathbf{N}_2$ by removing open sets of a compact set, so it remains a compact set; moreover, the fixed points of the vector fields stay inside $\mathbf{N}_2$. On each wall of the rectangular sets $\widetilde{C}_i$, at most, two vector fields are defined except the points belonging to a piece of a straight line, in which three vector fields are defined. An example of this, the vector fields defined in the walls of the rectangular set $\widetilde{C}_1$ are X_1, X_2, and/or X_7, see Figure 3.6. The following step corresponds to remove these pieces of a straight line in each $\widetilde{C}_i$. Also, note that if $p \in \left(\widetilde{\Sigma}_3 \cap \widetilde{C}_1\right) \setminus \widetilde{C}_2$ then $p \in \mathbf{n}_2^-$; in fact, $X_1 f_3(p) = -\theta_{1,3} + u_{3,2} > 0$. This also happens for other points in $\mathbf{n}_2$, but we want an isolating block with $\mathbf{n}_2^- = \emptyset$.

We make some cuts in each $\widetilde{C}_i$ so that $\mathbf{n}_i^-$ and in the walls of each cube, at most two vector fields are defined.

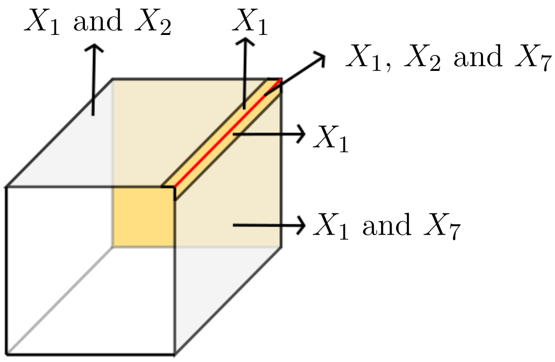

Figure 3.6: The vector fields are defined in some walls of the rectangular set $\widetilde{C}_1$.

Let $\varepsilon_{i,j} > 0$ with $i = 1,2,3$ and $j = 1,2,3,5,6,7$, and consider the following functions

$$\begin{aligned}
f_{\widetilde{C}_1} &= x_3 + \tfrac{\varepsilon_{3,1}}{\varepsilon_{2,1}} x_2 - (\theta_{1,3} + \tfrac{\varepsilon_{3,1}}{\varepsilon_{2,1}}\theta_{3,2} - \varepsilon_{3,1}),\\
f_{\widetilde{C}_2} &= x_3 + \tfrac{\varepsilon_{3,2}}{\varepsilon_{1,2}} x_1 - (\theta_{1,3} + \tfrac{\varepsilon_{3,2}}{\varepsilon_{1,2}}\theta_{2,1} + \varepsilon_{3,2}),\\
f_{\widetilde{C}_3} &= x_2 + \tfrac{\varepsilon_{2,3}}{\varepsilon_{1,3}} x_1 - (\theta_{3,2} + \tfrac{\varepsilon_{2,3}}{\varepsilon_{1,3}}\theta_{2,1} - \varepsilon_{2,3}),\\
f_{\widetilde{C}_5} &= x_3 + \tfrac{\varepsilon_{3,5}}{\varepsilon_{2,5}} x_2 - (\theta_{1,3} + \tfrac{\varepsilon_{3,5}}{\varepsilon_{2,5}}\theta_{3,2} + \varepsilon_{3,5}),\\
f_{\widetilde{C}_6} &= x_3 + \tfrac{\varepsilon_{3,6}}{\varepsilon_{1,6}} x_1 - (\theta_{1,3} + \tfrac{\varepsilon_{3,6}}{\varepsilon_{1,6}}\theta_{2,1} - \varepsilon_{3,6}),\\
f_{\widetilde{C}_7} &= x_2 + \tfrac{\varepsilon_{2,7}}{\varepsilon_{1,7}} x_1 - (\theta_{3,2} + \tfrac{\varepsilon_{2,7}}{\varepsilon_{1,7}}\theta_{2,1} + \varepsilon_{2,7}).
\end{aligned}$$

We denote by $X_i f_{\widetilde{C}_i}$ the Lie derivative of the vector field X_i with the function $f_{\widetilde{C}_i}$ for each $i = 1,2,3,5,6,7$. In order to construct the attracting isolating block, the sets $f^{-1}_{\widetilde{C}_i}(0)$ have to be planes such that $X_i f_{\widetilde{C}_i}(p) < 0$ for all $p \in f^{-1}_{\widetilde{C}_i}(0) \cap \widetilde{C}_i$ with $i = 1,3,6$ and $X_i f_{\widetilde{C}_i}(p) > 0$ for all $p \in f^{-1}_{\widetilde{C}_i}(0) \cap \widetilde{C}_i$ with $i = 2,5,7$. Then, each $\varepsilon_{i,j}$ satisfies the same conditions of ε_i of the equations (3-5), together with the following conditions

$$\begin{aligned}
\varepsilon_{2,1} &< -\tfrac{(-\theta_{3,1}+l_{2,1})\varepsilon_{3,1}}{(-\theta_{1,3}+u_{3,2})+\varepsilon_{3,1}}, & \varepsilon_{1,2} &> -\tfrac{(-\theta_{2,1}+l_{1,3})\varepsilon_{3,2}}{(-\theta_{1,3}+u_{3,2})+\varepsilon_{3,2}},\\
\varepsilon_{1,3} &< -\tfrac{(-\theta_{2,1}+l_{1,3})\varepsilon_{2,3}}{(-\theta_{3,2}+u_{2,1})+\varepsilon_{2,3}}, & \varepsilon_{2,5} &< -\tfrac{(-\theta_{3,2}+u_{2,1})\varepsilon_{3,5}}{(-\theta_{1,3}+l_{3,2})+\varepsilon_{3,5}},\\
\varepsilon_{1,6} &> -\tfrac{(-\theta_{2,1}+u_{1,3})\varepsilon_{3,6}}{(-\theta_{1,3}+l_{3,2})+\varepsilon_{3,6}}, & \varepsilon_{1,7} &> -\tfrac{(-\theta_{2,1}+u_{1,3})\varepsilon_{2,7}}{(-\theta_{3,2}+l_{2,1})+\varepsilon_{2,7}}.
\end{aligned}$$

Now, use the functions $f_{\widetilde{C}_i}$, $i = 1,2,3,5,6,7$, to remove the open rectangular sets b_i of each of the $\widetilde{C}_i$.

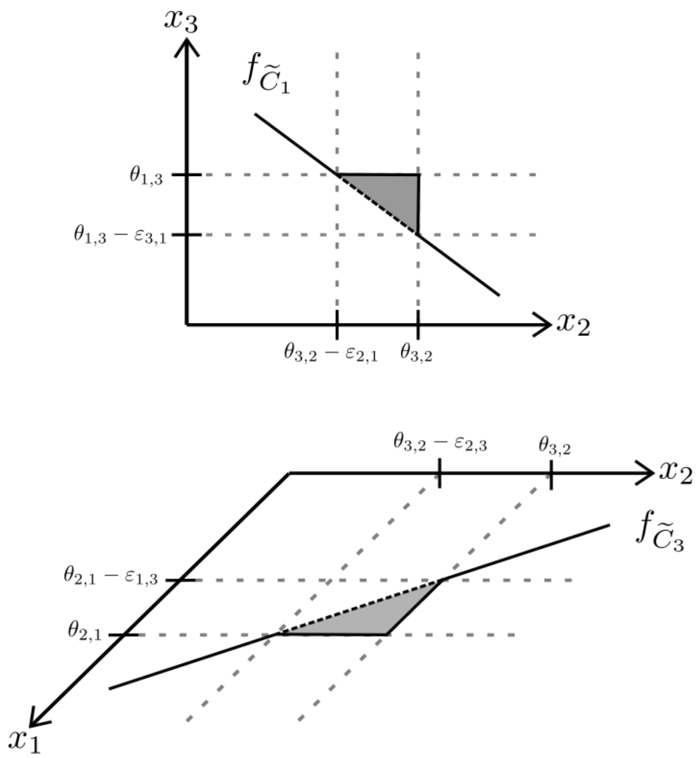

Figure 3.7: Graphical representation of $f_{\widetilde{C}_1}$ and $f_{\widetilde{C}_3}$.

$$
\begin{aligned}
b_1 = \Big\{&(x_1,x_2,x_3)\in\widetilde{C}_1;\ \theta_{2,1}\le x_1\le\theta_{2,1}+\eta_1 \text{ and } \theta_{3,2}-\varepsilon_{2,1}<x_2\le\theta_{3,2}\\
&\text{and } -\tfrac{\varepsilon_{3,1}}{\varepsilon_{2,1}}x_2-\Big(\theta_{1,3}+\tfrac{\varepsilon_{3,1}}{\varepsilon_{2,1}}\theta_{3,2}-\varepsilon_{3,1}\Big)<x_3\le\theta_{1,3}\Big\},\\
b_2 = \Big\{&(x_1,x_2,x_3)\in\widetilde{C}_2;\ \theta_{2,1}\le x_1<\theta_{2,1}+\varepsilon_{1,2} \text{ and } 0\le x_2\le\theta_{3,2} \text{ and}\\
&\theta_{1,3}\le x_3<-\tfrac{\varepsilon_{3,2}}{\varepsilon_{1,2}}x_1+\Big(\theta_{1,3}+\tfrac{\varepsilon_{3,2}}{\varepsilon_{1,2}}\theta_{2,1}+\varepsilon_{3,2}\Big)\Big\},\\
b_3 = \Big\{&(x_1,x_2,x_3)\in\widetilde{C}_3;\ \theta_{2,1}-\varepsilon_{1,3}<x_1\le\theta_{2,1} \text{ and}\\
&-\tfrac{\varepsilon_{2,3}}{\varepsilon_{1,3}}x_1+\Big(\theta_{3,2}+\tfrac{\varepsilon_{2,3}}{\varepsilon_{1,3}}\theta_{2,1}-\varepsilon_{2,3}\Big)<x_2\le\theta_{3,2} \text{ and } \theta_{1,3}\le x_3\le\theta_{1,3}+\eta_3\Big\},\\
b_5 = \Big\{&(x_1,x_2,x_3)\in\widetilde{C}_5;\ 0\le x_1\le\theta_{2,1} \text{ and } \theta_{3,2}\le x_2<\theta_{3,2}+\varepsilon_{2,5} \text{ and}\\
&\theta_{1,3}\le x_3<-\tfrac{\varepsilon_{3,5}}{\varepsilon_{2,5}}x_2+\Big(\theta_{1,3}+\tfrac{\varepsilon_{3,5}}{\varepsilon_{2,5}}\theta_{3,2}+\varepsilon_{3,5}\Big)\Big\},\\
b_6 = \Big\{&(x_1,x_2,x_3)\in\widetilde{C}_6;\ \theta_{2,1}-\varepsilon_{1,6}<x_1\le\theta_{2,1} \text{ and } \theta_{3,2}\le x_2\le\theta_{3,2}+\eta_2 \text{ and}\\
&-\tfrac{\varepsilon_{3,6}}{\varepsilon_{1,6}}x_1+\Big(\theta_{1,3}+\tfrac{\varepsilon_{3,6}}{\varepsilon_{1,6}}\theta_{2,1}-\varepsilon_{3,6}\Big)<x_3\le\theta_{1,3}\Big\},\\
b_7 = \Big\{&(x_1,x_2,x_3)\in\widetilde{C}_7;\ \theta_{2,1}\le x_1<\theta_{2,1}+\varepsilon_{1,7} \text{ and}\\
&\theta_{3,2}\le x_2<-\tfrac{\varepsilon_{2,7}}{\varepsilon_{1,7}}x_1+\Big(\theta_{3,2}+\tfrac{\varepsilon_{2,7}}{\varepsilon_{1,7}}\theta_{2,1}+\varepsilon_{2,7}\Big) \text{ and } 0\le x_3\le\theta_{1,3}\Big\}.
\end{aligned}
$$

We define $B_i := \widetilde{C}_i\setminus b_i$ and $\mathrm{N} := B_1\cup B_2\cup B_3\cup B_5\cup B_6\cup B_7$, see Figures 3.8 and 3.9. By using the previous constructions, the have the following results.

Proposition 3.6 *The set* N *is an attracting isolating block for the semiflow* ϕ *and homotopic to a solid torus.*

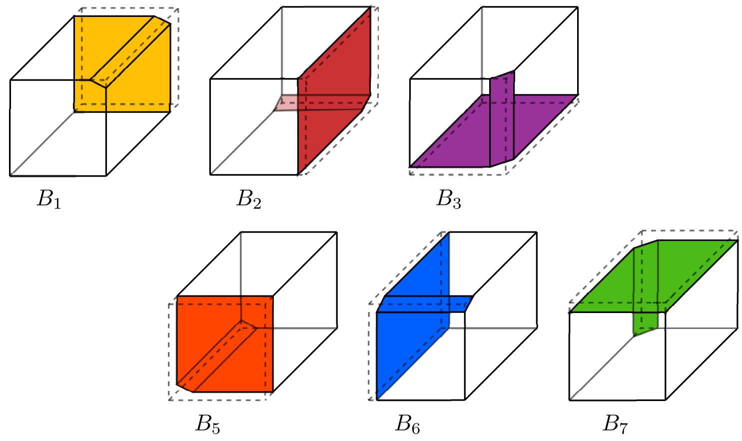

Figure 3.8: The cubes B_i.

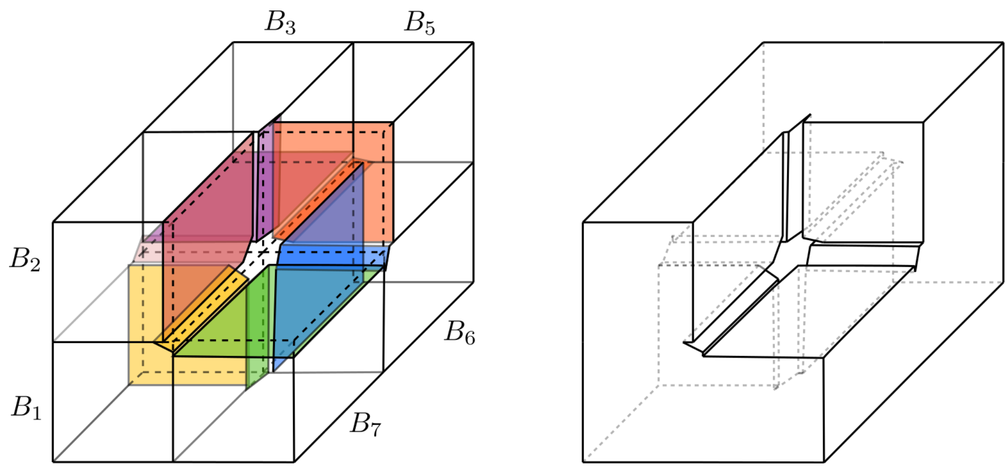

Figure 3.9: The isolating block $\mathrm{N} = B_1 \cup B_2 \cup B_3 \cup B_5 \cup B_6 \cup B_7$.

Proof. By the above construction, and Definition 1.4, we have that $n^- = \emptyset$ and N is an attracting block for ϕ, and it is homotopic to a solid torus. □

Since the isolating block N is homeomorphic to a solid torus and $n^- = \emptyset$, then the Conley index is the same as an unstable hyperbolic periodic orbit.

Proposition 3.7 *The Conley Index of the $S = \mathrm{inv}(\mathrm{N})$, where* N *is an isolating neighborhood corresponds to*

$$CH^k(S) \approx \begin{cases} \mathbb{Z}, & k = 1, 2, \\ 0, & \textit{otherwise}. \end{cases}$$

3.3.2 Construction of the Poincaré Section

In the following proposition consider the sets $\Sigma_{i,j}$ equal to the intersection between C_i and B_j.

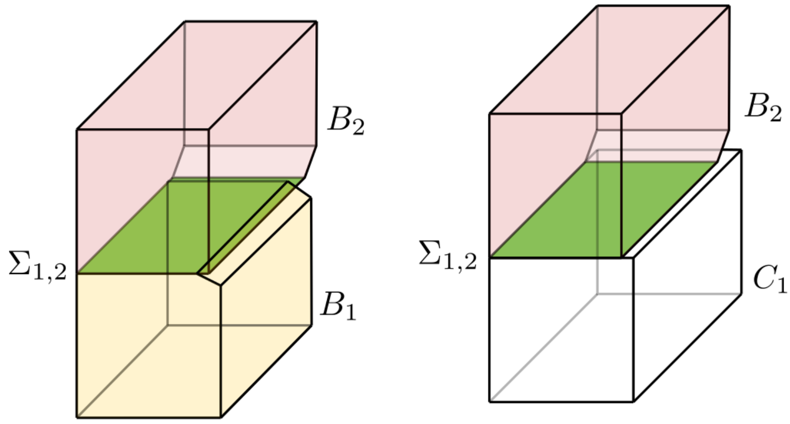

Figure 3.10: The set $\Sigma_{1,2}$.

Proposition 3.8 *The set $\Xi = \Sigma_{1,2} \cup \Sigma_{2,3} \cup \Sigma_{3,5} \cup \Sigma_{5,6} \cup \Sigma_{6,7} \cup \Sigma_{7,1}$ is a Poincaré section for ϕ in* N.

Proof. Let ϕ be the semiflow generated by the gluing of solution orbits of the system (3-4) with the conditions (3-3). Let $p \in C_{\xi}^{N}(cl\ \Sigma_{1,2})$ for $\xi > 0$ sufficiently small, then $p \cdot (0,\xi) \cap \Xi \neq \emptyset$, in particular $p \cdot (0,\xi) \cap \Sigma_{1,2} \neq \emptyset$. Since $p \in B_1 \setminus \Sigma_{1,2}$, then the local trajectory is defined by the vector field X_1. By Lemma 3.5 the ω-limit of (p) is a point in the set B_2 and so, there exist a unique $t(p) > 0$ such that $p \cdot t(p) \in \Sigma_{1,2}$. Analogously for the other sets $\Sigma_{i,j}$. In the induced topology by $\mathbb{R}^3$, the set $\Xi \cap$ N is closed, in fact, it is a finite union of closed sets. Furthermore $\Xi \cap$ N by construction is transverse to the semiflow (each set $\Sigma_{i,j}$ is a crossing region in the Filippov convention), therefore Ξ is a local section for the semiflow ϕ.

Now, we need show that the forward orbit of every point in N intersects Ξ. Let $p \in N$, without loss of generality, assume that $p \in B_1$. The vector field defined in B_1 is X_1 and, since X_1 has only one singular point $p_1 \in B_2$, then there exists a time $T_p > 0$ such that the forward orbit intersects $\Sigma_{1,2}$. The same behavior occurs with all elements in B_i.

□

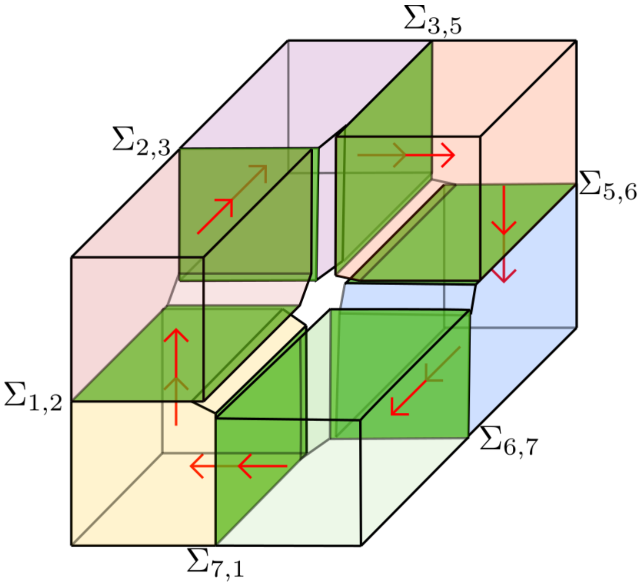

Figure 3.11: Behavior of semi-flow generate by system (3-4).

Theorem D *Let ϕ be a semiflow generated by the system (3-4) of the repressilator model. Then for the following combination of parameters*

$$\begin{cases} 0 < l_{1,3} < \theta_{2,1} < u_{1,3}, \\ 0 < l_{2,1} < \theta_{3,2} < u_{2,1}, \\ 0 < l_{3,2} < \theta_{1,3} < u_{3,2}, \end{cases}$$

the set inv(N) *contains a periodic orbit.*

Proof. To prove that the system (3-4) admits a periodic trajectory, we need to show that the assumptions of Theorem B are satisfied. Firstly, note that any isolating block is an isolating neighborhood, so by Proposition 3.6, the set N is an isolating neighborhood for the semiflow ϕ generated by system (3-4). By Proposition 3.7, the Conley Index of $S = \text{inv}(N)$ has the same homotopy type a periodic orbit. Finally, the existence of the Poincaré section for ϕ in N is guaranteed by Proposition 3.8. □

CHAPTER 4

Limit sets near a cusp-fold singularity occurring in a piecewise smooth vector field

In this chapter, we develop a local approach to study the behavior of the forward orbits of a family of piecewise smooth vector fields presenting a co-dimension one structure known as *cusp-fold singularity*. This chapter is structured as follows. In Section 4.1, we state the problem using the Filippov convention and we state the main results that will be proven throughout this chapter. Section 4.2 addresses to the construction of the crossing and sliding Poincaré maps. Section 4.3 describes the local structure around the cusp-fold singularity. Finally, Section 4.4 refers to the demonstrations of the propositions and theorems.

4.1 Setting the problem

4.1.1 The cusp-fold singularity

Consider the piecewise smooth vector field

$$Z_\gamma(x,y,z) = \begin{cases} X(x,y,z) = (0,\alpha,\beta(x+y^2)) & \text{if } z \geq 0, \\ Y_\gamma(x,y,z) = (\gamma,1,\delta y) & \text{if } z \leq 0, \end{cases} \tag{4-1}$$

where $\alpha,\beta,\delta,\gamma \in \mathbb{R}$. We notice that orbits of X and Y in system (4-1) can meet the separation region $\Sigma = \{(x,y,z) \in \mathbb{R}^3;\ z=0\}$ in distinct ways. Thus, Z_γ may not be smooth over Σ and may be piecewise smooth. In this chapter, we consider piecewise smooth vector fields from a Filippov theory perspective, see [19].

We notice that the origin $p=(0,0,0)$ is a fold singularity for the vector field Y_γ since it is on its fold curve $y=0$. On the other hand, p is a cusp singularity for the vector field X and splits the fold points of the curve $x=-y^2$ into visible and invisible ones. Thus, p is a fold-singularity for (4-1) regardless of the value of γ.

The main goal of this chapter is to employ a local approach for studying the forward orbits of (4-1). The incidence of a fold and a cusp singularity is a co-dimension one phenomenon, see for instance [24], where the authors unfold a planar cusp-fold singularity. In this chapter, we do not unfold the cusp-fold but analyze the local dynamics around that singularity for small values of the parameter γ. We emphasize that the piecewise smooth vector field (4-1) is a particular system presenting a cusp-fold singularity, that is, so far it is simply a model presenting the desired object of study. Future work consists in checking whether equation (4-1) is a normal form for the three-dimensional cusp-fold. Another paper considering a model of a cusp-fold can be found in [7]. Theorem 1 in [7] presents the asymptotic stability of a PSVF whenever it has a cusp-fold singularity.

The parameters α, β, γ, and δ of system (4-1) characterize the contact of X and Y with Σ and the different behavior of their orbits near the origin. In this direction, we notice that the system (4-1) can present 8 possible behaviors in the local dynamic. In all cases, the origin represents a tangential contact of cusp with X and invisible fold with Y. The first four cases correspond to $\delta > 0$, where $Yh(x_0, y_0, 0) < 0$ when $y_0 < 0$ and $Yh(x_0, y_0, 0) > 0$ when $y_0 > 0$. The remaining four cases correspond to $\delta < 0$. The parameter β changes the visibility of the cusp, and the parameter α changes the visibility of the fold of the tangential contact of X with Σ. The parameter γ changes the behavior of the vector field Z^s_γ but does not create cases other than those mentioned. The behavior of the 8 cases is described in Figures 4.1 and 4.2. We call "case A_i" and "case B_i" with $i = 1, \ldots, 4$, the cases presented in Figures 4.1 and 4.2.

We are interested in the case A_2 since the sliding motion points toward the curve of visible fold points. In such a scheme, we can define a Poincaré map using not only the flows associated to X and Y but also, the Filippov vector field which produces a rich dynamics near the cusp-fold singularity.

4.1.2 Results

In the following we present some auxiliary results that are important on their own as well as the main results of the chapter. The next proposition concerns the non-existence of periodic orbits of any kind around a cusp-fold singularity.

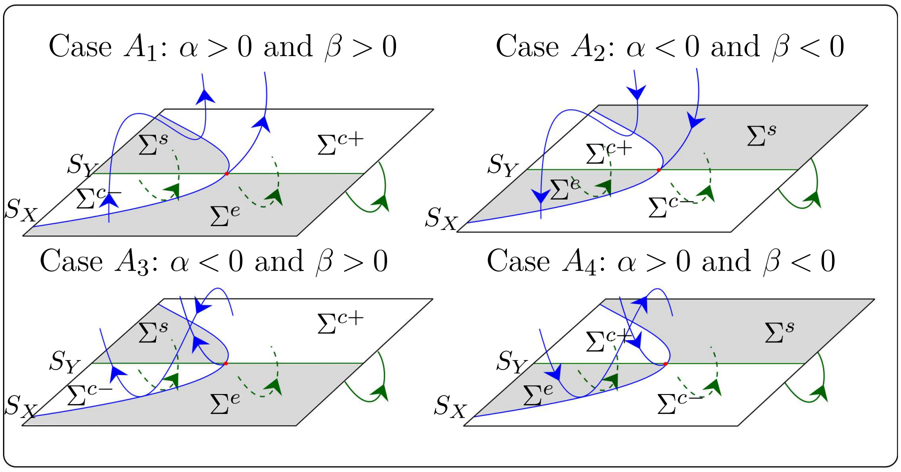

Figure 4.1: Local behavior of the cusp-fold singularity in the system (4-1) when $\delta > 0$.

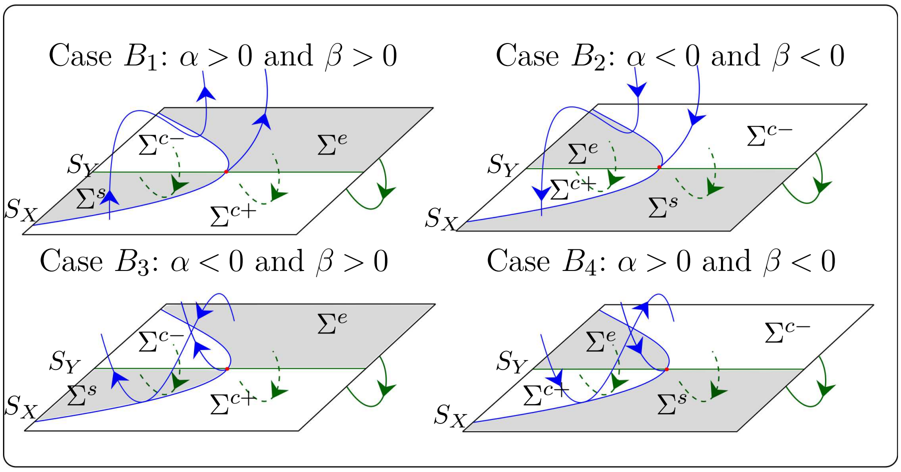

Figure 4.2: Local behavior of the cusp-fold singularity in the system (4-1) when $\delta < 0$.

Proposition 4.1 *The piecewise smooth vector field Z_γ has neither crossing nor crossing-sliding periodic orbits around the cusp-fold singularity regardless the values of α, β, γ and δ.*

The next two propositions describe some invariant center manifolds for the case $\gamma = 0$ and stable/unstable ones for the case $\gamma \neq 0$ small.

Proposition 4.2 *Consider the cusp-fold system associated to Z_0 such that $\alpha < 0$, $\beta < 0$, and $\delta > 0$. Let Z_0^s be the respective normalized sliding vector field. The following statements hold.*

(i) The cusp-fold singularity belongs to the curve of equilibrium points $\mathfrak{M}_0^1$ of Z_0^s denote by $\mathfrak{M}_0^1 = \{(x,y,0) \in B_\ell(0); x = \frac{1}{\beta}(-\beta y^2 + \alpha\delta y)\} \subset \Sigma^s \cup \Sigma^e\}$;

(ii) The orbits of Z_0 form a piecewise smooth manifold;

$\mathfrak{M}_0^2 = \{(0,0,0)\} \cup \Gamma_{X_\ell} \cup \Pi(\Gamma_{X_\ell})$, *where* $\Gamma_{X_\ell} = \{(x,y,0) \in S_X;\ 0 < y < \frac{2\alpha\delta}{3\beta}\}$ *and* Π *is sliding Poincaré map;*

(iii) $\mathfrak{M}_0^2$ *is filled with periodic orbits, and every other orbit outside* $\mathfrak{M}_0^1$ *is a regular orbit.*

Proposition 4.3 *Consider the cusp-fold system associated to* Z_γ *with* $\gamma \neq 0$, $\alpha < 0$, $\beta < 0$, *and* $\delta > 0$. *Let* Z_γ^s *be the respective normalized sliding vector field. The following statements hold.*

(i) *The cusp-fold singularity of* Z_γ *is an equilibrium point of* Z_γ^s *which is a saddle if* $\gamma > 0$; *a proper node if* $\gamma < 0$ *with* $\gamma \neq \frac{\alpha\delta}{\beta}$; *and an improper node if* $\gamma < 0$ *with* $\gamma = \frac{\alpha\delta}{\beta}$.

(ii) *The stable and unstable manifolds associated to* p *on* Σ^s *are tangent at* p *to* $E^s = \{(x,y,0); x = 0\}$ *and* $E^\omega = \{(x,y,0);\ x = (\gamma - \frac{\alpha\delta}{\beta})y\}$ *with* $\omega = s$ *if* $\gamma < 0$ *and* $\omega = u$ *if* $\gamma > 0$.

For the next theorems we introduce the following notation: $\omega(p)$ corresponds to the ω-limit set of the point p, the point $\mathfrak{p}$ is an equilibrium point of Z_0^s and ℓ is a closed periodic orbit of the center manifold $\mathfrak{M}_0^2$ associated to Z_0. The main results of chapter are the following.

Theorem E *Consider the cusp-fold system associated to* Z_0 *such that* $\alpha < 0$, $\beta < 0$, *and* $\delta > 0$. *Let* $\mathfrak{B}_0$ *be a ball of radius* $r < \frac{\alpha\delta}{\beta}$ *centered at the cusp-fold. Then, the positive orbit of an arbitrary point* $q \in \mathfrak{B}_0$ *satisfies one of the following statements:*

(i.) $\omega(q) = \mathfrak{p} \in \mathfrak{M}_0^1$;

(ii.) $\omega(q) = \ell \in \mathfrak{M}_0^2$;

(iii.) *there exists* $t_0 > 0$ *such that* $\{\phi_{Z_0}(q,t); t > t_0\} \cap \mathfrak{B}_0 = \emptyset$.

Theorem F *Consider the cusp-fold system associated to* Z_γ *such that* $\alpha < 0$, $\beta < 0$, *and* $\delta > 0$. *Assume that* $\gamma > 0$ *is sufficiently small. Let* $\widetilde{\mathfrak{B}}$ *be the ball of radius* $r < 12\gamma^2$ *centered at the origin. Then, the positive orbit of an arbitrary point* $q \in \widetilde{\mathfrak{B}}$ *satisfies one of the following statements:*

(i.) $\omega(q) = p = (0,0,0)$;

(iii.) *there exists* $\tilde{t} > 0$ *such that* $\{\phi_{Z_0}(q,t); t > \tilde{t}\} \cap \widetilde{\mathfrak{B}} = \emptyset$.

Theorem G *Consider the cusp-fold system associated to* Z_γ *such that* $\alpha < 0$, $\beta < 0$, *and* $\delta > 0$. *Assume that* $\gamma < 0$ *is sufficiently small. Let* $\widehat{\mathfrak{B}}$ *be the ball of radius* $r < \frac{4}{3}\gamma$ *centered at the origin. Then for the forward orbits of the points in* $q \in \widehat{\mathfrak{B}}$, *satisfies one of the following statements:*

(i.) $\omega(q) = p = (0,0,0)$;

(iii.) there exists $\widehat{t} > 0$ *such that* $\{\phi_{Z_0}(q,t); t > \widehat{t}\} \cap \widehat{\mathfrak{B}} = \emptyset$.

4.2 Crossing and sliding Poincaré maps

The Poincaré map, or first return map, is a classical, very powerful tool in studying periodic behavior in a dynamical system. Based on it, in this chapter we make use of the crossing Poincaré map, but also of what we call a sliding Poincaré map which is a composition of two half return maps and a transition function. It turns out that such a composition is a diffeomorphism between two semi-local transversal sections as we describe next. We define the sliding Poincaré map on the invariant manifold $\mathfrak{M}^\gamma$ for any value of γ.

4.2.1 The crossing Poincaré map

Note that, by time rescaling we can reduce the eight cases in Figures 4.1 and 4.2 to the four cases A_i. Hence, without loss of generality, we only consider the cases A_1 through A_4. Let V be an open neighborhood of the origin and $p_0 = (x_0, y_0, 0) \in \Sigma^{c+} \cap V$, so in particular $y_0 > 0$. The flow $\varphi_X(t, p_0)$ associated to X is given by

$$\varphi_X(t, x_0, y_0, z_0) = \left(x_0, y_0 + \alpha t, \frac{1}{3}\left(3z_0 + 3\beta t x_0 + 3\beta t {y_0}^2 + 3\alpha\beta t^2 y_0 + \alpha^2 \beta t^3\right)\right).$$

Now, let $t_{1,i}(p_0)$ be the smallest positive time such that $\varphi_X(t_{1,i}(p_0), p_0) \in \Sigma$ where $i = 1,2,3,4$ depending on the cases A_i. Denoting $\Delta = \sqrt{-3(4x_0 + y_0^2)}$ we have the following:

(i) $t_{1,2}(p_0) = \frac{-3y_0 - \Delta}{2\alpha}$ which is positive since in the case A_2 we have $\alpha < 0$;

(ii) $t_{1,3}(p_0) = \frac{-3y_0 + \Delta}{2\alpha}$ which is positive and less than $\frac{-3y_0 - \Delta}{2\alpha}$ since in the case A_3 we have $\alpha < 0$;

(iii) $t_{1,4}(p_0) = \frac{-3y_0 + \Delta}{2\alpha}$ which is positive since in the case A_4 we have $\alpha > 0$;

(iv) there is no positive time such that $\varphi_X(t_{1,1}(p_0), p_0) \in \Sigma$. Then the case A_1 is not considered in the construction of the crossing Poincaré map.

In all cases, because $t_{1,i}(p_0) \in \mathbb{R}$ we assume that $x_0 < -\frac{y_0^2}{4}$.

Now set $p_{1,i} = \phi_X(t_{1,i}(p_0), p_0)$. The point $p_{1,i}$ either belongs to the regions Σ^{c-}, Σ^s or is a tangency point. But we are only interested in the cases such that $p_{1,i} \in \Sigma^{c-}$. The case A_4 cannot occur because $p_{1,4} = \varphi_X(t_{1,4}(p_0), p_0) \notin \Sigma^{c-}$ and so it is not possible the return to the region Σ^{c+}. Moreover, for the case A_3, $p_{1,3} \in \Sigma^{c-}$ if, and only if, $-\frac{{y_0}^2}{3} < x_0 < -\frac{{y_0}^2}{4}$.

Now, notice that the flow $\varphi_{Y_\gamma}(t, p_0)$ associated to Y_γ is given by

$$\varphi_{Y_\gamma}(t, x_0, y_0, z_0) = (x_0 + \gamma t, t + y_0, \frac{1}{2}(2z_0 + t^2\delta + 2t\delta y)).$$

Let $t_{2,i}$ be the smallest positive time such that $\varphi_{Y_\gamma}(t_{2,i}(p_{1,i}), p_{1,i}) \in \Sigma$ where $i = 2,3$ depending on the cases A_2 or A_3. Then $t_{2,2}(p_0) = y + \Delta$ and $t_{2,3}(p_0) = y - \Delta$. Finally, assuming $\varphi_{Y_\gamma}(t_{2,i}(p_{1,i}), p_{1,i}) \in \Sigma^{c+}$, for the case A_2 we obtain $\widetilde{\Pi}_2 : \Sigma^{c+} \cap V \longrightarrow \Sigma$ where $\widetilde{\Pi}_2 = \phi_Y(t_{2,2}(p_0), \phi_X(t_{1,2}(p_0), p_0))$ which can be written as:

$$\widetilde{\Pi}_2(x_0, y_0, 0) = \left(x_0 + \gamma\,(y_0 + \Delta)\,, \frac{1}{2}\,(y_0 + \Delta)\,, 0\right).$$

For the case A_3, we obtain $\widetilde{\Pi}_3 : \{(x_0, y_0, 0) \in V; -\frac{{y_0}^2}{3} < x_0 < -\frac{{y_0}^2}{4} \quad \} \longrightarrow \Sigma$ where $\widetilde{\Pi}_3 = \phi_Y(t_{2,3}(p_0), \phi_X(t_{1,3}(p_0), p_0))$ which can be written as:

$$\widetilde{\Pi}_3(x_0, y_0, 0) = \left(x_0 + \gamma\,(y_0 - \Delta)\,, \frac{1}{2}\,(y_0 - \Delta)\,, 0\right).$$

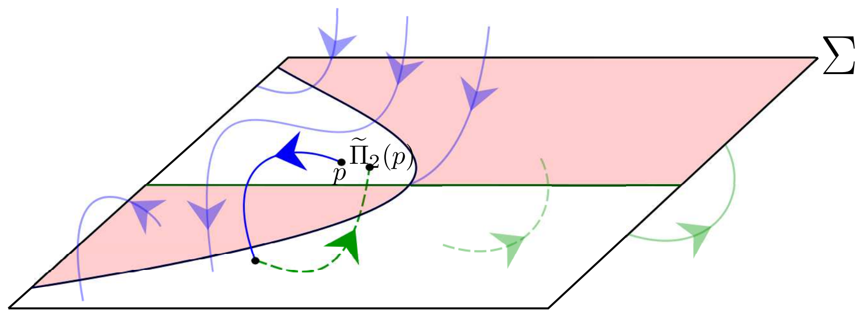

Figure 4.3: Local behavior of the crossing Poincaré map for the case A_2.

We say that an orbit γ_Z of Z_γ is a *crossing orbit* if $\widetilde{\Pi}_i(q) = q$ for $q \in \gamma_Z$.

4.2.2 The sliding Poincaré map

Let V be an open neighborhood of the origin and $p_0 = (x_0, y_0, 0) \in \Gamma_{X_\ell} \cap V$ a visible fold point contained in $\Gamma_{X_\ell} = \{(x, y, 0) \in S_X;\ 0 < y < \frac{2\alpha\delta}{3\beta}\}$. We define the sliding Poincaré map Π as a map defined on a one-dimensional region Γ_{X_ℓ} as described next. Let V be an open neighborhood of the origin and $p_0 = (x_0, y_0, 0) \in \Gamma_{X_\ell} \cap V$ a visible fold point contained in $\Gamma_{X_\ell} = \{(x, y, 0) \in S_X;\ 0 < y < \frac{2\alpha\delta}{3\beta}\}$. We define the sliding Poincaré map Π as a map defined on a one-dimensional region

Γ_{X_ℓ} as the composition of the flows ϕ_X, ϕ_Y, and $\phi_{Z_\gamma^s}$. Here we only consider the case A_2 because it is the only case where $\phi_{Y_\gamma}\left(\phi_X\left(\Gamma_{X_\ell}\right)\right)$ is well defined and belongs to the sliding region.

First, consider the flow φ_X of X having $p_0=(-y_0^2,y_0,0)$ as initial condition, from which we obtain

$$\varphi_X(t,p_0)=\left(-y_0^2,y_0+\alpha t,\frac{1}{3}(3\alpha\beta t^2y_0+\alpha^2\beta t^3)\right).$$

Let $t_1(p_0)$ be the smallest positive time such that $p_1=\varphi_X(t_1(p_0),p_0)\in\Sigma$. We get $t_1(p_0)=-\frac{3y_0}{\alpha}$ which is positive since $\alpha<0$ and moreover $p_1=\varphi_X(t_1(p_0),p_0)=(-y_0^2,-2y_0,0)$. Similarly, using the flow associated to the vector field Y, we have that $t_2(p_0)=4y_0$ and $p_2=(-y_0^2+4\gamma y_0,2y_0,0)$.

From now on, since p_2 belongs to the sliding region, the orbit from p_2 must follow the flow of the normalized Filippov vector field. Nevertheless, as we are doing a local study, we consider the linearized flow denoted by $\varphi_{Z_\gamma^s}$. One can see that $\varphi_{Z_\gamma^s}(t,p_2)$ can be written as

$$\varphi_{Z_\gamma^s}(t,p_2)=\left(e^{-\beta\gamma t}(4\gamma y_0-{y_0}^2),\frac{e^{-\beta\gamma t}\beta y_0(4\gamma-y_0)+e^{-\alpha\delta t}y_0\left(\beta y_0+2(\alpha\delta-\beta\gamma)\right)}{\beta\gamma+\alpha\delta},0\right).$$

Moreover, $\beta\gamma+\alpha\delta$ does not vanish. Indeed, if $\gamma=-\frac{\alpha\delta}{\beta}$, then $p_2=(-y_0^2-\frac{4\alpha\delta y_0}{\beta},2y_0,0)$ is a point of the sliding region only if $y_0>\frac{4\alpha\delta}{3\beta}$. Let $t_3(p_0)$ be the positive time such that $p_3=\varphi_{Z_\gamma^s}(t_3(p_0),p_2)\in\Gamma_{X_\ell}$. One can see that $t_3(p_0)=\log[1-\frac{4\gamma}{y_0}](\beta\gamma)^{-1}$, in fact, $t_3(p_0)$ is the solution of the equations $x_2(t,y_0)=-y_0^2$ and $y_2(t,y_0)=y_0$. As $\beta<0$ and $y_0>0$ then $t_3(p_0)>0$ only for the cases $\gamma=0$ and $\gamma<0$. If $\gamma>0$ then $t_3(p_0)>0$ if and only if $y_0>4\gamma$ which again does not correspond to a local study.

Let σ be a parametrization of the curve Γ_{X_ℓ}. Therefore, composing the three maps obtained previously we get the map $\Pi(y_0)=[\varphi_{Z_\gamma^s}\circ\varphi_Y\circ\varphi_X](-y_0^2,y_0,0)$ with the conditions $\gamma\leq 0$ and $\gamma\neq-\frac{\alpha\delta}{\beta}$, where $\Pi(y_0)$ is actually $\Pi(\sigma(y_0))$. It can be written as

$$\Pi(y_0)=\left(-y_0^2,\frac{y_0\left(-\beta y_0+\left(1-\frac{4\gamma}{y_0}\right)^{\frac{\alpha\delta}{\beta\gamma}}(\beta y_0-2\beta\gamma\alpha+\delta)\right)}{\beta\gamma+\alpha\delta},0\right).$$

Notice that Π is a diffeomorphism despite the fact that Z_γ is a piecewise smooth vector field.

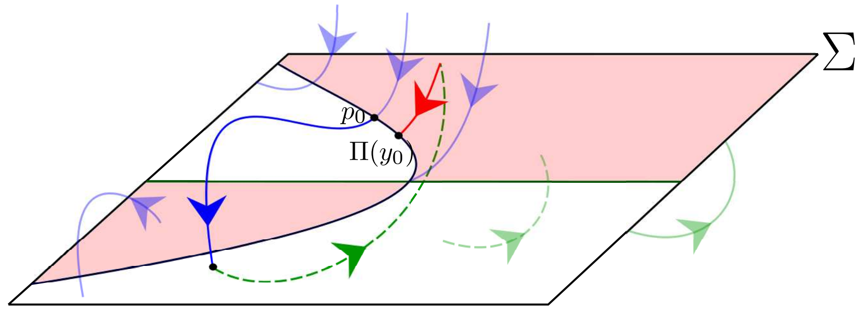

Figure 4.4: Local behavior of the sliding Poincaré map.

We say that an orbit γ_Z of Z_γ is a *crossing-sliding orbit* if $\Pi(y) = (-y^2, y, 0)$ for $(-y^2, y, 0) \in \gamma_Z$.

4.3 The local structure of the cusp-fold

The approach in proving Theorems E, F, and G is based on the following steps:

- First we consider a small ball $\mathfrak{B}$ of radius r centered at the cusp-fold that we are assuming to be the origin of $\mathbb{R}^3$;
- Then we intersect the ball $\mathfrak{B}$ with Σ to obtain a disk $\mathfrak{D}$ around the cusp-fold contained in Σ. This disk is split by some particular curves associated with tangency points and crossing and sliding Poincaré maps, generating some regions on the disk;
- Once we have fully analyzed the orbits of equation (4-1) on the disks in each case, we study the limit sets of its orbits in order to prove the main results.

In order to follow the previous steps, we construct the disk $\mathfrak{D}$ and the respective regions forming it. We analyze only the case A_2 that corresponds to $\alpha < 0$, $\beta < 0$ and $\delta > 0$. Firstly, we analyze system (4-1) with $\gamma = 0$, then we study the case $\gamma \neq 0$ with γ small.

Analyze of the case $\gamma = 0$

Consider a disk $\mathfrak{D} \subset \mathbb{R}^2$ centered at the origin and with radius $r < \frac{\alpha\delta}{\beta}$ which is the intersection of a ball $\mathfrak{B} \subset \mathbb{R}^3$ of same radius. We split the disk D into 4 regions from R_1 to R_4 divided by the tangency curves S_X and S_Y of the vector fields X and Y, namely the crossing regions R_1 and R_3, the sliding region R_2 and the escape region R_4. In order to divide these regions into smaller ones, we use the following curves and segments.

Description of curves and segments

Let C_1 be the parabola of equilibrium points associated to the normalized Filippov vector field, see the dotted curve in Figure 4.1. We notice that C_1 is given by the parametric equation

$$\sigma_1(s) = \left(-s^2 + \frac{\alpha\delta}{\beta}s, s\right). \tag{4-2}$$

As $\frac{\alpha\delta}{\beta} > 0$, The curve C_1 is tangent at the origin, with the straight line having an angular coefficient $\frac{\alpha\delta}{\beta}$. Then C_1 is entirely contained in the sliding and escaping regions, passing through the origin. Taking the flow of the vector field Y_γ at the points $p_0 = (x_0, y_0, z_0)$ in C_1 such that $y_0 \geq 0$, and assuming $p_1 = \varphi_{Y_\gamma}(t(p_0), p_0) \in \Sigma^{c-}$ and $p_2 = \varphi_X(t_1(p_1), p_1) \in \Sigma$, we obtain that $t_0(p_0) = -2y_0 < 0$ and $t_1(p_1) = (3\beta y_0 + \sqrt{3\left(3\beta^2 y_0^2 - 4\alpha\beta\delta y_0\right)})(2\alpha\beta)^{-1} < 0$. The set of points as p_1 forms the curve C_2 and the set of points as p_2 forms the curve C_5. The parametric equations of those curves are given respectively by

$$\sigma_2(s) = \left(-s^2 + \frac{\alpha\delta}{\beta}s, -s\right), \tag{4-3}$$

and

$$\sigma_5(s) = \left(-s^2 + \frac{\alpha\delta}{\beta}s, \frac{\beta s + \sqrt{3\beta s\,(3\beta s - 4\alpha\delta)}}{2\beta}\right). \tag{4-4}$$

The curve C_3 is composed by the points satisfying $x = 0$. Now, notice that the curves C_1 and C_2 are parabolas with vertices at $A = (\frac{\alpha^2\delta^2}{4\beta^2}, \frac{\alpha\delta}{2\beta})$ and $B = (\frac{\alpha^2\delta^2}{4\beta^2}, -\frac{\alpha\delta}{2\beta})$, respectively. Consider the segment $\overline{AB}$. This segment is used to divide in the regions Σ^- and Σ^s into subregions.

Now, for the construction of curve C_4, we take the points S_X such that $y > 0$, the visible fold points of the vector field X, and apply the crossing Poincaré-map. The curve C_4 is given by the parametric equation $\sigma_4(s) = \left(-s^2, -2s\right)$ obtained by the image of the visible fold points belonging to the disk. Applying the implicit function theorem for the curve σ_5, we obtain that the graph of σ_5 is given by $(x, f(x))$ where

$$f(x) = \frac{\alpha\delta - \sqrt{-4x\beta^2 + \alpha^2\delta^2} + \sqrt{6(-6x\beta^2 + \alpha\delta(-\alpha\delta + \sqrt{-4x\beta^2 + \alpha^2\delta^2}))}}{4\beta}$$

and $x < -\frac{4\alpha^2\delta^2}{9\beta^2}$. Taking the flow associated to vector field Y_γ at the points on C_5

with $-\frac{3\alpha^2\delta^2}{4\beta^2} < x < -\frac{4\alpha^2\delta^2}{9\beta^2}$ we obtain:

$$\widetilde{\sigma}_5(x) = (x, \widetilde{f}(x)) \tag{4-5}$$

where

$$\widetilde{f}(x) = \frac{-\alpha\delta + \sqrt{-4x\beta^2+\alpha^2\delta^2} - \sqrt{6(-6x\beta^2+\alpha\delta(-\alpha\delta+\sqrt{-4x\beta^2+\alpha^2\delta^2}))}}{4\beta}$$

and $-\frac{3\alpha^2\delta^2}{4\beta^2} < x < -\frac{4\alpha^2\delta^2}{9\beta^2}$.

Now let us consider some distinguished points in the disk $\mathfrak{D}$:

- C is the point that belongs to quadrant III with $x = -\frac{4\alpha^2\delta^2}{9\beta^2}$, that is, $C = \left(-\frac{4\alpha^2\delta^2}{9\beta^2}, y_C\right)$,
- $D = \left(-\frac{4\alpha^2\delta^2}{9\beta^2}, -\frac{2\alpha\delta}{3\beta}\right)$ is the intersection between $\widetilde{\sigma}_5$ and S_X,
- $E = \left(-\frac{4\alpha^2\delta^2}{9\beta^2}, \frac{2\alpha\delta}{3\beta}\right)$ is the intersection between C_5 and S_X,
- $F = \left(-\frac{4\alpha^2\delta^2}{9\beta^2}, y_F\right)$ the point on the disk $\mathfrak{D}$ belong to quadrantII,
- $G = \{-\frac{3\alpha^2\delta^2}{4\beta^2}, 0\}$ is the intersection point between C_5 and $\widetilde{\sigma}_5$.

The above curves and points can be seen in Figure 4.5. This information will be useful in the proofs of the results in this chapter.

Description of the subregions in the disk $\mathfrak{D}$:

Next we fully describe the subregions of each region from R_1 to R_4. The regions $R_{1,1}$ to $R_{1,5}$ belongs to the crossing region Σ^{c-}:

$R_{1,1} = \left\{(x,y,z) \in \mathfrak{D};\ -y^2 \le x \le -\frac{4\alpha^2\delta^2}{9\beta^2} \text{ and } y \le 0\right\}$,
$R_{1,2} = \left\{(x,y,z) \in \mathfrak{D};\ \left(x \ge -y^2 \text{ and } -\frac{2\alpha\delta}{3\beta} \le y \le 0\right) \text{ or } \left(x \ge -\frac{4\alpha^2\delta^2}{9\beta^2} \text{ and } y \le -\frac{2\alpha\delta}{3\beta}\right)\right\}$,
$R_{1,3} = \left\{(x,y,z) \in \mathfrak{D};\ 0 \le x \le -y^2 - \frac{\alpha\delta}{\beta}y \text{ and } y \le 0\right\}$,
$R_{1,4} = \left\{(x,y,z) \in \mathfrak{D};\ -y^2 - \frac{\alpha\delta}{\beta}y \le x \le \frac{\alpha^2\delta^2}{4\beta^2} \text{ and } y \le 0\right\}$,
$R_{1,5} = \left\{(x,y,z) \in \mathfrak{D};\ \left(x \ge \frac{\alpha^2\delta^2}{4\beta^2} \text{ and } -\frac{\alpha\delta}{2\beta} \le y \le 0\right) \text{ or } \left(x \ge -y^2 - \frac{\alpha\delta}{\beta}y \text{ and } y \le -\frac{\alpha\delta}{2\beta}\right)\right\}$.

The regions $R_{2,1}$ to $R_{2,5}$ belong to the sliding region Σ^s:

$R_{2,1} = \left\{(x,y,z) \in \mathfrak{D};\ \left(x \ge \frac{\alpha^2\delta^2}{4\beta^2} \text{ and } 0 \le y \le \frac{\alpha\delta}{2\beta}\right) \text{ or } \left(x \ge -y^2 + \frac{\alpha\delta}{\beta}y \text{ and } y \ge \frac{\alpha\delta}{2\beta}\right)\right\}$,
$R_{2,2} = \left\{(x,y,z) \in \mathfrak{D};\ -y^2 + \frac{\alpha\delta}{\beta}y \le x \le \frac{\alpha^2\delta^2}{4\beta^2} \text{ and } 0 \le y \le \frac{\alpha\delta}{2\beta}\right\}$,
$R_{2,3} = \left\{(x,y,z) \in \mathfrak{D};\ 0 \le x \le -y^2 + \frac{\alpha\delta}{\beta}y \text{ and } y \ge 0\right\}$,

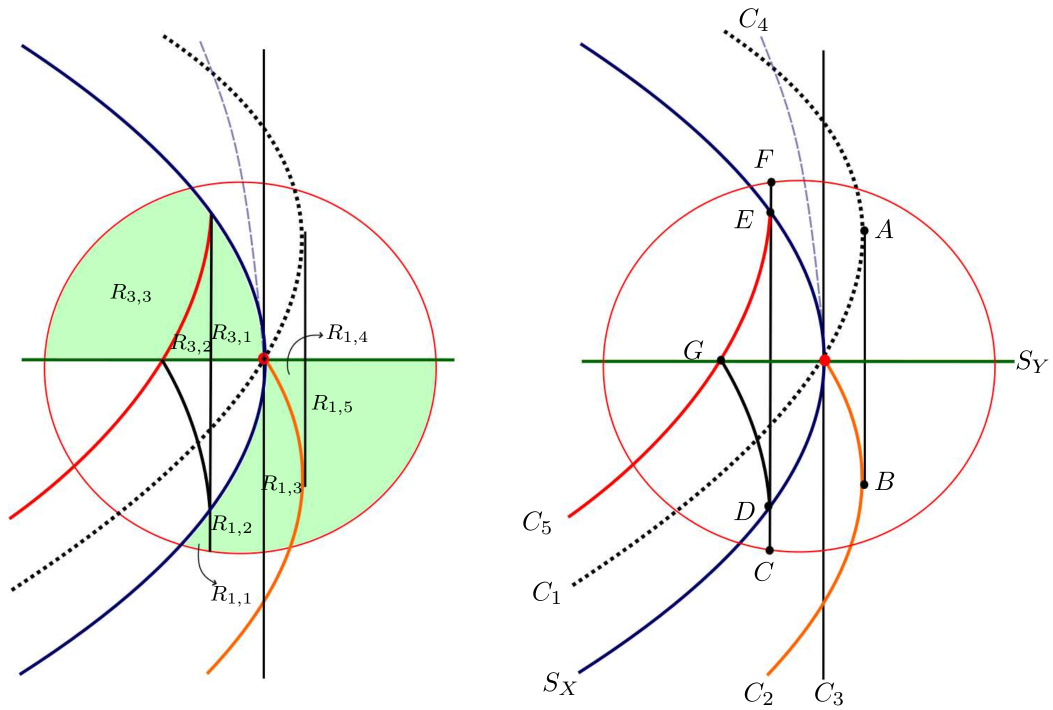

Figure 4.5: The subregions formed in the crossing region.

$R_{2,4} = \left\{(x, y, z) \in \mathfrak{D};\ \left(-y^2 \le x \le 0 \text{ and } 0 \le y \le \frac{2\alpha\delta}{3\beta}\right) \text{ or } \left(-\frac{4\alpha^2\delta^2}{9\beta^2} \le x \le 0 \text{ and } y \ge \frac{2\alpha\delta}{3\beta}\right)\right\},$
$R_{2,5} = \left\{(x, y, z) \in \mathfrak{D};\ -y^2 \le x \le -\frac{4\alpha^2\delta^2}{9\beta^2} \text{ and } y \ge 0\right\}.$

The regions $R_{3,1}$ to $R_{3,3}$ belong to the crossing region Σ^{c+}

$R_{3,1} = \left\{(x, y, z) \in \mathfrak{D};\ -\frac{4\alpha^2\delta^2}{9\beta^2} \le x \le -y^2 \text{ and } y \ge 0\right\},$
$R_{3,2} = \left\{(x, y, z) \in \mathfrak{D};\ -\frac{3\alpha^2\delta^2}{4\beta^2} \le x \le -\frac{4\alpha^2\delta^2}{9\beta^2} \text{ and } 0 \le y \le f(x)\right\},$
$R_{3,3} = \left\{(x, y, z) \in \mathfrak{D};\ x \le -\frac{3\alpha^2\delta^2}{4\beta^2} \text{ or } \left(-\frac{3\alpha^2\delta^2}{4\beta^2} \le x \le -\frac{4\alpha^2\delta^2}{9\beta^2} \text{ and } f(x) \le y \le \sqrt{-x}\right)\right\}.$

The regions $R_{4,1}$ to $R_{4,4}$ belong to the escaping region Σ^e.

$R_{4,1} = \left\{(x, y, z) \in \mathfrak{D};\ -\frac{4\alpha^2\delta^2}{9\beta^2} \le x \le -y^2 \text{ and } y \le 0\right\},$
$R_{4,2} = \left\{(x, y, z) \in \mathfrak{D};\ -\frac{3\alpha^2\delta^2}{4\beta^2} \le x \le -\frac{4\alpha^2\delta^2}{9\beta^2} \text{ and } \widetilde{f}(x) \le y \le 0\right\},$
$R_{4,3} = \left\{(x, y, z) \in \mathfrak{D};\ \left(x \le -\frac{3\alpha^2\delta^2}{4\beta^2} \text{ and } y \le 0\right) \text{ or } ,\right.$

$\left.\left(-\frac{3\alpha^2\delta^2}{4\beta^2} \le x \le -\frac{4\alpha^2\delta^2}{9\beta^2} \text{ and } -\sqrt{-x} \le y \le \widetilde{f}(x)\right)\right\}.$

Remark 4.4 *As the vector field Z_0^s has a curve of equilibrium points C_1 and it is defined on $\Sigma^s \cup \Sigma^e$, then near the origin we have the following behavior (see Figure 4.7):*

(i.) The positive orbit of the flow $\phi_{Z_0^s}$ at the points belonging to the sector S_0 (same region $R_{2,2}$), tends to one point of the curve C_1.

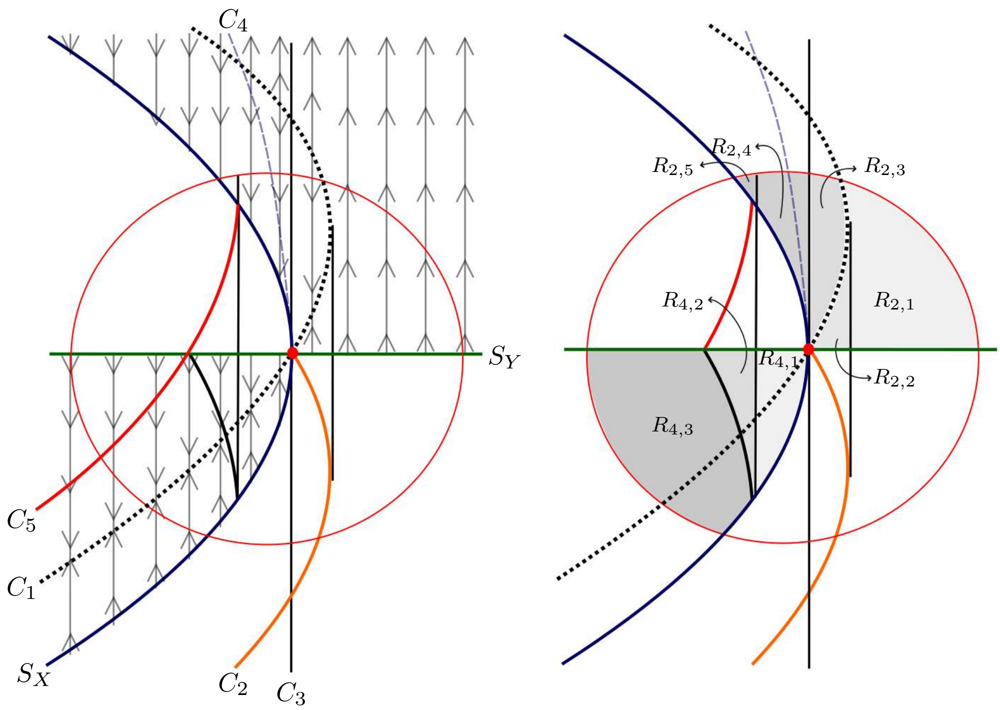

Figure 4.6: The subregions formed to the sliding and escaping region

(ii.) *Consider the set*

$$S_1 = \left\{(x,y,z)\in R^3;\ \left(x > \frac{\alpha^2\delta^2}{4\beta^2}\ and\ 0<y<\frac{\alpha\delta}{2\beta}\right) or \left(x > -y^2+\frac{\alpha\delta}{\beta}y\ and\ y>\frac{\alpha\delta}{2\beta}\right)\right\}$$

For the points belonging to the sector S_1, the positive orbit generated by the flow $\phi_{Z_0^s}$ escapes from any neighborhood of the origin.

(iii.) *Consider the set*

$$S_2 = \left\{(x,y,z)\in R^3;\ 0<x<-y^2+\frac{\alpha\delta}{\beta}y\ and\ y>0\right\}.$$

The positive orbit of the flow $\phi_{Z_0^s}$ for the points belonging to this sector tends to one point of the curve C_1.

(iv.) *Consider the set*

$$S_3 = \left\{(x,y,z)\in R^3;\ x<0\ and\ \sqrt{-x}<y<\frac{\alpha\delta-\sqrt{\alpha^2\delta^2-4\beta^2x}}{2\beta}\right\}.$$

If $q\in S_3$, then there exists $t(q)>0$ such that $\phi_Z^s(t(q),q)\in S_X$. Moreover, if $q\in S_2\cap S_3$, then there exists $t(q)>0$ such that $\phi_Z^s(t(q),q)$ is located at the origin.

(v.) *The positive orbit of the flow $\phi_{Z_0^s}$ at the points belonging to sector R_4, tends*

to one point of the curve C_1.

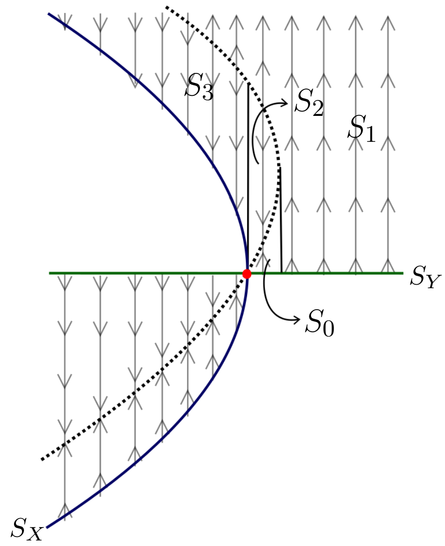

Figure 4.7: Behavior of the Z_0^s in some sectors near the origin.

Analyze of the case $\gamma \neq 0$

For the analysis of this case we assume that γ is small. The normalized Filippov vector fields are given by $Z_\gamma^s = \left(-\beta\gamma\left(x+y^2\right), -\beta\left(x+y^2\right)+\alpha\delta y\right)$ and it has only the origin as equilibrium point. Using the Hartman-Grobman theorem and the conditions $\alpha < 0$, $\beta < 0$, and $\delta > 0$, we have that the origin is locally topologically equivalent to a stable node when $\gamma < 0$ and to a saddle when $\gamma > 0$. The eigenvalues of the linearization of Z_γ^s are $\lambda_1 = -\beta\gamma$ and $\lambda_2 = \alpha\delta$ and the eigenspaces associated with λ_1 and λ_2 are, respectively:

$$E_1 = \left\{(x,y,z) \in \Sigma;\ y = \frac{\beta}{\beta\gamma+\alpha\delta}x\right\} \text{ and } E_2 = \{(x,y,z) \in \Sigma;\ x = 0\}.$$

Similarly to the case $\gamma = 0$, for the cases $\gamma > 0$ and $\gamma < 0$ we consider a disk $\mathfrak{D} \subset \mathbb{R}^2$ centered at the origin where $\mathfrak{D} = \mathfrak{B} \cap \Sigma$. Later we describe once more distinguished curves and subregions.

Description of the dynamic when $\gamma > 0$:

In this case, we consider the disk $\mathfrak{D}$ of radius $r < 12\gamma^2$. We split the disk $\mathfrak{D}$ into 4 regions from R_1^+ to R_4^+ divided by the curves S_X and S_Y. The crossing region R_1^+ and R_3^+, the sliding region R_2^+ and the escape region R_4^+. The sliding region is divided in the subregions $R_{2,1}^+$ and $R_{2,2}^+$. This subregions are separated by the curve $x = 0$. The escaping region is divided into the subregions $R_{4,1}^+$ and $R_{4,2}^+$ which are separated by the curve C^+ which is formed by the points p_0 in Σ such that for some time $t_0(p_0) > 0$, the point $p_1 = (x_1, y_1, z_1) = \phi_{Y_\gamma}(t_0(p_0), p_0) \in \Sigma$ and $x_1 = 0$. The curve C^+ has the following parametric equation:

$$\sigma_6(s) = (-2\gamma s, -s). \tag{4-6}$$

Remark 4.5 *Note that, $p_1 = \phi_{Y_\gamma}(t_0(p_0), p_0) \in \Sigma$ belong to the first quadrant for all points $p_0 = (x_0, y_0, z_0)$ such that $x_0 > 2\gamma y_0$ and $y_0 < 0$. Moreover, when we take a point $p_0 = (-y_0^2, y_0, 0) \in S_X$ such that $0 < y_0 < 4\gamma$ and we apply the crossing Poincaré map, we obtain a point $\widetilde{\Pi}(p_0) = (-y_0^2 + 4\gamma y_0, 2y_0, 0) = (\tilde{x}, \tilde{y}, 0)$ with $\tilde{x} > 0$ and $0 < \tilde{y} < 8\gamma$.*

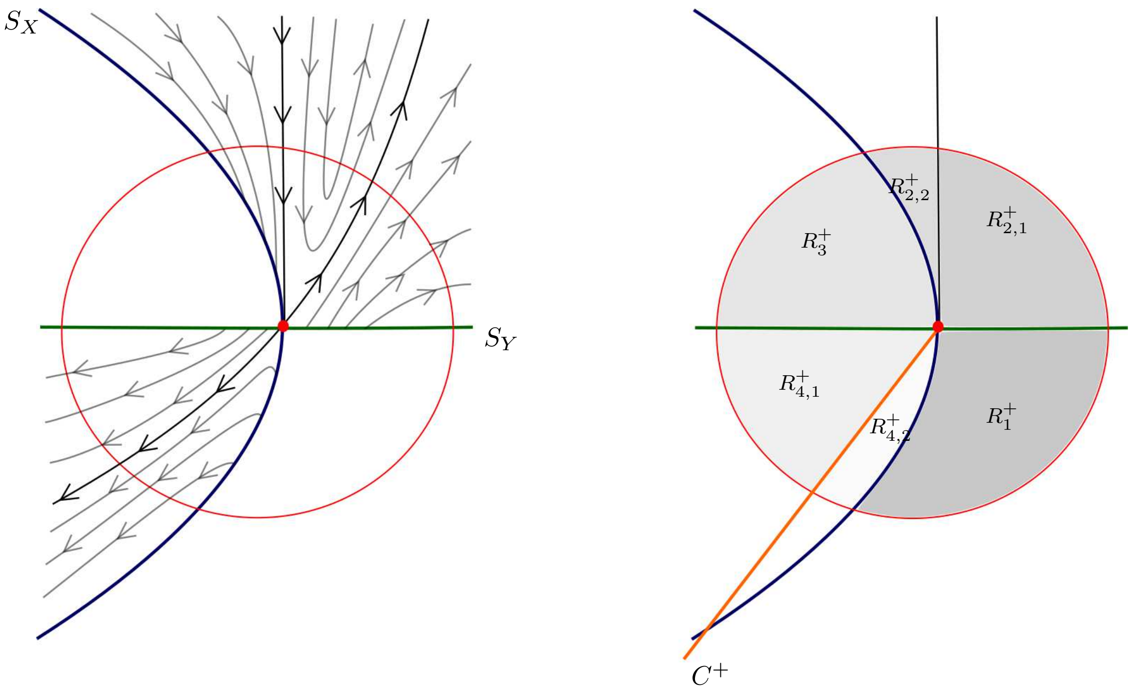

Figure 4.8: Behavior of the Z^s_γ on the disk $\mathfrak{D}$ when $\gamma > 0$ and subregions formed in this case.

Description of the dynamic when $\gamma < 0$:

Now, we consider the disk $\mathfrak{D}$ with radius $r < \frac{4}{3}\gamma$. We divided the disk $\mathfrak{D}$ into 4 regions R_1^- through R_4^- divided by the curves S_X and S_Y. The crossing regions R_1^- and R_3^-, the sliding region R_2^- and the escape region R_4^-. The crossing region R_1^- is divided in three subregions, $R_{1,1}^-$, $R_{1,2}^-$ and $R_{1,3}^-$ which are separated by the curves C_1^- and C_2^-. The curve C_1^- is formed by the points p_0 in Σ^{c-} such that for a time $t_0(p_0) > 0$, the point $p_1 = (x_1, y_1, z_1) = \phi_{Y_\gamma}(t_0(p_0), p_0) \in S_X$, C_1^- is given by the parametric equation:

$$\sigma_7(s) = \left(-2\gamma s - s^2, -s\right). \tag{4-7}$$

The curve C_2^- is formed by the points $\widehat{p}_0$ in Σ^{c-} such that for a time $\widehat{t}_0(\widehat{p}_0) > 0$, the point $\widehat{p}_1 = \phi_{Y_\gamma}(\widehat{t}_0(\widehat{p}_0), \widehat{p}_0) \in \Sigma$ and $x_1 = 0$. The curve C_2^- is given by the parametric equation (4-6). Finally, the sliding region is divided in the subregions $R_{2,1}^-$ and $R_{2,2}^-$ which are separated by the curve $x = 0$.

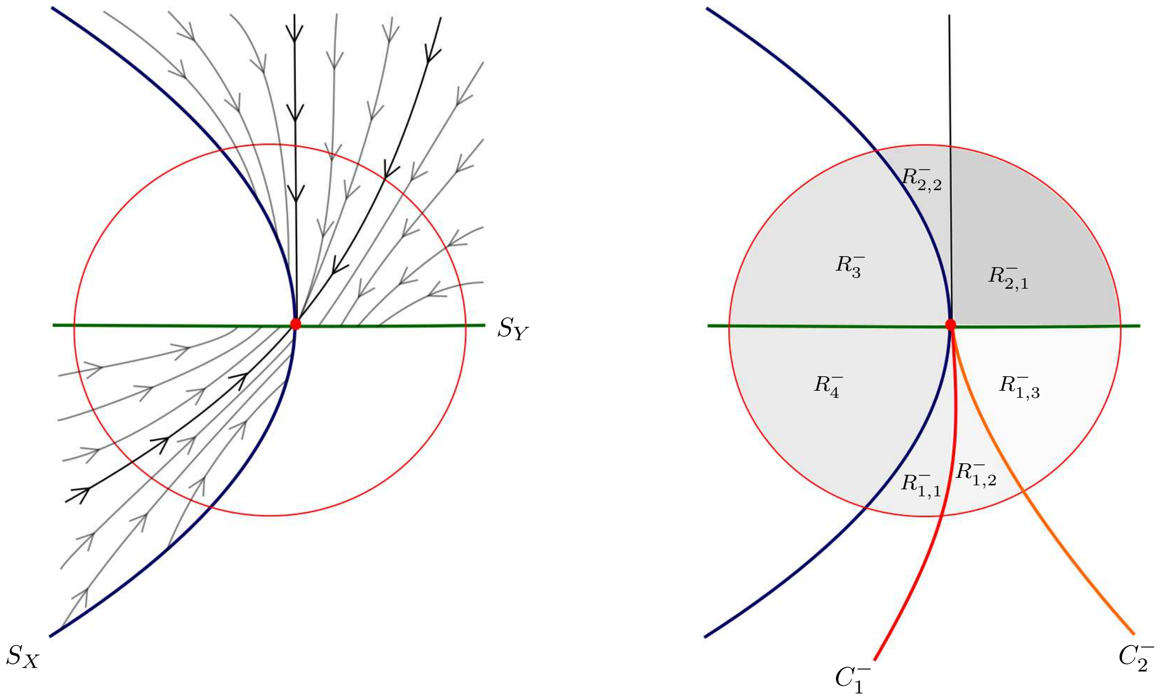

Figure 4.9: Behavior of the Z_γ^s in the disk $\mathfrak{D}$ when $\gamma < 0$ and subregions formed in this case.

4.4 Proof of the results

In this section we prove the results of this chapter.

Proof.[Proposition 4.1] In order to show that no crossing periodic orbits occur in equation (4-1), we use the Poincaré map $\widetilde{\Pi}_2(x_0, y_0, 0) = (x_2, y_2, 0)$ associated with (4-1), where $x_2 = x_0 + \gamma(y_0 + \Delta)$, $y_2 \frac{1}{2}(y_0 + \Delta)$, and $\Delta = \sqrt{-3(4x_0 + y_0^2)}$. We need to show that $\widetilde{\Pi}$ has no fixed points other than from the origin. Fix $\Phi(x_0, y_0) = (x_2 - x_0, y_2 - y_0)$, then we are going to show that $\Phi(x_0, y_0) = (0, 0)$ if, and only if, $(x_0, y_0) = (0, 0)$.

It is clear that $\Phi(0,0)=(0,0)$ and so we just need to solve the system of equations $x_2-x_0=0$ and $y_2-y_0=0$. By $y_2-y_0=0$, we have that $x_0=-\frac{y_0^2}{3}$. Notice that $(-\frac{y_0^2}{3},y_0,0)\notin\Sigma^{c+}$, so setting $x_2-x_0=0$, we conclude that $y_0=0$. Thus, the origin is the unique fixed point of $\widetilde{\Pi}(x_0,y_0)$. The same conclusion can be obtained for $\widetilde{\Pi}_3$ using similar arguments.

Now consider the sliding Poincaré map $\Pi(y_0)=(-y_0^2,\Delta(y_0),0)$ with

$$\Delta(y_0)=\frac{y_0\left(-\beta y_0+\left(1-\frac{4\gamma}{y_0}\right)^{\frac{\alpha\delta}{\beta\gamma}}(\beta y_0-2\beta\gamma\alpha+\delta)\right)}{\beta\gamma+\alpha\delta}-y_0.$$

Observe that, the domain of the function Δ is $[0,\infty)$ if $\gamma<0$, and $[4\gamma,\infty)$ if $\gamma>0$. The concavity of Δ does not change independent of the case considered. In fact,

$$\Delta''(y_0)=\frac{32\alpha\delta\left(1-\frac{4\gamma}{y_0}\right)^{\frac{\alpha\delta}{\beta\gamma}}(-\beta\gamma+\alpha\delta)}{y_0^2\beta^2(y_0-4\gamma)^2}$$

where $1-\frac{4\gamma}{y_0}>0$. Then $\Delta''(y_0)>0$ if $-\beta\gamma+\alpha\delta<0$ and $\Delta''(y_0)<0$ if $-\beta\gamma+\alpha\delta>0$. Since γ is small, we have that $\Delta''(y_0)>0$ in the cases A_3, A_4, B_1, and B_2 and $\Delta''(y_0)<0$ in the cases A_1, A_2, B_3, and B_4. Finally, we analyze the function Δ depending on the sign of γ:

- if $\gamma>0$ then $\lim_{y_0\to\infty}\Delta=-3<0$ and $\lim_{y_0\to4\gamma}\Delta=-\frac{5\beta\gamma+\alpha\delta}{\beta\gamma+\alpha\delta}<0$, similarly
- if $\gamma<0$ then $\lim_{y_0\to\infty}\Delta=-3<0$ and $\lim_{y_0\to0}\Delta=-1<0$.

Then, we claim that there are no real roots for the function Δ. Indeed, assume that $\gamma>0$ and that there exists $\widetilde{y}_0$ such that $\Delta(\widetilde{y}_0)=0$. In this case, as $\lim_{y_0\to4\gamma}\Delta<0$ the graph of Δ would increase to touch the y_0-axis at $\widetilde{y}_0$ either transversally or tangentially. If $\Delta''>0$ then the contact is transversal and $\Delta(y_0)>0$ for all $y>\widetilde{y}_0$ but this is a contradiction to the fact that $\lim_{y_0\to\infty}\Delta=-3<0$, that is, Δ should assume negative values again. If $\Delta''<0$ then whatever the contact is at $\widetilde{y}_0$, there exists $\widehat{y}_0>\widetilde{y}_0$ such that $\Delta(\widehat{y}_0)=-3$ and such that $\Delta(y_0)<-3$ for all $y_0>\widehat{y}_0$, but this is again a contradiction to the fact that $\lim_{y_0\to\infty}\Delta=-3<0$. Thus, there is no such $\widetilde{y}_0$ satisfying $\Delta(\widetilde{y}_0)=0$, that is, $\Delta(y_0)<0$ for all $y_0\in[4\gamma,\infty)$.

A similar proof can be carried out for the case $\gamma<0$ to show that $\Delta(y_0)<0$ for all $y_0\in[0,\infty)$ so the claim is proved. Therefore, there are no values of $\alpha<0$, $\beta<0$, $\delta>0$ and γ for which Z_γ has a crossing-sliding periodic trajectories. $\square$

Proof.[Proposition 4.2] Firstly, notice that the origin belongs to the curve C_1 defined in section 4.3, thus item (i) is proved. In order to prove item (ii), let p be a point belonging to the visible fold tangency curve, then $p_0 = (-y_0^2, y_0, 0)$ with $y_0 > 0$. Using the construction of the sliding Poincaré map with $\gamma = 0$ for these points we obtain that $p_1 = (-y_0^2, -2y_0, 0)$ and $p_2 = (-y_0^2, 2y_0, 0) \in Z^s$. Then p_2 is a point of the curve $x = -\frac{1}{4}y^2$, that is, $p_2 \in C_4$. Now, the curve of equilibrium points of the normalized Filippov vector field is $x = y\left(-y + \frac{\alpha\delta}{\beta}\right)$ which is called C_1 and so the intersection point between C_1 and C_4 is $q = \left(-\frac{4\alpha^2\delta^2}{9\beta^2}, \frac{2\alpha\delta}{3\beta}\right)$. In addition, observe that, for all visible fold points p such that $p_x < q_x$ the vector field Z_0^s points in a positive direction with respect to the y-axis, while for all $p_x > q_x$ and $p_x < 0$ the vector field Z_0^s points a negative direction with respect to the y-axis. Now, as the orbits of this normalized Filippov field are straight vertical lines and the coordinate of the x-axis of p and p_2 are equal, then for all visible fold points p such that $q_x < p_x < 0$ there exists one positive time such that the orbit of p_2 intersects S_X at the point p. Finally, using the sliding Poincaré map for the points in $\Gamma_{X_\ell} = \{(x, y, 0) \in S_X;\ 0 < y < \frac{2\alpha\delta}{3\beta}\}$, we have that $\mathfrak{M}_0^2 = \{(0,0,0)\} \cup \Gamma_{X_\ell} \cup \Pi(\Gamma_{X_\ell})$. The proof of item (iii) follows from Proposition 4.1 and also from the fact that there are no equilibrium points in the vector fields X and Y. □

The behavior of some orbits in $\mathfrak{M}_0^2$ is represented in Figure 4.10.

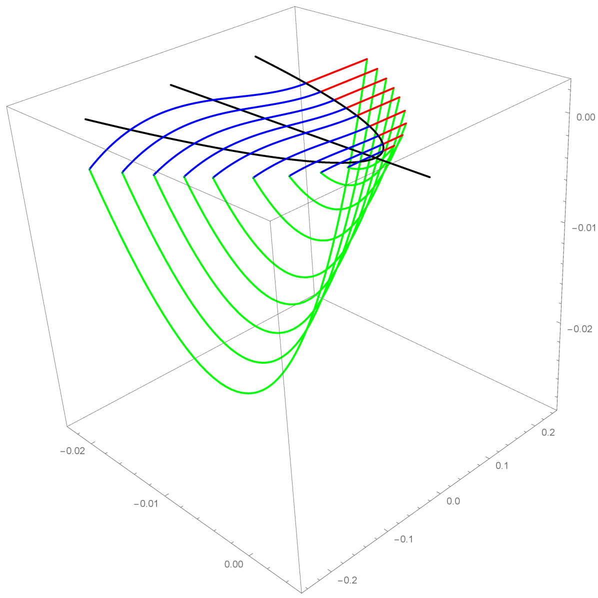

Figure 4.10: Local behavior of the orbits in $\mathfrak{M}_0^2$, for specific values for $\alpha < 0$, $\beta < 0$ and $\delta > 0$.

Proof.[Theorem 4.3] The proof of the items (i) and (ii) follows from the application of the Hartman-Grobman theorem to system (4-1), assuming the conditions $\alpha < 0$, $\beta < 0$, and $\delta > 0$. See Subsection 4.3. □

Now we prove the main results of this chapter.

Proof.[Theorem E] Fix a point p_0 in $\mathfrak{B}$. There exists two possibilities concerning its forward orbit. Either there exists a time $t' > 0$ such that $\phi_{Z_0}(p,t') \in \mathfrak{D}$, or the forward orbit never reaches the disk $\mathfrak{D}$. Then, it is sufficient to prove the result only for the points in $\mathfrak{D}$. Denote by ℓ a closed periodic orbit of $\mathfrak{M}_0^2$ and let $\mathfrak{p}$ be a point of the curve C_1, the singularities points of the normalized Filippov vector field. Let $f_1(x_0) = (\alpha\delta - \sqrt{-4\beta^2 x_0 + \alpha^2\delta^2})(2\beta)^{-1}$ be the image of x_0 on the curve C_1 in the second quadrant, that is, $f_1(x_0) > 0$. We are going to consider the 13 cases which depend on each one of the subregions of the disk $\mathfrak{D}$. We begin with the points in the region R_1 as follows.

- *Case (1).* Fix $p_0 = (x_0, y_0, 0) \in \text{int}(R_{1,1})$. Assuming $\phi_Y(t_0(p_0), p_0) \in \Sigma$, we obtain $t_0(p_0) = -2y_0$ and $p_1 = \phi_Y(t_0(p_0), p_0) = (x_0, -y_0, 0) = (x_1, y_1, 0) \in \Sigma^s$. As $x_0 < -\frac{4\alpha^2\delta^2}{9\beta^2}$ and $-y_0 < f_1(x_0)$, using Remark 4.4 there exists $t_1(p_1) > 0$ such that $\phi_Z^s(t_1(p_1), p_1) \in S_X$. If $p_2 = \phi_Z^s(t_1(p_1), p_1)$, then applying the crossing Poincaré map $\widetilde{\Pi}(p_2)$, we obtain one point $p_4 \in C_4$ with $x_4 = x_0 < -\frac{4\alpha^2\delta^2}{9\beta^2}$ (the point $p_3 = \phi_X(t_2(p_2), p_2)$ belongs to Σ, this point is generated by the flow of X). If $t_2(p_2) + t_3(p_3)$ is a positive time such that $p_4 = \widetilde{\Pi}(p_2)$, it follows that for all $t > t_0(p_0) + t_1(p_1) + t_2(p_2) + t_3(p_3)$ we have $\phi_{Z_0}(p_0, (t, \infty)) \cap \mathfrak{D} = \emptyset$.

- *Case (2).* Fix $p_0 = (x_0, y_0, 0) \in \text{int}(R_{1,2})$. As in *Case (1)*, $p_1 = \phi_Y(t_0(p_0), p_0) = (x_0, -y_0, 0) = (x_1, y_1, 0) \in \Sigma^s$, and as $-\frac{4\alpha^2\delta^2}{9\beta^2} < x_0 < 0$ and $-y_0 < f_1(x_0)$, there exists $t_1(p_1) > 0$ such that $\phi_Z^s(t_1(p_1), p_1) \in S_X$. If $p_2 = \phi_Z^s(t_1(p_1), p_1)$, then applying the crossing Poincaré map $\widetilde{\Pi}(p_2)$, we obtain one point $p_4 \in C_4$ with $-\frac{4\alpha^2\delta^2}{9\beta^2} < x_0 < 0$, and by the Proposition 4.2, it follows that $\omega(p_0) = \ell$.

- *Case (3).* Fix $p_0 = (x_0, y_0, 0) \in \text{int}(R_{1,3})$. As in *Case (1)*, $p_1 = (x_0, -y_0, 0) = (x_1, y_1, 0)$ with $p_1 \in R_{2,3} \in \Sigma^s$. As $R_{2,3} \subset S_2$, then by Remark 4.4, we have that $\omega(p_0) = \mathfrak{p}$.

- *Case (4).* Fix $p_0 = (x_0, y_0, 0) \in \text{int}(R_{1,4})$. Then $p_1 = (x_0, -y_0, 0) = (x_1, y_1, 0) \in R_{2,2} = S_0 \subset \Sigma^s$, that is, $\omega(p_0) = \mathfrak{p}$ from Remark 4.4.

- *Case (5).* Fix $p_0 = (x_0, y_0, 0) \in \text{int}(R_{1,5})$. Then $p_1 = (x_0, -y_0, 0) =$

$(x_1, y_1, 0) \in S_1 \subset \Sigma^s$, and by Remark 4.4, if $t_1(p_1) > 0$ is such that $\phi_Y(t_0(p_0), p_0) = p_1$, then for all $t > t_0(p_0) + \widehat{t}$ we have that $\phi_{Z_0}(p_0, (t, \infty)) \cap \mathfrak{D} = \emptyset$, where $\widehat{t} > 0$ is the necessary time for p_1 to exit the disk $\mathfrak{D}$ following the flow of Z_0^s.

Considering the cases in which p_0 belongs to the intersection of the previous subregions, we follow the next approach: for the points $p_0 = (x_0, y_0, 0) \in R_{1,1} \cap R_{1,2}$, we use the same analysis of cases *(1)* and *(2)* and so $\omega(p_0) = \mathfrak{p}$. In this case, $\mathfrak{p} = q$ (q is the intersection between C_1 and C_4). For the points $p_0 \in R_{1,2} \cap R_{1,3}$, $\omega(p_0)$ is the cusp-fold point, in fact $p_1 = (0, y_1, 0)$ with $y_1 < \frac{\alpha\delta}{\beta}$. Continuing with a similar analysis, we have that $\omega(p_0) = \mathfrak{p}$ for all $p_0 \in R_{1,3} \cap R_{1,4}$ or $p_0 \in R_{1,4} \cap R_{1,5}$ or $p_0 \in R_{1,3} \cap R_{1,5}$.

Now, we consider the cases where the points belong to the region R_2; then, the local orbit is given by $\phi_{Z_0^s}$. Using the Remark 4.4, we obtain that $p_0 = (x_0, y_0, 0) \in \mathrm{int}(R_{2,1})$ then there exists $t_0(p_0) > 0$ such that, for all $t > t_0(p_0)$, we have that $\phi_{Z_0}(p_0, (t, \infty)) \cap \mathfrak{D} = \emptyset$. On the other hand, $\omega(p_0) = \mathfrak{p}$ for all $p_0 = (x_0, y_0, 0) \in \mathrm{int}(R_{2,2} \cup R_{2,3})$. Notice that the points belonging to $R_{2,1} \cap R_{2,3}$ or $R_{2,2} \cap R_{2,3}$ are equilibrium points of Z_0^s, furthermore, for all $p_0 \in R_{2,1} \cap R_{2,1}$ we have that $\omega(p_0) = \mathfrak{p}$ or, more precisely, $\omega(p_0) = \{A\}$.

- *Case (6).* Fix $p_0 = (x_0, y_0, 0) \in \mathrm{int}(R_{2,4})$. Using Remark 4.4, there exists $t_0(p_0) > 0$ such that $\phi_{Z_0^s}(t_0(p_0), p_0) \in S_X$. Let $p_1 = (x_1, y_1, 0)$ be this point. Note that $-\frac{4\alpha^2\delta^2}{9\beta^2} < x_1 < 0$, then, similarly as in case *(2)*, we have that $\omega(p_0) = \ell$. It is clear that, the cusp-fold point in the omega set of the points p_0 belong to $R_{2,3} \cap R_{2,4}$.

- *Case (7).* Fix $p_0 = (x_0, y_0, 0) \in \mathrm{int}(R_{2,5})$. The forward orbit of p_0 behaves like the one of p_1 in *Case (1)*, then p_0 belongs to the orbit of some point as in *Case (1)*. Therefore there exists $\widehat{t} > 0$ such that, for all $t > \widehat{t}$, $\phi_{Z_0}(p_0, (t, \infty)) \cap \mathfrak{D} = \emptyset$. Notice that $\omega(p_0) = \mathfrak{p}$ for all $p_0 \in R_{2,4} \cap R_{2,5}$. Again, in this case $\mathfrak{p} = q$.

Now, the local orbit is given by ϕ_X for the points belonging to R_3.

- *Case (8).* Fix $p_0 = (x_0, y_0, 0) \in \mathrm{int}(R_{3,1})$. Applying the crossing Poincaré map, we obtain $\widetilde{\Pi}(x_0, y_0, 0) = (x_0, \frac{\alpha y - \sqrt{-3\alpha^2(4x+y^2)}}{2\alpha}, 0) = (x_2, y_2, z_2) = p_2$. As $x_2 = x_0$ and $-\frac{4\alpha^2\delta^2}{9\beta^2} < x_0 < 0$, by Remark 4.4 there exists $t_2(p_2) > 0$ such that $\varphi_{Z_0^s}(t_2(p_2), p_2) \in S_X$. So by Proposition 4.2, it follows that $\omega(p_0) = \ell$.

- *Case (9).* Fix $p_0 = (x_0, y_0, 0) \in \mathrm{int}(R_{3,2})$. Applying the crossing Poincaré map, we obtain $p_2 = (x_0, \frac{\alpha y - \sqrt{-3\alpha^2(4x+y^2)}}{2\alpha}, 0) = (x_2, y_2, z_2)$ with $y_2 < f_1(x_2)$.

Employing Remark 4.4, there exists $t_2(p_2)>0$ such that $\phi_{Z_0^s}(t_2(p_2),p_2)=p_3\in S_X$ with $x_3=x_0<-\frac{4\alpha^2\delta^2}{9\beta^2}$. Again, we apply the crossing Poincaré map to obtain $p_5=(x_5,y_5,z_0)$ with $y_5>f_1(x_5)$ and $x_2<-\frac{4\alpha^2\delta^2}{9\beta^2}$. As $p_5\in S_1$, for all $t>\sum_{i=1}^{4}t_i(p_i)$ we get $\phi_{Z_0}(p_0,(t,\infty))\cap\mathfrak{D}=\emptyset$. It is clear that $\omega(p_0)=\mathfrak{p}=q$ for the points p_0 in $R_{3,1}\cap R_{3,2}$.

- *Case (10).* Fix $p_0=(x_0,y_0,0)\in\mathrm{int}(R_{3,3})$. Using the crossing Poincaré map, we obtain $p_2=(x_0,\frac{\alpha y-\sqrt{-3\alpha^2(4x+y^2)}}{2\alpha},0)\in S_1$, and by Remark 4.4, we obtain $\phi_{Z_0}(p_0,(t,\infty))\cap\mathfrak{D}=\emptyset$ for all $t>t_0(p_0)+t_1(p_1)$. By the definition of the curve C_5, the omega limit set of the points in C_5 is $\mathfrak{p}$.

There are three possibilities for the positive local orbit of the points belonging to the escape region. So, in the following cases, we analyze all of those options using the vector fields X, Y, and Z^s. Firstly, using Remark 4.4 again, it follows that $\omega(p_0)=\mathfrak{p}$ for all $p_0\in R_4$ when we use the local orbit given by ϕ_{Z^s}.

- *Case (11).* Fix $p_0=(x_0,y_0,0)\in\mathrm{int}(R_{4,1})$ with the local orbit given by ϕ_X. Using the crossing Poincaré map we obtain that $p_2\in S_3$ with $-\frac{4\alpha^2\delta^2}{9\beta^2}<x_2<0$. Consequently, use the analysis of the *Case (9)* to obtain $\omega(p_0)=\ell$. Now, using the local orbit given by ϕ_Y, we have that $p_1\in R_{3,1}$ and from *Case (8)*, $\omega(p_0)=\ell$.

- *Case (12).* Fix $p_0=(x_0,y_0,0)\in\mathrm{int}(R_{4,2})$ with the local orbit given by ϕ_X. We use the crossing Poincaré map to obtain $p_2\in S_3$ with $x_2<-\frac{4\alpha^2\delta^2}{9\beta^2}$ and $y_2<f_1(x_2)$. Using Remark 4.4, there exisst $t_2(p_2)>0$ such that $\phi_{Z_0^s}(t_2(p_2),p_2)=p_3\in S_X$ with $x_3=x_0<-\frac{4\alpha^2\delta^2}{9\beta^2}$. Again, with the crossing Poincaré map, we obtain $p_5\in S_1$ and $\phi_{Z_0}(p_0,(t,\infty))\cap\mathfrak{D}=\emptyset$ for all $t>\sum_{i=1}^{4}t_i(p_i)$. Now, using the local orbit given by ϕ_Y, we get $p_1\in R_{3,2}$ and by *Case (9)*, there exists $\widehat{t}>0$ such that, for all $t>\widehat{t}$, $\phi_{Z_0}(p_0,(t,\infty))\cap\mathfrak{D}=\emptyset$.

Regardless of the chosen local orbit ϕ_X or ϕ_Y, we have $p_0\in R_{4,1}\cap R_{4,2}$. It is easy to see that $\omega(p_0)=\mathfrak{p}$, where $\mathfrak{p}=q$.

- *Case (13).* Fix $p_0=(x_0,y_0,0)\in\mathrm{int}(R_{4,3})$ with the local orbit given by ϕ_X. Then, $\widetilde{\Pi}(p_0)=p_2\in S_3$ with $x_2<-\frac{4\alpha^2\delta^2}{9\beta^2}$ and $y_2<f_1(x_2)$. So, by Remark 4.4, there exists $t_2(p_2)>0$ such that $\phi_{Z_0^s}(t_2(p_2),p_2)=p_3\in S_X$ with $x_3<-\frac{4\alpha^2\delta^2}{9\beta^2}$. We apply the crossing Poincaré map for p_3 and obtain $p_5=(x_5,y_5,z_0)\in C_4$ with $y_5>f_1(x_5)$ and $x_2<-\frac{4\alpha^2\delta^2}{9\beta^2}$. Therefore, there exists $\widehat{t}>0$ such that $\phi_{Z_0}(p_0,(t,\infty))\cap\mathfrak{D}=\emptyset$ for all $t>\widehat{t}$. On the other hand, using the local orbit given by ϕ_Y we obtain $p_1\in R_{3,3}$ and from *Case (10)*, there exists $\widehat{t}>0$ such that $\phi_{Z_0}(p_0,(t,\infty))\cap\mathfrak{D}=\emptyset$ for all $t>\widehat{t}$.

The same happens with the points $p_0 \in R_{4,2} \cap R_{4,3}$, regardless of which local path (ϕ_X or ϕ_Y) is taken.

Finally, we analyze the forward orbits of the points in the disk $\mathfrak{D}$ that also belong to $S_X \cup S_Y$. Notice that, if $p_0 \in S_Y$ with $x_0 > 0$ or $p_0 \in S_X$ with $y > 0$, then $P_0 \in \partial\Sigma^s$; if $p_0 \in S_Y$ with $x_0 < 0$, then $P_0 \in \partial\Sigma^{c+}$; and if $p_0 \in S_X$ with $y_0 < 0$ then $P_0 \in \partial\Sigma^{c-}$. Therefore, we get the following conclusion.

(i) There exists $\widehat{t} > 0$ such that $\phi_{Z_0}(p_0,(t,\infty)) \cap \mathfrak{D} = \emptyset$, for all $t > \widehat{t}$ when p_0 satisfies the following conditions:
   - if $p_0 \in S_Y \cap \mathfrak{D}$ with $x_0 > \frac{\alpha^2\delta^2}{4\beta^2}$ or $x_0 < -\frac{4\alpha^2\delta^2}{9\beta^2}$, or
   - if $p_0 \in S_X \cap \mathfrak{D}$ with $x_0 < -\frac{4\alpha^2\delta^2}{9\beta^2}$.

(ii) $\omega(p_0) = \mathfrak{p}$, for all $p_0 \in S_Y$ such that $0 < x_0 < \frac{\alpha^2\delta^2}{4\beta^2}$ and $\omega(p_0) = \mathfrak{p} = A$, for $p_0 = (\frac{\alpha^2\delta^2}{4\beta^2}, 0, 0)$.

(iii) $\omega(p_0) = \ell$, for all $p_0 \in S_X$ such that $-\frac{4\alpha^2\delta^2}{9\beta^2} < x_0 < 0$, and $\omega(p_0) = \mathfrak{p} = q$, for $p_0 = (-\frac{4\alpha^2\delta^2}{9\beta^2}, 0, 0)$.

(iv) Lastly, the omega set of the cusp-fold point is itself.

Thus, we have proved Theorem E. □

Proof.[Theorem F] Similar to the case $\gamma = 0$, it is sufficient to prove the result only for the points in $\mathfrak{D}$. Fix $p_0 = (x_0, y_0, 0) \in \text{int}(R_1^+)$, the local orbit is given by ϕ_Y. Assuming $\phi_Y(t_0(p_0), p_0) \in \Sigma$, we obtain $t_0(p_0) = -2y_0$ and $p_1 = \phi_{Y_\gamma}(t_0(p_0), p_0) = (x_0 - 2\gamma y_0, -y_0, 0) = (x_1, y_1, 0) \in \Sigma^s$. As γ is small, then $p_1 \in R_{2,1}^+$. For the points $p_0 \in \text{int}(R_{2,1}^+)$, the local orbit is then given by ϕ_{Z_γ} and so, for all $t > t'$ we have that $\phi_{Z_\gamma}(p_0,(t,\infty)) \cap \mathfrak{D} = \emptyset$ (where t' is the time needed for p_0 to exit the disk $\mathfrak{D}$ by the flow on Z_γ^s). The same analysis is made for the points $p_0 \in R_1^+ \cap R_{2,1}^+$.

Let p_0 be a point of $\text{int}(R_{2,2}^+)$ and using the vector field on Z_γ^s, there exist a time $t_0(p_0) > 0$ such that $p_1 = \phi_{Z_\gamma^s}(t_0(p_0), p_0) \in S_X$ with $y_1 > 0$. Employing the crossing Poincaré map $p_3 = (x_3, y_3, 0) = \widetilde{\Pi}(p_1)$, there exist two possibilities for p_3, either $y_3 \leq 8\gamma$ or $y_3 > 8\gamma$. If $y_3 \leq 8\gamma$, then there exist $t' > 0$ such that for all $t > t'$ we have $\phi_{Z_\gamma}(p_0,(t,\infty)) \cap \mathfrak{D} = \emptyset$. On the other hand, if $y_3 > 8\gamma$, then again there exist a time $t_3(p_3) > 0$ such that $p_4 = \phi_{Z_\gamma^s}(t_3(p_3), p_3) \in S_X$ with $y_1 > 0$ and $x_1 < x_4 < 0$. Since the origin is the single singular point of the Filippov vector field, and by Proposition 4.1, Z_γ has no periodic orbits, then there exist $t' > 0$ such that for all $t > t'$ we have $\phi_{Z_\gamma}(p_0,(t,\infty)) \cap \mathfrak{D} = \emptyset$. Is clear that, if $p_0 \in R^+_{2,1} \cap R^+_{2,2}$, then $\omega(p_0) = (0,0,0)$.

Fix $p_0 \in R^+_{2,2} \cap R^+_3$. Then $p_2 \in R^+_{2,1}$, where $p_2 = (x_2,y_2,0) = \widetilde{\Pi}(p_0)$. In fact, this is a consequence of Remark 4.5 because $y_2 < 8\gamma$. Taking into account the behavior of the vector field Z^s_γ in the first quadrant, if $p_0 \in R^+_{2,2} \cap R^+_3$ then there exists $t' > 0$ such that for all $t > t'$ we get $\phi_{Z_\gamma}(p_0,(t,\infty)) \cap \mathfrak{D} = \emptyset$. In fact, $t' = t_0(p_0) + t_1(p_1) + \widetilde{t}$, where $\widetilde{t}$ is the time needed for p_2 to exit the disk $\mathfrak{D}$.

Fix $p_0 \in \text{int}(R^+_3)$, the local orbit is given by ϕ_X, then $p_1 = (x_1,y_1,0)$ is such that $x_1 = x_0$ and $y_1 = -\frac{y_0+\sqrt{-3(4x_0+y_0^2)}}{2}$. As the intersection between C^+ and S_X is $(-4\gamma^2,-2\gamma)$, for $x_1 > -4\gamma^2$ and $y_1 < 0$, then by Remark 4.5, $p_2 = \phi_{Y_\gamma}(t_1(p_1),p_1) \in \Sigma$ belongs to the first quadrant, that is $p_2 \in R^+_{2,1}$.

Now, there are three possibilities for the positive local orbit of the points $p_0 \in R^+_4$. Assuming that the local orbit is given by $\phi_{Z^s_\gamma}$, then there exist a time $t_0(p_0) > 0$ such that for all $t > t_0(p_0)$, $\phi_{Z_\gamma}(p_0,(t,\infty)) \cap \mathfrak{D} = \emptyset$. Assuming that the local orbit is given by the vector field X, similar to what happens in the region R^+_3, we obtain $p_2 \in R^+_{2,1}$. Finally, taking the local orbit given by ϕ_{Y_γ}, we have one of the following possibilities.

(i) If $p_0 \in \text{int}(R^+_{4,1})$, then the forward orbit of p_0 escapes from the disk $\mathfrak{D}$. In fact, if the local orbit is given by ϕ_X, we apply the crossing Poincaré map to obtain $p_2 = \widetilde{\Pi}(p_0) \in \Sigma^s$ with $x_2 < 0$. Otherwise, if the local orbit is given by ϕ_{Y_γ}, then $p_1 \in \Sigma^{c+}$ or $p_1 \in \Sigma^s$ with $x_1 < 0$. The assertion follows because the two possibilities are analogous to the analysis of the regions $R^+_{2,2}$ and R^+_3.

(ii) If $p_0 \in \text{int}(R^+_{4,2})$, then there exists $t' > 0$ such that for all $t > t'$ it holds that $\phi_{Z_\gamma}(p_0,(t,\infty)) \cap \mathfrak{D} = \emptyset$. The same occurs for the points $p_0 \in R^+_{4,2} \cap R^+_1$.

(iii) By the definition of the curve C^+, if $p_0 \in R^+_{4,1} \cap R^+_{4,2}$ then $\omega(p_0) = (0,0,0)$, that is, a cusp-fold point.

The previous arguments proves Theorem F. □

Proof.[Theorem G] We prove the result only for the points in $\mathfrak{D}$. Let p_0 be a point in $\text{int}(R^-_3)$. Then

$$p_2 = (x_2,y_2,0) = \left(x_0 + \gamma\left(y_0 + \sqrt{-3(4x_0+y_0^2)}\right), \frac{y_0+\sqrt{-3(4x_0+y_0^2)}}{2}, 0 \right) = \widetilde{\Pi}(p_0).$$

There are two possibilities for the forward orbit of the point p_2, either $p_2 \in \Sigma^{c+}$ or $p_2 \in \Sigma^s$. In any case, $x_2 < x_0$. If $p_2 \in \Sigma^{c+}$, we repeat the same approach to obtain p_4 such that $x_4 < x_2 < x_0$. If $p_2 \in \Sigma^s$, then there exist $t_2(p_2) > 0$ such that $p_3 = \phi_{Z_\gamma^s}(t_2(p_2), p_2) \in S_X$ and again we use the crossing Poincaré map getting $p_5 = (x_5, y_5, 0)$ with $x_5 < x_0$. Therefore, regardless of the case, the forward orbit of the point p_0 escapes from the disk $\mathfrak{D}$. With a similar analysis, we obtain the same consequence for the points $p_0 \in \text{int}(R_{2,2}^-)$ and $R_3^- \cap \mathbb{R}_{2,2}^-$.

Notice that $p_1 = \phi_{Y_\gamma}(t_0(p_0), p_0) \in \Sigma$ belongs to $R_{2,1}^-$ for all the points $p_0 = (x_0, y_0, z_0)$ such that $x_0 \geq 2\gamma y_0$ and $y_0 \leq 0$. Therefore, $\omega(p_0) = (0,0,0)$ for all $p_0 \in R_{1,3}^- \cup R_{2,1}^-$.

Fix $p_0 \in \text{int}(R_{1,2}^-)$, then $p_1 = \phi_{Y_\gamma}(t_0(p_0), p_0) \in \Sigma^s$ is such that $x_1 < 0$. So, analogously to what happens in the region $R_{2,2}^-$, there exist $t' > 0$ such that for all $t > t'$ we have $\phi_{Z_\gamma}(p_0, (t, \infty)) \cap \mathfrak{D} = \emptyset$. The points $p_0 \in R_{1,1}^- \cap R_{1,2}^-$ are such that $p_1 = \phi_{Y_\gamma}(t_0(p_0), p_0) \in S_X$. Then their forward orbits escapes the disk $\mathfrak{D}$. The same happens for the points $p_0 \in \text{int}(R_{1,1}^-)$.

Fix $p_0 \in R_4^+$. Assuming that the local orbit is given by $\phi_{Z_\gamma^s}$, then $\omega(p_0) = (0,0,0)$. Assuming that the local orbit is given by the vector field Y_γ, so $p_1 = \phi_{Y_\gamma}(t_0(p_0), p_0) \in \Sigma^{c+}$ and their forward orbits escapes the disk $\mathfrak{D}$. The same conclusion is obtained when we use the local orbit ϕ_X and also for the points $p_0 \in R_{1,1}^- \cup R_4^-$ and $p_0 \in R_3^- \cup R_4^-$. □

CHAPTER 5

Closing remarks, future directions and work in progress

In this thesis, we prove that the existence of an underlying semiflow given by PSVF using Filippov's convention, assuming no escaping region. Under the same condition, in Chapter 2, we use topological tools, more specifically Conley theory, to guarantee the existence of periodic orbits in a PSVF of dimension 3. Under the condition that the switching manifold does not have tangencies then the result is generalized for higher dimensions. Furthermore, the results in this thesis bring us one step closer to generalizing the tool in [31] for piecewise smooth vector fields using Filippov's convention. The next step is to include tangential singularities of order greater than 3 in systems defined on manifolds of dimension $n > 3$. The main importance of the obtained result in Theorem B is to state a new topological tool to find periodic orbits of piecewise smooth vector fields.

In Chapter 3, we present an application of the Theorem B obtained in Chapter 2 for biological systems. We defined a discontinuous system using switching systems given by Dynamic Signatures Generated by Regulatory Networks (DSGRN), and we employ the Filippov's convention for switching systems. We describe one method for guaranteeing the existence of a periodic orbit in a gene regulatory network called the Repressilator. As future direction to this work, we are interested in extending this method for other gene regulatory networks with more than 3 genes.

Other results in this thesis are in line with the Poincaré Bendixson Theorem. For the system 4-1, the origin corresponds to the co-dimension one tangential singularity called cusp-fold, regardless of the values of α, β, δ, or γ. We prove that in the PSVF given by the system 4-1 there exists neither crossing periodic orbits nor sliding crossing periodic orbits. We assume that $\alpha < 0$, $\beta < 0$, $\delta > 0$, and $\gamma = 0$, and we prove that there exists a region filled with periodic orbits passing through the origin. We show three results concerning Poincaré-Bendixson type theorems for PSVF. We perform a local study for the forward trajectories of the system 4-1 assuming that $\alpha < 0$, $\beta < 0$ and $\delta > 0$. Also, the following work is in progress along the same lines of the work done in the Chapter 4.

1. The local study of the forward trajectories of the system 4-1 regardless of the values of α, β, δ, or γ. As there exists neither crossing periodic orbits nor sliding crossing periodic orbits, then the analysis for each point near the origin concerning the ω limit does not have many possibilities.

   Conjecture 5.1 *Consider the fold-cusp system associated to*

$$Z_\gamma(x,y,z) = \begin{cases} X(x,y,z) = (0,\alpha,\beta(x+y^2)) & \textit{if } z \geq 0, \\ Y_\gamma(x,y,z) = (\gamma,1,\delta y) & \textit{if } z \leq 0. \end{cases} \tag{5-1}$$

   Assume that γ is sufficiently small, and α, β, and δ in one of the cases from A_1 to B_4. Let $\mathfrak{B}$ be the ball of radius $\widetilde{r}$ centered at the origin. Then, the positive orbit of an arbitrary point $q \in \mathfrak{B}$ satisfies one of the following statements.

   (i.) $\omega(q) = p = (0,0,0)$;

   (iii.) there exists $\widetilde{t} > 0$ *such that* $\{\phi_{Z_0}(q,t); t > \widetilde{t}\} \cap \widetilde{\mathfrak{B}} = \emptyset$.

2. In [45], the author defines a Shilnikov homoclinic orbit, which is a trajectory connecting a hyperbolic saddle-focus equilibrium to itself bi-asymptotically. Some years later, with the qualitative study of piecewise smooth vector fields, in paper [5], they define a sliding Shilnikov orbit for 3D Filippov systems. Consider the following perturbation of the system (4-1) of Chapter 4.

$$Z_{0,\epsilon}(x,y,z) = \begin{cases} X(x,y,z) = (0,\alpha,\beta(x+y^2)) & \text{If } z \geq 0, \\ Y_{0,\epsilon}(x,y,z) = (\epsilon^2+\epsilon y,1,\delta y) & \text{If } z \leq 0, \end{cases} \tag{5-2}$$

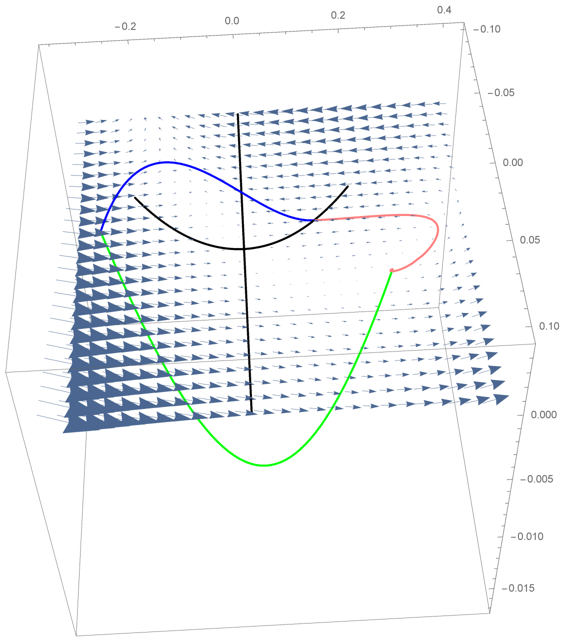

Figure 5.1: $\alpha = -\frac{1}{2}$, $\beta = -\frac{75}{100}$, $\delta = \frac{1}{2}$ e $\epsilon = \frac{1}{16}\left(3 - \sqrt{\frac{155}{3}}\right)$.

We have the following conjecture.

Conjecture 5.2 *The system 5-2 with* $\beta < 0, \alpha < 0, \delta > 0$ *and* $\epsilon = \frac{1}{16}\left(3 - \sqrt{\frac{9\beta + 128\alpha\delta}{\beta}}\right)$ *contains a Shilnikov-type orbit.*

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