UNIVERSIDADE FEDERAL DE GOIÁS (UFG)
INSTITUTO DE FÍSICA (IF)
PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA (PPGF)

ALEXANDRE BORGES BIDINOTTO

Relativistic Symmetry in Quantum Information Theory

GOIÂNIA
2022

UNIVERSIDADE FEDERAL DE GOIÁS
INSTITUTO DE FÍSICA

TERMO DE CIÊNCIA E DE AUTORIZAÇÃO (TECA) PARA DISPONIBILIZAR VERSÕES ELETRÔNICAS DE TESES

E DISSERTAÇÕES NA BIBLIOTECA DIGITAL DA UFG

Na qualidade de titular dos direitos de autor, autorizo a Universidade Federal de Goiás (UFG) a disponibilizar, gratuitamente, por meio da Biblioteca Digital de Teses e Dissertações (BDTD/UFG), regulamentada pela Resolução CEPEC nº 832/2007, sem ressarcimento dos direitos autorais, de acordo com a Lei 9.610/98, o documento conforme permissões assinaladas abaixo, para fins de leitura, impressão e/ou download, a título de divulgação da produção científica brasileira, a partir desta data.

O conteúdo das Teses e Dissertações disponibilizado na BDTD/UFG é de responsabilidade exclusiva do autor. Ao encaminhar o produto final, o autor(a) e o(a) orientador(a) firmam o compromisso de que o trabalho não contém nenhuma violação de quaisquer direitos autorais ou outro direito de terceiros.

1. Identificação do material bibliográfico

[X] Dissertação [ ] Tese

2. Nome completo do autor

Alexandre Borges Bidinotto

3. Título do trabalho

Relativistic symmetry in quantum information theory

4. Informações de acesso ao documento (este campo deve ser preenchido pelo orientador)

Concorda com a liberação total do documento [X] SIM [ ] NÃO[1]

[1] Neste caso o documento será embargado por até um ano a partir da data de defesa. Após esse período, a possível disponibilização ocorrerá apenas mediante:
a) consulta ao(à) autor(a) e ao(à) orientador(a);
b) novo Termo de Ciência e de Autorização (TECA) assinado e inserido no arquivo da tese ou dissertação.
O documento não será disponibilizado durante o período de embargo.
Casos de embargo:
- Solicitação de registro de patente;
- Submissão de artigo em revista científica;
- Publicação como capítulo de livro;
- Publicação da dissertação/tese em livro.

Obs. Este termo deverá ser assinado no SEI pelo orientador e pelo autor.

---

sei! assinatura eletrônica

Documento assinado eletronicamente por **ALEXANDRE BORGES BIDINOTTO**, **Discente**, em 18/04/2022, às 17:19, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

Documento assinado eletronicamente por **Ardiley Torres Avelar**, **Professor do Magistério Superior**, em 19/04/2022, às 21:32, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

A autenticidade deste documento pode ser conferida no site https://sei.ufg.br/sei/controlador_externo.php?acao=documento_conferir&id_orgao_acesso_externo=0, informando o código verificador **2840872** e o código CRC **DE474A9F**.

ALEXANDRE BORGES BIDINOTTO

Relativistic Symmetry in Quantum Information Theory

Dissertação apresentada ao Programa de Pós-Graduação em Física (PPGF), do Instituto de Física (IF), da Universidade Federal de Goiás (UFG), como requisito para obtenção do título de Mestre em Física.

Área de concentração: Física
Linha de pesquisa: Óptica Quântica e Informação Quântica

Orientador: Ardiley Torres Avelar
Coorientador: Emile Raymond Ferreira Taillebois

GOIÂNIA
2022

Ficha de identificação da obra elaborada pelo autor, através do
Programa de Geração Automática do Sistema de Bibliotecas da UFG.

Bidinotto, Alexandre Borges
Relativistic Symmetry in Quantum Information Theory [manuscrito] / Alexandre Borges Bidinotto. - 2022.
vii, 49 f.

Orientador: Prof. Dr. Ardiley Torres Avelar; co-orientador Dr. Emile Raymond Ferreira Taillebois.
Dissertação (Mestrado) - Universidade Federal de Goiás, Instituto de Física (IF), Programa de Pós-Graduação em Física, Goiânia, 2022.
Bibliografia.
Inclui gráfico, lista de figuras.

1. Simetria. 2. Teoria de Grupos. 3. Relatividade Restrita. 4. Tomografia de Estados Quânticos. I. Avelar, Ardiley Torres, orient. II. Título.

CDU 53

UNIVERSIDADE FEDERAL DE GOIÁS

INSTITUTO DE FÍSICA

ATA DE DEFESA DE DISSERTAÇÃO

Ata nº 200 da sessão de Defesa de Dissertação de Alexandre Borges Bidinotto, que confere o título de Mestre em Física, na área de concentração em Física.

Aos 04 dias do mês de abril de 2022, a partir das 14h00min, por meio de videoconferência, realizou-se a sessão pública de Defesa de Dissertação intitulada “Relativistic symmetry in quantum information theory”. Os trabalhos foram instalados pelo Orientador, Professor Doutor Ardiley Torres Avelar (IF/UFG), com a participação dos demais membros da Banca Examinadora: Professor Doutor Ademir Eugênio de Santana (IF/UnB), membro titular externo; e Professor Doutor Wesley Bueno Cardoso (IF/UFG), membro titular interno. Durante a arguição, os membros da banca não fizeram sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Dissertação, tendo sido o candidato APROVADO pelos seus membros. Proclamados os resultados pelo Professor Doutor Ardiley Torres Avelar, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos membros da Banca Examinadora, aos 04 dias do mês de abril de 2022.

TÍTULO SUGERIDO PELA BANCA

Documento assinado eletronicamente por **Ardiley Torres Avelar**, **Professor do Magistério Superior**, em 04/04/2022, às 16:19, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

Documento assinado eletronicamente por **Wesley Bueno Cardoso**, **Professor do Magistério Superior**, em 04/04/2022, às 16:20, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

Documento assinado eletronicamente por **ADEMIR EUGENIO DE SANTANA**, **Usuário Externo**, em 05/04/2022, às 15:07, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

A autenticidade deste documento pode ser conferida no site https://sei.ufg.br/sei/controlador_externo.php?acao=documento_conferir&id_orgao_acesso_externo=0, informando o código verificador **2756123** e o código CRC **9E8F71E8**.

Referência: Processo nº 23070.012288/2022-51 SEI nº 2756123

Acknowledgements

Agradeço à UFG pela estrutura e ao CNPq pelo financiamento, por permitirem que minha formação fosse possível.

Agradeço ao meu orientador, o professor Dr. Ardiley Torres Avelar pela orientação e confiança desde os tempos de iniciação científica e discussões esclarecedoras e seu imenso entusiasmo inspirador pela física. Agradeço também outros professores que foram importantes em minha formação, em especial ao Dr. Emile Raymond Ferreira Taillebois pela coorientação no mestrado e aos professores Dr. Wesley Bueno Cardoso e Dr. Rafael de Morais Gomes pela discussão muito construtiva em minha qualificação deste projeto e também ao professor Dr. Norton Gomes de Almeida pelas aulas e conversas sobre física.

Agradeço aos colegas e amigos do instituto de física, em especial à Gabriella Damas por todo o auxílio acadêmico e de amizade e pelas inúmeras vezes que leu este documento antes de qualquer outra pessoa.

Agradeço aos meus pais, Alessandra e Joni por apoiarem meus sonhos, por sempre priorizarem minha educação e por entenderem quando precisei estar ausente, amo vocês. Agradeço também à minha família, em especial meus irmãos, Murilo, Henry e Luana.

Agradeço à minha rede de suporte emocional, meus amigos, que me ajudaram a manter a disposição para seguir em frente, por mais sombrio que os tempos fossem.

Abstract

This dissertation begins with a succinct introduction to group theory to enable the understanding of special relativity theory from a symmetrical point of view. From this perspective, the notion of hierarchy of dynamical variables, which is present in relativity, is introduced to quantum mechanical formalism in order to construct the irreducible unitary representations of the Poincaré group for secondary variables. Given this discussion, the questions regarding the existence of a subset of transformations in which the Wigner rotations associated do not depend on the momenta. In sequence, the relativistic Stern-Gerlach observable is presented. This observable also depends on the momenta, so the initial problem on the density matrices for spin remains unsolved. The open question on the Stern-Gerlach observable is what is the minimum number of measurements in order to fully determine a relativistic density matrix. Both questionings on this dissertation can lead to the determination of the quantum state tomography of relativistic particles, which is the further goal of this document.

Keywords: Symmetry, Group Theory, Special Relativity, Quantum State Tomography.

Resumo

Esta dissertação começa com uma introdução sucinta à teoria dos grupos para permitir a compreensão da teoria da relatividade especial de um ponto de vista simétrico. Nessa perspectiva, a noção de hierarquia de variáveis dinâmicas, que está presente na relatividade, é introduzida no formalismo da mecânica quântica a fim de construir as representações unitárias irredutíveis do grupo de Poincaré para variáveis secundárias. Diante dessa discussão, questiona-se a existência de um subconjunto de transformações em que as rotações de Wigner associadas independem dos momentos. Em seqüência, o observável do experimento de Stern-Gerlach relativístico é apresentado. Este observável também depende dos momentos, então o problema inicial nas matrizes de densidade para spin permanece. A questão em aberto no observável Stern-Gerlach é: Qual é o número mínimo de medições para determinar completamente uma matriz de densidade relativística? Ambos os questionamentos desta dissertação podem levar à determinação da tomografia de estados quânticos de partículas relativísticas, que é o objetivo futuro deste documento.

Palavras-chave: Simetria, Teoria de Grupos, Relatividade Restrita, Tomografia de Estados Quânticos.

Contents

List of Figures

Nomenclature

In this document, the greek letters, $\alpha, \beta, \gamma, ...$ represent indices from 0 to 3 and the latin letters indicate indices from 1 to 3.

Another convention adopted is that the metric tensor $g^{\alpha\beta}$ will be written in the form $g^{\alpha\beta} = diag(1, -1, -1, -1)$.

1 Introduction

Mathematically, in a given structured object of any sort, when there is a mapping from this object to itself, which preserves the given structure, we say that this structured object has a symmetry [2]. Physics uses mathematical artifices to model nature interactions in an understandable way to human minds. Therefore, it is natural that we search for symmetries in physical interactions for a better understanding of the fundamental constituents of the universe.

The laws of nature can be very difficult to identify under the infinite variety of different possible initial conditions of a system. Changing initial conditions can change a lot the behavior of the physical system in question. Even with hard to predict changes, some general properties are always present in interactions of the same type, these are the invariances due to the laws of nature. Symmetry is directly connected to the laws of nature, representing certain regularities that are independent of the above-mentioned initial conditions.

A useful mathematical tool for understanding and representing symmetries is the group theory. Group theory predicted the existence of some elementary particles prior to their experimental observation [3]. Even more recreational activities that involve symmetry can be studied by group theory, e.g., the Rubik's cube [4]. This powerful mathematical apparatus is efficient in predicting characteristics of systems without considering the specific nuances for every system that behaves in a certain way, e.g., each differential symmetry in the action of a physical system with conservative forces corresponds to a conserved quantity [5].

In the special relativity frame of work, the symmetries are related to the direct properties of the space in the interactions take place (as in any frame of work). The isotropy of space, as in the Newtonian interactions, is preserved. Unlike the Newtonian homogeneity of time, the time-related homogeneity takes account of the spatial dimensions too, resulting in a spacetime homogeneity[6]. In a situation where the time dimension is constant (the Minkowski spacetime is a 4-dimensional manifold formed by the three Euclidean spatial dimensions and a time dimension in which the spacetime interval between two events is independent of changes from one inertial frame to another), we have a Euclidean space and the classical symmetries hold. When bearing the general theory of relativity in mind, the number of symmetries of the system increases, as the geometric properties of general relativity spacetime are more intricate. There are symmetries related to curvature, matter, and geodesics [7].

In a quantum mechanical environment, one can observe several kinds of symmetries. Considering a rest frame, the space and the time translations are symmetric and lead to conservation laws for energy and momentum [8]. Rotations also have symmetry (which results in angular momentum conservation). And beyond other symmetries, there is a curious set of symmetries that are a consequence of the above time and space characteristics said above. The CPT symmetry [9] consists of the symmetry in charge, parity, and time and, in certain regimes, there are breaks in parity and time symmetry, although the CPT symmetry remains unbroken. This combined symmetry is one of the foundations of quantum field theory.

When considering the special relativistic symmetry in quantum mechanics, one must impose the Poincaré group as a fundamental symmetry group for the interactions that are being studied. This imposition results in a commutation relation that the particles wave states must obey.

In this dissertation, some symmetry-related aspects of the relativistic theory are introduced and discussed, and further, the application of such concepts on the quantum mechanic framework is analyzed.

The adjacent chapter is a succinct initiation to some necessary aspects of group theory that will be needed in the further discussions on this dissertation [10, 11, 12]. Subsequently, the theory of special relativity is discussed from the point of view of group theory. The Stern-Gerlach experiment is the opening gambit of the quantum spin state tomography[13]. In [14, 15] the relativistic Stern-Gerlach experiment is described and the authors leave some open points still to be answered. Then, the last chapter discusses the formalism that takes account spin and momentum for relativistic particles [16, 17], where the notion of a hierarchy of dynamic variables from relativity is introduced to the quantum mechanical formalism.

2 A Brief Introduction to Group Theory

2.1 Introduction

When a physical system has a feature, local or global, that remains unchanged by some variety of transformations, we say that this system has some symmetry. The mathematical description that helps us to understand such invariance is the group theory.

The element of physical systems that we know as conservation is a concept directly associated with symmetry. Each conservation law is related to some symmetry of physical interactions due to a category of transformations. For instance, the conservation of linear momentum is a direct consequence of the translational symmetry of the space, which means that the physical interactions do not take into account where the origin of the coordinates system was placed.

Another definition of symmetry can be expressed as any geometry of a field is associated with a particular transformation group $\mathcal{G}$[11]. Elements of such geometry which are equivalent in respect to $\mathcal{G}$ are considered to be the same ones[18].

Therefore, the understanding of group theory is a cornerstone of symmetry-related interactions. Group theory is the most useful mathematical description of symmetries and their consequences. In this chapter, the essential group theory principles needed for the understanding of the following chapters will be provided.

2.2 Groups and Some Basic Definitions

2.2.1 Group

A set $\{\mathcal{G} : a, b, c, ...\}$ is said to form a group[10] if there is an operation, $\cdot$, called group product that maps any ordered pair of elements $a, b \in \mathcal{G}$ with a well-defined product $a \cdot b$ which is also in $\mathcal{G}$. Some conditions must be satisfied[1]:

· Group product is associative, i.e.: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;

· There is an element in $\mathcal{G}$ called *Identity*, which we will denote by e, that has the following property: $a \cdot e = a$, $\forall \ a \in \mathcal{G}$;

· For each $a \in \mathcal{G}$, there is an *Inverse Element of* a, denoted by a^{-1} so that $a \cdot a^{-1} = e$.

A cyclic group $\mathcal{C}_n$ is a group that has general structure $\{\mathcal{C}_n = e, a, a^2, ..., a^{n-1};\ a^n = e\}$, where n is any positive integer. If one makes a table where the table elements are the products between the cyclic group elements, the lines on the table will be cyclical permutations of each other, hence the name.

An *Abelian* group is a group $\mathcal{G}$ in which, for every element $a, b \ \in \ \mathcal{G}$[2], we have $ab = ba$. If finite, we say that a group $\mathcal{G}$ has order n, where n is the group's cardinality, the number of elements in $\mathcal{G}$. The cyclic groups, $\mathcal{C}_n$, are Abelian groups of order n ($n = 1, 2, ...$).

2.2.2 Subgroups

A subset $\mathcal{H}$ of a group $\mathcal{G}$ forms a subgroup if $\mathcal{H}$ itself forms a group (according to the group definition) with the same product law as $\mathcal{G}$. If this holds, we say that $\mathcal{H}$ forms a subgroup of $\mathcal{G}$.

Any set of $n \times n$ invertible matrices which contains the $n \times n$ identity matrix and is closed in multiplication forms a group. Some important matrix groups are the general linear group, $\mathcal{GL}(n)$(the group containing all the $n \times n$ invertible matrices), the unitary group, $\mathcal{U}(n)$ (the group containing all the unitary matrices), the special unitary group, $\mathcal{SU}(n)$ (containing all the unitary matrices with determinant 1) and the orthogonal group,

[1] From these conditions we can also derive that: i) $e^{-1} = e$; ii) $a^{-1} \cdot a = e$.

[2] From now on, we will omit the $\cdot$ symbol when dealing with group multiplication, unless the opposite is said.

$\mathcal{O}(n)$ (real matrices where $OO^T = I$).

2.2.3 Symmetric Group and Invariant Subgroups

A direct consequence of the existence of the inverse element of each group element is the rearrangement lemma, which says, if $p,\ b,\ c,\ \in\ \mathcal{G}$ and $pb = pc$, then $b = c$, i.e., if b and c are distinct elements of a group, then pc and pb are also distinct. Therefore, if all the elements of a group are ordered in a sequence and we multiply all of them by an arbitrary element, said p, the result of such operation is a rearrangement of the original sequence.

Considering a finite, n-order, group with elements denoted by $\{g_1,\ g_2,\ ...,\ g_n\}$, if we multiply each of these elements by a fixed element, said h, we have $\{hg_1,\ hg_2,\ ...,\ hg_n\} = \{g_{h1},\ g_{h2},\ ...,\ g_{hn}\}$, where the sequence $\{h1,\ h2,\ ...,\ hn\}$ is just a permutation of the original sequence, $\{1,\ 2,\ ...,\ n\}$, determined by the element h.

Therefore, a relation between any element of a group (h) and a permutation of the group elements ($\{h1,\ h2,\ ...,\ hn\}$). The set of $n!$ permutations of n elements forms a group, called the Symmetric Group or Permutation Group, S_n. The group product for this group is given by successive permutations. The identity element is leaving the sequence unchanged. The inverse element of an arbitrary permutation p is given by the permutation from p to the identity element.

2.2.4 Isomorphisms

Two groups, $\mathcal{G}$ and $\mathcal{G}'$, are said to be isomorphic to each other if there is a one to one correpondence between group elements preserving the multiplication law of the groups, i.e., if $g_i\ \in\ \mathcal{G} \leftrightarrow\ g'_i$, and $g_1 g_2 = g_3$ in $\mathcal{G}$, then $g'_1 g'_2 = g'_3$ and *vice versa*.

Cayley's theorem[19] states that every group $\mathcal{G}$ of order n is isomorphic to a subgroup of S_n. Using Lagrange's theorem for groups, that states that if $\mathcal{G}$ is a finite group, the order of its subgroups is divisible by the order of $\mathcal{G}$. Therefore, if $\mathcal{G}$ is a group with order n and n is a prime number, the only possible subgroups are the identity element and $\mathcal{G}$ itself. Thus, if the order of a finite group is a prime number n, this group is isomorphic to C_n.

2.2.5 Classes and Invariant Subgroups

A group element, $b \ \in \mathcal{G}$ is said to be conjugated to other group element, $a \ \in \mathcal{G}$ if there is other group element $p \ \in \mathcal{G}$ such that $b = pap^{-1}$. The notation for conjugation is $b \sim a$. Some equivalence relations follows from the conjugation conceptc (i) Identity: each element is conjugated to itself; (ii) Simmetry: if $b \sim a$, $a \sim b$; and (iii) Transitivity: if $b \sim a$ and $b \sim c$, then $c \sim a$.

Group elements conjugated to each other are said to form a conjugation class. Each element belongs to one and only one conjugation class (a direct consequence of the transitivity relation of conjugation). The identity element forms a class by itself. A general result for symmetric groups is that permutations with the same cyclic structure[3] belongs in the same class.

If $\mathcal{H}$ is a subgroup of $\mathcal{G}$ and $a \ \in \ \mathcal{G}$, then $\mathcal{H}' = \{aha^{-1}; h \in H\}$ also forms a subgroup of $\mathcal{G}$ and $\mathcal{H}'$ is said to be a conjugated subgroup to $\mathcal{H}$. $\mathcal{H}$ and $\mathcal{H}'$ have the same amount of elements and both groups are isomorphic to each other or the only element in common is the identity.

If a subgroup of $\mathcal{G}$ is identical to every conjugated subgroup of $\mathcal{G}$, this subgroup is called invariant or normal subgroup. A requirement to invariant subgroups is that the subgroup must contain all elements of the classes on it, e.g., for the symmetrical group S_3, the subgroup $e,\ (123),\ (321)$ forms an invariant subgroup because it contains the identity and the 3-cycle class. Every group $\mathcal{G}$ has at least two invariant subgroups, e and $\mathcal{G}$ itself.

Every subgroup of an Abelian group is an invariant subgroup, because, for $a,\ b\ in\ \mathcal{G}$ if $\mathcal{G}$ is Abelian, $ab = ba$. Equation-wise, the requirement for a subgroup to be invariant is, if $\mathcal{H}$ is a invariant subgroup of $\mathcal{G}$, then $\mathcal{H} = g\mathcal{H}g^{-1},\ \forall g \ \in \ \mathcal{G}$, therefore, for Abelian groups, the requirement always holds.

When the only invariant subgroups that a group $\mathcal{G}$ has are the trivial subgroups, it is said that $\mathcal{G}$ is a simple group. If $\mathcal{G}$ does not have Abelian invariant subgroups, it is said

[3]Cyclic representation is a more compact form of representing permutations. Consider the following permutation: (1,2,3,4,5,6)→ (3,5,4,1,2,6). 1 gives place to 3, 3 in its turn gives place to 4, which gives place to 1, therefore is said that these three elements constitute a 3-cycle. Similarly, 2 and 5 forms a 2-cycle and 6 forms a 1-cycle. This permutation can be expressed as (134)(25)(6) in cycle notation.

to be a semisimple subgroup.

2.2.6 Cosets and Homomorphisms

A left coset is a set of elements $p\mathcal{H}$ where $\mathcal{H}$ is a nontrivial subgroup of $\mathcal{G}$ and $p \in \mathcal{G}$ but is not in $\mathcal{H}$. Correspondingly, the set $\mathcal{H}p$ is the right coset of $\mathcal{H}$. Cosets do not form groups, because the identity element is not present if $p \notin \mathcal{H}$. If $p \in \mathcal{H}$ the new set is just a new rearrangement of $\mathcal{H}$. Either two cosets of a subgroup absolutely coincide or they do not have any elements in common.

When $\mathcal{H}$ is an invariant subgroup of $\mathcal{G}$, the cosets with multiplication law $p\mathcal{H} \cdot q\mathcal{H} = (pq)\mathcal{H}$ form a group called quotient group, denoted by $\mathcal{G}/\mathcal{H}$ with order $n_{\mathcal{G}}/n_{\mathcal{H}}$.

A homomorphism is a mapping from one group, $\mathcal{G}$, to another, $\mathcal{G}'$, where the product law is conserved, i.e, if $\mathcal{G}$ and $\mathcal{G}'$ have a homomorphism, if $g_i \in \mathcal{G}$, then exists $g_i' \in \mathcal{G}'$ and if $g_1 g_2 = g_3$ in $\mathcal{G}$, then $g_1' g_2' = g_3'$ in $\mathcal{G}'$. For a homomorphism , the mapping between the groups does not need to be one-to-one (i.e., if the mapping is bijective), if it is, then the homomorphism is also an isomorphism.

If $f : \mathcal{G} \rightarrow \mathcal{G}'$ is a homomorphism from $\mathcal{G}$ to $\mathcal{G}'$ and $\mathcal{K}$ is the set of elements on $\mathcal{G}$ that are mapped into the identity of $\mathcal{G}'$ on this homomorphism, then $\mathcal{K}$, called kernel of $\mathcal{G}$ or $ker(\mathcal{G})$, form an invariant subgroup of $\mathcal{G}$ and if the mapping covers all elements of $\mathcal{G}'$, i.e., the mapping is surjective, the quotient group $\mathcal{G}/\mathcal{K}$ is isomorphic to $\mathcal{G}'$ and $\mathcal{G}$ is called covering of $\mathcal{G}'$.

2.3 Group Representations

2.3.1 Automorphism group

In order to define group representation, the notion of automorphism will be introduced first. Given a homomorphism f of a group $\mathcal{G}$ into $\mathcal{G}'$, $f : \mathcal{G} \rightarrow \mathcal{G}'$. If f is bijective, then f is an isomorphism, and if it is so, the inverse of f is also an isomorphism[12]. Then, if the domain and the set of destination (codomain) are the same, it is said that f is an automorphism. That is to say, an automorphism is an isomorphism of a group on itself. The automorphism of a group $\mathcal{G}$ is commonly denoted by $Aut(\mathcal{G}$. An automorphism of a group permutes elements of the given group.

The set of all automorphisms of a mathematical object, e.g. a group $\mathcal{G}$, forms a group called the automorphisms group of $\mathcal{G}$. The group composition law is the composition operation, $\circ$.

To prove the statement above, the main properties of a group will be shown to the set of all automorphisms of a given group $\mathcal{G}$:

→ Given two automorphisms, f and g, the composition $f \circ g$ is also an automorphism: if f and g are automorphisms, they permute the elements of $\mathcal{G}$, and the combination of two permutations is also a permutation (2.2.3). More formally, since f and g are bijective, $g \circ f$ is also bijective,

$$\begin{aligned}(g \circ f)(\phi\theta) &= g(f(\phi\theta)) \\ &= g(f(\phi)f(\theta)) \\ &= g(f(\phi))g(f(\theta)) \\ &= (g \circ f)(\phi)(g \circ f)(\theta)\end{aligned}$$

for any ϕ and $\theta \in \mathcal{G}$;

→ $\circ$ is associative, i.e. $(g \circ f) \circ h = g \circ (f \circ h)$,

$$\begin{aligned}(g \circ f) \circ h(\phi\theta) &= (g \circ f)(h(\phi\theta)) \\ &= g(f(h(\phi\theta))) \\ &= g \circ (f \circ h)(\phi\theta)\end{aligned}$$

for any ϕ and $\theta \in \mathcal{G}$;

→ The identity element is the mapping $\mathcal{G} \mapsto \mathcal{G} : a \mapsto a, \forall\, a \in \mathcal{G}$.

→ There is an inverse element for each automorphism $f \in Aut(\mathcal{G})$ which is also a morphism. If f^{-1} is the inverse of f, then $f \circ f^{-1} = \mathbb{I} = f^{-1} \circ f$. If $\theta, \phi \in \mathcal{G}$, then there is a unique $\alpha, \beta \in \mathcal{G}$ such that $f(\theta) = \alpha$ and $f(\phi) = \beta$, so

$$\begin{aligned} f^{-1}(\alpha\beta) &= f^{-1}(f(\theta)f(\phi)) \\ &= f^{-1}(f(\theta\phi)) \\ &= \theta\phi \end{aligned}$$

and

$$\begin{aligned} f^{-1}(\alpha)f^{-1}(\beta) &= f^{-1}(f(\theta))f^{-1}(f(\phi)) \\ &= \theta\phi. \end{aligned}$$

Therefore, $f^{-1}(\alpha\beta) = f^{-1}(\alpha)f^{-1}(\beta)$.

2.3.2 Representations of groups

The representation of a group $\mathcal{G}$ is picturing $\mathcal{G}$ elements by a subgroup of the automorphism group of $\mathcal{G}$. The complex representation of $\mathcal{G}$ is the homomorphism $f : \mathcal{G} \to \mathcal{GL}(V)$ where V is a vector space over $\mathbb{C}$.

For an arbitrary group $\mathcal{G}$ and any vector space of $\mathbb{C}$ there is a representation $f : \mathcal{G} \to \mathcal{GL}(V)$ where $f(g) = \mathbb{I}_V$ for $\forall g \in \mathcal{G}$ called trivial representation of $\mathcal{G}$.

If the homomorphism is also an isomorphism, i.e., $Ker(f) = \mathbb{I}_{\mathcal{G}}$, the representation is said to be faithful if the representation is not faithful, it is said to be a degenerate representation.

Linear operators, or linear transformations, are multiplication associative but not necessarily commutative. If there is a set of invertible linear transformations closed in respect of multiplication and if this set satisfies the group formation axioms, the linear transformation group is formed.

2.3.3 Reducible and Irreducible Representations

Being $U(\mathcal{G})$ a representation of the group $\mathcal{G}$ on the space V and being V_1 a subspace of V such that

$$U(g)|x\rangle \in V_1, \; \forall\, x \in V_1, \; \forall\, g \in \mathcal{G}. \tag{1}$$

It is said that V_1 is an invariant subspace of V in respect to $\mathcal{G}$. A representation $U(\mathcal{G})$ in V of $\mathcal{G}$ is said to be irreducible if there is not a invariant subspace in V in relation to $U(\mathcal{G})$. If the opposite is true, the said representation is said to be reducible.

For a reducible representation, if the orthogonal complement of the invariant subspace is also an invariant subspace in respect to $U(\mathcal{G})$, it is said that the representation is completely reducible.

Unitary Representation

A unitary representation of a group $\mathcal{G}$ is a homomorphism of $\mathbb{G}$, say f, where $f : \mathcal{G} \to U(V)$ is inside the set of automorphisms of $\mathcal{G}$, $aut(\mathcal{G})$ for a set of unitary operators where $U^\dagger = U^{-1}$[20].

When a space cannot be represented as the union of two or more disjoint, non-empty open subsets, the space is said to be connected. The identity component of a group $\mathcal{G}$ is the largest connected subgroup of $\mathcal{G}$ that contains the identity element.

2.4 Lie Groups

Given a continuous (also called topological) group, $\mathcal{G}$, one can parametrize its elements, g, by a set of continuous variables[20], λ,

$$g = g(\lambda), \; (\lambda = \lambda^a, \; a = 1, 2, ...). \tag{2}$$

When a group forms a differentiable manifold, it is called a Lie group. In other words, a Lie group $\mathcal{G}(\mathbb{A}, f)$ is a infinite group whose elements can be parametrized smoothly and analytically [10], where $\mathbb{A}$ is a continuous set and $f : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is a smooth map.

For a Lie group, the continuous parameters form a continuous map, that is, the set of parameters that forms the group correspond to the points of a continuous space. Lie groups use a geometrical point of view of groups.

Taking the product of two elements of a group,

$$g(\lambda^1)g(\lambda^2) = g(\lambda^3), \tag{3}$$

the product parameter is a function of the parameters of the first two elements, that is,

$$\lambda^3 = f(\lambda^1, \lambda^2). \tag{4}$$

Taking the inverse of g, g^{-1}, we have

$$gg^{-1} = e, \tag{5}$$

then,

$$\lambda = f(\lambda^{-1}). \tag{6}$$

If the elements of $\mathcal{G}$ form a topological space, that is, a geometrical space which accepts closeness, and the functions of the λ parameters are continuous, $\mathcal{G}$ is a topological group. In a Lie group, the elements of $\mathcal{G}$ form a manifold, the functions of the λ parameters are analytic functions and the λ parameters are called Lie group coordinates.

Lie groups are an important tool because they represent well continuous symmetry[21], which is the kernel of the quantum field theory and has an essential role in many physical and mathematical fields.

Connectivity

A set is said to be connected if there exists a smooth path from any point of the given set to any other point of the set. It is said to be simply connected if all loops on the set can be contracted smoothly to a point. If the set in question is not connected, we say that the set is disconnected. Figure 2.1 illustrates the concepts of simply-connected, connected, and disconnected sets.

The component of a Lie group element g is the set of all the connected subgroups of $\mathcal{G}$ which contain g. A relevant component of a group is the identity component, which is the set containing all the connected subgroups that include the identity element of the group. If the entire group is not connected, the connected component of the identity forms an invariant subgroup.

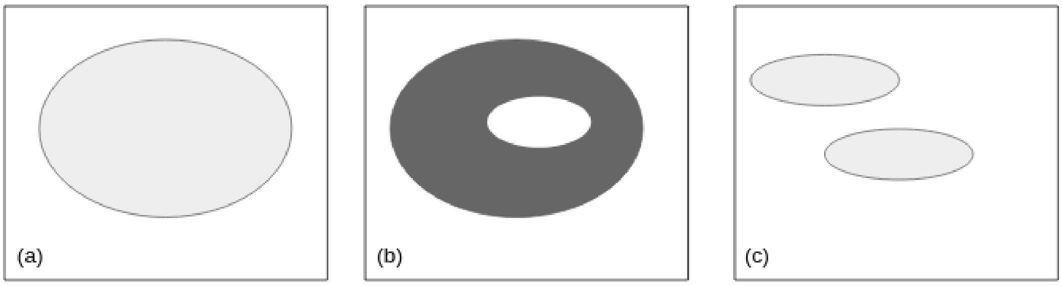

Figure 2.1: Connectivity of three sets: (a) Simply connected set; (b) Connected set; and (c) Disconnected set.

Homotopy Classes

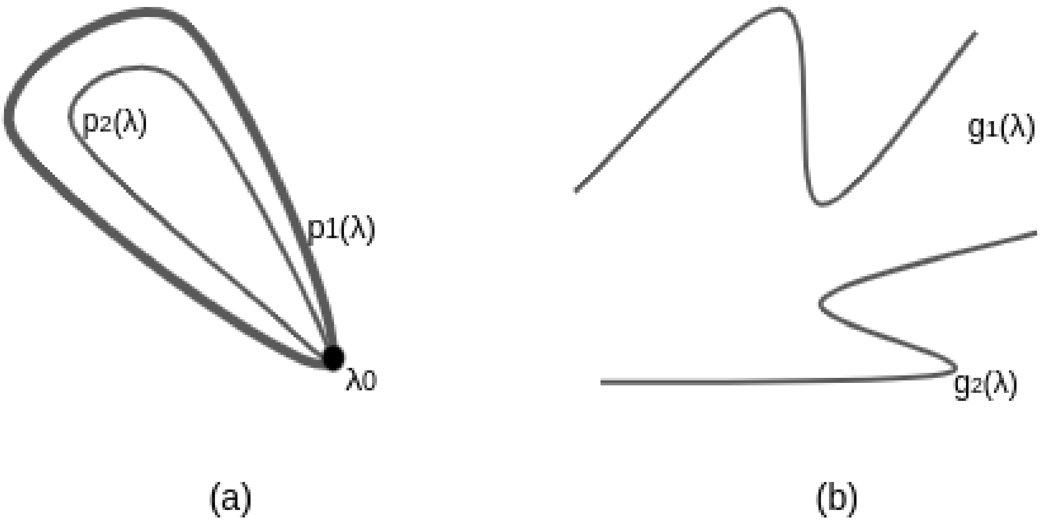

Figure 2.2: (a)Two closed paths on the Lie group, $p_1(\lambda)$ and $p_2(\lambda)$ with same end (and starting) point, p_0. If this paths can be continuously deformed to take the same shape of the other, they are said to be on the same homotopy class. (b) Two open curves on the Lie group, $g_1(\lambda)$ and $g_2(\lambda)$, if the said curves can be continuously deformed into one another, they are in the same homotopy class.

Given two closed paths in a Lie Group, $p_1(\lambda)$ and $p_2(\lambda)$ with the same point of end, p_0. This paths can be represented by continuous maps in $\mathcal{G}$ such that λ takes value in $[0,1]$ and $p(0) = p(1) = p_0$. If these two paths can be continuously deformed into one another, they belong in the same homotopy class. The Figure (5.1) illustrates the notion of homotopy classes for closed paths and open curves.

When two paths are homotopic, their inverse paths are also homotopic (because they are the same path but in inverse direction), the composition between two paths also contains homotopy [22].

The set of homotopy classes around a point of end p_0 respects the conditions to form a group, with the composition as the group operation. The identity component of the homotopy group is the path $p(\lambda) = p_0$, called the trivial path. In a connected space, the sets of homotopies around different points of end are isomorphic. Therefore, if the space is connected, exists a homotopy group that is composed of all the homotopy classes on the given space.

Universal Covering

When the homotopy group of a space (in this case, the Lie group) is a trivial group, i.e. all the paths on the group can be reduced to the trivial path, the space is simply connected. Dealing with simply connected spaces is less laborious than dealing with connected ones. In order to a better comprehension of a connected group (space), $\mathcal{G}$, it is possible to create a group that is simply connected and locally topological isomorphic to $\mathcal{G}$, called the universal covering of $\mathcal{G}$, denoted by $\tilde{\mathcal{G}}$.

The universal cover of a group $\mathcal{G}$ is a group $\tilde{\mathcal{G}}$ which has a mapping, said p, that for every element x on the group $\tilde{\mathcal{G}}$ there is a vicinity V_x such that $V_x \mapsto p(V_x)$ and is a homomorphism. This map and its inverse are both continuous.

2.4.1 Lie Algebra

A Lie algebra is composed of a Lie group $\mathcal{G}$ mapped in a vector space and an operation called Lie product, $[\ ;\ ]$, which is an alternating bilinear map. That is, for any g_i and g_j in $\mathcal{G}$, the operation $[g_i, g_j]$ is also a group element and if $i = j$, the result is zero. A Lie algebra also satisfies the Jacobi identity, that is

$$[g_i, [g_j, g_k]] + [g_j, [g_k, g_i]] + [g_k, [g_i, g_j]] = 0. \tag{7}$$

From this identity, it is possible to show that

$$[g_i, g_j] = -[g_j, g_i]. \tag{8}$$

This product is also bilinear, which is, for a and b scalars,

$$[ag_i + bg_j, g_k] = a[g_i, g_k] + b[g_j, g_k], \tag{9}$$

and

$$[g_k, ag_i + bg_j] = a[g_k, g_i] + b[g_k, g_j]. \tag{10}$$

Tangent space of a Lie Group

Manifolds are locally $\mathbb{R}^n$ spaces. Using the differentiable structure of the manifolds, one can approximate the neighborhood of a given point in a Lie group by a tangent Euclidean space to the point in question.

Considering a Lie group $\mathcal{G}$ with a one-parameter subgroup, said $g(\lambda)$. $g(\lambda)$ is a continuous and differentiable curve passing through the identity element of $\mathcal{G}$, with $g(0) = e$[4] (and $\lambda \in \mathbb{R}$). On the vicinity of e, we have

$$g(\lambda_1)g(\lambda_2) = g(\lambda_1 + \lambda_2), \tag{11}$$

and

$$g^{-1}(\lambda) = g(-\lambda). \tag{12}$$

Given the condition on (11), locally this subgroup is isomorphic to $\mathbb{R}$. Then we can write, for a infinitesimal increment $\delta\lambda$,

$$g(\delta\lambda) = 1 + \delta\lambda X + \mathcal{O}(2) \tag{13}$$

[4]Through a normalization, curve can always have $g(0) = e$.

Where X is the definition of a vector in the tangent space,

$$X = \left. \frac{dg(\lambda)}{d\lambda} \right|_{\lambda=0}. \tag{14}$$

This is the tangent vector to the curve $g(\lambda)$ on e. The set of all tangent vectors of all curves passing through e forms the tangent space of $\mathcal{G}$, denoted by $Te\mathcal{G}$[20].

The $Te\mathcal{G}$, usually called $\mathfrak{g}$ has important features for studying the Lie group that generates it. It has a lie algebra structure, $dim\ \mathcal{G} = dim\ \mathfrak{g}$. The base vectors of $\mathfrak{g}$ given by

$$X_a = \left. \frac{\partial g}{\partial \lambda} \right|_{\lambda^a=0} \tag{15}$$

are called generators and a combination of generators can describe every group element.

Invariant Measure

Recalling the rearrangement lemma for finite groups, when performing a summation for all elements in a finite group $\mathcal{G}$ in the form

$$\sum_{g' \in \mathcal{G}} f(g'g). \tag{16}$$

It is the same thing as writing

$$\sum_{g \in \mathcal{G}} f(g). \tag{17}$$

This result also works for the gg' product and g^{-1}. This result implies in an invariance. It is possible to construct invariances for continuous groups as well, introducing an integration measure that is also invariant for inversion, $g'g$ and gg'. This can be written as

$$d\mu(g) = d\mu(g'g) = d\mu(gg') = d\mu(g^{-1}) \tag{18}$$

and the invariance of the continuous function $f(g)$ is measured with

$$\int f(g) d\mu(g). \tag{19}$$

For compact[5] groups such invariance exists and is called Haar measure [23].

2.5 Conclusion

This chapter provided some group theory notions that will be necessary for the upcoming discussions on the next chapters. The next chapter brings the theory of special relativity into the optics of group theory, that is, the symmetrical approach to special relativity will be presented and discussed.

[5] A compact group is a continuous group in which the domain of the group parameter (on this dissertation, λ) is compact.

3 A Group Theory Approach to Special Relativity

3.1 Introduction

The group containing all the Lorentz transformations on Minkowski Spacetime is called the Lorentz Group. This group is a subgroup of the Poincaré group, which contains all isometries of Minkowski Spacetime. Isometry is a mapping of a metric space onto another space (or onto itself) that conserves distance.

In this chapter, the Poincaré group, its properties, and its algebra will be discussed in order to enable the discussions in Chapters 4 and 5.

3.2 Lorentz Group

3.2.1 Lorentz Transformations

When there are two inertial frames of reference, that is, reference frames with a constant relative velocity between each other, one can determine a set of linear transformations between the coordinates of one frame, said X to the other, said X'. This set of linear transformations is the so-called Lorentz transformations. Each transformation is composed of six components that depend on the velocity between the frames and their spatial and temporal components.

The Lorentz transformations can be, as any linear transformation, represented by matrices, and we will denote such matrices as Λ. Therefore, for two inertial reference frames, with components $\vec{X}$ and $\vec{X}'$, we can relate their coordinates by

$$\vec{X}' = \lambda \vec{X}, \tag{1}$$

where Λ is the Lorentz transformation between the frames X and X'.

Lorentz transformation matrix Λ^{μ}_{λ} which satisfies $\Lambda^{\mu}_{\lambda}\Lambda^{\nu}_{\rho}g_{\mu\nu} = g_{\lambda\rho}$

3.2.2 Lorentz Transformations form a group

In this subsection will be shown that the set of all Lorentz transformations forms a group with respect to the usual matrix product.

→ If $\Lambda_1 = \Lambda^{\mu}_{1\nu}$ and $\Lambda_2 = \Lambda^{\nu}_{2\rho}$ are arbitrary Lorentz transformations and $\Lambda^{\mu}_{3\rho} = \Lambda^{\mu}_{1\nu}\Lambda^{\nu}_{2\rho}$, then it yields

$$\begin{aligned}
\Lambda^{\mu}_{3\lambda}\Lambda^{\nu}_{3\rho}g_{\mu\nu} &= \Lambda^{\mu}_{1\alpha}\Lambda^{\alpha}_{2\lambda}\Lambda^{\nu}_{1\beta}\Lambda^{\beta}_{2\rho}g_{\mu\nu} \\
&= \left(\Lambda^{\mu}_{1\alpha}\Lambda^{n}_{1\beta}ug_{\mu\nu}\right)\Lambda^{\alpha}_{2\lambda}\Lambda^{\beta}_{2\rho} \\
&= g_{\alpha\beta}\Lambda^{\alpha}_{2\lambda}\Lambda^{\beta}_{2\rho} \\
&= g_{\rho},
\end{aligned}$$

therefore, Λ_3 is also a Lorentz Transformation.

→ Associativity is straightfoward from the matrix group, $\Lambda_1(\Lambda_2\Lambda 3) = (\Lambda_1\Lambda_2)\Lambda_3$.

→ The identity group element is defined as $e \equiv g^{\mu}_{\nu}$ and it is also a element of the group, following the condition on 3.2.1:

$$g^{\mu}_{\lambda}g^{\nu}_{\rho}g_{\mu\nu} = g_{\lambda\rho},$$

and for an arbitrary group element Λ,

$$g^{\mu}_{\nu}\Lambda^{\nu}_{\rho} = \Lambda^{\mu}_{\nu}g^{\nu}_{\rho} = \Lambda^{\mu}_{\rho}.$$

→ Given an arbitrary Lorentz Transformation Λ, there is an inverse element Λ^{-1} defined as $\left(\Lambda^{-1}\right)^{\mu}_{\nu} = g_{\nu\alpha}\Lambda^{\alpha}_{\beta}g^{\beta\mu}$, and

$$\begin{aligned}
\left(\Lambda^{-1}\right)^{\mu}_{\nu}\Lambda^{\nu}_{\rho} &= g_{\nu\alpha}\Lambda^{\alpha}_{\beta}g^{\beta\mu}\Lambda^{\nu}_{\rho} \\
&= \left(\Lambda^{\nu}_{\rho}\Lambda^{\alpha}_{\beta}g_{\nu\alpha}\right)g^{\beta\mu} \\
&= g_{\rho\beta}g^{\rho\mu} \\
&= g^{\mu}_{\rho}
\end{aligned}$$

and Λ^{-1} is also a group element,

$$\begin{aligned}
\left(\Lambda^{-1}\right)^{\mu}_{\lambda}\left(\Lambda^{-1}\right)^{\nu}_{\rho}g_{\mu\nu} &= g_{\lambda\alpha}\Lambda^{\alpha}_{\beta}g^{\beta\mu}g_{\rho\gamma}\Lambda^{\gamma}_{\delta}g^{\delta\nu}g_{\mu\nu} \\
&= g_{\lambda\alpha}\Lambda^{\alpha}_{\beta}\Lambda^{\gamma}_{\delta}g_{\rho\gamma}\left(g^{\beta\mu}g^{\delta\nu}g_{\mu\nu}\right) \\
&= g_{\lambda\alpha}\Lambda^{\alpha}_{\beta}\left(g^{\rho\gamma}\Lambda_{d}elta^{\gamma}g^{\beta\delta}\right) \\
&= g_{\lambda\alpha}\left(\Lambda^{\alpha}_{\beta}\left(\Lambda^{-1}\right)^{\beta}_{\rho}\right) \\
&= g_{\lambda\rho}.
\end{aligned}$$

The Lorentz transformation group is not composed only by boosts, any 3-dimensional rotation in $\mathbb{R}^3$ keeps the Lorentz invariance, i.e., keeps the inner product outcome unaltered, therefore is also part of the Lorentz group. Although, pure 3-dimensional rotations (rotations only on the Latin indexes, 1, 2, and 3) are not boosts by definition, therefore, 3-D rotations are Lorentz transformations but not boosts. The set of boosts does not form a group because the composition of two consecutive boosts is not a boost, but a boost combined with a rotation. This kind of combination is called Wigner rotation.

Defining the transformations of time-reversal $\mathfrak{T}$ and space inversion $\mathfrak{P}$ (parity) by

$$T^{\mu}_{\nu} \equiv \mathfrak{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P^{\mu}_{\nu} \equiv \mathfrak{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

even though the parity transformation is numerically identical to the metric, $g_{\mu\nu}$, it is not the same, since $\mathfrak{P}$ is a transformation and $g_{\mu\nu}$ is a metric and the tensors have different ranks.

With successive infinitesimal transformations starting from the identity, it is possible to build any boosts and 3-dimensional rotations, but $\mathfrak{T}$ and $\mathfrak{P}$ cannot be constructed in the same way, i.e., $\mathfrak{T}$ and $\mathfrak{P}$ are not connected with the identity. A continuously connected Lorentz transformation to the identity is a boost, a 3-dimensional rotation, or a combination of both.

The Lorentz group can be summarized by four elements: boosts, 3-dimensional rotations, the $\mathfrak{P}$ transformation, and the $\mathfrak{T}$ transformation.

3.3 Restricted Lorentz Group $\mathcal{L}_+^\uparrow$

Lorentz group is the group containing all linear endomorphisms on $\Re^4$ preserving the quadratic form bellow

$$(t, x, y, z) \to t^2 - x^2 - y^2 - z^2, \quad \forall x, y, z, t \in \Re. \tag{2}$$

The Restricted Lorentz Group is the identity component of the Lorentz Group, i.e., the subgroup containing all the Lorentz transformations that can be connected to the identity element by a continuous curve inside the said subgroup.

The generators of the Restricted Lorentz Group are an ordinary spatial rotation and a Lorentz boost. A proper Lorentz transformation can be written as a product of a rotation and a boost, hence, it is needed 6 components to fully specify such transformation (3 spatial and 3 from the boost), which is the same amount of components needed to specify elements in Lorentz group.

The set of all rotations forms a Lie subgroup, isomorphic to the ordinary rotation group SO(3). On the other hand, the set of all boosts does not form a Lie subgroup because generally, the combination of two boosts does not result in another boost.

3.4 Poincaré Group

The addition of a spatial translation on an infinitesimal line element, $dx^2 = g_{\mu\nu} dx^\mu dx^\nu$ does not invalidate the invariance of the interval. Hence, in order to describe completely the interactions, a combination of Lorentz transformations and spatial translations, described by

$$x' = T(\Lambda, a)x = \Lambda x + a, \tag{3}$$

where, Λ is an arbitrary Lorentz transformation and a, an arbitrary spatial translation. The Lorentz Group is just a subgroup of the symmetry group for relativistic interactions.

The subgroup generated by the spatial translations and rotations is the Euclidean group, and it is also a symmetry subgroup for non-relativistic interactions.

The group containing rotations, boosts, and spatial translations is the Poincaré group, consisting of a semi-direct product between the Lorentz group and the translation group ($\mathcal{P} = \mathcal{T} \rtimes \mathcal{L}$), and will be denoted by $\mathcal{P}$.

Two consecutive Poincaré transformations result in another Poincaré transformation,

$$T(\Lambda', b)T(\Lambda, a) = T(\Lambda'\Lambda, b + \Lambda' a). \tag{4}$$

From the equation above, it is possible to determine the inverse element of a Poincaré transformation,

$$T^{-1}(\Lambda, a) = T(\Lambda^{-1}, -\Lambda^{-1}a), \tag{5}$$

because

$$\begin{aligned} T^{-1}(\Lambda, a)T(\Lambda, a) &= T(\Lambda^{-1}, -\Lambda^{-1}a)T(\Lambda, a) \\ &= T(\Lambda^{-1}\Lambda, -\Lambda^{-1}a + \Lambda^{-1}a) \\ &= T(\mathbb{I}, 0), \end{aligned} \tag{6}$$

which is the identity element for the Poincaré group.

3.5 Restricted Poincaré Group and Poincaré Algebra

The restricted Poincaré group is composed by the semi-direct product between the translation group and the restricted Lorentz group,

$$\mathcal{P}_+^\uparrow = \mathcal{T} \rtimes \mathcal{L}_+^\uparrow. \tag{7}$$

The universal cover of $\mathcal{P}_+^\uparrow$ uses the universal cover of $\mathcal{L}_+^\uparrow$, the special linear group of degree 2 over the $\mathbb{C}$ field, $SL(2, \mathbb{C})$, therefore

$$\tilde{\mathcal{P}}_+^\uparrow = \mathcal{T} \rtimes SL(2, \mathbb{C}). \tag{8}$$

The translational generators, P have a trivial commutation relation, that is, [24]

$$[P_\mu, P_\nu] = 0. \tag{9}$$

The commutation relation between two Lorentz group generators is

$$[\mathbb{M}_{\mu\nu}, \mathbb{M}_{\rho\sigma}] = -i(g_{\mu\rho}\mathbb{M}_{\nu\sigma} - g_{\mu\sigma}\mathbb{M}_{\nu\rho} - g_{\nu\rho}\mathbb{M}_{\mu\sigma} + g_{\nu\sigma}\mathbb{M}_{\mu\rho}) \tag{10}$$

The commutator between the generators of the Lorentz group and the translational group is

$$[\mathbb{M}_{\mu\nu}, P_\rho] = -i(g_{\mu\rho}P_\nu - g_{\nu\rho}P_\mu). \tag{11}$$

This three equations above constitutes the Poincaré algebra, a extension of the Lie algebra of the Lorentz group.

Casimir Operators of Poincaré Algebra

There are two Casimir operators for the Poincaré algebra[1], the $P^2 = P_\mu P^\mu$ operator and the squared Pauli-Ljubanski vector, $W^2 = W_\mu W^\mu$.

The Pauli-Ljubanski vector, also called Pauli-Ljubanski polarisation vector, is given by

[1] The number of Casimir operators for an algebra is equal to the rank of this algebra.

[24]

$$W_\mu := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu \mathbb{M}^{\rho\sigma}, \tag{12}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor[2].

Being Casimir operators, P^2 and W^2 have the following properties

$$[P_\mu, P^2] = 0, \tag{13}$$

$$[P_\mu, W^2] = 0, \tag{15}$$

and

$$[\mathbb{M}_{\mu\nu}, P^2] = 0, \tag{14}$$

$$[\mathbb{M}_{\mu\nu}, W^2] = 0. \tag{16}$$

From the definition of the Pauli Ljubanski vector, we have

$$W_\mu P^\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu (\mathbb{M}^{\rho\sigma} P^\mu), \tag{17}$$

where, the term in parenthesis can be written using Equation (11) as

$$\mathbb{M}^{\rho\sigma} P_\mu = P^\mu \mathbb{M}^{\rho\sigma} + [\mathbb{M}^{\rho\sigma}, P^\mu] \tag{18}$$

So Equation (17) becomes

$$W_\mu P^\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu P^\mu M^{\rho\sigma} + \frac{i}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu g^{\sigma\mu} P^\rho - \frac{i}{2}\epsilon_{\mu\nu\rho\sigma} P^\nu g^{\rho\mu} P^\sigma. \tag{19}$$

Raising the indexes changed by the metric and separating the first term in half and changing the dummy indexes, we obtain

$$W_\mu P^\mu = \frac{1}{4}\epsilon_{\mu\nu\rho\sigma} P^\nu P^\mu \mathbb{M}^{\rho\sigma} + \frac{1}{4}\epsilon_{\nu\mu\rho\sigma} P^\mu P^\nu \mathbb{M}^{\rho\sigma} + \frac{i}{2}\epsilon_{\mu\nu\rho}{}^{\mu} P^\nu P^\rho - \frac{i}{2}\epsilon_{\mu\nu}{}^{\mu}{}_{\sigma} P^\nu P^\sigma. \tag{20}$$

[2] $\epsilon_{\mu\nu\rho\sigma} = \begin{cases} +1, & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } (0,1,2,3) \\ -1, & \text{if } \mu\nu\rho\sigma \text{ is an odd permutation of } (0,1,2,3) \\ 0, & \text{otherwise.} \end{cases}$

The two last terms cancel each other, and $\epsilon_{\mu\nu\rho\sigma} = -\epsilon_{\nu\mu\rho\sigma}$, therefore

$$W_\mu P^\mu = \frac{1}{4}\sigma_{\mu\nu\rho\sigma}[P^\nu, P^\mu]\mathbb{M}^{\rho\sigma} \tag{21}$$

but, from (9),

$$W_\mu P^\mu = 0. \tag{22}$$

3.6 Irreducible Representations of Poincaré Group

The Schur's Lemma [10] states that in an irreducible representation of a Lie algebra, the Casimir operators are proportional to the identity element. This lemma allows the classification of different irreducible representations. In quantum field theory, these IRU represents the one-particle states.

For the Poincaré group, there are three classes of irreducible representations [24]:

Case i) The state $|m\rangle$ is labeled by a number m^2 for P^2

$$P^2|m\rangle = -m^2|m\rangle \tag{23}$$

and for W^2

$$W^2|m,s\rangle = -m^2 s(s+1)|m,s\rangle \tag{24}$$

Case ii) The states for P^2 and W^2 results in

$$P^2|m\rangle = 0|m\rangle \tag{25}$$

and

$$W^2|m\rangle = 0|m\rangle; \tag{26}$$

Case iii) We have

$$P^2|m\rangle = 0|m\rangle, \tag{27}$$

and

$$W^2|m\rangle = -\alpha^2|m\rangle. \tag{28}$$

The Poincaré group irreducible representations that have correspondence to experimentally observable physical states are the first and second cases above, the third case describes a massless particle with infinite states of polarization.

The first case represents a massive particle with mass m, because the P^μ operator transforms in the same way as the 4-momentum operator. The s label can only admit integer and half-integer values. Knowing that in the rest frame of a particle the 4-momentum operator is

$$P^\mu = (m, 0, 0, 0). \tag{29}$$

Using (22), in the rest frame we have that $W_0 = 0$ and thus we write

$$W_i = \frac{1}{2}\epsilon_{i0jk}P^0\mathbb{M}^{jk}. \tag{30}$$

Where

$$S^i = \frac{1}{2}\epsilon_{ijk}\mathbb{M}^{jk} \tag{31}$$

is the spin operator. Therefore

$$W_i = mS_i. \tag{32}$$

The second case, where P^2 and W^2 yield in 0, imply that P_μ and W_μ are linearly dependent,

$$W_\mu = \lambda P_\mu. \tag{33}$$

Where the constant factor λ is called helicity of the particle and is equal in absolute value to its spin. This case describes states for massless particles.

The third case describes massless particles with infinite states of polarisation. This kind of state does not seem observable in nature [24] [20].

3.7 Conclusion

The above-mentioned symmetry groups of special relativity will have a crucial role in the construction of the projective and unitary representation for relativistic particles, which is the main topic in the next chapter. This crucial role is due to such matrices taking account of the relativistic symmetry on quantum particles. For the next chapter will be considered massive particles (the first case on the discussion above about irreducible representations of the Poincaré group).

4 Projective and Unitary Representation of $\mathcal{P}_+^\uparrow$

4.1 Introduction

A density matrix is the most useful way to represent a mixed quantum state. It is a matrix capable of describing the quantum state of a system. When describing a system in which there is some sort of entanglement or a system where the preparation of such a system is not completely known, it is said that the matrices are representing mixed states. When there are some characteristics of a system that are not known, or impossible to measure, the partial trace of the system's density matrix is made. This operation is useful to describe attributes of a system that only depends on parts of it. Another use for the partial trace is reducing the dimension of a system.

When dealing with quantum relativistic particles, e.g., quantum particles moving at a relativistic speed, one must impose the symmetry groups of relativity on a quantum mechanical frame of work. Performing this imposition, the notion of a hierarchy of dynamical variables, which is present on relativity, comes up in quantum mechanics.

On this chapter, in order of provide a representation of such density matrices taking account the relativistic properties the formalism for accurately describing . The problem on this representation lies on the result of a process called Wigner rotation, that is a combination of a boost and a 3D rotation (or two non-colinear boosts) in which the resulting state depends non-directly on the spin and momentum in a non-separable way. The direct consequence of this kind of rotation is that the usual quantum entropy is not Lorentz invariant.

4.2 Construction of the $\tilde{\mathcal{P}}_+^\uparrow$ Irreducible Unitary Representations

4.2.1 Orbit through a 4-momentum

Reproducing the calculations on [16] and [20], the orbit of the restricted Lorentz group, $\mathcal{L}_+^\uparrow$, through a 4-momentum p is

$$O(p) = \{L(A)p \,|A \in SL(2,\mathbb{C})\} \tag{1}$$

where $L(A)$ is a homomorphism from $SL(2,\mathbb{C})$ to $\mathcal{L}_+^\uparrow$. All the 4-momenta inside $O(p)$ are equivalent. The orbit of a 4-momentum p can be interpreted as a hypersurface of constant squared momentum [25], p^2. For the different physical cases in (3.6) the geometry of the hypersurface changes,

→ *For* $p^2 = -m^2$

Represents the massive IUR, the hypersurface is a two sheet hyperboloid and each sheet is called mass-shell.

→ *For* $p^2 = 0$

Represent the massless IUR, the hypersurface generated is a light cone.

On the following calculations the mass-shell orbit and all the previous discussion about its operators will be considered.

4.2.2 Construction of the basis states for $\tilde{\mathcal{P}}_+^\uparrow$

For each irreducible unitary representation of the universal cover of the restricted Poincaré group, $\tilde{\mathcal{P}}_+^\uparrow$ we must have dependency on only one orbit, therefore, we can construct the basis states with a set $\{|p,\alpha\rangle\}$ of the 4-momentum operator such that

$$\langle p,\alpha|q,\beta\rangle = 2\omega_q\delta(p-q)\delta_{\alpha,\beta}. \tag{2}$$

Where α and β are the sets of secundary variables, yet to be specified and

$$\omega_q = q^0 = \sqrt{||q||^2 + m}. \tag{3}$$

If the 4-momentum q is such that $q \in O(p)$, the little group of q is defined by the subgroup $G_q \subset SL(2,\mathbb{C})$ such that $L(M_q)q = q,\ \forall\ M_q \in G_q$. From this definition, the irreducible unitary representations of $\tilde{\mathcal{P}}_+^\uparrow$ are written as

$$U(M_q)|q,\alpha\rangle = \sum_\beta Q^q_{\beta\alpha}(M_q)|q,\beta\rangle. \tag{4}$$

The little groups for elements on the same orbit, said p and q are isomorphic. This can be shown constructing a Lorentz transformation such that $L(A)q = p$. Knowing that exists a $M_q \in G_q$ such that $L(M_q)q = q$ (the definition of little group), then [20]

$$L(AM_qA^{-1})p = p, \tag{5}$$

and AM_qA^{-1} is an element of the little group of p. This relation holds for any M_q and the relation is conserved on multiplication of different M_q's,

$$M_p^1 M_p^2 = AM_q^1A^{-1}AM_q^2A^{-1} = AM_q^1M_q^2A^{-1}. \tag{6}$$

Where the $Q^q(M_q)$ matrices form irreducible unitary representations of the little group of q, given the group product rule

$$Q^q(M_q^1M_q^2) = Q^q(M_q^1)Q^q(M_q^2), \tag{7}$$

and the α indexes relate these representations to finite Hilbert spaces $\mathcal{H}_q$.

The entire Hilbert space is a fiber bundle[1] in which the base space is the space of

[1]Fiber bundle is a topological space that is locally (in its fibers) a product space but globally has other topological structure

square-integrable functions with respect to the measure

$$d\mu(p) = \frac{dp}{2\omega_p}, \tag{8}$$

and each fiber of the fiber bundle is a $\mathcal{H}_q$ space. It is also necessary to say that the $\mathcal{H}_q$ Hilbert spaces and the little groups to equivalent momenta are isomorphic.

It is required to construct a way to relate the α labels for all the 4-momenta in the orbit $O(p)$ in order to establish the infinite-dimensional set of irreducible unitary representations of $\tilde{\mathcal{P}}_+^\uparrow$.

In order to construct such relation, we choose a fundamental vector $k \in O(p)$ and a complementary set $\{C(p,k)\}$ such that as for any vector $q \in O(p)$, there is only one transformation $C(p,k) \in \{C(p,k)\}$ with $L(C(q,k))k = q$. When dealing with massive particles, the k vector is the 4-momentum of the rest frame $(m,0,0,0)$. Then we write the relation

$$\mathcal{G}_b(p)|0,\alpha_k\rangle = \sum_\beta (g_b)_{\beta\alpha}|0,\beta_k\rangle. \tag{9}$$

For the rest 4-momentum, $k = (m,0,0,0)$, $G_k = SU(2)$, therefore, the $J = (J_1, J_2, J_3)$ are the generators for the infinite-dimensional irreducible unitary representations of $\tilde{\mathcal{P}}_+^\uparrow$ associated with $SU(2)$ and g the generators of the finite-dimensional irreducible unitary representations of G_k.

It is a rational choice relate $|0,\alpha_k\rangle$ states in a form that the α label is related to the J_3 generator by

$$J_3|0,\alpha_k\rangle = \alpha|0,\alpha_k\rangle, \tag{10}$$

that is, α is the total angular momentum of the z direction for the particle at rest. Then we can relate the J_1 and J_2 generators by

$$(g_1)_{\beta\alpha} = \frac{1}{2}[\sqrt{s(s+1)-\alpha(\alpha+1)}\delta_{\beta,\alpha+1} + \sqrt{s(s+1)-\alpha(\alpha+1)}\delta_{\beta,\alpha-1}] \tag{11}$$

and

$$(g_2)_{\beta\alpha} = \frac{1}{2i}[\sqrt{s(s+1)-\alpha(\alpha+1)}\delta_{\beta,\alpha+1} - \sqrt{s(s+1)-\alpha(\alpha+1)}\delta_{\beta,\alpha-1}]. \tag{12}$$

And for α, β the J_3 is written as (given (10))

$$(g_3)_{\alpha\beta} = \beta\delta_{\beta,\alpha}. \tag{13}$$

Defined the complementary set, $\{C(p,k)\}$, the base states for the 4-momenta equivalent to k are such

$$|p, \alpha_p\rangle^C \equiv U[C(p,k)]|0, \alpha_k\rangle, \tag{14}$$

where $U[C(p,k)]$ is the unitary operator related to $C(p,k)$. The C index on the left part of Equation 14 is due to the fact that any arbitrary set which satisfies condition 4.2.2 is a valid complementary set.

The irreducible unitary representations of $\tilde{\mathcal{P}}_+^\uparrow$ can be fully defined by

$$U(A)|p, \alpha_p\rangle^C = \sum_\beta Q^k_{\alpha\beta}(M^C_k(A,p))|L(A)p, \beta_{L(A)p}\rangle^C, \tag{15}$$

where $A \in SL(2, \mathbb{C})$ and

$$M^c_k(A,p) = C^{-1}(L(A)p,k)AC(p,k). \tag{16}$$

The expression on (16) is called Generalized Wigner rotation. When a particle goes through a Wigner rotation, the final state is not a direct product of functions depending on spin and momentum. The spin and the momentum of the particle on the same particle appear to be in an entangled state (on means that the functions describing the state have a not direct dependence on spin and momentum just like a mixed state for a set of subsystems)[17].

4.2.3 Construction of the secondary variable observables for $\tilde{\mathcal{P}}_+^\uparrow$

The sets of secondary variables $\{\alpha,\ \beta,\ ...\}$ depend on each p and can be written as

$$\mathcal{G}_C(p)|p, \alpha_p\rangle^C = \sum_\beta (g)_{\beta\alpha}|p, \beta_p\rangle^C, \tag{17}$$

where the $\mathcal{G}_C(p)$ parameter relates the transformations on the little group G_p and the generators of $\tilde{\mathcal{P}}_+^\uparrow$. The α indexes depends on each p inside the orbit, different indexes can be related using the completeness relation for each orbit,

$$\sum_{\alpha}\int \frac{dp}{2\omega(p)}|p,\alpha_p\rangle\langle p,\alpha_p| = \mathbb{I} \tag{18}$$

we can then write

$$\mathcal{G}_b^C(p) = \sum_{\alpha}\int \frac{dp}{2\omega(p)}\mathcal{G}_b^C(p)|p,\alpha_p\rangle^{CC}\langle p,\alpha_p| \tag{19}$$

so that it is possible to write a relation between α_q and β_q for $\forall q \in O(p)$

$$\mathcal{G}_C|q,\alpha_q\rangle^C = \sum_{\beta}(g)_{\beta\alpha}|q,\beta_q\rangle^C, \tag{20}$$

and

$$U(C(p,k))J_iU^\dagger(C(p,k))|p,\alpha_p\rangle^C = \sum_{\beta}(g_i)_{\beta\alpha}|p,\beta_p\rangle^C. \tag{21}$$

4.3 Examples: Helicity and Spin Basis

Choosing different complementary sets and fundamental vectors results on a different basis for the system, that is, for each complementary set and fundamental vector chosen, the IUR is represented differently. Yet the fiber space $\mathcal{H}_q$ is the same.

In this section, two different complementary sets will be used for the same fundamental vector k, ($k = (m,0,0,0)$), which is the 4-momentum on the rest frame of the particle, as said in (4.2.2). For the Helicity basis, composed by a boost in the z direction with velocity ξ and a spatial rotation $R(\phi,\theta,\psi)$, the complementary set has the following form [10, 20, 26]

$$C(p,k) = R(\alpha,\beta,0)B_z(\xi), \tag{22}$$

where given a 4-vector p, a transformation like the one above will first boost the fundamental vector k from $k = (m,0,0,0)$ to $B_z(\xi)k = (\sqrt{\|p\| + m^2}, 0, 0, \|p\|)$ and then, the rotation completes the transformation taking $B_z(\xi)k$ to $p = (\sqrt{\|p\| + m^2}, p)$, where the spatial part of p is (p_x, p_y, p_z).

The secondary variables for the Helicity basis, as the basis name suggests, are related to the particle helicity, which is the projection of the total angular momentum onto the direction of the momentum. From Equation (10) and the definition of Helicity and [20], we define the operator $H_3(p)$ as

$$H_3(p) = \frac{J \cdot p}{\|p\|}, \tag{23}$$

where

$$H_3(p)|p, \alpha_p\rangle = \alpha|p, \alpha_p\rangle. \tag{24}$$

The spin basis is the resulting base of the complementary set composed only by pure boosts, where the boosts $B(p, k)$ are defined by [20]

$$B(p, k) = \left(\frac{p^0 + m}{2m}\right)^{\frac{1}{2}} \sigma_0 + \left(\frac{p^0 - m}{2m}\right)^{\frac{1}{2}} \frac{\sigma \cdot p}{\|p\|}. \tag{25}$$

The $\vec{S}(p)$ operator that comes on the Spin basis in a similar way to the $H_3(p)$ on the Helicity basis is [20]

$$m\vec{S}(p) = Jp^0 - \frac{(J \cdot p)}{m + p^0} p + (K \times p), \tag{26}$$

where K is a boost on the $\hat{e}$ direction.

Then we have the eigenvalue relation for S_3,

$$S_3|p, \alpha_p\rangle = \alpha|p, \alpha_p\rangle. \tag{27}$$

The interpretation of the physical meaning of the observables on this representation is not straightforward, the $\vec{S}$ operator for the Spin basis represents the spin of the particle on the rest frame. This result is discussed on [10, 20, 27].

4.4 Conclusion

The formalism explained in this chapter will be the basis to answer the question that will permit quantum state tomography for relativistic particles. The main next goal is to find if there is a complementary set $C(k, p)$ that relates the generators of the effective matrices to a subgroup of generators of rotations in the Stern Gerlach apparatus in the real world that does not depend on momentum.

It has been shown in [14, 17, 28], that the quantization axis for spin measurement changes from observer to observer. Therefore, determining such relation between the rotations on the measuring apparatus and the quantization axis in a measurable manner is not a trivial task. The complementary set in question may even not exist. The determination of the existence and, if it exists, is the next objective of this study.

The next and last chapter is an intuitive derivation of the spin operator for the Stern-Gerlach experiment and brings up other relevant discussions on the executability of the reconstruction of quantum relativistic states via quantum state tomography.

5 Relativistic Stern-Gerlach Experiment

5.1 Introduction

The Stern-Gerlach experiment is an apparatus that is capable of determining the spin state for non-relativistic particles. In this chapter, the relativistic Stern-Gerlach observable is obtained from the relativistic transformation properties of the magnetic field that constitutes the experiment.

When there is a possibility of performing a various number of Stern-Gerlach measurements in quantum systems described by the same density matrix, one can, statistically, determine a density matrix that successfully describes the system in question [29]. For a non-relativistic system, it is enough for determining the density matrix, three linearly independent measurements. When considering a relativistic system, the minimal number of measurements in order to determine the density matrix is not known.

5.2 The non-relativistic Stern-Gerlach experiment

The Stern-Gerlach experiment was first idealized by Otto Stern in 1921 and executed in 1922 by Walther Gerlach [1]. The main result of this experiment is the spatial quantization of the electron intrinsic magnetic moment, unraveling the discreet nature of microscopic interactions. [30].

The original experiment consists of a beam of silver atoms submitted to a spatially inhomogeneous magnetic field perpendicular to the beam trajectory which deflected the atoms. The silver atomic number is 47, which means it has 46 paired electrons and one 5s electron unpaired. The s orbital has zero orbital angular momentum, therefore, the only interaction between the electron and the magnetic field is attributable to spin. Essentially, the silver atom is a massive neutral object with a total magnetic moment equal to the intrinsic magnetic moment of one single electron.

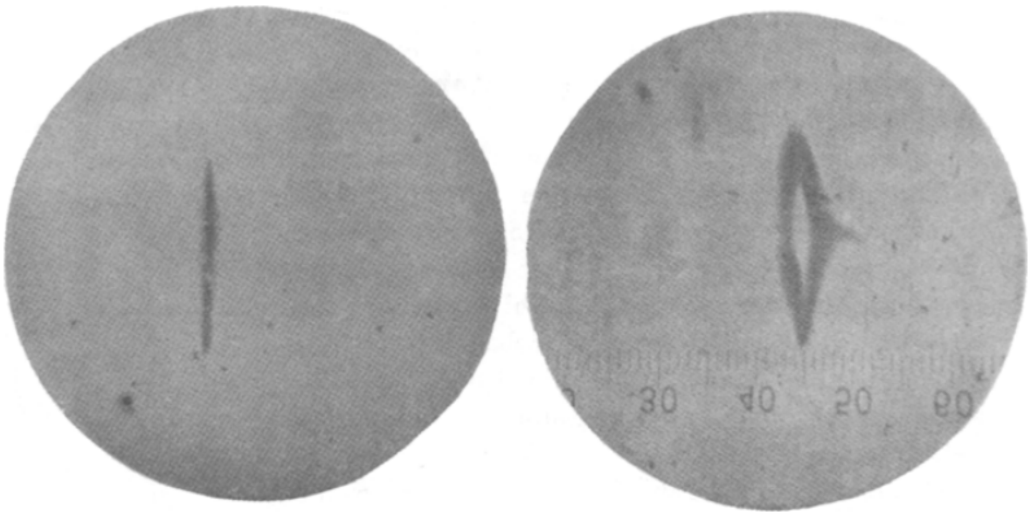

Fig. 2. **Fig. 3.**

Figure 5.1: This figure is from the original paper [1] and shows on the first circle the beam of silver atoms unaltered and the other circle is the result of the application of the magnetic field to the beam. This elongated shape is due to the wideness of the beam of silver atoms being greater than the wideness of the magnetic field.

A detector installed far from the deflection region registered the position of each atom. The classical theory [31] predicted that the magnetic moment of the silver atoms should be randomly oriented, in other words, the magnetic moment could assume any value from $-\mu_B$ to μ_B and the expected result on the detector would be a Gaussian distribution centered in 0. However the experiment yielded two discreet points of accumulation on the detector due to the discretization of the spin.

For a non-relativistic measurement, the measurement data obtained in a Stern-Gerlach measurement is independent of the momenta. Therefore, it is enough to perform three measurements in linearly independent directions in order to reconstruct the density matrix for the system studied, in a procedure called quantum state tomography.

5.3 Mathematical description of spin qubits

In order to describe the relativistic representation of spin, we must define a formalism that deals well with Lorentz group transformations and with the mathematical nature of quantum mechanics. This formalism must be Lorentz covariant, in other words, one must be capable of determining the consequence of a change from one inertial reference frame to another on the interactions described by such formalism. Furthermore, the formalism must obey the postulates of quantum mechanics in order to provide a proper portrayal of quantum interactions.

Generally, one can use Wigner representations [32] to describe the relativistic behavior of spin. Wigner representations are infinite-dimensional unitary representations of the Poincaré group. This group in question has an added symmetry of translational invariation therefore Wigner states are labeled with spin and momentum, $|p, \sigma\rangle$. The momentum transforms under the Lorentz group, i.e. $p^\alpha \to \Lambda^\alpha_\beta p^\beta$.

However, there is another representation called $SL(2, \mathbb{C})$ spinor. A special linear group, $SL(n, F)$, of degree n over a field F is the group of $n \times n$ invertible unitary matrices equipped with the group operations of ordinary matrix multiplication and inversion [33]. A $SL(2, \mathbb{C})$ spinor is a two-component complex vector in a two-dimensional complex vector space W, ψ_A, which transforms covariantly under the spin $1/2$ representation of the Lorentz group. In contrast to the Wigner rotations, in which the transformation law is a spatial rotation regarding a specific reference frame, the transformation law for $SL(2, \mathbb{C})$ spinors is frame independent.

The spinors here treated are irreducible representations of $SL(2, \mathbb{C})$. This irreducible representation forms the double cover[1] of the restricted Lorentz group, $SO^+(1, 3)$. The complex conjugation operation takes an spinor $\psi_A \in W$ to an spinor $\overline{\psi_A} = \psi_{A'}$ in the conjugate space of W, $\overline{W}$.

We will work with the spin of a massive fermion, like an electron. Such object is generally represented as a four-component Dirac field, $\psi(x)$, which constitutes a reducible spin $1/2$ representation of the Lorentz group [14]. A four-component Dirac field, also called Dirac spinor, is the solution to the plane wave of the Dirac equation. Dirac fields

[1] Double cover is a two-to-one mapping from one topological space to another.

can be represented in the Weyl or Chiral representation, in which, the field is split in two $SL(2,\mathbb{C})$ spinors, $\psi(x) = (\phi_A(x), \xi'^{A'}(x))$ which represent the left and right-handed irreducible spin $1/2$ representations of the Lorentz group [34]. In this work and in [14], the qubits will be represented in the two-component left-handed Weyl spinor field, $\phi_A(x)$, but if we worked with the right-handed Weyl spinor field, the results would be the same as the obtained.

The Dirac γ matrices in the Weyl representation are defined by

$$\gamma^\mu = \begin{bmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{bmatrix}, \tag{1}$$

in which, $\sigma^\mu = (1, \vec{\sigma})$ and $\overline{\sigma}^\mu = (1, -\vec{\sigma})$. It satisfies the Clifford[2] algebra

$$\{\gamma^\alpha, \gamma^\beta\} = 2\eta^{\alpha\beta}, \tag{2}$$

from which we can obtain the left-handed Clifford algebra,

$$\sigma^\alpha_{AA'}\overline{\sigma}^{\beta A'B} + \sigma^\beta_{AA'}\overline{\sigma}^{\alpha A'B} = 2\eta^{\alpha\beta}\delta^B_A. \tag{3}$$

The generators of this group in spinor notation for the left-handed 2-spinor are given by

$$L^{\alpha\beta B}{}_{A} = \frac{i}{4}\big(\sigma^\alpha_{AA'}\overline{\sigma}^{\beta A'B} - \sigma^\beta_{AA'}\overline{\sigma}^{\alpha A'B}\big). \tag{4}$$

The operators for the spinors have a index structure in the form D^A_B, but on this notation will be represented by $\hat{D}$. The generators are written in the form[3]

$$\hat{L}^{0j} = \frac{i}{2}\hat{\sigma}^j \tag{5}$$

$$\hat{L}^{ij} = \frac{1}{2}\epsilon^{ij}_k\hat{\sigma}^k, \tag{6}$$

in which, the $\hat{L}^{0j}$ components generate boosts and the $\hat{L}^{ij}$ components generate rotations.

[2]The Clifford algebra in question is the $C\ell_{1,3}(\mathbb{C})$ Clifford algebra, also known as the Dirac algebra [35].

[3]For the 4-component formalism, the generators are written as $S^{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta]$

In order to accommodate the quantum nature of the spin interactions, we must provide the convenient measurement formalism. The W space is a complete space, therefore, to promote W to the status of a Hilbert space, we must define an inner product and, equipped with this product, construct a prediction formalism based on the quantum mechanics postulates. The 2-component irreducible representation of $SL(2, \mathbb{C})$ is not a unitary spin $1/2$ representation of the Lorentz group, however, the unitarity property can be recovered by the choice of an adequate inner product and the relinquish of the use of representations.

In order to a sesquilinear[4] inner product in spinor notation, it is required a spinorial object with index structure $I^{A'A}$. This spinorial object has the form $I_u^{A'A} \equiv u_\alpha \overline{\sigma}^{\alpha A'A}$ where u_α denotes the 4-velocity of the particle. The inner product between two spinors ψ_A^1 and ψ_A^2 is therefore written as

$$\langle \psi^1 | \psi^2 \rangle = I_u^{A'A} \overline{\psi}^1 \psi^2 = u_\alpha \overline{\sigma}^{\alpha A'A} \overline{\psi}^1 \psi^2, \tag{7}$$

where the correspondence between Dirac notation and spinor notation is[15]

$$|\psi\rangle \to \psi_A \quad \langle \psi | \to I_u^{A'A} \overline{\psi}_{A'}. \tag{8}$$

Looking closely to equation 7 and remembering the correspondence cited above, one can see that all indices are contracted, therefore, the inner product is Lorentz invariant.

The object that enables the inner product, $I_u^{A'A}$, depends on the 4-velocity, correspondingly, depends on the 4-momentum. So each $I_u^{A'A}$ belongs to a specific Hilbert space, that we shall call $\mathbb{H}_p$. Formally, we do not have a representation of the Lorentz group, since a Lorentz transformation Λ will result in a mapping from the Hilbert space $\mathbb{H}_p$ to Hilbert space $\mathbb{H}_{\Lambda p}$, $\Lambda : \mathbb{H}_p \to \mathbb{H}_{\Lambda p}$, consequently, when we make a linear transformation like Λ in this formalism we are not dealing with the same group anymore.

Another key aspect of quantum formalism that must be reevaluated for this analysis is the general form of Hermitian operators and observables. For a Hermitian operator $\hat{A}$,

[4]Being $f : V \times V \to \mathbb{C}$. f is a sesquilinear form if it is linear in respect to the first coordenate ($f(\lambda u + v, w) = \lambda f(u, w) + f(v, w)$) and antilinear in respect to the second coordenate ($f(u, \lambda v + w) = \overline{\lambda} f(u, v) + f(u, w)$).

we have

$$\langle \chi | \hat{A} \psi \rangle - \langle \chi \hat{A} | \psi \rangle = 0. \tag{9}$$

Using the formalism described above,

$$I_u^{A'A} \bar{\chi}_{A'} A_A^B \psi_B - I_u^{A'A} \bar{A}_{A'}^{B'} \bar{\chi}_{B'} \psi_A = 0. \tag{10}$$

A convenient form of making Equation (10) hold is introducing a way to describe Hermitian operators depending on the Pauli-Ljubanski vector, in the following way

$$\hat{A} = n_\alpha \left(-\frac{2\hat{W}^\alpha(p)}{m} + u^\alpha \hat{\mathbb{I}} \right) \tag{11}$$

where each operator depends on a 4-vector n_α.

On the rest frame of the particle, the spin operator must reduce itself to the usual spin operator, therefore the n_α vector must be orthogonal to u^α,

$$n_\alpha u^\alpha = 0. \tag{12}$$

Normalizing n^α so that we have $n^2 = -1$, we can write the Lorentz invariant expectation value for and operator $\hat{A}$ and a spinor $|\psi_A\rangle$ as

$$\langle \psi | \hat{A} | \psi \rangle = -I_u^{A'A} \hat{\psi}_{A'} n_\alpha \frac{2W_A^{\alpha B}}{m} \psi_B. \tag{13}$$

Using the relation

$$I_u^{A'B} \frac{2W_B^{\alpha A}}{m} = \bar{\sigma}^{alphaA'A} - u^\alpha u_\beta \bar{\sigma}^{\beta A'A}, \tag{14}$$

and (12), we write

$$\langle \psi | \hat{A} | \psi \rangle = -n_\alpha \bar{\sigma}^{\alpha A'A} \bar{\psi}_{A'} \psi_A \tag{15}$$

which is a invariant quantity, since all the indexes contract.

5.4 Derivation of the Stern-Gerlach observable

Given the form of Hermitian operators presented in the previous section, to determine the suitable spin observable for a relativistic Stern-Gerlach measurement it is enough to resolve the relation between n_α and the direction of the Stern-Gerlach apparatus. The first step to acknowledge this observable is a non-relativistic analysis of the Stern-Gerlach experiment on the formalism presented above.

The arbitrary observable for the non-relativistic Stern-Gerlach measurement is simply $n_i\sigma^i$. The Fermion is exposed to an inhomogeneous magnetic field, B_i^{RF}, for a brief period of time in its own reference frame. The field is the same on the Stern-Gerlach reference frame, $B_i^{RF} = B_i^{SG} = |B^{SG}|b_i^{SG}$. The direction of the magnetic field defines the direction of the spin quantization. On other hand, the field gradient, $\nabla_i|B^{SG}|$ is responsible for the rate and direction in which the wave packet will divide into the different eigenstates of $b_i^{SG}\sigma^i$. Thus, in the non-relativistic scenario, the direction of the magnetic field, b_i^{SG}, is the measurement direction, n_α.

When the Fermion moves with relativistic velocity through the apparatus, on its reference frame the spin observable is indistinct from the non-relativistic scenario, being its direction b_i^{RF}, but the magnetic field is not the same on the Stern-Gerlach apparatus frame. The transformed magnetic field will be denoted by B^{RF}, thus, the measurement direction will be $n^i = b^i_{RF}$.

On the Stern-Gerlach apparatus reference frame, the purely-magnetic field is written as

$$F_{\alpha\beta} = -\epsilon_{\alpha\beta\gamma\delta}v^\gamma B^\delta_{SG} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ij} \end{pmatrix} \tag{16}$$

where v^γ is the 4-velocity of the apparatus and B_{SG} its magnetic field. For the magnetic field on the particle's rest frame, it is necessary to take account the relativistic transformations of the magnetic field.

The measurement direction on the particle rest frame is given by

$$b^\alpha_{RF} = \frac{B^\alpha_{RF}}{|B_{RF}|} \tag{17}$$

where B_{RF} relates to B_{SG} as [14]

$$B^{\alpha}_{RF} = B^{\alpha}_{SG}(v_{\lambda}u^{\lambda}) - v^{\alpha}(B_{SG\lambda}u^{\lambda}), \tag{18}$$

where v^{λ} is the 4-velocity of the apparatus and u^{λ} the 4-velocity of the particle.

Then we can write the spin observable depending on the Pauli-Ljubanski vector as in (11)

$$S^{B}_{A} = -b^{RF}_{\alpha}\frac{2W^{\alpha B}_{A}}{m}. \tag{19}$$

5.5 Conclusion

The work on [14] defines two different spin observables for particles that are moving in a relativistic relative velocity in relation to the Stern Gerlach apparatus. The first observable is described in this chapter, the second one, based on a WKB analysis can be found on [14]. Both observables also depend on momentum. The authors debate that for non-relativistic particles, the set of three linearly independent measurements suffice enough data to reconstruct the state, but given the relativistic transformation properties of the magnetic field and the spin operator it is not known what is the size of the minimal set of measurements to fully specify the relativistic state.

On the rest frame of the particle, the proposed spin observables are reduced to the usual and well-defined non-relativistic spin observable, however this does not assure that the relativistic observable really has a physical usability.

6 Conclusion

This document begins with a brief introduction to group theory, which is necessary to understand the symmetry implications of the special relativity theory. The symmetry groups regarding the characteristics of the special relativity mechanic are presented and discussed, and besides that, the notion of the covering maps and the connection of the Poincaré group are explained. Both this notions are crucial on the further analysis on this dissertation. Given the symmetry groups of special relativity theory, the notion of a projective and unitary representation of relativistic particles, where the calculations on [16, 20] are shown and the hierarchy of dynamic variables in relativity is introduced on the quantum mechanical framework.

From these discussions, the issue of finding a subset of transformations in which the Wigner rotations do not depend on the momenta for an arbitrary complementary set is brought to light. This issue is fundamental in order to determine if there is a possibility to define a process of quantum state tomography for relativistic particles.

Further on the document, the relativistic adaptation of the Stern-Gerlach experiment is described under the optics of [14] where the final procedure for fully determining the quantum state of a relativistic particle is not complete, because the minimal number of measurements is not known. In essence, the problem brought in [14] is the same problem that is stated in chapter 4.

The massive relativistic particle with spin is described on Chapters 4 and 5 differently, with the irreducible representation of $\tilde{\mathcal{P}}_{+}^{\uparrow}$ for massive particles and the Weyl representation.

From the crucial works of [14, 16, 17] and [15] the open points on the quantum state tomography for relativistic particles are discussed and hopefully will be further resolved.

This dissertation discusses such points and the main goal of the continuation of this work is to solve the problem that constitutes on the determination of a subset of Wigner rotations that do not depend on the momenta.

7 Perspectives and further steps

In chapter 5 it was shown that there is a set of transformations in which the Wigner rotations are associated with the complementary boots set [10] and with an arbitrary complementary set [16]. The main further goal is to show, given an arbitrary complementary set, whether there is or there is not a subset of transformations in which the associated Wigner rotations do not depend on the momenta. In other words, we are searching for a correspondence between the set of generators of effective matrices transformations and a subset of generators of physical $\mathcal{R}^3$ rotations.

If the above correspondence exists, it is possible to fully determine the relativistic quantum state tomography.

Bibliography

[1] Walther Gerlach and Otto Stern. Der experimentelle nachweis der richtungsquantelung im magnetfeld. *Zeitschrift für Physik*, 9(1):349–352, Dec 1922.

[2] B.C. Van Fraassen, P.P.B.C. Van Fraassen, and Oxford University Press. Laws and symmetry. 1989.

[3] S. Weinberg. *The Discovery of Subatomic Particles Revised Edition.* Cambridge University Press, 2003.

[4] JJ Chen. Group theory and the rubik's cube, 2004.

[5] E. Noether. Invariante variationsprobleme. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1918:235–257, 1918.

[6] Yaakov Friedman and Tzvi Scarr. Symmetry and special relativity. *Symmetry*, 11(10), 2019.

[7] GS Hall. The mathematical study of symmetry in general relativity. *Gravity and Cosmology*, 2(4):270–276, 1996.

[8] R.P. Feynman, R.B. Leighton, and M. Sands. The feynman lectures on physics, vol. i: The new millennium edition: Mainly mechanics, radiation, and heat. (v. 1), 2015.

[9] Robert Bluhm, V Alan Kostelecký, Charles D Lane, and Neil Russell. Clock-comparison tests of lorentz and CPT symmetry in space. *Phys Rev Lett*, 88(9):090801, February 2002.

[10] W.K. Tung. *Group Theory in Physics.* World Scientific, 1985.

[11] Felix Klein. Vergleichende betrachtungen über neuere geometrische forschungen. *Mathematische Annalen*, 43(1):63–100, Mar 1893.

[12] D. Robinson, S.J. Axler, F.W. Gehring, and P.R. Halmos. *A Course in the Theory of Groups*. Graduate Texts in Mathematics. Springer New York, 1996.

[13] V. I. Man'ko and O. V. Man'ko. Spin state tomography. *Journal of Experimental and Theoretical Physics*, 85(3):430–434, Sep 1997.

[14] Palmer et al. Wkb analysis of relativistic stern–gerlach measurements. *Annals of Physics (New York)*, 336, 9 2013.

[15] Matthew C. Palmer, Maki Takahashi, and Hans F. Westman. Localized qubits in curved spacetimes. *Annals of Physics*, 327(4):1078–1131, 2012.

[16] E. R. F. Taillebois and A. T. Avelar. Spin-reduced density matrices for relativistic particles. *Phys. Rev. A*, 88:060302, Dec 2013.

[17] Asher Peres, Petra F. Scudo, and Daniel R. Terno. Quantum entropy and special relativity. *Phys. Rev. Lett.*, 88:230402, 2002.

[18] H. Weyl. Quantenmechanik und gruppentheorie. *Zeitschrift für Physik*, 46(1):1–46, Nov 1927.

[19] Stephen L. Bloom, Zoltan Esik, and Ernest G. Manes. A cayley theorem for boolean algebras. *The American Mathematical Monthly*, 97(9):831–833, 1990.

[20] Emile Raymond Ferreira Taillebois et al. Traço parcial em sistemas relativísticos: uma nova visão. 2013.

[21] Jiaqi Huang. Lie Groups and their applications to Particle Physics: A Tutorial for Undergraduate Physics Majors. 11 2020.

[22] Jean-Bernard Zuber. Invariances in Physics and Group Theory. *IRMA Lect. Math. Theor. Phys.*, 23:307–324, 2015.

[23] J. Diestel and A. Spalsbury. *The Joys of Haar Measure*. Graduate studies in mathematics. American Mathematical Society, 2014.

[24] H.J.W. Müller-Kirsten, A. Wiedemann, and A. Wiedemann. *Introduction to Supersymmetry*. World Scientific lecture notes in physics. World Scientific, 2010.

[25] Xavier Bekaert and Nicolas Boulanger. The unitary representations of the Poincar\'e group in any spacetime dimension. *SciPost Phys. Lect. Notes*, 30:1, 2021.

[26] J. Beckers. Helicity and canonical spin operators of the poincaré algebra. *Journal of Mathematical Physics*, 13(6):915–918, 1972.

[27] S. Weinberg, W. S, and T. de campos. The quantum theory of fields. (v. 1), 1995.

[28] Pablo L. Saldanha and Vlatko Vedral. Spin quantum correlations of relativistic particles. *Phys. Rev. A*, 85:062101, Jun 2012.

[29] Giacomo Mauro D'Ariano. Quantum tomography: General theory and new experiments. *Fortschritte der Physik*, 48(5-7):579–588, 2000.

[30] J.J. Sakurai and J. Napolitano. Modern quantum mechanics, 2011.

[31] A.P. French and E.F. Taylor. *An Introduction to Quantum Physics*. M.I.T. introductory physics series. Norton, 1978.

[32] S. Weinberg, W. S, and T. de campos. *The Quantum Theory of Fields*. Number v. 1 in Quantum Theory of Fields, Vol. 2: Modern Applications. Cambridge University Press, 1995.

[33] B. Hall and B.C. Hall. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*. Graduate Texts in Mathematics. Springer, 2003.

[34] M.E. Peskin and D.V. Schroeder. *An Introduction To Quantum Field Theory*. Frontiers in Physics. Avalon Publishing, 1995.

[35] J. Jost. *Riemannian Geometry and Geometric Analysis*. Universitext. Springer Berlin Heidelberg, 2013.