UNIVERSIDADE FEDERAL DE GOIÁS (UFG)
INSTITUTO DE FÍSICA (IF)
PROGRAMA DE PÓS-GRADUAÇÃO EM FÍSICA (PPGF)

THIAGO HENRIQUE MOREIRA

Strong magnetic field corrections to the NJL coupling constant from vacuum polarization

GOIÂNIA
2022

UNIVERSIDADE FEDERAL DE GOIÁS
INSTITUTO DE FÍSICA

TERMO DE CIÊNCIA E DE AUTORIZAÇÃO (TECA) PARA DISPONIBILIZAR VERSÕES ELETRÔNICAS DE TESES E DISSERTAÇÕES NA BIBLIOTECA DIGITAL DA UFG

Na qualidade de titular dos direitos de autor, autorizo a Universidade Federal de Goiás (UFG) a disponibilizar, gratuitamente, por meio da Biblioteca Digital de Teses e Dissertações (BDTD/UFG), regulamentada pela Resolução CEPEC nº 832/2007, sem ressarcimento dos direitos autorais, de acordo com a Lei 9.610/98, o documento conforme permissões assinaladas abaixo, para fins de leitura, impressão e/ou download, a título de divulgação da produção científica brasileira, a partir desta data.

O conteúdo das Teses e Dissertações disponibilizado na BDTD/UFG é de responsabilidade exclusiva do autor. Ao encaminhar o produto final, o autor(a) e o(a) orientador(a) firmam o compromisso de que o trabalho não contém nenhuma violação de quaisquer direitos autorais ou outro direito de terceiros.

1. Identificação do material bibliográfico

[X] Dissertação [ ] Tese

2. Nome completo do autor

Thiago Henrique Moreira

3. Título do trabalho

Strong magnetic field corrections to the NJL coupling constant from vacuum polarization

4. Informações de acesso ao documento (este campo deve ser preenchido pelo orientador)

Concorda com a liberação total do documento [X] SIM [ ] NÃO[1]

[1] Neste caso o documento será embargado por até um ano a partir da data de defesa. Após esse período, a possível disponibilização ocorrerá apenas mediante:
a) consulta ao(à) autor(a) e ao(à) orientador(a);
b) novo Termo de Ciência e de Autorização (TECA) assinado e inserido no arquivo da tese ou dissertação.
O documento não será disponibilizado durante o período de embargo.
Casos de embargo:
- Solicitação de registro de patente;
- Submissão de artigo em revista científica;
- Publicação como capítulo de livro;
- Publicação da dissertação/tese em livro.

Obs. Este termo deverá ser assinado no SEI pelo orientador e pelo autor.

Documento assinado eletronicamente por **THIAGO HENRIQUE MOREIRA**, **Discente**, em 02/02/2022, às 10:46, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

Documento assinado eletronicamente por **Fábio Luis Braghin**, **Professor do Magistério Superior**, em 02/02/2022, às 11:24, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

THIAGO HENRIQUE MOREIRA

Strong magnetic field corrections to the NJL coupling constant from vacuum polarization

Dissertação apresentada ao Programa de Pós-Graduação em Física, do Instituto de Física, da Universidade Federal de Goiás (UFG), como requisito para obtenção do título de Mestre em Física.
Área de concentração: Física.
Linha de pesquisa: Partículas e Campos.

Orientador: Professor Doutor Fábio Luis Braghin.

GOIÂNIA
2022

Ficha de identificação da obra elaborada pelo autor, através do
Programa de Geração Automática do Sistema de Bibliotecas da UFG.

Moreira, Thiago Henrique
Strong magnetic field corrections to the NJL coupling constant from vacuum polarization [manuscrito] / Thiago Henrique Moreira. - 2022.
xiv, 85 f.: il.

Orientador: Prof. Dr. Fábio Luis Braghin.
Dissertação (Mestrado) - Universidade Federal de Goiás, Instituto de Física (IF), Programa de Pós-Graduação em Física, Goiânia, 2022.
Bibliografia. Apêndice.
Inclui gráfico, tabelas, lista de figuras.

1. Nambu-Jona-Lasinio. 2. Campo magnético. 3. Catálise magnética. 4. Acoplamento dependente do campo magnético. I. Braghin, Fábio Luis, orient. II. Título.

CDU 53

UNIVERSIDADE FEDERAL DE GOIÁS

INSTITUTO DE FÍSICA

ATA DE DEFESA DE DISSERTAÇÃO

Ata nº 197 da sessão de Defesa de Dissertação de Thiago Henrique Moreira, que confere o título de Mestre em Física, na área de concentração em Física.

Aos 28 dias do mês de janeiro de 2022, a partir das 14h00min, por meio de videoconferência, realizou-se a sessão pública de Defesa de Dissertação intitulada “Strong magnetic field corrections to the NJL coupling constant from vacuum polarization”. Os trabalhos foram instalados pelo Orientador, Professor Doutor Fábio Luis Braghin (IF/UFG), com a participação dos demais membros da Banca Examinadora: Professor Doutor Igor Shovkovy (Arizona State University / EUA), membro titular externo; e Professor Doutor Gergely Markó (University of Bielefeld / Alemanha), membro titular externo. Durante a arguição, os membros da banca não fizeram sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Dissertação, tendo sido o candidato aprovado pelos seus membros. Proclamados os resultados pelo Professor Doutor Fábio Luis Braghin, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos membros da Banca Examinadora, aos 28 dias do mês de janeiro de 2022.

TÍTULO SUGERIDO PELA BANCA

Documento assinado eletronicamente por **Fábio Luis Braghin**, **Professor do Magistério Superior**, em 28/01/2022, às 15:28, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

Documento assinado eletronicamente por **Lucas Chibebe Celeri**, **Coordenador de Pós-graduação**, em 31/01/2022, às 17:44, conforme horário oficial de Brasília, com fundamento no § 3º do art. 4º do Decreto nº 10.543, de 13 de novembro de 2020.

A autenticidade deste documento pode ser conferida no site https://sei.ufg.br/sei/controlador_externo.php?acao=documento_conferir&id_orgao_acesso_externo=0, informando o código verificador **2629062** e o código CRC **49A709C2**.

Referência: Processo nº 23070.001041/2022-17 SEI nº 2629062

Agradecimentos

Após esses dois longos anos, que com certeza foram preenchidos por inúmeras dificuldades, devo meus mais sinceros agradecimentos a todos que me ajudaram a chegar até aqui. Embora não haja espaço suficiente para os nomes de todos que me apoiaram nessa jornada, mesmo que de maneira simples, porém marcante, não posso deixar de fazer algumas menções especiais.

Primeiramente, agradeço a CAPES pelo financiamento ao longo desses dois anos e aos professores e servidores do Instituto de Física da Universidade Federal de Goiás que colaboraram com minha formação. Em especial, agradeço ao Prof. Dr. Fábio Luis Braghin pela orientação, pelos ensinamentos e pelas excelentes oportunidades.

Um dos ensinamentos mais importantes que aprendi nos últimos anos, além daqueles relacionados à física, foi com relação à necessidade de manter a saúde mental para a realização de um projeto como esse. Por isso, não há palavras que descrevem o quanto sou grato a todas as pessoas que me apoiaram em todo esse tempo: em especial, aos meus amigos Maykon e Edson, pelas visitas nas quais nenhuma cadeira foi consertada; ao Pedro Augusto, que além de ser meu mentor de soulsborne, me manteve motivado com a física em várias ocasiões através de sua própria animação pessoal; ao Lúcio Flávio, pela companhia nos primeiros meses da pandemia e por ter sido, provavelmente, o que mais me ouviu reclamar de tudo; a Sarah Borges, minha companheira de viagens oficial, pelos mais de dez anos de amizade e companhia, e por ter sido alguém com quem sempre pude contar.

Vivendo na realidade do isolamento social, preciso reconhecer hoje que eu não teria conseguido me dedicar a esse projeto se eu não soubesse que estaria com meus amigos todos os dias às 18h, mesmo que online. Por isso, devo enormes agradecimentos ao meu esquadrão do Warzone: meu irmão de outra mãe, Arthur Bernardes; meu companheiro de caminhadas sob a chuva, Matheus Mazetti; e, por fim, meu parceiro de fofocas, Rafael Gratão.

E, é claro, eu não seria ninguém e não teria chegado a lugar algum sem o apoio, carinho, dedicação e amor das pessoas que eu mais amo nessa vida: meu pai Jeneir, minha mãe Nádia e minha irmã Fernanda (e a Fioninha, minha companhia diária). Vocês têm grandes méritos em minhas conquistas!

Resumo

Nesse trabalho calculamos correções para a constante de acoplamento do modelo NJL na presença de um campo magnético uniforme. Começando com a lagrangiana do modelo de Nambu-Jona-Lasinio acoplada com um campo magnético externo, separamos os campos de quarks em duas componentes: uma correspondendo aos quarks que condensam e outra correspondendo às quase-partículas interagentes. Integrando a primeira componente, uma ação efetiva em termos dos quarks interagentes é obtida. Em seguida, expandimos o determinante em termos das correntes de quarks e mostramos que o termo de primeira ordem produz uma correção para as massas dos quarks dada pela equação do gap. Os valores obtidos para as massas dos quarks constituintes aumentam com o campo magnético, o que sinaliza o aprimoramento da quebra de simetria quiral pelo campo externo e, portanto, mostrando que o sistema exibe catálise magnética. O termo de segunda ordem da expansão do determinante produz uma correção para a constante de acoplamento do modelo NJL que decresce com o campo magnético para as interações escalares e cresce com o campo magnético para as pseudoescalares. Em seguida consideramos o modelo NJL com os acoplamentos dependentes do sabor e de B obtidos da polarização do vácuo e calculamos as massas dos quarks e dos mésons. Enquanto os acoplamentos escalares parecem melhorar a conciliação com os resultados da rede para as massas dos quarks, o mesmo não pode ser dito sobre os acoplamentos pseudoescalares que alteram as massas dos mésons de uma maneira diferente daquela conhecida na literatura devido a seu comportamento com o campo aplicado.

Palavras - chave: Nambu-Jona-Lasinio, Campo magnético, Catálise magnética, Acoplamento dependente do campo magnético.

Abstract

In this work we calculate corrections to the NJL model coupling constant in the presence of a uniform magnetic field. Starting with the Nambu-Jona-Lasinio Lagrangian coupled with an external magnetic field, we separate the quark field into two components: one corresponding to the condensed quarks and the other corresponding to the interacting quasiparticle quarks. By integrating out the former, an effective action in terms of the interacting quarks is obtained. We then expand the quark determinant in terms of the quark currents and show that the first-order term provides a correction to the quark masses given by the gap equation. The values obtained for the constituent quark masses increase with the magnetic field, which signals the enhancement of chiral symmetry breaking by the external field and therefore showing that the system exhibits magnetic catalysis. The second-order term of the quark determinant expansion provides a correction to the NJL coupling constant, which decreases with increasing magnetic field for the scalar interactions and increases with increasing magnetic field for the pseudoscalar ones. We then consider a NJL model with the flavor- and B-dependent couplings obtained from vacuum polarization and compute quark and meson masses. While the scalar couplings seem to improve the conciliation with lattice results for the quark masses, the same cannot be said about the pseudoscalar couplings which alters the pseudoscalar meson masses in a different way than what is known in the literature due to its behavior with the applied field.

Key - words: Nambu-Jona-Lasinio, Magnetic field, Magnetic catalysis, Magnetic field dependent coupling.

List of Figures

Notations and conventions

We will work in natural units, where

$$\hbar = c = 1.$$

Here $\hbar$ is Planck constant divided by 2π and c is the speed of light. With our choice of units, we have

$$[\text{length}] = [\text{time}] = [\text{energy}]^{-1} = [\text{mass}]^{-1},$$

where the bracket refers to the unit of the quantity inside it.

The metric tensor that we use has sign convention

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix},$$

with Greek indices running over $0, 1, 2, 3$. Roman indices, i, j, etc., run over $1, 2, 3$. Repeated indices are summed unless said otherwise. Contractions with the Dirac matrices γ^μ will sometimes be denoted by the Feynman slash,

$$\not{p} = \gamma^\mu p_\mu.$$

Contents

CHAPTER 1

Introduction

In the past years, the properties of strongly interacting magnetized matter have been attracting a lot of attention. The complicated nature of quantum chromodynamics (QCD) inspires the investigation of this theory under external conditions which provide a controllable dynamics [1, 2]. An applied magnetic field is one example of such external parameters and one which is relevant to many applications. Magnetic fields play an important role in physical systems such as noncentral heavy ion collisions, neutron stars and the early universe [3]. In heavy ion collisions, for example, the magnetic field produced can be of the order of $eB \sim 15m_\pi^2$ for LHC [4], where e is the proton electric charge, B is the magnetic field strength and $m_\pi \simeq 135\,\mathrm{MeV}$ is the π^0 meson mass. Such value of the magnetic field is strong enough to influence the strongly interacting matter significantly [1].

One very important property of QCD is chiral symmetry breaking, for which the chiral condensate $\left\langle \bar{\psi}\psi \right\rangle$ is an order parameter. In the theory of strong interactions, dynamical breaking of chiral symmetry leads to the definition of massive constituent quarks which are responsible for most part of the hadron masses, as described by the constituent quark model [5].

It has been established that an applied magnetic field has a tendency to enhance spin-zero fermion-antifermion condensates, which are associated with the breaking of global symmetries and lead to a dynamical generation of masses [6, 7]. This mechanism is called *magnetic catalysis* [6, 8]. It is a model-independent effect since its essence is the dimensional reduction in the dynamics of fermion pairing in a magnetic field. In fact, a constant magnetic field was shown to be a strong catalyst of dynamical chiral symmetry breaking even at the weakest attractive interaction between fermions [7].

Serving as a low energy effective theory for QCD, the Nambu-Jona-Lasinio (NJL) model [9, 10] is excellent for describing the effects of the approximate chiral symmetry and its breaking to generate the dynamical quark masses [5, 11, 12]. This makes the model great for giving a clear illustration of the general effect that is magnetic catalysis.

Additionally, the NJL model successfully describes meson properties allowing us to study the effects of the magnetic field on the masses of hadrons, for example.

The NJL model is one of the many approaches to understand the effects of a magnetic field on strong interactions. Particularly, the structure of the QCD phase diagram in a magnetic field is a topic that have been receiving increasing attention, both in effective models predictions and lattice calculations [3]. While effective models exhibiting the magnetic catalysis phenomenon predict that the critical temperature for chiral symmetry restoration should increase with B, lattice simulations support its decrease with increasing magnetic field strength [13, 14]. This latter effect is known as inverse magnetic catalysis.

It seems like this contradiction between effective models and lattice predictions lies in the fact that the couplings in those models are fixed and independent of the applied magnetic field, which is not the case for QCD [2, 15, 16]. It has been established that the lattice results can be reproduced by the NJL model, in particular, if the coupling constant of the theory decreases with both the magnetic field strength and the temperature [16, 17]. Particularly, a B-dependent effective coupling in the NJL model has been shown to reproduce results in good agreement with lattice QCD simulations [18, 19, 20, 21].

In this work, our main goal is to obtain a mechanism for which the NJL coupling acquires a B-dependence. This will be done by investigating vacuum polarization effects in the presence of a strong magnetic field [22, 23], like it was done in Ref. [24] for the weak magnetic field case. Consequently, one may expect to obtain effective B-dependent couplings that receive flavor-dependent contributions [25] since quarks have different masses and electric charges, and thus may respond to the applied field with different intensities. The U(3) NJL Lagrangian minimally coupled with an Abelian gauge field, namely a uniform magnetic field in the $z-$direction, $\mathbf{B} = B\hat{\mathbf{e}}_z$, will be our departure point. Working at zero temperature, we split the quark field bilinears into one component that condenses and another for the interacting quarks. We then integrate out the former and obtain an effective action in terms of interacting quark fields [22]. Then, the quark determinant can be expanded in terms of the quark field bilinears. Particularly of interest to this work will be the second-order term, which provides an effective coupling that depends on the magnetic field.

This work is organized as follows. In Chapter 2 we present a brief discussion on the properties of QCD and introduce the NJL model in the absence of external fields. Applying the background field method, we obtain the effective action, the chiral condensate and the gap equation for the dynamically generated quark masses. A word on different regularization schemes is mentioned and we compute some meson properties at $B = 0$ in order to fix the free parameters of the model. In Chapter 3, we couple the NJL Lagrangian with the uniform magnetic field and obtain expressions for the chiral condensate and the gap equation as functions of B. The effect of the magnetic field in enhancing the chiral condensate is discussed. We then expand the quark determinant in terms of the quark field bilinears and obtain expressions for the $B-$dependent effective couplings. In Chapter 4 we consider the effects of the new flavor- and B-dependent couplings in quark and meson masses. Finally, Chapter 5 concludes with some final remarks. Some useful mathematical results are presented in Appendix A as well as some properties regarding Pauli, Dirac and Gell-Mann matrices in Appendix B. Appendix C presents a derivation of the fermion propagator in a uniform magnetic field following Schwinger proper time method [26, 27], which is used several times in the text. Lastly, Appendix D lists some details on the flavor trace of the second order term in the quark determinant expansion.

CHAPTER 2

The Nambu-Jona-Lasinio model

2.1 The theory of strong interactions

One may say that high-energy physics is one of mankind greatest efforts to pursue the ultimate structure of matter. This is done by looking for the smallest constituents of matter, the elementary particles, and describing how they interact with one another to construct the world we live in. Our current knowledge tells us that the fundamental building blocks of matter are quarks and leptons, which are fermions having spin 1/2, and that the interactions between particles are mediated by gauge bosons. The four known fundamental interactions are the electromagnetic, weak, strong and gravitational forces (with the latter being negligibly weak at elementary particle level). While all particles are affected by the weak force, only electrically charged particles interact with the electromagnetic field. Furthermore, leptons are not affected by the strong force, which is the interaction that binds quarks to form composite states called hadrons, and also binds protons and neutrons to form the nucleus.

The theoretical framework of particle physics falls under the dominion of quantum field theory (QFT) since it must be guided by the principles of special relativity and quantum mechanics if is to describe the microscopic world of particles moving at speeds comparable to the speed of light. Except for gravity, all of the known elementary particle interactions are described by the Standard Model, which encompasses quantum electrodynamics (QED), the Glashow-Weinberg-Salam theory of electroweak processes and quantum chromodynamics (QCD) [28, 29].

Quantum chromodynamics is the theory describing the strong interaction between quarks, which is based on the gauge group SU(3) acting on a degree of freedom called color. Quarks come in six flavors, up (u), down (d), strange (s), charm (c), bottom (b) and top (t), and they are equipped with what we call *color charge*, which generates the force field just as the electromagnetic field is generated by an electric charge. There are three kinds of color charges, which we label by R (red), B (blue) and G (green), and only color neutral combinations are observed. We then describe the strong interaction by a SU(3) Yang-Mills theory with each quark flavor transforming as the fundamental triplet representation [30, 31].

The QCD Lagrangian is given by [31, 32]

$$\mathcal{L} = \bar{\psi}\left(i\not{D} - m\right)\psi - \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a, \tag{2.1}$$

where $\psi = (u, d, s, \dots)^T$ is the quark field with three colors, $m = \text{diag}(m_u, m_d, m_s, \dots)$ is the current quark mass matrix and

$$D_\mu = \partial_\mu - ig_s\frac{\lambda^a}{2}A^a_\mu,$$

is the covariant derivative, with g_s representing the strong coupling constant and the colored gauge field A^a_μ representing the gluon field. The gluon field strength tensor is given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f_{abc}A^b_\mu A^c_\nu.$$

Here a, b, c are SU(3) adjoint representation indices running from 1 to 8 and a sum over repeated indices is implicit. The matrices λ^a are the Gell-Mann matrices and the coefficients f_{abc} are the SU(3) structure constants.

The computation of the propagator for the gauge boson follows the usual procedure of introducing the gauge fixing term and the associated Faddeev-Popov ghost fields [33, 34]. However, the usual calculational procedure of renormalized perturbation theory that works so well for QED cannot be used for QCD since it is a strong interaction theory. The most crucial difference between the electromagnetic and strong interactions is the *confinement of color* exhibited by the latter. While electrons and photons are observed as free particles, quarks and gluons are not; instead, they are bound into hadrons and the only observed states are those that are singlets of color SU(3) [32, 33].

A renormalization group analysis shows that the QCD coupling effectively decreases with energy, which is a property exhibited by non-Abelian gauge theories called *asymptotic freedom*[1]. This property was the reason for the success of the parton model in explaining deep inelastic scattering phenomena by treating the particles inside hadrons as freely moving; it also enables us to use the QED perturbative techniques in the high energy regime [30].

But perturbation theory does not work in the regime of strong coupling and al-

1. For the discovery of asymptotic freedom in the theory of the strong interaction the Nobel prize in physics was awarded to D. J. Gross, H. D. Politzer and F. Wilczek in 2004 [31, 35].

ternative approaches are required. One such example is lattice QCD, an approximation scheme in which the continuum gauge theory is replaced by a discrete statistical mechanical system on a four-dimensional Euclidean lattice [33]. However, limited computer power and difficulty in transforming numerical data into essential physics is enough inspiration for the investigation of another approach. That is where the effective models of QCD come in.

The construction of an effective model of QCD must be guided by the symmetry properties of the theory, and so it is instructive to list those properties before introducing such a model.

2.1.1 QCD symmetries

Let us now restrict ourselves to the case of QCD with three quark flavors, so that the quark field is given by

$$\psi(x) = \begin{pmatrix} u(x) \\ d(x) \\ s(x) \end{pmatrix}, \tag{2.2}$$

and each quark flavor exits in three colors. The local color SU(3) gauge symmetry of the Lagrangian (2.1), which is exact, will not be important for our considerations on the effective model to be introduced [11, 12], and thus it will not be discussed.

The QCD Lagrangian can be written as

$$\mathcal{L} = \mathcal{L}^{(0)} - \Big(m_u \bar{u}u + m_d \bar{d}d + m_s \bar{s}s\Big), \tag{2.3a}$$

where

$$\mathcal{L}^{(0)} = \bar{\psi} i \not{D} \psi - \frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a. \tag{2.3b}$$

The Lagrangian $\mathcal{L}^{(0)}$ is invariant under the group

$$\mathrm{U}_V(3) \otimes \mathrm{U}_A(3) = \mathrm{SU}_V(3) \otimes \mathrm{SU}_A(3) \otimes \mathrm{U}_V(1) \otimes \mathrm{U}_A(1), \tag{2.4}$$

where the transformations are specified by

$$\begin{aligned} \mathrm{U}_V(1):&\quad \psi \to e^{-i\alpha}\psi, \\ \mathrm{SU}_V(3):&\quad \psi \to e^{-i\lambda^a \alpha^a/2}\psi, \\ \mathrm{U}_A(1):&\quad \psi \to e^{-i\beta\gamma_5}\psi, \\ \mathrm{SU}_A(3):&\quad \psi \to e^{-i\lambda^a \beta^a \gamma_5/2}\psi, \end{aligned}$$

with α and β denoting arbitrary transformation parameters. The unitary transformations $\mathrm{U}_V(1)$ correspond to baryon conservation while $\mathrm{SU}_V(3)$ is associated with the eightfold way [11]. The $U_A(1)$ and $SU_A(3)$ transformations, which include the γ_5 matrix, are called *chiral* or *axial transformations*.

However, the axial symmetries are not manifest in particle degeneracies, and so we must assume that the dynamics is such that the QCD vacuum breaks these symmetries [31]. Now, we know that the spontaneous breaking of a continuous global symmetry implies the existence of associated massless bosons as stated by the Goldstone theorem. In fact, the spontaneous breaking of the chiral symmetry $SU_A(3)$ is corroborated by the observation of the Goldstone modes of the π octet. However, the breaking of $U_A(1)$ requires a ninth pseudoscalar meson that is not observed in nature. This is the so-called axial U(1) problem. It was solved by 't Hooft who showed that, due to instanton effects, the $U_A(1)$ symmetry should not result in physical manifestations [11]. Nevertheless, chiral symmetry breaking is an important QCD feature that we expect our effective model to share with the fundamental theory.

Let us note that the total Lagrangian $\mathcal{L}$, which includes the mass term, breaks the chiral symmetry explicitly and also the symmetry under $SU_V(3)$ if $m_u \neq m_d \neq m_s$. For this reason, we say that $\mathcal{L}^{(0)}$ is the QCD Lagrangian in the *chiral limit*, and sometimes the notation $\mathcal{L}_{\text{chiral}}$ is used. However, current quark masses are known to have the values listed below [36]

$$\begin{aligned} m_u &= 2.16^{+0.49}_{-0.26}\,\text{MeV}, \\ m_d &= 4.67^{+0.48}_{-0.17}\,\text{MeV}, \\ m_s &= 93^{+11}_{-5}\,\text{MeV}, \end{aligned}$$

which are relatively small when compared to typical hadron mass scales of about 1 GeV. This means that chiral symmetry is a good approximation in the regime of low energy where we consider only the three quark flavors listed above. Furthermore there is evidence both from low-energy hadron phenomenology and from lattice QCD that chiral symmetry is *spontaneously* broken [12], which is why we emphasize that it is important that an effective model exhibits this property.

Spontaneous chiral symmetry breaking is signaled by non-vanishing quark pair condensates, $\left\langle \bar{\psi}\psi \right\rangle$ [12]. If we consider two flavors, u and d, the Gell-Mann-Oakes-Renner relation gives [5]

$$f_\pi^2 m_\pi^2 \simeq -\frac{(m_u + m_d)}{2} \left\langle \bar{u}u + \bar{d}d \right\rangle,$$

where f_π is the pion decay constant and m_π is the pion mass. The pion is the Goldstone boson associated with the chiral symmetry breaking, but it is not massless because the chiral symmetry is not exact for $m_u \neq 0$ and $m_d \neq 0$. The equation above then indicates that $\left\langle \bar{u}u + \bar{d}d \right\rangle \neq 0$ as a consequence of the spontaneous breaking of chiral symmetry. This means that the QCD ground state has condensation of quark and antiquark pairs [5]. The vacuum expectation value $\left\langle \bar{\psi}\psi \right\rangle$ is called the *chiral condensate* and the fact that

$\left\langle \bar{\psi}\psi \right\rangle \neq 0$ suggests the generation of dynamical quark masses. As a matter of fact, the constituent quark model describes many of hadron properties by assuming that massive quarks with $M_u \simeq M_d \simeq 300\,\mathrm{MeV}$ and $M_s \simeq 500\,\mathrm{MeV}$ interact weakly inside hadrons [5].

Having briefly discussed some QCD properties, we are now ready to introduce a low-energy effective model, which we expect to be simpler to work with than the fundamental theory. The effective Lagrangian must contain the same symmetry structure of QCD, namely the $\mathrm{SU}_V(3) \otimes \mathrm{U}_V(1)$ symmetry in the chiral limit as well as chiral symmetry $\mathrm{SU}_A(3)$ and its spontaneous breakdown. Furthermore we want the effective theory to be a basis of the constituent quark model, so we write a Lagrangian with no gluons and where the basic degrees of freedom will be the quarks u, d and s. We should also keep in mind that we are interested in the low-energy properties of the quark dynamics where the energy scale is smaller that some cutoff scale $\Lambda \simeq 1\,\mathrm{GeV}$ [5, 12]. At last, let us now introduce such an effective theory, namely the Nambu-Jona-Lasinio (NJL) model of quantum chromodynamics.

2.2 The NJL Lagrangian

Historically, the Nambu-Jona-Lasinio model was introduced as a theory of nucleons in analogy with the BCS theory of superconductivity [9, 10]. Its construction came before the development of quantum chromodynamics and, after the arrival of the fundamental theory, it was soon realized that the NJL model shares some conceptually important features with low energy QCD [12]; in particular, the dynamic generation of fermion masses due to the spontaneous chiral symmetry breaking is one important property exhibited by the model [11].

Although the nucleonic NJL model is still used to this day [37], it is usually reinterpreted as a theory with quark degrees of freedom instead of the original nucleons. It shall serve as an effective low energy model of the strong interaction, but it should be kept in mind that it does not exhibit color confinement as does QCD. This limits its application to hadronic and nuclear phenomena that do not depend on details of the confinement mechanism [12]. In the NJL model the interaction between quarks and antiquarks are assumed to be attractive, without specifying the complicated processes of gluon exchange, and leads to a quark-antiquark pair condensation in the vacuum [11].

The U(3) NJL (chiral) Lagrangian is given by [5, 11, 12]

$$\mathcal{L}^{(0)}_{\mathrm{NJL}} = \bar{\psi} i \partial\!\!\!/ \psi + \frac{g}{2}\Big[\Big(\bar{\psi}\lambda^a\psi\Big)^2 + \Big(\bar{\psi} i\gamma_5\lambda^a\psi\Big)^2\Big], \tag{2.5}$$

where g is the NJL coupling constant, the quark field ψ is given in (2.2) and the index a takes on the values $0,\ldots,8$. As we can see, the coupling constant has dimension of $[\mathrm{energy}]^{-2}$ and the model is, in fact, nonrenormalizable, meaning that we need to add an infinite number of terms to the Lagrangian in order to absorb the divergences that arise [38]. However, it is worth mentioning that even a nonrenormalizable theory is able to make useful predictions at energies below some ultraviolet cutoff, like $\Lambda \simeq 1\text{GeV}$ for the NJL model.

It should be noted that the Lagrangian (2.5) exhibits the same symmetries as $\mathcal{L}^{(0)}$ in Eq. (2.3b), namely the invariance under the group (2.4). The unwanted $\text{U}_A(1)$ symmetry can be removed by the addition of the 't Hooft determinant interaction [12, 39] or equivalently by considering a third order interaction term from vacuum polarization [22]. Since our goal is to obtain a four-fermion effective coupling from polarization effects in the quark determinant expansion up to second order in the quark currents, the 't Hooft term will not play an important role in this work, as it would if we were to compute the third order term in the expansion. Therefore, we shall not consider this term throughout this work.

Adding the mass term, which breaks the chiral symmetry as before, as well as the $\text{SU}_V(3)$ symmetry if $m_u \neq m_d \neq m_s$, the NJL Lagrangian reads

$$\mathcal{L}_{\text{NJL}} = \bar{\psi}\Big(i\partial\!\!\!/ - m\Big)\psi + \frac{g}{2}\Big[\Big(\bar{\psi}\lambda^a\psi\Big)^2 + \Big(\bar{\psi}i\gamma_5\lambda^a\psi\Big)^2\Big]. \tag{2.6}$$

The interaction term can also be rewritten in terms of the fundamental SU(3) representation as

$$\mathcal{L}_{\text{NJL}} = \bar{\psi}\Big(i\partial\!\!\!/ - m\Big)\psi + g\sum_{f,g=u,d,s}\Big[\Big(\bar{\psi}_f\psi_g\Big)\Big(\bar{\psi}_g\psi_f\Big) + \Big(\bar{\psi}_f i\gamma_5\psi_g\Big)\Big(\bar{\psi}_g i\gamma_5\psi_f\Big)\Big]. \tag{2.7}$$

Having introduced the Nambu-Jona-Lasinio Lagrangian, we now proceed to obtain a 1-loop effective action for the NJL model in terms of the interacting quark fields. Since this procedure does not depend on the minimally coupled U(1) gauge field, namely the magnetic field we are interested in, we will compute the effective action from the Lagrangian (2.6). Then, in Chapter 3, we add the contribution of the magnetic field to the NJL Lagrangian and make the necessary modifications.

2.3 The effective action and the gap equation

2.3.1 The effective action

The generating functional of the local U(3) NJL model is given by

$$Z[\bar{\eta},\eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, \exp\left\{i\left[S_{\text{NJL}}[\bar{\psi},\psi] + \int d^4x\,(\bar{\psi}\eta + \bar{\eta}\psi)\right]\right\}, \tag{2.8}$$

where $S_{\rm NJL} = \int d^4x\, \mathcal{L}_{\rm NJL}$, $\mathcal{D}\psi$, $\mathcal{D}\bar{\psi}$ are the fermion functional measures and $\bar{\eta}$ and η are fermion and antifermion sources, respectively [11].

In order to obtain an effective action in terms of interacting quark fields we separate the quark field bilinears into two components [22, 24, 23]: one corresponding to the condensed quarks, $\left(\bar{\psi}\psi\right)_c$, and the other to the interacting quasiparticle quarks, $\bar{\psi}\psi$,

$$\bar{\psi}\psi \to \left(\bar{\psi}\psi\right)_c + \bar{\psi}\psi. \tag{2.9}$$

We also consider that the functional measure of the generating functional will be decomposed analogously. The separation of the kinetic and mass terms of $S_{\rm NJL}$ is straightforward, while the resulting interaction term in the Lagrangian can be written as $\mathcal{L}_I = \mathcal{L}_q + \mathcal{L}_c + \mathcal{L}_{\rm int}$ [22], where

$$\mathcal{L}_c = \frac{g}{2}\left[(\bar{\psi}\lambda^a\psi)_c^2 + (\bar{\psi}i\gamma_5\lambda^a\psi)_c^2\right],$$

$$\mathcal{L}_q = \frac{g}{2}\left[(\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi}i\gamma_5\lambda^a\psi)^2\right],$$

$$\begin{aligned}\mathcal{L}_{\rm int} =& \frac{g}{2}\Big[(\bar{\psi}\lambda^a\psi)_c\cdot(\bar{\psi}\lambda^a\psi) + (\bar{\psi}\lambda^a\psi)\cdot(\bar{\psi}\lambda^a\psi)_c \\ &+(\bar{\psi}i\gamma_5\lambda^a\psi)_c\cdot(\bar{\psi}i\gamma_5\lambda^a\psi) + (\bar{\psi}i\gamma_5\lambda^a\psi)\cdot(\bar{\psi}i\gamma_5\lambda^a\psi)_c\Big].\end{aligned}$$

We may now integrate out the quark component $(\bar{\psi}\psi)_c$ with the help of the usual SU(3) auxiliary fields S_a, P_a by recalling that the path integral of Gaussian functions can be performed exactly [11, 40],

$$N\int \mathcal{D}\Phi\, \exp\Big[i\int d^4x\left(\pm A\Phi - B\Phi^2\right)\Big] = \exp\Big(i\int d^4x\, \frac{A^2}{4B}\Big),$$

where N is an unimportant normalization constant. We may then write the identities

$$\exp\left[i\int d^4x\,\frac{g}{2}\,(\bar{\psi}\lambda^a\psi)_c^2\right] = N\int \mathcal{D}S_a \exp\left\{i\int d^4x\,\left[-(\bar{\psi}\lambda^a\psi)_c\,S_a - \frac{1}{2g}S_a^2\right]\right\},$$

$$\exp\left[i\int d^4x\,\frac{g}{2}\,(\bar{\psi}i\gamma_5\lambda^a\psi)_c^2\right] = N\int \mathcal{D}P_a \exp\left\{i\int d^4x\,\left[-(\bar{\psi}i\gamma_5\lambda^a\psi)_c\,P_a - \frac{1}{2g}P_a^2\right]\right\},$$

which allow us to drop the fourth order quark interaction $\mathcal{L}_c$. The generating functional becomes

$$\begin{aligned}Z[\bar{\eta},\eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}S_a\mathcal{D}P_a\, \exp\Big\{i\int d^4x\,\Big[&\bar{\psi}_c\Big(i\partial\!\!\!/ - m - S_a\lambda^a - i\gamma_5 P_a\lambda^a\Big)\psi_c \\ +\mathcal{L}_{\rm int} - \frac{1}{2g}(S_a^2 + P_a^2) + \bar{\psi}(i\partial\!\!\!/ - m)\psi &+ \frac{g}{2}\left[(\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi}i\gamma_5\lambda^a\psi)^2\right]\Big]\Big\}.\end{aligned}$$

As we can see, the introduction of the auxiliary fields into the generating functional leads to contributions to the mass term. We then define the constituent quark mass matrix by the expression

$$M = m + S_a\lambda^a + i\gamma_5 P_a\lambda^a. \tag{2.10}$$

Defining

$$S^{-1}(x-y) = \left\{i\partial\!\!\!/ - M + g\left[\lambda^a(\bar{\psi}\lambda^a\psi) + \lambda^a i\gamma_5(\bar{\psi}i\gamma_5\lambda^a\psi)\right]\right\}\delta^4(x-y), \tag{2.11}$$

the generating functional can be written as

$$\begin{aligned} Z[\bar{\eta},\eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}S_a\mathcal{D}P_a\, Z_c[\bar{\eta},\eta]\, \exp\left\{i\int d^4x\left[-\frac{1}{2g}(S_a^2+P_a^2)\right.\right. \\ \left.\left.+\bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{g}{2}\left[(\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi}i\gamma_5\lambda^a\psi)^2\right] + \bar{\psi}\eta + \bar{\eta}\psi\right]\right\}, \end{aligned} \tag{2.12}$$

where

$$Z_c[\bar{\eta},\eta] = \int \mathcal{D}\bar{\psi}_c\mathcal{D}\psi_c\, \exp\left\{i\int d^4x\left[\int d^4y\, \bar{\psi}_c(x)S^{-1}(x-y)\psi_c(y) + \bar{\psi}_c\eta + \bar{\eta}\psi_c\right]\right\}. \tag{2.13}$$

Eq. (2.13) is a Gaussian integral on the Grassmann variables $\bar{\psi}_c$, ψ_c of the form [40]

$$\int\prod_j d\theta_j^* d\theta_j\, e^{i\theta^\dagger\mathcal{M}\theta + i\eta^\dagger\theta + i\theta^\dagger\eta} = \det(-i\mathcal{M})\, e^{-i\eta^\dagger\mathcal{M}^{-1}\eta}, \tag{2.14}$$

where $\theta_1,\ldots,\theta_n,\theta_1^*,\ldots,\theta_n^*$ are the generators of a $2n$-dimensional Grassmann algebra. Using (2.14) to integrate out the component $(\bar{\psi}\psi)_c$, with the identification $\mathcal{M} \equiv S^{-1}$, where $S^{-1} \equiv \int d^4y\, S^{-1}(x-y)$, we get

$$\begin{aligned} Z_c[\bar{\eta},\eta] &= \det\left(-iS^{-1}\right)\, e^{-i\int d^4x\, d^4y\, \bar{\eta}(x)S(x-y)\eta(y)} \\ &= e^{\mathrm{Tr}\ln\left(-iS^{-1}\right)}\, e^{-i\int d^4x\, d^4y\, \bar{\eta}(x)S(x-y)\eta(y)}, \end{aligned} \tag{2.15}$$

where Tr stands for trace over flavor, color and Dirac indices, and also an integration over the spacetime coordinates (or, equivalently, momentum coordinates).

By plugging the result (2.15) into the generating functional (2.12) one finds the resulting effective action for the quasiparticle quarks with the auxiliary variables to be given by

$$\begin{aligned} S_{\text{eff}} = &-i\mathrm{Tr}\,\ln\left(-iS^{-1}\right) \\ &+ \int d^4x\left\{-\frac{1}{2g}(S_a^2+P_a^2) + \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{g}{2}\left[(\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi}i\gamma_5\lambda^a\psi)^2\right]\right\}. \end{aligned} \tag{2.16}$$

The expansion of the fermion determinant in terms of the quark bilinears (in the presence of the uniform magnetic field) will be left for Chapter 3. In the remainder of this section we shall provide a further discussion on the constituent quark masses defined back in Eq. (2.10).

2.3.2 The mass matrix

Assuming that only the scalar auxiliary fields S_a have nonvanishing vacuum expectation value and dropping the contribution of the pseudoscalar fields P_a, we write the constituent quark mass matrix as [11, 22]

$$M = m + S_a\lambda^a. \tag{2.17}$$

Regardless of its form, the mass matrix can be brought to diagonal form through flavor-mixing transformations [11]. Since we are interested in a diagonal matrix, we may only consider the contributions of the term $S_a\lambda^a$ that come from the diagonal Gell-Mann matrices, namely, $a = 0, 3, 8$. Therefore, we have

$$\begin{aligned} M &= m + S_a\lambda^a = m + S_0\lambda^0 + S_3\lambda^3 + S_8\lambda^8 \\ &= \begin{pmatrix} m_u + \sigma_u & 0 & 0 \\ 0 & m_d + \sigma_d & 0 \\ 0 & 0 & m_s + \sigma_s \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \sigma_u &= \sqrt{\frac{2}{3}}S_0 + S_3 + \sqrt{\frac{1}{3}}S_8, \\ \sigma_d &= \sqrt{\frac{2}{3}}S_0 - S_3 + \sqrt{\frac{1}{3}}S_8, \\ \sigma_s &= \sqrt{\frac{2}{3}}S_0 - 2\sqrt{\frac{1}{3}}S_8, \end{aligned}$$

from which we find

$$S_a^2 = S_0^2 + S_3^2 + S_8^2 = \frac{1}{2}(\sigma_u^2 + \sigma_d^2 + \sigma_s^2).$$

Thus, we see how the effective action depends explicitly on the corrections to the quark masses.

2.3.3 The gap equation

In order to obtain the effective quark masses, we impose the stationary condition

$$\left.\frac{\partial S_{\text{eff}}}{\partial \sigma_f}\right|_{\sigma_f \to \langle\sigma_f\rangle} = 0.$$

Since quasiparticle fields are zero in the ground state, $\psi, \bar{\psi} \to 0$, the stationary condition yields

$$i \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}_{DC} \left(\frac{1}{\not{p} - M_f} \right) - \frac{1}{2g} \langle \sigma_f \rangle = 0,$$

or, writing $\langle \sigma_f \rangle = M_f - m_f$,

$$M_f = m_f + 2g\, \mathrm{tr}_{DC} \Big[i S_f^{(0)}(0) \Big], \tag{2.18}$$

where tr_{DC} denotes the trace over color and Dirac indices, and $S_f^{(0)}(x-y)$ is the free quark propagator of flavor f, with $f = u, d, s$. This is the gap equation of the NJL model.

In the mean field approximation, the chiral condensate is given by [12]

$$\left\langle \bar{\psi}_f \psi_f \right\rangle = -\mathrm{tr}_{DC} \Big[i S_f^{(0)}(0) \Big], \tag{2.19}$$

and so the gap equation can be written as

$$M_f = m_f - 2g \left\langle \bar{\psi}_f \psi_f \right\rangle. \tag{2.20}$$

Eq. (2.20) shows the dynamical quark mass generation due to chiral symmetry breaking, which is signaled by the nonvanishing chiral condensate. We thus need to obtain an expression for the condensate $\left\langle \bar{\psi}_f \psi_f \right\rangle$ in order to solve the gap equation for the constituent quark masses.

The free quark propagator in the absence of external fields is given by

$$S_f^{(0)}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{\not{p} + M_f}{p^2 - M_f^2 + i\epsilon} e^{ip\cdot(x-y)}. \tag{2.21}$$

It then follows from Eq. (2.19) that

$$\left\langle \bar{\psi}_f \psi_f \right\rangle = -4iN_c \int \frac{d^4p}{(2\pi)^4} \frac{M_f}{p^2 - M_f^2}, \tag{2.22}$$

where the $i\epsilon$ prescription is now implicit. The Dirac trace resulted in a factor of 4 while the color trace resulted in a factor of N_c, which is the number of colors ($N_c = 3$).

The integral in (2.22) is divergent and thus a regularization scheme must be adopted. Since the NJL model is nonrenormalizable, it depends on the regularization procedure chosen, which is not unique. Here we present two different schemes: the four-momentum cutoff in Euclidean space and the regularization in proper time. Although we wish to adopt the former, the latter will prove to be the most convenient choice when dealing with magnetic field effects. However, since the magnetic field dependent terms introduces no new divergences to the equations, we will be able to separate these contributions from the divergent ones and then adopt any other regularization scheme. It is important, then, to obtain the expression for the chiral condensate by adopting both procedures.

Four-momentum cutoff In order to solve the integral in (2.22) we rotate the momentum coordinates to Euclidean space by letting $p^0 \to ip_E^0$ and $p^i \to p_E^i$, so that $d^4p \to id^4p_E$ and $p^2 \to -p_E^2$. Hence, the condensate becomes

$$\left\langle \bar{\psi}_f \psi_f \right\rangle = -4N_c \int \frac{d^4p_E}{(2\pi)^4} \frac{M_f}{p_E^2 + M_f^2} = -\frac{M_f N_c}{4\pi^4} \int d\Omega_4 \int_0^\infty dp_E \, \frac{p_E^3}{p_E^2 + M_f^2},$$

where $d\Omega_4$ is the differential solid angle in four dimensions in Euclidean space. The angular integral yields

$$\begin{aligned} \int d\Omega_4 &= \int_0^\pi d\varphi_1 \sin^2\varphi_1 \int_0^\pi d\varphi_2 \sin\varphi_2 \int_0^{2\pi} d\varphi_3 \\ &= 2\pi^2, \end{aligned}$$

and, by letting $\xi = p_E^2$, we are left with

$$\left\langle \bar{\psi}_f \psi_f \right\rangle = -\frac{M_f N_c}{4\pi^2} \int_0^\infty d\xi \, \frac{\xi}{\xi + M_f^2}.$$

Since this integral diverges in its upper limit, we introduce a cutoff on $\xi = p_E^2 < \Lambda^2$:

$$\begin{aligned} \left\langle \bar{\psi}_f \psi_f \right\rangle &= -\frac{M_f N_c}{4\pi^2} \int_0^{\Lambda^2} d\xi \, \frac{\xi}{\xi + M_f^2} \\ &= -\frac{M_f N_c}{4\pi^2} \left[\Lambda^2 - M_f^2 \ln\left(\frac{\Lambda^2 + M_f^2}{M_f^2} \right) \right]. \end{aligned} \tag{2.23}$$

Regularization in proper time The integral in (2.22) can also be solved by introducing a proper time variable s and writing

$$\left\langle \bar{\psi}_f \psi_f \right\rangle = -4N_c M_f \int \frac{d^4p}{(2\pi)^4} \int_0^\infty ds \, e^{is(p^2 - M_f^2)}.$$

As it was done for the regularization using a four-momentum cutoff, the momentum integral can be performed by rotating to Euclidean space. The result is

$$\left\langle \bar{\psi}_f \psi_f \right\rangle = \frac{iM_f N_c}{4\pi^2} \int_0^\infty \frac{ds}{s^2} e^{-isM_f^2}.$$

We now make the variable change $s \to -is$ and introduce a proper time cutoff $\Lambda_{\rm pt}$ to account for the integral divergence in its lower limit to obtain

$$\left\langle \bar{\psi}_f \psi_f \right\rangle = -\frac{M_f N_c}{4\pi^2} \int_{1/\Lambda_{\rm pt}^2}^\infty \frac{ds}{s^2} e^{-sM_f^2}.$$

By using the result (A.22) with $n = 1$ as well as the expansion in Eq. (A.12), we find

$$\int_{1/\Lambda_{\rm pt}^2}^{\infty} \frac{ds}{s^2} e^{-M_f^2 s} = \Lambda_{\rm pt}^2 e^{-M_f^2/\Lambda_{\rm pt}^2} + M_f^2 {\rm Ei}\left(-\frac{M_f^2}{\Lambda_{\rm pt}^2}\right)$$
$$= \Lambda_{\rm pt}^2 + M_f^2\left(\ln\frac{M_f^2}{\Lambda_{\rm pt}^2} + \gamma_E - 1\right) + O\left(\frac{1}{\Lambda_{\rm pt}^2}\right), \tag{2.24}$$

and the chiral condensate becomes

$$\left\langle \bar{\psi}_f \psi_f \right\rangle = -\frac{M_f N_c}{4\pi^2}\left[\Lambda_{\rm pt}^2 + M_f^2\left(\ln\frac{M_f^2}{\Lambda_{\rm pt}^2} + \gamma_E - 1\right)\right]. \tag{2.25}$$

As we can see, both regularization procedures exhibit the expected quadratic and logarithmic divergences. The gap equation is written in terms of the chiral condensate in Eq. (2.20). Then, by using Eqs. (2.23) and (2.25) we find that, in the four-momentum cutoff and proper time regularization procedures, respectively, the constituent quark masses are the solutions of

$$M_f - m_f = \frac{g N_c M_f}{2\pi^2}\left[\Lambda^2 - M_f^2 \ln\left(\frac{\Lambda^2 + M_f^2}{M_f^2}\right)\right],$$

$$M_f - m_f = \frac{g N_c M_f}{2\pi^2}\left[\Lambda_{\rm pt}^2 + M_f^2\left(\ln\frac{M_f^2}{\Lambda_{\rm pt}^2} + \gamma_E - 1\right)\right],$$

We notice that the gap equations depend on five free parameters, namely the cutoff Λ, or $\Lambda_{\rm pt}$, the coupling constant g and the current quark masses m_u, m_d and m_s. Since $m_u \simeq m_d$, we can take $m_u = m_d \equiv m_{ud}$ which then leaves us with four parameters to be fixed. This can be done by computing observables and setting them to their experimental values, such as meson masses and decay constants. This will be the subject of Section 2.4, where we will compute the neutral pion and kaon masses, as well as the pion and kaon decay constants. However, in order to do so, we need to choose a regularization procedure and stick to it. In this work, we choose the four-momentum cutoff regularization scheme, so our gap equation in the absence of external fields reads

$$M_f - m_f = \frac{g_\Lambda N_c M_f}{2\pi^2}\left[1 - \frac{M_f^2}{\Lambda^2} \ln\left(\frac{\Lambda^2 + M_f^2}{M_f^2}\right)\right], \tag{2.26}$$

where we have defined the dimensionless coupling constant $g_\Lambda = g\Lambda^2$.

Eq. (2.26) is a self-consistent or transcendental equation for M_f and one must look for its solution by using numerical methods. Figure 2.1 shows the solution for the three quark flavors as a function of the dimensionless coupling constant g_Λ. For the current quark masses we used $m_{ud} = 6\,\text{MeV}$ and $m_s = 165.7\,\text{MeV}$, and for the cutoff we have set $\Lambda = 914.6\,\text{MeV}$ (see Section 2.4).

From Figure 2.1 we see how the constituent quark masses increase with increasing

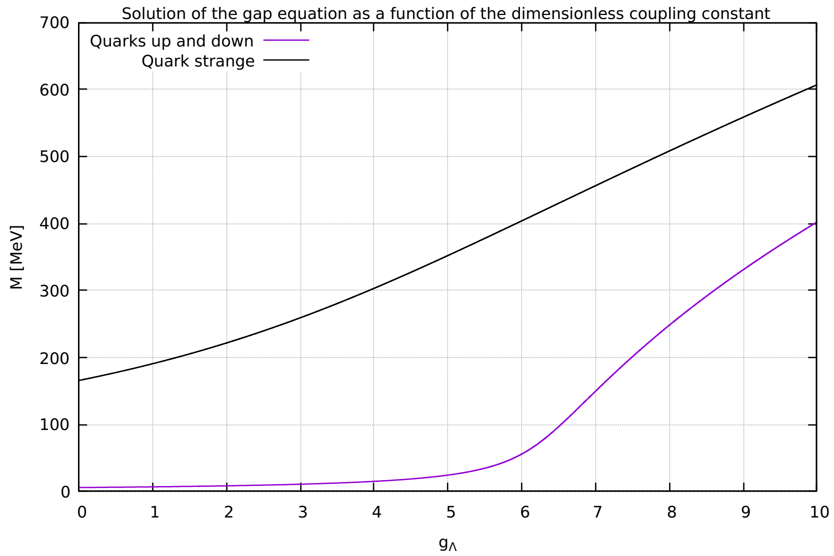

Figure 2.1: Solution of the gap equation (2.26). Here we used $\Lambda = 914.6\,\text{MeV}$, $m_{ud} = 6\,\text{MeV}$ and $m_s = 165.7\,\text{MeV}$.

coupling constant. Particularly for $g_\Lambda \simeq 8.16$ ($g \simeq 9.76\,\text{GeV}^{-2}$ for $\Lambda = 914.6\,\text{MeV}$) we find $M_{ud} \simeq 262.8\,\text{MeV}$ and $M_s \simeq 516.4\,\text{MeV}$. The nonvanishing values of the constituent quark masses show that the NJL model indeed exhibits the dynamical generation of fermion masses due to the spontaneous chiral symmetry breaking. Once we include the magnetic field in the NJL Lagrangian in Chapter 3 we shall see how this field contributes to enhance the values of the dynamically generated masses, and thus the breaking of chiral symmetry, which is the phenomenon of magnetic catalyses. But before that, let us now discuss about the procedure of parameter fixing.

2.4 Meson properties and parameter fixing

We saw in Section 2.3 that the NJL model contains some free parameters, such as the regularization parameter, which here is the four-momentum cutoff Λ, the coupling constant g and the current quark masses. Since we are considering $m_u = m_d \equiv m_{ud}$, we have, together with m_s, four free parameters in our model. To fix those we are required to compute four observable quantities and set them equal to their experimental values. That is where meson properties come in. For the three-flavor NJL model, the pseudoscalar-meson spectrum consists of the π, K, η and η' mesons [11]. In this work, we shall treat only the first two cases. Specifically, we choose to fix our parameters by computing the masses of the neutral pion and kaon mesons, as well as pion and kaon decay constants. Furthermore, the procedure developed in this section may also be extended to compute the pseudoscalar neutral meson masses in the presence of a uniform magnetic field in Chapter 4.

2.4.1 Bound state equation

In the framework of the NJL model, the basic idea is to consider the mesons as collective excitations of quark/antiquark pairs. Methods to examine these excitations include the auxiliary-field path integrals, Bethe-Salpeter equations, usually in the framework of the Random Phase Approximation (RPA) [11, 41, 42], and so on [5]. Naturally, we will employ the auxiliary-field method.

In Section 2.3 we obtained an effective action given by

$$\begin{aligned} S_{\text{eff}} = &-i\text{Tr}\ln\Big(-i\Big\{i\partial\!\!\!/ - m - S_a\lambda^a - i\gamma_5 P_a\lambda^a + g\Big[\lambda^a(\bar\psi\lambda^a\psi) + \lambda^a i\gamma_5(\bar\psi i\gamma_5\lambda^a\psi)\Big]\Big\}\Big) \\ &+\int d^4x\left\{-\frac{1}{2g}(S_a^2+P_a^2) + \bar\psi(i\partial\!\!\!/ - m)\psi + \frac{g}{2}\Big[(\bar\psi\lambda^a\psi)^2 + (\bar\psi i\gamma_5\lambda^a\psi)^2\Big]\right\}. \end{aligned}$$

Since the scalar auxiliary fields have nonvanishing vacuum expectation value, we can shift the variables S_a to a field $S'_a = S_a - \sigma_a$, where σ_a stands for the expectation value of S_a [11, 12]. Then, while the $\sigma_a\lambda^a$ term leads to the contribution to the effective mass, we can relabel the field $S'_a \to S_a$ and write and the effective action, in the ground state, as

$$\begin{aligned} S_{\text{eff}}(S_a, P_a) &= -i\text{Tr}\ln\Big[-i\Big(i\partial\!\!\!/ - M - S_a\lambda^a - i\gamma_5 P_a\lambda^a\Big)\Big] + \int d^4x\left[-\frac{1}{2g}(S_a^2+P_a^2)\right] \\ &= -i\text{Tr}\ln\left[1 + \frac{1}{i\partial\!\!\!/ - M}(S_a + i\gamma_5 P_a)\lambda^a\right] + \int d^4x\left[-\frac{1}{2g}(S_a^2+P_a^2)\right], \end{aligned}$$

where we dropped a constant term which would lead to an unimportant constant in the generating functional. We may then use the expansion (A.10) to write

$$\begin{aligned} S_{\text{eff}}(S_a, P_a) = &\int d^4x\left[-\frac{1}{2g}(S_a^2+P_a^2)\right] - i\text{Tr}\Big[S^{(0)}(S_a + i\gamma_5 P_a)\lambda^a\Big] \\ &+\frac{i}{2}\text{Tr}\Big[S^{(0)}(S_a + i\gamma_5 P_a)\lambda^a S^{(0)}(S_b + i\gamma_5 P_b)\lambda^b\Big] + \dots, \end{aligned}$$

where $S^{(0)} = \Big(i\partial\!\!\!/ - M\Big)^{-1}$. Dropping the terms with vanishing Dirac trace leads to

$$\begin{aligned} S_{\text{eff}}(S_a, P_a) = &\int d^4x\left[-\frac{1}{2g}(S_a^2+P_a^2)\right] - i\text{Tr}\Big(S^{(0)}S_a\lambda^a\Big) + \frac{i}{2}\text{Tr}\Big(S^{(0)}S_a\lambda^a S^{(0)}S_b\lambda^b\Big) \\ &+\frac{i}{2}\text{Tr}\Big(S^{(0)}i\gamma_5 P_a\lambda^a S^{(0)}i\gamma_5 P_b\lambda^b\Big) + \dots . \end{aligned}$$

Note that the first order term is of the form $\text{Tr}[F(P)H(X)]$, that is, the trace of a product of operators that are local in p and x [40]. Therefore, the trace can be separated,

$$\begin{aligned}
\mathrm{Tr}\left[F(P)H(X)\right] &= \mathrm{tr}\int \frac{d^4p}{(2\pi)^4}\langle p|F(P)H(X)|p\rangle \\
&= \mathrm{tr}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^4p'}{(2\pi)^4}\int d^4x \int d^4x' \,\langle p|F(P)|p'\rangle\langle p'|x'\rangle\langle x'|H(X)|x\rangle\langle x|p\rangle \\
&= \mathrm{tr}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^4p'}{(2\pi)^4}\int d^4x \int d^4x' \,F(p')\langle p|p'\rangle e^{-ip'\cdot x'}H(x')\langle x|x'\rangle e^{ip\cdot x} \\
&= \mathrm{tr}\int \frac{d^4p}{(2\pi)^4}F(p)\int d^4x\, H(x),
\end{aligned}$$

where now the trace tr no longer denotes an integral over spacetime (or momentum) variables. Similarly, for the second-order terms we have

$$\begin{aligned}
\mathrm{Tr}[F(P)H(X)F(P)H(X)] &= \mathrm{tr}\int \frac{d^4p}{(2\pi)^4}\,\langle p|F(P)H(X)F(P)H(X)|p\rangle \\
&= \mathrm{tr}\int \frac{d^4p}{(2\pi)^4}\int \frac{d^4p'}{(2\pi)^4}\int d^4x \int d^4x' \,F(p)F(p')H(x)H(x') \\
&\times e^{-i(p-p')\cdot x}e^{i(p-p')\cdot x'}.
\end{aligned}$$

If we assume the local limit on H, $H(x') \simeq H(x)$, we write

$$\mathrm{Tr}[F(P)H(X)F(P)H(X)] = \mathrm{tr}\int d^4x \int \frac{d^4p}{(2\pi)^4}F^2(p)H^2(x).$$

Applying the results above and considering only the terms dependent on the pseudoscalar auxiliary fields (since we are interested only in the pseudoscalar meson spectrum) we write

$$S_{\text{eff}}(P_a) = -\frac{1}{2g}\int d^4x \left[1 - g\Pi_a^{\text{ps}}(k^2)\right]P_a^2 + \ldots, \tag{2.27}$$

where we are considering only diagonal terms for the meson polarization functions,

$$\Pi_a^{\text{ps}}(k^2) = -i\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\Big[iS^{(0)}(p)i\gamma_5\lambda_a iS^{(0)}(p-k)i\gamma_5\lambda_a\Big],$$

which is given by the diagram shown in Figure 2.2, with k being the meson four-momentum.

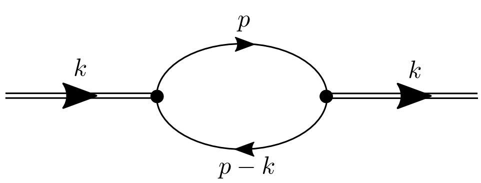

Figure 2.2: Meson polarization diagram.

Therefore, we have found a Lagrangian for the pseudoscalar auxiliary fields given by

$$\mathcal{L}_{\text{eff}}(P_a) = -\frac{1}{2g} F_a(k^2) P_a^2, \tag{2.28}$$

where $F_a(k^2) \equiv 1 - g\Pi_a^{\text{ps}}(k^2)$. For the physical propagating meson fields P_a', we wish to write a Lagrangian (in momentum space) of the form

$$\mathcal{L}_{\text{eff}}(P_a') = \frac{1}{2}\Big(k^2 - m_{\text{ps}}^2\Big) P_a'^2, \tag{2.29}$$

where m_{ps}^2 stands for the particle mass. To do so, we expand $F_a(k^2)$ and write (2.28) as

$$\begin{aligned}
\mathcal{L}_{\text{eff}}(P_a) &= -\frac{1}{2g}\left[F_a(m_{\text{ps}}^2) + \Big(k^2 - m_{\text{ps}}^2\Big) \frac{\mathrm{d}F_a}{\mathrm{d}k^2}\bigg|_{k^2=m_{\text{ps}}^2} + \ldots\right] P_a^2 \\
&= -\frac{1}{2}\left[k^2 - m_{\text{ps}}^2 + \frac{F_a(m_{\text{ps}}^2)}{F_a'(m_{\text{ps}}^2)} + \ldots\right] \frac{1}{g}\frac{\mathrm{d}F_a}{\mathrm{d}k^2}\bigg|_{k^2=m_{\text{ps}}^2} P_a^2 \\
&= \frac{1}{2}\left[k^2 - m_{\text{ps}}^2 + \frac{F_a(m_{\text{ps}}^2)}{F_a'(m_{\text{ps}}^2)} + \ldots\right] \frac{\mathrm{d}\Pi_a^{\text{ps}}}{\mathrm{d}k^2}\bigg|_{k^2=m_{\text{ps}}^2} P_a^2.
\end{aligned}$$

Defining

$$g_{\text{ps}qq}^{-2} = \frac{\mathrm{d}\Pi^{\text{ps}}}{\mathrm{d}k^2}\bigg|_{k^2=m_{\text{ps}}^2} \tag{2.30}$$

and re-scaling the pseudoscalar field, $P_a \to P_a' = P_a/g_{\text{ps}qq}$, we find

$$\mathcal{L}_{\text{eff}}(P_a') = \frac{1}{2}\left[k^2 - m_{\text{ps}}^2 + \frac{F_a(m_{\text{ps}}^2)}{F_a'(m_{\text{ps}}^2)} + \ldots\right] P_a'^2.$$

Comparison with the Lagrangian in Eq. (2.29) leads to the conclusion $F(m_{\text{ps}}^2) = 0$, or,

$$1 - g\Pi^{\text{ps}}(m_{\text{ps}}^2) = 0. \tag{2.31}$$

This is the bound state equation, from which we can obtain the pseudoscalar meson masses. To do so, we first need to obtain an expression for the meson polarization tensors.

2.4.2 Meson polarization tensors

The pseudoscalar meson polarization tensor is given by

$$\Pi_{ij}^{\text{ps}}(k^2) = -i\int \frac{d^4p}{(2\pi)^4} \text{tr}\Big[T_i i S^{(0)}(p) i\gamma_5 T_j i S^{(0)}(p-k) i\gamma_5\Big], \tag{2.32}$$

where $k_\mu = (k_0, \mathbf{0})$ is the meson four-momentum, tr is the trace over Dirac, flavor and color indices, and $S^{(0)}(p)$ is the quark propagator matrix in momentum space. Here, T_i

and T_j select the appropriate flavor channel [11],

$$T_i = \begin{cases} \frac{1}{\sqrt{2}}(\lambda_1 \pm i\lambda_2) & \text{for } \pi^\pm \\ \lambda_3 & \text{for } \pi^0 \\ \frac{1}{\sqrt{2}}(\lambda_4 \pm i\lambda_5) & \text{for } K^\pm \\ \frac{1}{\sqrt{2}}(\lambda_6 \pm i\lambda_7) & \text{for } K^0, \bar{K}^0 \end{cases}.$$

For the π^0 meson, we set $T_i = T_j = \lambda_3$ and denote $\Pi^{\rm ps}_{33}(k^2) = \Pi^{\rm ps}_{\pi^0}(k^2)$. Then, we have

$$\Pi^{\rm ps}_{\pi^0}(k^2) = iN_c \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}_D\Big[S_u^{(0)}(p)i\gamma_5 S_u^{(0)}(p-k)i\gamma_5 + S_d^{(0)}(p)i\gamma_5 S_d^{(0)}(p-k)i\gamma_5\Big],$$

where tr_D denotes the Dirac trace. Similarly, for the K^0 meson, we set $T_i = \frac{1}{\sqrt{2}}(\lambda_6 - i\lambda_7)$ and $T_j = \frac{1}{\sqrt{2}}(\lambda_6 + i\lambda_7)$ in Eq. (2.32), which leads to

$$\Pi^{\rm ps}_{K^0}(k^2) = 2iN_c \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}_D\Big[S_s^{(0)}(p)i\gamma_5 S_d^{(0)}(p-k)i\gamma_5\Big].$$

The meson polarization tensors can be written as

$$\Pi^{\rm ps}_{\pi^0}(k^2) = \frac{1}{2}\Big[\Pi^{\rm ps}_{uu}(k^2) + \Pi^{\rm ps}_{dd}(k^2)\Big], \tag{2.33a}$$

$$\Pi^{\rm ps}_{K^0}(k^2) = \Pi^{\rm ps}_{sd}(k^2), \tag{2.33b}$$

where

$$\Pi^{\rm ps}_{fg}(k^2) = 2iN_c \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}_D\Big[S_f^{(0)}(p)i\gamma_5 S_g^{(0)}(p-k)i\gamma_5\Big]. \tag{2.34}$$

In momentum space, the quark propagator is given by

$$S_f(p) = \frac{\not{p} + M_f}{p^2 - M_f^2},$$

from which we find

$$\mathrm{tr}_D[S_f(p)i\gamma_5 S_g(p-k)i\gamma_5] = -4\frac{p^2 - p\cdot k + M_f M_g}{\big(p^2 - M_f^2\big)\big[(p-k)^2 - M_g^2\big]}.$$

Then, rewriting

$$\frac{p^2 - p\cdot k + M_f M_g}{\big(p^2 - M_f^2\big)\big[(p-k)^2 - M_g^2\big]} = \frac{p^2 - p\cdot k + M_f M_g}{2\Big[p^2 - p\cdot k + \frac{k^2}{2} - \frac{1}{2}\Big(M_f^2 + M_g^2\Big)\Big]}\left[\frac{1}{p^2 - M_f^2} + \frac{1}{(p-k)^2 - M_g^2}\right],$$

and

$$p^2 - p\cdot k + M_f M_g = p^2 - p\cdot k + \frac{k^2}{2} - \frac{1}{2}\Big(M_f^2 + M_g^2\Big) - \frac{k^2}{2} + \frac{1}{2}(M_f - M_g)^2,$$

we find

$$\begin{aligned}\Pi^{\rm ps}_{fg}(k^2) &= 4iN_c \int \frac{d^4p}{(2\pi)^4}\left[\frac{1}{p^2 - M_f^2} + \frac{1}{(p-k)^2 - M_g^2}\right] - 4iN_c\Big[k^2 - (M_f - M_g)^2\Big] I_{fg}(k^2) \\ &= 4iN_c \int \frac{d^4p}{(2\pi)^4}\left[\frac{1}{p^2 - M_f^2} + \frac{1}{p^2 - M_g^2}\right] - 4iN_c\Big[k^2 - (M_f - M_g)^2\Big] I_{fg}(k^2), \quad (2.35)\end{aligned}$$

where

$$I_{fg}(k^2) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{\Big(p^2 - M_f^2\Big)\Big[(p-k)^2 - M_g^2\Big]}. \quad (2.36)$$

The first two integrals in (2.35) can be written in terms of the gap equation by noticing from Eqs. (2.20), (2.22) and (2.23) that

$$4iN_c \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - M_f^2} = \frac{M_f - m_f}{2gM_f} = \frac{N_c}{4\pi^2}\left[\Lambda^2 - M_f^2 \ln\left(\frac{\Lambda^2 + M_f^2}{M_f^2}\right)\right]. \quad (2.37)$$

In order to solve Eq. (2.36), we rotate the momenta to Euclidean space, $p_0 \to ip_{0E}$ and $k_0 \to ik_{0E}$, so that

$$I_{fg}(k_E^2) = i \int \frac{d^4p_E}{(2\pi)^4} \frac{1}{\Big(p_E^2 + M_f^2\Big)\Big[(p_E - k_E)^2 + M_g^2\Big]}.$$

We can solve the integral above with the aid of the Feynman parameter x, introduced via the formula

$$\frac{1}{AB} = \int_0^1 \frac{dx}{\left[xA + (1-x)B\right]^2}.$$

This allows us to write

$$\begin{aligned}\frac{1}{\Big(p_E^2 + M_f^2\Big)\Big[(p_E - k_E)^2 + M_g^2\Big]} &= \int_0^1 dx\left\{x\Big[(p_E - k_E)^2 + M_f^2\Big] + (1-x)\Big(p_E^2 + M_g^2\Big)\right\}^{-2} \\ &= \int_0^1 dx\left[p_E^2 - 2xp_E\cdot k_E + xk_E^2 + xM_f^2 + (1-x)M_g^2\right]^{-2} \\ &= \int_0^1 dx\, \frac{1}{\Big(\ell^2 + D_{fg}^2\Big)^2},\end{aligned}$$

where

$$\ell \equiv p_E - xk_E,$$

$$D_{fg}^2 \equiv x(1-x)k_E^2 + xM_f^2 + (1-x)M_g^2.$$

Since $d^4p_E = d^4\ell$ we have

$$
\begin{aligned}
I_{fg}(k_E^2) &= i\int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4}\frac{1}{\left(\ell^2+D_{fg}^2\right)^2} \\
&= \frac{i}{16\pi^2}\int_0^1 dx \int_0^\infty d\left(\ell^2\right)\frac{\ell^2}{\left(\ell^2+D_{fg}^2\right)^2} \\
&= \frac{i}{16\pi^2}\int_0^1 dx \left[\ln\left(\frac{\Lambda^2+D_{fg}^2}{D_{fg}^2}\right)+\frac{D_{fg}^2}{\Lambda^2+D_{fg}^2}-1\right],
\end{aligned} \tag{2.38}
$$

where Λ is the usual four-momentum cutoff.

Substituting Eqs. (2.37) and (2.38) back in Eq. (2.35), and rotating the momentum k back to Minkowski space, we find

$$
\begin{aligned}
\Pi_{fg}^{\mathrm{ps}}(k^2) =& \frac{M_f-m_f}{2gM_f}+\frac{M_g-m_g}{2gM_g} \\
&+\frac{N_c}{4\pi^2}\left[k^2-(M_f-M_g)^2\right]\int_0^1 dx\left\{\ln\left[\frac{\Lambda^2+D_{fg}^2(k^2)}{D_{fg}^2(k^2)}\right]+\frac{D_{fg}^2(k^2)}{\Lambda^2+D_{fg}^2(k^2)}-1\right\},
\end{aligned} \tag{2.39}
$$

with

$$
D_{fg}^2(k^2) = -x(1-x)k^2 + xM_f^2 + (1-x)M_g^2. \tag{2.40}
$$

Therefore, to find the neutral pseudoscalar meson masses one is required to solve the equations

$$
\begin{aligned}
1-g\Pi_{\pi^0}^{\mathrm{ps}}(m_\pi^2) &= 0 \\
1-g\Pi_{K^0}^{\mathrm{ps}}(m_K^2) &= 0
\end{aligned} \tag{2.41}
$$

with $\Pi_{\pi^0}^{\mathrm{ps}}(k^2)$ and $\Pi_{K^0}^{\mathrm{ps}}(k^2)$ given by Eqs. (2.33) in terms of the quark polarization functions given by Eq. (2.39). Before we obtain the values of those masses, let us first tackle the question of obtaining the other two necessary observables quantities: the pion and kaon decay constants.

2.4.3 Meson decay constants

A minimal local interaction Lagrangian capable of describing the coupling of the meson fields to quark fields is given by [11]

$$
\begin{aligned}
\mathcal{L}_{\mathrm{psqq}} &= \bar{\psi}i\gamma_5\lambda^a P_a\psi \\
&= g_{\mathrm{psqq}}\bar{\psi}i\gamma_5\lambda^a P'_a\psi,
\end{aligned} \tag{2.42}
$$

where g_{psqq} is the coupling strength of the mesons to the quarks defined back in Eq. (2.30).

The pseudoscalar meson decay constants, $f_{\rm ps}$ with ps $= \pi, K$, are calculated from the vacuum to one-meson axial-vector matrix element [5, 11, 31],

$$if_{\rm ps} k_\mu \delta^{ab} = \left\langle 0 \middle| A^a_\mu(0) \middle| {\rm ps}^b \right\rangle, \qquad {\rm ps} = \pi, K.$$

The matrix element on the right-hand side is given by the diagram shown in Figure 2.3, with $\Gamma^b = i g_{{\rm ps}qq} \gamma_5 \lambda^b$, in the framework of the interaction Lagrangian (2.42).

$$\langle 0 | A^a_\mu(0) | {\rm ps}^b \rangle = i\gamma_\mu \gamma_5 \tfrac{\lambda^a}{2}$$

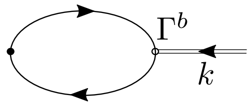

Figure 2.3: Vacuum to one pseudoscalar meson axial-vector current matrix element as a Feynman diagram. Here $\Gamma^b = g_{{\rm ps}qq} i\gamma_5 \lambda^b$ is the vertex factor.

Translating the diagram in Figure 2.3 according to the Feynman rules leads to

$$\begin{aligned} if_{\rm ps} k_\mu \delta^{ab} &= -\int \frac{d^4p}{(2\pi)^4} {\rm tr}\left[i\gamma_\mu \gamma_5 \frac{\lambda^a}{2} iS\left(p + \frac{k}{2}\right) i g_{{\rm ps}qq} \gamma_5 \lambda^b iS\left(p - \frac{k}{2}\right)\right] \\ &= -\frac{g_{{\rm ps}qq}}{2} \int \frac{d^4p}{(2\pi)^4} {\rm tr}\left[\gamma_\mu \gamma_5 \lambda^a S\left(p + \frac{k}{2}\right) \gamma_5 \lambda^b S\left(p - \frac{k}{2}\right)\right], \end{aligned} \tag{2.43}$$

where tr denotes the trace over Dirac, flavor and color indices. It is clear from Eq. (2.43) that we need to obtain an expression for the meson-quark-quark couplings $g_{{\rm ps}qq}$.

2.4.3.1 Meson-quark-quark couplings

The meson-quark-quark couplings are given by Eq. (2.30). For the π^+ meson we choose $T_i = \frac{1}{\sqrt{2}}(\lambda_1 - i\lambda_2)$ and $T_j = \frac{1}{\sqrt{2}}(\lambda_1 + i\lambda_2)$ in Eq. (2.32), so that the polarization tensor becomes

$$\Pi^{\rm ps}_{\pi^+}(k^2) = \Pi^{\rm ps}_{ud}(k^2),$$

with $\Pi^{\rm ps}_{fg}(k^2)$ given by Eq. (2.39). Then, at zero momentum transfer, one finds

$$g^{-2}_{\pi qq} = \frac{N_c}{4\pi^2}\left[\tilde{\mathcal{I}}_{ud}(0) + (M_d - M_u)^2 \left.\frac{\partial \tilde{\mathcal{I}}_{ud}}{\partial k^2}\right|_{k^2=0}\right], \tag{2.44}$$

where

$$\tilde{\mathcal{I}}_{fg}(k^2) = \int_0^1 dx \left\{ \ln\left[\frac{\Lambda^2 + D^2_{fg}(k^2)}{D^2_{fg}(k^2)}\right] + \frac{D^2_{fg}(k^2)}{\Lambda^2 + D^2_{fg}(k^2)} - 1\right\},$$

with $D^2_{fg}(k^2)$ given by Eq. (2.40). A direct calculation yields

$$\left.\frac{\partial \tilde{\mathcal{I}}_{fg}}{\partial k^2}\right|_{k^2=0} = -\int_0^1 dx\, x(1-x) \frac{1}{D^2_{fg}(0)} \left[\frac{\Lambda^2}{\Lambda^2 + D^2_{fg}(0)}\right]^2.$$

Similarly, for the K^+ meson we choose $T_i = \frac{1}{\sqrt{2}}(\lambda_4 - i\lambda_5)$ and $T_j = \frac{1}{\sqrt{2}}(\lambda_4 + i\lambda_5)$, which corresponds to the same calculations for the pion, but with the exchange $d \to s$. Thus, we conclude

$$g_{Kqq}^{-2} = \frac{N_c}{4\pi^2}\left[\tilde{\mathcal{I}}_{us}(0) + (M_s - M_u)^2 \left.\frac{\partial \tilde{\mathcal{I}}_{us}}{\partial k^2}\right|_{k^2=0}\right], \tag{2.45}$$

at zero momentum transfer.

2.4.3.2 Decay constants

Eq. (2.43) can be written as

$$i f_{\text{ps}} k_\mu = -\frac{g_{\text{ps}qq}}{2} \int \frac{d^4p}{(2\pi)^4} \text{tr}\left[\gamma_\mu \gamma_5 \lambda^a S\left(p + \frac{k}{2}\right)\gamma_5 \lambda^a S\left(p - \frac{k}{2}\right)\right].$$

Choosing the appropriate channels for computing the flavor trace, we find

$$i f_{\text{ps}} k_\mu = -g_{\text{ps}qq} N_c \int \frac{d^4p}{(2\pi)^4} \text{tr}_D\left[\gamma_\mu \gamma_5 S_f\left(p + \frac{k}{2}\right)\gamma_5 S_g\left(p - \frac{k}{2}\right)\right],$$

where $f = u$ and $g = d$ for ps $= \pi$, and $f = u$ and $g = s$ for ps $= K$.

The computation of the Dirac trace is analogous to the one for the polarization tensors in Section 2.4.2, only with the extra γ_μ matrix. The result is

$$i f_{\text{ps}} k_\mu = 4 g_{\text{ps}qq} N_c \frac{(M_f + M_g)}{2} k_\mu \int \frac{d^4p}{(2\pi)^4} \frac{1}{\left[\left(p + \frac{k}{2}\right)^2 - M_f^2\right]\left[\left(p - \frac{k}{2}\right)^2 - M_g^2\right]},$$

and we find

$$f_\pi = \frac{g_{\pi qq} N_c}{4\pi^2} \frac{(M_u + M_d)}{2} \tilde{\mathcal{I}}_{ud}(0), \tag{2.46a}$$

$$f_K = \frac{g_{Kqq} N_c}{4\pi^2} \frac{(M_u + M_s)}{2} \tilde{\mathcal{I}}_{us}(0), \tag{2.46b}$$

at zero-momentum transfer, with $g_{\pi qq}$ given by Eq. (2.44) and g_{Kqq} given by Eq. (2.45).

2.4.4 Parameter fixing

Now that we have the expressions for obtaining four observables quantities from our model, we can fix the free NJL parameters. The neutral meson masses are found by solving Eqs. (2.41) while the pion and kaon decay constants are given by Eqs. (2.46). Table 2.1 shows, in the first column, the values for the NJL parameters that reproduce the values shown in the second column for the observable meson properties.

Parameters		Meson properties	
Λ	914.6 MeV	m_π	135.0 MeV
g	9.76 GeV^{-2}	m_K	498.0 MeV
m_{ud}	6.0 MeV	f_π	93.0 MeV
m_s	165.7 MeV	f_K	111.0 MeV

Table 2.1: Parameter fixing and meson properties.

Having fixed the free parameters by setting meson properties to their physical values, we shall now proceed to consider the NJL model in the presence of a uniform magnetic field with the free parameters set to the values from Table 2.1.

CHAPTER 3

NJL model at strong uniform magnetic field

3.1 Introduction

With quarks interacting with one another via the NJL Lagrangian, let us now investigate their coupling with a uniform magnetic field. Quarks interact with the electromagnetic field since they possess flavor-dependent electric charges. While the influence of an electric field serves to restore chiral symmetry by destroying the condensate [43], the magnetic field tends to enhance its breaking by aiding in antialigning the helicities which are bound by the NJL interaction [11]. In this work, we shall consider only the effects of a uniform magnetic field.

By minimally coupling a U(1) gauge field A_μ into the NJL Lagrangian, the action becomes

$$S_{\rm NJL}[\bar{\psi},\psi] = \int d^4x \left\{ \bar{\psi}(i\not{D} - m)\psi + \frac{g}{2}\left[(\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi}i\gamma_5\lambda^a\psi)^2\right] \right\}, \tag{3.1}$$

with $a = 0, \ldots, 8$. Here $m = \mathrm{diag}\,(m_u, m_d, m_s)$ is the current quark mass matrix as before, $D_\mu = \partial_\mu - iQA_\mu$ is the covariant derivative with $Q = \mathrm{diag}\left(\frac{2e}{3}, -\frac{e}{3}, -\frac{e}{3}\right)$ being the electric charge matrix and we are considering the gauge field A_μ to be an electromagnetic field.

The only thing that is new in the action (3.1) is the minimal coupling with the electromagnetic field through the covariant derivative in the kinect term. Since we are not considering any *a priori* dependence of the coupling constant g on the external field, the only difference with respect to our analysis from Chapter 2 will be on the propagation of the quark fields. Therefore, we only need to modify the expression for the quark propagator to apply to the case of a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{e}}_z$. For that purpose, we derive an expression for the free fermion propagator in Appendix C following Schwinger proper time method [26, 27], both in coordinate and momentum spaces. With these expressions at hand, we can compute the gap equation in the presence of the magnetic field similar to what we did in Chapter 2.

3.2 Magnetic catalyses in the NJL model

Recall that the chiral condensate is given by Eq. (2.19) as

$$\left\langle \bar{\psi}_f \psi_f \right\rangle = -\mathrm{tr}_{DC}\left[iS_f^{(0)}(0) \right],$$

where tr_{DC} denotes the trace over color and Dirac indices, and $S_f^{(0)}(x-y)$ is the free quark propagator of flavor f, with $f = u, d, s$.

In order to obtain an expression for the condensate we need the quark propagator computed in the presence of a uniform magnetic field. Using the proper time representation for the propagator from Eqs. (C.11a) and (C.18), we see that

$$\begin{aligned} S_f^{(0)}(0) &= -\left(4\pi\right)^{-2} M_f \int_0^\infty \frac{ds}{s^2} \frac{|q_f B| s}{\sin\left(|q_f B| s\right)} \exp\left(-iM_f^2 s + i\,\mathrm{sign}(q_f B)|q_f B| s \boldsymbol{\sigma}_3\right) \\ &= -\frac{|q_f B| M_f}{(4\pi)^2} \int_0^\infty \frac{ds}{s} e^{-iM_f^2 s} [\cot(|q_f B| s) - \gamma_1\gamma_2\,\mathrm{sign}(q_f B)]. \end{aligned}$$

Then, substituting this result back into the expression for the chiral condensate and taking the Dirac trace using the properties listed in Appendix B, and also the color trace which yields a factor N_c, we find

$$\left\langle \bar{\psi}_f \psi_f \right\rangle_B = \frac{iN_c M_f |q_f B|}{4\pi^2} \int_0^\infty \frac{ds}{s} e^{-iM_f^2 s} \cot(|q_f B| s).$$

Changing to imaginary proper-time, $s \to -is$, leaves us with

$$\begin{aligned} \left\langle \bar{\psi}_f \psi_f \right\rangle_B &= -\frac{N_c M_f |q_f B|}{4\pi^2} \int_0^\infty \frac{ds}{s} e^{-M_f^2 s} \coth(|q_f B| s) \\ &= -\frac{N_c M_f |q_f B|}{4\pi^2} \int_0^\infty \frac{d\tau}{\tau} e^{-M_f^2 \tau / |q_f B|} \coth\tau, \end{aligned} \tag{3.2}$$

where $\tau = |q_f B| s$ and we used $\cot(ix) = -i\coth x$.

However, the proper time integral does not converge, since its integrand is undetermined for $\tau = 0$. It is important to isolate the divergences into the vacuum term before introducing the regularization parameter [44, 45, 46]. By using the expansion (A.9), we can separate the integrand of Eq. (3.2) into a divergent and a finite part as $\tau \to 0$,

$$\int_0^\infty \frac{d\tau}{\tau} e^{-a_f^2 \tau} \coth\tau = \int_{|q_f B|/\Lambda_{\mathrm{pt}}^2}^\infty \frac{d\tau}{\tau^2} e^{-a_f^2 \tau} + \int_0^\infty \frac{d\tau}{\tau^2} e^{-a_f^2 \tau} (\tau\coth\tau - 1),$$

where we introduced a proper time cutoff Λ_{pt}^2 to account for the integral divergence and we are denoting $a_f = M_f/\sqrt{|q_f B|}$. From Eq. (2.24), we know that

$$\int_{|q_fB|/\Lambda_{\rm pt}^2}^{\infty} \frac{d\tau}{\tau^2} e^{-a_f^2\tau} = \frac{1}{|q_fB|}\left[\Lambda_{\rm pt}^2 + M_f^2\left(\ln\frac{M_f^2}{\Lambda_{\rm pt}^2} + \gamma_E - 1\right)\right] + O\left(\frac{1}{\Lambda_{\rm pt}^2}\right).$$

Similarly, using Eq. (A.24), we obtain

$$\int_0^{\infty} \frac{d\tau}{\tau^2} e^{-a_f^2\tau}(\tau\coth\tau - 1) = \frac{1}{|q_fB|}\left[|q_fB|\ln\frac{|q_fB|}{\pi M_f^2} + \left(2|q_fB| - M_f^2\right)\ln\frac{M_f^2}{2|q_fB|}\right.$$
$$\left. +2|q_fB|\ln\Gamma\left(\frac{M_f^2}{2|q_fB|}\right) + M_f^2\right].$$

Using the results above, the chiral condensate becomes

$$\left\langle\bar{\psi}_f\psi_f\right\rangle_B = -\frac{N_cM_f}{4\pi^2}\left[\Lambda_{\rm pt}^2 - M_f^2\left(\ln\frac{\Lambda_{\rm pt}^2}{2|q_fB|} - \gamma_E\right) + |q_fB|\ln\frac{M_f^2}{4\pi|q_fB|}\right.$$
$$\left. +2|q_fB|\ln\Gamma\left(\frac{M_f^2}{2|q_fB|}\right)\right]. \tag{3.3}$$

One may check if the expression above reproduces the known result for $B = 0$ by taking the limit $B \to 0$ in (3.3). In this limit, the argument of the Gamma function goes to infinity and we may use the expansion (A.18) to write

$$2|q_fB|\ln\Gamma\left(\frac{M_f^2}{2|q_fB|}\right) \to M_f^2\left(\ln\frac{\Lambda_{\rm pt}^2}{2|q_fB|} + \ln\frac{M_f^2}{\Lambda_{\rm pt}^2} - 1\right) - |q_fB|\ln\frac{M_f^2}{4\pi|q_fB|} \quad \text{as} \quad B \to 0.$$

This leads to

$$\lim_{B\to 0}\left\langle\bar{\psi}_f\psi_f\right\rangle = -\frac{N_cM_f}{4\pi^2}\left[\Lambda_{\rm pt}^2 + M_f^2\left(\ln\frac{M_f^2}{\Lambda_{\rm pt}^2} + \gamma_E - 1\right)\right],$$

which matches the expression for the chiral condensate in the absence of external fields that we found in Eq. (2.25) by using the proper time regularization scheme.

It is worth noticing that the magnetic field introduces no new divergences in the chiral condensate, as the expression in the $B \to 0$ limit contains all cutoff-dependent terms. Therefore, a cutoff independent quantity is obtained if we subtract the expression for $\left\langle\bar{\psi}_f\psi_f\right\rangle_{B=0}$ from Eq. (3.3). The result is

$$\left\langle\bar{\psi}_f\psi_f\right\rangle_B - \left\langle\bar{\psi}_f\psi_f\right\rangle_{B=0} = -\frac{N_cM_f}{4\pi^2}\left[M_f^2\left(1 - \ln\frac{M_f^2}{2|q_fB|}\right) + |q_fB|\ln\frac{M_f^2}{4\pi|q_fB|}\right.$$
$$\left. +2|q_fB|\ln\Gamma\left(\frac{M_f^2}{2|q_fB|}\right)\right], \tag{3.4}$$

which contains no explicit dependence on the regularization scheme. We also note that this quantity indeed vanishes in the limit $B \to 0$. This allows us to write the chiral condensate as a sum of two terms: one divergent contribution that needs to be regularized and another that depends on the applied magnetic field but is finite and regularization independent, and that vanishes in the limit $B \to 0$. Although we have computed the chiral condensate by using the proper time regularization scheme, which was appropriate since our departure point was the Schwinger proper time representation for the propagator, we now have the freedom to choose the expression for $\left\langle \bar{\psi}_f \psi_f \right\rangle_{B=0}$ in any other regularization scheme we would like. For instance, we can take

$$\left\langle \bar{\psi}_f \psi_f \right\rangle_{B=0} = -\frac{N_c M_f}{4\pi^2}\left[\Lambda^2 - M_f^2 \ln\left(\frac{\Lambda^2 + M_f^2}{M_f^2}\right)\right],$$

which was obtained using the four-momentum cutoff regularization, and write

$$\begin{aligned}\left\langle \bar{\psi}_f \psi_f \right\rangle_{B} = &-\frac{N_c M_f}{4\pi^2}\left[\Lambda^2 - M_f^2 \ln\left(\frac{\Lambda^2 + M_f^2}{M_f^2}\right)\right] \\ &- \frac{N_c M_f}{4\pi^2}\left[M_f^2\left(1 - \ln\frac{M_f^2}{2|q_f B|}\right) + |q_f B| \ln\frac{M_f^2}{4\pi|q_f B|} + 2|q_f B| \ln\Gamma\left(\frac{M_f^2}{2|q_f B|}\right)\right].\end{aligned} \tag{3.5}$$

The gap equation in the presence of the magnetic field is given by

$$M_f = m_f - 2g\left\langle \bar{\psi}_f \psi_f \right\rangle_{B}.$$

Then, substituting Eq. (3.5) gives

$$\begin{aligned}M_f = m_f &+ \frac{gN_c M_f}{2\pi^2}\left[\Lambda^2 - M_f^2 \ln\left(\frac{\Lambda^2 + M_f^2}{M_f^2}\right)\right] \\ &+ \frac{gN_c M_f}{2\pi^2}\left[M_f^2\left(1 - \ln\frac{M_f^2}{2|q_f B|}\right) + |q_f B| \ln\frac{M_f^2}{4\pi|q_f B|} + 2|q_f B| \ln\Gamma\left(\frac{M_f^2}{2|q_f B|}\right)\right].\end{aligned} \tag{3.6}$$

Just like Eq. (2.26), Eq. (3.6) is a self-consistent equation for M_f and we must look for its solution by employing numerical methods. Figure 3.1 shows the solution for the three quark flavors as a function of eB. The four-momentum cutoff was taken as $\Lambda = 914.6\,\text{MeV}$ once again and the coupling constant value was chosen to be $g = 9.76\,\text{GeV}^{-2}$.

As we can see from Figure 3.1, the magnetic field breaks the degenerescence between the effective masses of the quarks up and down due to their different electric charges. Furthermore, the response of the quark up to the magnetic field is greater than that of the other light quarks due to the value of its electric charge, $|q_u| = 2|q_d| = 2|q_s|$. In general, the increasing values of the constituent masses with eB is a clear sign of the enhancement of chiral symmetry breaking by the presence of the external magnetic field.

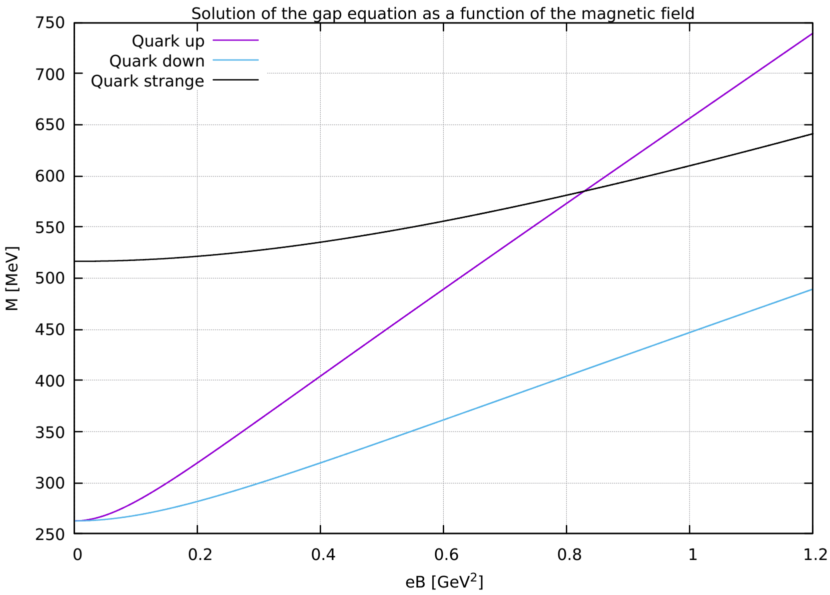

Figure 3.1: Solution of the gap equation as a function of the applied magnetic field, Eq. (3.6). Here we used $\Lambda = 914.6\,\text{MeV}$, $g = 9.76\,\text{GeV}^{-2}$, $m_{ud} = 6\,\text{MeV}$ and $m_s = 165.7\,\text{MeV}$.

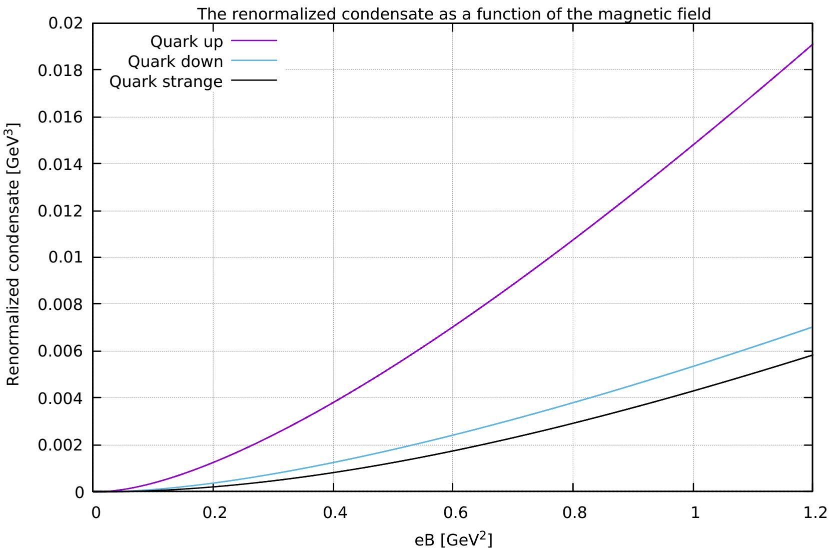

Figure 3.2: Absolute value of the renormalized condensate, Eq. (3.4), as a function of the magnetic field. The values of the effective quark masses used were the ones displayed in Figure 3.1.

The effect of magnetic catalysis is more evident in Figure 3.2, where we plot the absolute value of the renormalized condensate, $\left|\left\langle\bar{\psi}_f\psi_f\right\rangle_B - \left\langle\bar{\psi}_f\psi_f\right\rangle_{B=0}\right|$ given in Eq. (3.4), against the magnetic field. The increasing value of the chiral condensate with B signals the enhancement of chiral symmetry breaking.

This concludes our discussion on dynamical mass generation by the breaking of chiral symmetry and its enhancement by the external magnetic field. Let us now turn our attention to the quark determinant and its expansion in terms of quark field bilinears.

3.3 The quark determinant expansion

In Chapter 2 we obtained an expression for the effective action for the quasiparticle quarks by separating the quark fields into a component corresponding to the quarks that condense in the ground state and another corresponding to the interacting quarks. In the presence of a uniform magnetic field, this effective action reads

$$S_{\text{eff}} = S_{\text{det}} + \int d^4x \left\{ -\frac{1}{2g}(S_a^2 + P_a^2) + \bar{\psi}(i\not{D} - m)\psi + \frac{g}{2}\left[(\bar{\psi}\lambda^a\psi)^2 + (\bar{\psi}i\gamma_5\lambda^a\psi)^2\right]\right\}, \quad (3.7)$$

where

$$S_{\text{det}} = -i\text{Tr}\ln\left(-i\left\{i\not{D} - M + g\left[\lambda^a(\bar{\psi}\lambda^a\psi) + \lambda^a i\gamma_5(\bar{\psi}i\gamma_5\lambda^a\psi)\right]\right\}\right) \quad (3.8)$$

is the quark determinant. Here we have included the coupling with the external magnetic field through the covariant derivative. Our next task is to expand this determinant in terms of the quark field bilinears and to interpret the lowest order terms of the expansion as corrections to the original NJL Lagrangian.

3.3.1 Expansion and first-order term

The quark determinant is of the form

$$S_{\text{det}} = -i\text{Tr}\ln\left[-i\left(S^{(0)}\right)^{-1} - ig\lambda^a j^a(x)\right],$$

where $\left(S^{(0)}\right)^{-1} = i\not{D} - M$ and

$$j^a = (\bar{\psi}\lambda^a\psi) + i\gamma_5(\bar{\psi}i\gamma_5\lambda^a\psi) \equiv j_s^a + i\gamma_5 j_p^a$$

stands for the quark-current terms. Here the scalar and pseudoscalar currents are represented by j_s^a and j_p^a, respectively. The determinant can be written as

$$S_{\text{det}} = -i\text{Tr}\ln\left[-i\left(S^{(0)}\right)^{-1}\right] - i\text{Tr}\ln\left(1 + gS^{(0)}\lambda^a j^a\right)$$

and the first term can be dropped since it yields a constant in the generating functional. We now proceed to expand the determinant in powers of the quark currents j^a by using Eq. (A.10),

$$S_{\text{det}} = -ig\text{Tr}\left(S^{(0)}\lambda^a j^a\right) + \frac{i}{2}g^2\text{Tr}\left(S^{(0)}\lambda^a j^a S^{(0)}\lambda^b j^b\right) + \dots .$$

Following a similar procedure to the one applied in Section 2.4 for dealing with the trace over continuum indices, we find the first order term to be given by

$$\begin{aligned}
S^{(1)}_{\text{det}} &= -ig\text{Tr}\Big(S^{(0)}\lambda^a j^a\Big) \\
&= -g\,\text{tr}\bigg\{\int d^4x \int \frac{d^4p}{(2\pi)^4}\frac{i}{\not{\Pi} - M}\,\lambda^a\Big[\Big(\bar{\psi}(x)\lambda^a\psi(x)\Big) + i\gamma_5\Big(\bar{\psi}(x)i\gamma_5\lambda^a\psi(x)\Big)\Big]\bigg\} \\
&= -g\text{tr}\Big\{iS^{(0)}(0)\int d^4x\,\lambda^a\Big[\Big(\bar{\psi}(x)\lambda^a\psi(x)\Big) + i\gamma_5\Big(\bar{\psi}(x)i\gamma_5\lambda^a\psi(x)\Big)\Big]\Big\},
\end{aligned} \tag{3.9}$$

where $\Pi_\mu = p_\mu + QA_\mu$, $S^{(0)}(x-y)$ is the quark propagator matrix and tr denotes the trace over Dirac, flavor and color indices. Explicitly, the flavor structure of the quark propagator reads

$$S^{(0)}(x-y) = \begin{pmatrix} S^{(0)}_u(x-y) & 0 & 0 \\ 0 & S^{(0)}_d(x-y) & 0 \\ 0 & 0 & S^{(0)}_s(x-y) \end{pmatrix}.$$

A direct calculation of the flavor trace yields

$$\begin{aligned}
\text{tr}_F\Big[S^{(0)}(0)\lambda^0\Big] &= \sqrt{\frac{2}{3}}\Big[S^{(0)}_u(0) + S^{(0)}_d(0) + S^{(0)}_s(0)\Big], \\
\text{tr}_F\Big[S^{(0)}(0)\lambda^3\Big] &= S^{(0)}_u(0) - S^{(0)}_d(0), \\
\text{tr}_F\Big[S^{(0)}(0)\lambda^8\Big] &= \frac{1}{\sqrt{3}}\Big[S^{(0)}_u(0) + S^{(0)}_d(0) - 2S^{(0)}_s(0)\Big],
\end{aligned} \tag{3.10}$$

while the traces for $a = 1, 2, 4, 5, 6, 7$ all vanish. Then, the only (scalar) currents that will contribute to the first-order term are

$$\begin{aligned}
j^0_s &= \bar{\psi}\lambda^0\psi = \sqrt{\frac{2}{3}}\Big(\bar{u}u + \bar{d}d + \bar{s}s\Big), \\
j^3_s &= \bar{\psi}\lambda^3\psi = \bar{u}u - \bar{d}d, \\
j^8_s &= \bar{\psi}\lambda^8\psi = \frac{1}{\sqrt{3}}\Big(\bar{u}u + \bar{d}d - 2\bar{s}s\Big),
\end{aligned} \tag{3.11}$$

while the term with the pseudoscalar currents has vanishing Dirac trace. The first-order term becomes

$$\begin{aligned}
S^{(1)}_{\text{det}} = -\frac{1}{2}\int d^4x\,\Bigg[&\frac{2}{3}\sum_f \Delta m_f \sum_f \bar{\psi}_f\psi_f + (\Delta m_u - \Delta m_d)\Big(\bar{u}u - \bar{d}d\Big) \\
&+\frac{1}{3}(\Delta m_u + \Delta m_d - 2\Delta m_s)\Big(\bar{u}u + \bar{d}d - 2\bar{s}s\Big)\Bigg],
\end{aligned}$$

where

$$\Delta m_f = M_f - m_f = 2g\,\text{tr}_{DC}\Big[iS^{(0)}_f(0)\Big] = -2g\left\langle\bar{\psi}_f\psi_f\right\rangle_B, \tag{3.12}$$

is simply the gap equation in the presence of a uniform magnetic field. By rearranging the terms in $S^{(1)}_{\text{det}}$ one finds

$$S^{(1)}_{\text{det}} = -\int d^4x \left(\Delta m_u \bar{u}u + \Delta m_d \bar{d}d + \Delta m_s \bar{s}s\right) = -\int d^4x \sum_f \Delta m_f \bar{\psi}_f \psi_f. \qquad (3.13)$$

Therefore the first-order term of the expansion produces a correction to the quark masses given by the gap equation, Eq. (3.6), which was already discussed in Section 3.2. We now proceed to compute the contribution of the second-order term.

3.3.2 Second-order term

The second-order term of the quark determinant reads

$$S^{(2)}_{\text{det}} = \frac{i}{2} g^2 \text{Tr}\left(S^{(0)} \lambda^a j^a S^{(0)} \lambda^b j^b\right).$$

Assuming the local limit, we find

$$\begin{aligned} S^{(2)}_{\text{det}} = \frac{i}{2} g^2 \text{tr} \int d^4x \int \frac{d^4p}{(2\pi)^4} S^{(0)}(p) \lambda^a \Big[(\bar{\psi}\lambda^a\psi) + i\gamma_5(\bar{\psi}i\gamma_5\lambda^a\psi)\Big] \\ \times S^{(0)}(p)\lambda^b \Big[(\bar{\psi}\lambda^b\psi) + i\gamma_5(\bar{\psi}i\gamma_5\lambda^b\psi)\Big], \end{aligned} \qquad (3.14)$$

where $S^{(0)}(p)$ is the quark propagator in momentum space. The integrand in Eq. (3.14) can be written as

$$\begin{aligned} S^{(0)}(p)\lambda^a j^a S^{(0)}(p)\lambda^b j^b = {} & S^{(0)}(p)\lambda^a S^{(0)}(p)\lambda^b j^a_s j^b_s + S^{(0)}(p)\lambda^a i\gamma_5 S^{(0)}(p)\lambda^b j^a_p j^b_s \\ & + S^{(0)}(p)\lambda^a S^{(0)}(p)\lambda^b i\gamma_5 j^a_s j^b_p + S^{(0)}(p)\lambda^a i\gamma_5 S^{(0)}(p)\lambda^b i\gamma_5 j^a_p j^b_p. \end{aligned}$$

Then, by dropping the terms with vanishing Dirac trace, we are left with

$$\begin{aligned} S^{(2)}_{\text{det}} = \frac{i}{2} g^2 \text{tr} \int d^4x \int \frac{d^4p}{(2\pi)^4} \Big[S^{(0)}(p)\lambda^a S^{(0)}(p)\lambda^b \left(\bar{\psi}\lambda^a\psi\right)\left(\bar{\psi}\lambda^b\psi\right) \\ + S^{(0)}(p)\lambda^a i\gamma_5 S^{(0)}(p)\lambda^b i\gamma_5 \left(\bar{\psi}i\gamma_5\lambda^a\psi\right)\left(\bar{\psi}i\gamma_5\lambda^b\psi\right)\Big]. \end{aligned} \qquad (3.15)$$

We see how the second-order term in the determinant expansion contributes to an effective coupling for an interaction of the form

$$\mathcal{L}^{\text{eff}}_{\text{int}} = \frac{1}{2} G^{\text{s}}_{ab}(B) \left(\bar{\psi}\lambda^a\psi\right)\left(\bar{\psi}\lambda^b\psi\right) + \frac{1}{2} G^{\text{ps}}_{ab}(B) \left(\bar{\psi}i\gamma_5\lambda^a\psi\right)\left(\bar{\psi}i\gamma_5\lambda^b\psi\right), \qquad (3.16)$$

with

$$G^{\text{s}}_{ab}(B) = g\delta_{ab} + ig^2 \int \frac{d^4p}{(2\pi)^4} \text{tr}\Big[S^{(0)}(p)\lambda^a S^{(0)}(p)\lambda^b\Big], \qquad (3.17a)$$

$$G^{\text{ps}}_{ab}(B) = g\delta_{ab} + ig^2 \int \frac{d^4p}{(2\pi)^4} \text{tr}\Big[S^{(0)}(p)\lambda^a i\gamma_5 S^{(0)}(p)\lambda^b i\gamma_5\Big], \tag{3.17b}$$

and we recall that tr denotes the trace over Dirac, flavor and color indices.

The next task is to compute the trace over flavor indices, which involves long and tedious calculations (see Appendix D for some details). The result is

$$\begin{aligned} S^{(2)}_{\text{det}} = 2ig^2 N_c \text{tr}_D \int d^4x \int \frac{d^4p}{(2\pi)^4} \sum_{f,g=u,d,s} \Big[& S^{(0)}_f(p) S^{(0)}_g(p) \Big(\bar{\psi}_f \psi_g\Big)\Big(\bar{\psi}_g \psi_f\Big) \\ & + S^{(0)}_f(p) i\gamma_5 S^{(0)}_g(p) i\gamma_5 \Big(\bar{\psi}_f i\gamma_5 \psi_g\Big)\Big(\bar{\psi}_g i\gamma_5 \psi_f\Big)\Big], \end{aligned} \tag{3.18}$$

where tr_D stands for the trace over Dirac indices. Since the original NJL Lagrangian is given by

$$\mathcal{L} = \bar{\psi}\Big(i\not{D} - m\Big)\psi + \sum_{f,g=u,d,s} \Big[g\Big(\bar{\psi}_f \psi_g\Big)\Big(\bar{\psi}_g \psi_f\Big) + g\Big(\bar{\psi}_f i\gamma_5 \psi_g\Big)\Big(\bar{\psi}_g i\gamma_5 \psi_f\Big)\Big],$$

we denote the effective couplings in the fundamental SU(3) representation as

$$G^{\text{s}}_{fg}(B) = g + g^2 \Pi^{\text{s}}_{fg}(B), \tag{3.19a}$$

$$G^{\text{ps}}_{fg}(B) = g + g^2 \Pi^{\text{ps}}_{fg}(B), \tag{3.19b}$$

where

$$\Pi^{\text{s}}_{fg}(B) = 2iN_c \int \frac{d^4p}{(2\pi)^4} \text{tr}_D\Big[S^{(0)}_f(p) S^{(0)}_g(p)\Big], \tag{3.20a}$$

$$\Pi^{\text{ps}}_{fg}(B) = 2iN_c \int \frac{d^4p}{(2\pi)^4} \text{tr}_D\Big[S^{(0)}_f(p) i\gamma_5 S^{(0)}_g(p) i\gamma_5\Big], \tag{3.20b}$$

with $S^{(0)}_f(p)$ representing the quark propagator in momentum space in the presence of the uniform magnetic field B.

Therefore, up to second order in the quark currents the effective action (3.7) becomes

$$\begin{aligned} S_{\text{eff}} = \int d^4x \Bigg\{ & -\frac{1}{2g}(S^2_a + P^2_a) + \bar{\psi}(i\not{D} - M)\psi \\ & + \sum_{f,g=u,d,s} \Big[G^{\text{s}}_{fg}(B)\Big(\bar{\psi}_f \psi_g\Big)\Big(\bar{\psi}_g \psi_f\Big) + G^{\text{ps}}_{fg}(B)\Big(\bar{\psi}_f i\gamma_5 \psi_g\Big)\Big(\bar{\psi}_g i\gamma_5 \psi_f\Big)\Big]\Bigg\}. \end{aligned} \tag{3.21}$$

Thus we have seen how the model acquires an effective flavor- and B-dependent coupling from vacuum polarization.

By using the definitions (3.17) and (3.19) as well the results listed in the Appendix

D, we may write the relations

$$G^{\rm s}_{00}(B) = \frac{1}{3}[G^{\rm s}_{uu}(B) + G^{\rm s}_{dd}(B) + G^{\rm s}_{ss}(B)], \tag{3.22a}$$

$$G^{\rm s}_{11}(B) = G^{\rm s}_{22}(B) = G^{\rm s}_{ud}(B), \tag{3.22b}$$

$$G^{\rm s}_{33}(B) = \frac{1}{2}[G^{\rm s}_{uu}(B) + G^{\rm s}_{dd}(B)], \tag{3.22c}$$

$$G^{\rm s}_{44}(B) = G^{\rm s}_{55}(B) = G^{\rm s}_{us}(B), \tag{3.22d}$$

$$G^{\rm s}_{66}(B) = G^{\rm s}_{77}(B) = G^{\rm s}_{ds}(B), \tag{3.22e}$$

$$G^{\rm s}_{88}(B) = \frac{1}{6}[G^{\rm s}_{uu}(B) + G^{\rm s}_{dd}(B) + 4G^{\rm s}_{ss}(B)], \tag{3.22f}$$

$$G^{\rm s}_{03}(B) = G^{\rm s}_{30}(B) = \frac{1}{\sqrt{6}}[G^{\rm s}_{uu}(B) - G^{\rm s}_{dd}(B)], \tag{3.22g}$$

$$G^{\rm s}_{08}(B) = G^{\rm s}_{80}(B) = \frac{1}{3\sqrt{2}}[G^{\rm s}_{uu}(B) + G^{\rm s}_{dd}(B) - 2G^{\rm s}_{ss}(B)], \tag{3.22h}$$

$$G^{\rm s}_{38}(B) = G^{\rm s}_{83}(B) = \frac{1}{2\sqrt{3}}[G^{\rm s}_{uu}(B) - G^{\rm s}_{dd}(B)], \tag{3.22i}$$

so that

$$\frac{1}{2}\sum_{a,b=0}^{8} G^{\rm s}_{ab}(B)\left(\bar{\psi}\lambda^a\psi\right)\left(\bar{\psi}\lambda^b\psi\right) = \sum_{f,g=u,d,s} G^{\rm s}_{fg}(B)\left(\bar{\psi}_f\psi_g\right)\left(\bar{\psi}_g\psi_f\right).$$

All the other couplings $G^{\rm s}_{ab}(B)$ vanish and analogous relations holds for the pseudoscalar couplings.

The relations listed in Eq. (3.22) will be useful in Chapter 4. For now, let us proceed to obtain explicit expressions for the couplings $G^{\rm s}_{fg}(B)$ and $G^{\rm ps}_{fg}(B)$.

3.4 The effective couplings

3.4.1 The polarization functions

In order to compute the effective couplings from Eqs. (3.19) we need to evaluate the polarization functions from Eqs. (3.20). This requires the expression for the quark propagator in momentum space in the presence of the uniform magnetic field. From Eq. (C.11a) in Appendix C, we know that the propagator is given by the product of a phase, called Schwinger phase, and a translation invariant term. In the general case, performing a complete momentum-space calculation is hard due to the presence of the Schwinger phases. Although there are ways to deal with these phases, like employing the Ritus eigenfunction method [47], deriving the correlation functions in coordinate space by adopting linear response theory based on imaginary-time path integral formalism [48], or simply discarding the phases, we will be focusing on the polarization functions for which

$f = g$, or, more generally, on the polarization functions that involve quark flavors with the same electric charge, since in those cases the Schwinger phases cancel out. Therefore, from now on we simply ignore the Schwinger phase and write the quark propagator in momentum space as [see Eq. (C.19)]

$$\begin{aligned} S_f^{(0)}(p) = -i \int_0^\infty ds\, \exp\left\{-is\left[M_f^2 - p_\parallel^2 + \frac{\tan(|q_f B|s)}{|q_f B|s} p_\perp^2\right]\right\} \\ \times\Big\{[1 - \mathrm{sign}(q_f B)\gamma_1\gamma_2 \tan(|q_f B|s)]\Big(M_f + \gamma\cdot p_\parallel\Big) - \gamma\cdot p_\perp\Big[1 + \tan^2(|q_f B|s)\Big]\Big\}, \end{aligned} \tag{3.23}$$

where, for two arbitrary 4−vectors a^μ and b^μ, we denote

$$\begin{aligned} (a\cdot b)_\parallel &= a^0 b^0 - a^3 b^3, \\ (a\cdot b)_\perp &= a^1 b^1 + a^2 b^2, \end{aligned}$$

and also

$$\mathrm{sign}(x) = \begin{cases} +1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Directly perfoming the Dirac trace leads to

$$\begin{aligned} &\mathrm{tr}_D\Big[S_f^{(0)}(p) S_g^{(0)}(p)\Big] \\ &= -4\int_0^\infty\int_0^\infty ds\, dr\, e^{-isM_f^2 - irM_g^2} \exp\left\{i(s+r)p_\parallel^2 - i\left[\frac{\tan(|q_f B|s)}{|q_f B|} + \frac{\tan(|q_g B|r)}{|q_g B|}\right]p_\perp^2\right\} \\ &\quad\times\Big\{[1 - \mathrm{sign}(q_f q_g)\tan(|q_f Bs|)\tan(|q_g B|r)]\Big(M_f M_g + p_\parallel^2\Big) \\ &\quad -p_\perp^2\Big[1 + \tan^2(|q_f B|s)\Big]\Big[1 + \tan^2(|q_g B|r)\Big]\Big\}, \end{aligned}$$

and

$$\begin{aligned} &\mathrm{tr}_D\Big[S_f^{(0)}(p) i\gamma_5 S_g^{(0)}(p) i\gamma_5\Big] \\ &= -4\int_0^\infty\int_0^\infty ds\, dr\, e^{-isM_f^2 - irM_g^2} \exp\left\{i(s+r)p_\parallel^2 - i\left[\frac{\tan(|q_f B|s)}{|q_f B|} + \frac{\tan(|q_g B|r)}{|q_g B|}\right]p_\perp^2\right\} \\ &\quad\times\Big\{[1 - \mathrm{sign}(q_f q_g)\tan(|q_f Bs|)\tan(|q_g B|r)]\Big(-M_f M_g + p_\parallel^2\Big) \\ &\quad -p_\perp^2\Big[1 + \tan^2(|q_f B|s)\Big]\Big[1 + \tan^2(|q_g B|r)\Big]\Big\}. \end{aligned}$$

The polarization functions become

$$\begin{aligned} \Pi_{fg}^{\overset{\mathrm{s}}{\mathrm{ps}}}(B) = -8iN_c \int \frac{d^4p}{(2\pi)^4} \int_0^\infty\int_0^\infty ds\, dr\, e^{-isM_f^2 - irM_g^2} \\ \times \exp\left\{i(s+r)p_\parallel^2 - i\left[\frac{\tan(|q_f B|s)}{|q_f B|} + \frac{\tan(|q_g B|r)}{|q_g B|}\right]p_\perp^2\right\} \\ \times\Big\{[1 - \mathrm{sign}(q_f q_g)\tan(|q_f Bs|)\tan(|q_g B|r)]\Big(\pm M_f M_g + p_\parallel^2\Big) \\ -p_\perp^2 \sec^2(|q_f B|s)\sec^2(|q_g B|r)\Big\}. \end{aligned}$$

After rotating the momentum to Euclidean space by letting $p_0 \to ip_{0E}$ so that $d^4p \to id^4p_E = id^2p_{\parallel E}d^2p_{\perp E}$, $p_\parallel^2 \to -p_{\parallel E}^2$ and $p_\perp^2 \to p_{\perp E}^2$, and introducing the imaginary proper time variables $s \to -is$ and $r \to -ir$, we find

$$\begin{aligned}\Pi_{fg}^{\overset{\mathrm{s}}{\mathrm{ps}}}(B) =& -8N_c \int_0^\infty \int_0^\infty ds dr\, e^{-sM_f^2 - rM_g^2} \int \frac{d^2p_{\parallel E}}{(2\pi)^2}\frac{d^2p_{\perp E}}{(2\pi)^2} \\ &\times \exp\left\{-(s+r)p_{\parallel E}^2 - \left[\frac{\tanh(|q_fB|s)}{|q_fB|} + \frac{\tanh(|q_gB|r)}{|q_gB|}\right]p_{\perp E}^2\right\} \\ &\times \Big\{[1 + \mathrm{sign}(q_fq_g)\tanh(|q_fB|s)\tanh(|q_gB|r)]\Big(\pm M_fM_g - p_{\parallel E}^2\Big) \\ &\quad -p_{\perp E}^2\,\mathrm{sech}^2(|q_fB|s)\,\mathrm{sech}^2(|q_gB|r)\Big\}\,,\end{aligned}$$

where we used the identities $\tan(-ix) = -i\tanh x$ and $\sec(-ix) = \mathrm{sech}\, x$. The momentum integrals are straightforward to compute,

$$\int \frac{d^2p_{\parallel E}}{(2\pi)^2}\, e^{-(s+r)p_{\parallel E}^2} = \frac{1}{4\pi(s+r)},$$

$$\int \frac{d^2p_{\parallel E}}{(2\pi)^2}\, p_{\parallel E}^2 e^{-(s+r)p_{\parallel E}^2} = \frac{1}{4\pi(s+r)^2},$$

$$\int \frac{d^2p_{\perp E}}{(2\pi)^2}\, \exp\left\{-[\mathcal{T}_f(s) + \mathcal{T}_g(r)]p_{\perp E}^2\right\} = \frac{1}{4\pi[\mathcal{T}_f(s) + \mathcal{T}_g(r)]},$$

$$\int \frac{d^2p_{\perp E}}{(2\pi)^2}\, p_{\perp E}^2 \exp\left\{-[\mathcal{T}_f(s) + \mathcal{T}_g(r)]p_{\perp E}^2\right\} = \frac{1}{4\pi[\mathcal{T}_f(s) + \mathcal{T}_g(r)]^2},$$

where we are denoting

$$\mathcal{T}_f(s) \equiv \frac{\tanh(|q_fB|s)}{|q_fB|} \qquad \text{and} \qquad \mathcal{T}_g(r) \equiv \frac{\tanh(|q_gB|r)}{|q_gB|}.$$

Then, we write the polarization functions as

$$\begin{aligned}\Pi_{fg}^{\overset{\mathrm{s}}{\mathrm{ps}}}(B) =& -\frac{N_c}{2\pi^2}\int_0^\infty\int_0^\infty ds dr\, \frac{e^{-sM_f^2 - rM_g^2}}{s+r} \\ &\times \left\{\frac{1 + \mathrm{sign}(q_fq_g)\tanh(|q_fB|s)\tanh(|q_gB|r)}{\mathcal{T}_f(s) + \mathcal{T}_g(r)}\left(\pm M_fM_g - \frac{1}{s+r}\right)\right. \\ &\left. -\left[\frac{\mathrm{sech}(|q_fB|s)\,\mathrm{sech}(|q_gB|r)}{\mathcal{T}_f(s) + \mathcal{T}_g(r)}\right]^2\right\}.\end{aligned} \tag{3.24}$$

As we mentioned, we shall be focusing on the polarization functions that involve quark flavors with the same electric charge, meaning that we will now set $q_f = q_g$. In that specific case, the polarization functions become

$$\Pi^{\overset{\text{s}}{\text{ps}}}_{fg}(B) = \frac{N_c|q_f B|}{2\pi^2}\int_0^\infty\int_0^\infty dsdr\,\frac{e^{-sM_f^2-rM_g^2}}{s+r}\left[\frac{1\mp M_f M_g(s+r)}{(s+r)\tanh\left(|q_f B|(s+r)\right)}\right.$$
$$\left.+\frac{|q_f B|}{\sinh^2\left(|q_f B|(s+r)\right)}\right], \tag{3.25}$$

where we used the relations (A.7) and (A.8).

It is interesting to take the limit $B\to 0$ in Eq. (3.25). From Eqs. (A.9) and (A.11) we see that

$$\lim_{x\to 0}\frac{x}{\tanh x} = \lim_{x\to 0}\frac{x^2}{\sinh^2 x} = 1,$$

leaving us with

$$\Pi^{\overset{\text{s}}{\text{ps}}}_{fg}(B=0) = \frac{N_c}{2\pi^2}\int_0^\infty\int_0^\infty dsdr\, e^{-sM_f^2-rM_g^2}\left[\frac{2\mp M_f M_g(s+r)}{(s+r)^3}\right]. \tag{3.26}$$

This result allows us to write Eq. (3.25) in a more convenient way,

$$\Pi^{\overset{\text{s}}{\text{ps}}}_{fg}(B) = \Pi^{\overset{\text{s}}{\text{ps}}}_{fg}(B=0) + \frac{N_c|q_f B|}{2\pi^2}\int_0^\infty\int_0^\infty dsdr\,\frac{e^{-sM_f^2-rM_g^2}}{s+r}$$
$$\times\left[\frac{1\mp M_f M_g(s+r)}{(s+r)\tanh\left(|q_f B|(s+r)\right)} + \frac{|q_f B|}{\sinh^2\left(|q_f B|(s+r)\right)} - \frac{2\mp M_f M_g(s+r)}{|q_f B|(s+r)^2}\right]. \tag{3.27}$$

When written like in Eq. (3.27), the polarization functions consist of the the sum of two terms, the vacuum and the pure magnetic contributions. Just as in the case of the chiral condensate, the vacuum contribution carries all the divergences that need to be regularized while the pure magnetic contribution is finite and regularization independent.

Before we proceed, let us now stop to evaluate the polarization functions for $B=0$.

3.4.2 The polarization functions for $B = 0$

In the absence of the external magnetic field, the polarization functions are given by Eqs. (3.20) with the quark propagator in momentum space written as

$$S_f^{(0)}(p) = \frac{\not{p}+M_f}{p^2-M_f^2}.$$

Then, a direct calculation yields

$$\Pi^{\overset{\text{s}}{\text{ps}}}_{fg}(B=0) = 8iN_c\int\frac{d^4p}{(2\pi)^4}\frac{p^2\pm M_f M_g}{\left(p^2-M_f^2\right)\left(p^2-M_g^2\right)}. \tag{3.28}$$

We can even check Eq. (3.26) by writing Eq. (3.28) as

$$
\begin{aligned}
\Pi^{\overset{\rm s}{\rm ps}}_{fg}(B=0) &= -8iN_c \int \frac{d^4p}{(2\pi)^4}\Big(p^2 \pm M_f M_g\Big) \int_0^\infty ds\, e^{is\left(p^2-M_f^2\right)} \int_0^\infty dr\, e^{ir\left(p^2-M_g^2\right)} \\
&= 8N_c \int_0^\infty \int_0^\infty dsdr\, e^{-sM_f^2-rM_g^2} \int \frac{d^4p_E}{(2\pi)^4}\Big(p_E^2 \mp M_f M_g\Big) e^{-(s+r)p_E^2} \\
&= \frac{N_c}{2\pi^2}\int_0^\infty dsdr\, e^{-sM_f^2-rM_g^2}\left[\frac{2 \mp M_f M_g (s+r)}{(s+r)^3}\right],
\end{aligned}
$$

where, in the second line, we rotated the momentum to Euclidean space and introduced the imaginary proper time variables. The result is exactly Eq. (3.26).

We can compute the polarization functions (3.28) by making an analogy with the result (2.39) from Section 2.4. For the pseudoscalar polarization function we simply set $k^2 = 0$ while for the scalar polarization function we additionally make the exchange $M_g \to -M_g$. Then, we find

$$
\begin{aligned}
\Pi^{\overset{\rm s}{\rm ps}}_{fg}(B=0) = &\frac{N_c}{4\pi^2}\left[\Lambda^2 - M_f^2 \ln\left(\frac{\Lambda^2+M_f^2}{M_f^2}\right)\right] + \frac{N_c}{4\pi^2}\left[\Lambda^2 - M_g^2 \ln\left(\frac{\Lambda^2+M_g^2}{M_g^2}\right)\right] \\
&-\frac{N_c}{4\pi^2}\left(M_f \pm M_g\right)^2 \int_0^1 dx \left\{\ln\left[\frac{\Lambda^2 + D_{fg}^2(0)}{D_{fg}^2(0)}\right] + \frac{D_{fg}^2(0)}{\Lambda^2 + D_{fg}^2(0)} - 1\right\},
\end{aligned}
\tag{3.29}
$$

where $D_{fg}^2(0) = xM_f^2 + (1-x)M_g^2$ [see Eq. (2.40)]. In the special case where $f = g$, Eq. (3.29) becomes

$$
\Pi^{\rm s}_{ff}(B=0) = \frac{N_c}{2\pi^2}\left[\Lambda^2 - 3M_f^2 \ln\left(\frac{\Lambda^2+M_f^2}{M_f^2}\right) + 2M_f^2 \frac{\Lambda^2}{\Lambda^2+M_f^2}\right], \tag{3.30a}
$$

$$
\Pi^{\rm ps}_{ff}(B=0) = \frac{N_c}{2\pi^2}\left[\Lambda^2 - M_f^2 \ln\left(\frac{\Lambda^2+M_f^2}{M_f^2}\right)\right]. \tag{3.30b}
$$

3.4.3 The polarization functions for $f = g$

Let us now turn our attention back to Eq. (3.27) and denote

$$
\left(\Pi^{\overset{\rm s}{\rm ps}}_{fg}\right)_B \equiv \Pi^{\overset{\rm s}{\rm ps}}_{fg}(B) - \Pi^{\overset{\rm s}{\rm ps}}_{fg}(B=0).
$$

Making the change of variables [7]

$$
s = \frac{u}{2}(1+v), \qquad r = \frac{u}{2}(1-v), \tag{3.31}
$$

with $0 \leq u < \infty$ and $-1 \leq v \leq 1$, so that $dsdr = (u/2)dudv$, leaves us with

$$\begin{aligned}\left(\Pi^{\overset{\text{s}}{\text{ps}}}_{fg}\right)_B = &\frac{N_c|q_fB|}{2\pi^2}\int_0^\infty du\int_{-1}^1 dv\,\frac{e^{-\frac{u}{2}(1+v)M_f^2-\frac{u}{2}(1-v)M_g^2}}{2}\\ &\times\left[\frac{1\mp uM_fM_g}{u\tanh\left(|q_fB|u\right)}+\frac{|q_fB|}{\sinh^2\left(|q_fB|u\right)}-\frac{2\mp uM_fM_g}{|q_fB|u^2}\right].\end{aligned} \tag{3.32}$$

The integrals in these new variables can be performed in closed form when $f = g$. In that case, we have

$$\left(\Pi^{\overset{\text{s}}{\text{ps}}}_{ff}\right)_B = \frac{N_c|q_fB|}{2\pi^2}\int_0^\infty du\,e^{-uM_f^2}\left[\frac{1\mp uM_f^2}{u\tanh\left(|q_fB|u\right)}+\frac{|q_fB|}{\sinh^2\left(|q_fB|u\right)}-\frac{2\mp uM_f^2}{|q_fB|u^2}\right],$$

which can be rewritten as

$$\begin{aligned}\left(\Pi^{\overset{\text{s}}{\text{ps}}}_{ff}\right)_B = \frac{N_c|q_fB|^2}{2\pi^2}\Bigg\{&\int_0^\infty du\,\frac{e^{-uM_f^2}}{|q_fB|u}\left[\coth(|q_fB|u)-\frac{1}{|q_fB|u}\right]\\ &\mp\frac{M_f^2}{|q_fB|}\int_0^\infty du\,e^{-uM_f^2}\left[\coth(|q_fB|u)-\frac{1}{|q_fB|u}\right]\\ &+\int_0^\infty du\,e^{-uM_f^2}\left[\frac{1}{\sinh^2\left(|q_fB|u\right)}-\frac{1}{\left(|q_fB|u\right)^2}\right]\Bigg\}.\end{aligned}$$

Now let $\tau = |q_fB|u$ so that

$$\begin{aligned}\left(\Pi^{\overset{\text{s}}{\text{ps}}}_{ff}\right)_B = \frac{N_c|q_fB|}{2\pi^2}\Bigg[&\int_0^\infty d\tau\,\frac{e^{-a_f^2\tau}}{\tau^2}(\tau\coth\tau-1)\mp a_f^2\int_0^\infty d\tau\,\frac{e^{-a_f^2\tau}}{\tau}(\tau\coth\tau-1)\\ &+\int_0^\infty d\tau\,e^{-a_f^2\tau}\left(\frac{1}{\sinh^2\tau}-\frac{1}{\tau^2}\right)\Bigg],\end{aligned}$$

where $a_f = M_f/\sqrt{|q_fB|}$. As we can see from the expansions (A.9) and (A.11), all the integrands in the expression above are finite for $\tau \to 0$, rendering the integrals convergent and avoiding the necessity of introducing a proper time cutoff. This would not be the case if we had not subtracted the vacuum contribution.

Using the results (A.24), (A.25) and (A.27) from Appendix A and rearranging the terms leads to

$$\begin{aligned}\left(\Pi^{\overset{\text{s}}{\text{ps}}}_{ff}\right)_B = \frac{N_cM_f^2}{2\pi^2}\Bigg[&1+\frac{|q_fB|}{M_f^2}\ln\left(\frac{M_f^2}{4\pi|q_fB|}\right)+\frac{2|q_fB|}{M_f^2}\ln\Gamma\left(\frac{M_f^2}{2|q_fB|}\right)\\ &+(1\pm1)\psi\left(\frac{M_f^2}{2|q_fB|}\right)-(2\pm1)\ln\left(\frac{M_f^2}{2|q_fB|}\right)+(1\pm1)\frac{|q_fB|}{M_f^2}\Bigg].\end{aligned}$$

More explicitly, we have, for the pure magnetic contribution to the scalar and pseudoscalar polarization functions, respectively,

$$\left(\Pi^{\rm s}_{ff}\right)_B = \frac{N_c M_f^2}{2\pi^2}\left[1+\frac{|q_fB|}{M_f^2}\ln\left(\frac{M_f^2}{4\pi|q_fB|}\right)+\frac{2|q_fB|}{M_f^2}\ln\Gamma\left(\frac{M_f^2}{2|q_fB|}\right)\right.$$
$$\left.+2\psi\left(\frac{M_f^2}{2|q_fB|}\right)-3\ln\left(\frac{M_f^2}{2|q_fB|}\right)+2\frac{|q_fB|}{M_f^2}\right], \tag{3.33a}$$

$$\left(\Pi^{\rm ps}_{ff}\right)_B = \frac{N_c M_f^2}{2\pi^2}\left[1+\frac{|q_fB|}{M_f^2}\ln\left(\frac{M_f^2}{4\pi|q_fB|}\right)+\frac{2|q_fB|}{M_f^2}\ln\Gamma\left(\frac{M_f^2}{2|q_fB|}\right)-\ln\left(\frac{M_f^2}{2|q_fB|}\right)\right]. \tag{3.33b}$$

Now that we have the expressions for the polarization functions, we can find the effective flavor- and B-dependent couplings.

3.4.4 Numerical results

The effective couplings, in the fundamental SU(3) representation, are given by Eqs. (3.19), with the polarization functions being given by Eq. (3.27) in our case of interest where $q_f = q_g$. Here we have two possible approaches: consider the full polarization functions, which includes both the vacuum and the pure magnetic contribution (V+B), or consider only the contribution from the external field (Only B) by simply dropping the vacuum regularization-dependent term, since we are interested only in the effects of the magnetic field in the NJL coupling. For completeness, we present both approaches.

3.4.4.1 V+B

Let us begin by considering the simplest case: the couplings $G^{\rm s}_{ff}(B)$ and $G^{\rm ps}_{ff}(B)$. From Eqs. (3.19), (3.30) and (3.33) we find

$$G^{\rm s}_{ff}(B) = g + \frac{g^2N_c}{2\pi^2}\left[\Lambda^2-3M_f^2\ln\left(\frac{\Lambda^2+M_f^2}{M_f^2}\right)+2M_f^2\frac{\Lambda^2}{\Lambda^2+M_f^2}\right]$$
$$+\frac{g^2N_cM_f^2}{2\pi^2}\left[1+\frac{|q_fB|}{M_f^2}\ln\left(\frac{M_f^2}{4\pi|q_fB|}\right)+\frac{2|q_fB|}{M_f^2}\ln\Gamma\left(\frac{M_f^2}{2|q_fB|}\right)\right.$$
$$\left.+2\psi\left(\frac{M_f^2}{2|q_fB|}\right)-3\ln\left(\frac{M_f^2}{2|q_fB|}\right)+2\frac{|q_fB|}{M_f^2}\right], \tag{3.34a}$$

$$G^{\rm ps}_{ff}(B) = g + \frac{g^2N_c}{2\pi^2}\left[\Lambda^2-M_f^2\ln\left(\frac{\Lambda^2+M_f^2}{M_f^2}\right)\right]$$
$$+\frac{g^2N_cM_f^2}{2\pi^2}\left[1+\frac{|q_fB|}{M_f^2}\ln\left(\frac{M_f^2}{4\pi|q_fB|}\right)+\frac{2|q_fB|}{M_f^2}\ln\Gamma\left(\frac{M_f^2}{2|q_fB|}\right)-\ln\left(\frac{M_f^2}{2|q_fB|}\right)\right]. \tag{3.34b}$$

These couplings are plotted in Figures 3.3 and 3.4 as functions of the magnetic field. Like before, we used $\Lambda = 914.6\,\text{MeV}$ and $g = 9.76\,\text{GeV}^{-2}$, as well as the solutions of the gap equations for the constituent quark masses displayed in Figure 3.1.

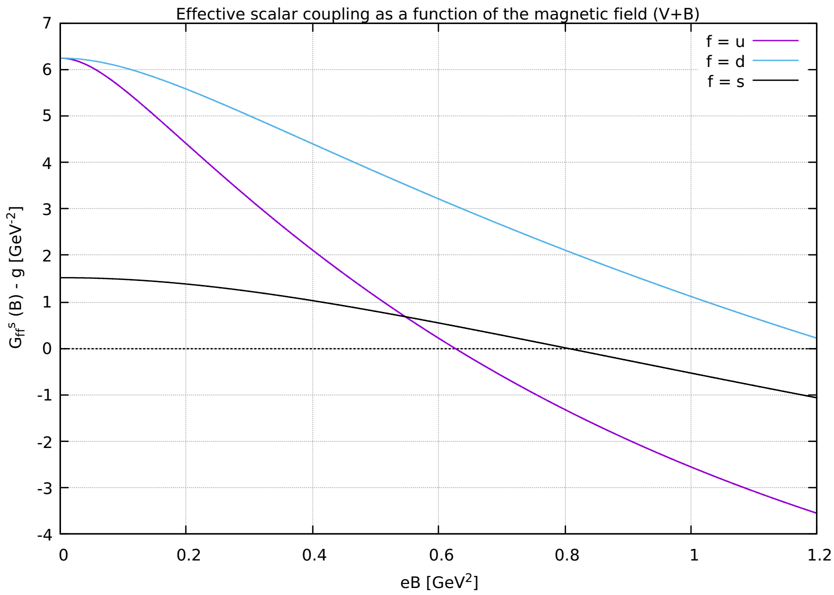

Figure 3.3: Correction to the effective scalar coupling as a function of the magnetic field, $G^{\text{s}}_{ff}(B) - g$, given by Eq. (3.34a), which considers both vacuum and pure magnetic contributions. Here we used $\Lambda = 914.6\,\text{MeV}$ and $g = 9.76\,\text{GeV}^{-2}$ as well as the values of the effective quark masses displayed in Figure 3.1.

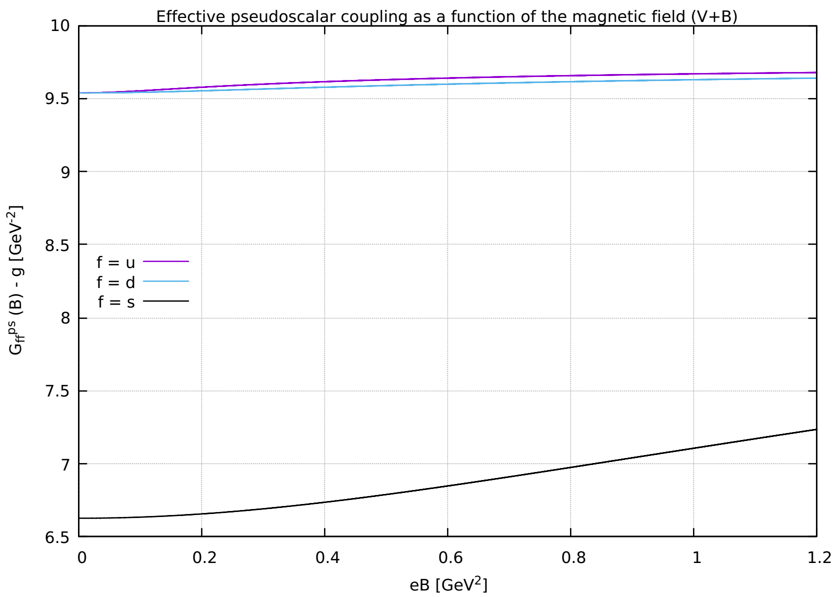

Figure 3.4: Correction to the effective pseudoscalar coupling as a function of the magnetic field, $G^{\text{ps}}_{ff}(B) - g$, given by Eq. (3.34b), which considers both vacuum and pure magnetic contributions. Here we used $\Lambda = 914.6\,\text{MeV}$ and $g = 9.76\,\text{GeV}^{-2}$ as well as the values of the effective quark masses displayed in Figure 3.1.

Figures 3.3 and 3.4 show that we obtain, from polarization effects, an effective scalar coupling which decreases and an effective pseudoscalar coupling that increases with the magnetic field. For the scalar coupling, this decrease is stronger for the quark up, which again exhibits a greater response to the magnetic field due to its electric charge being double (in absolute value) the electric charge of the quarks down and strange. Furthermore, we also note that the corrections to the couplings are greater for the quarks up and down for low field strengths. The reason for that can be inferred from the first line of Eq. (3.34a), specifically from the term with the minus sign that leads to a negative contribution proportional to the constituent quark mass, which is greater for the quark strange in the weak field regime. Same thing happens for the pseudoscalar coupling. However, we now have an increase which is greater for the quark strange, signaling that, in this case, the behavior of the correction is mostly dictated by the vacuum contribution, as there is little difference between the curves for the quarks up and down.

Another case of interest is the one of the couplings $G^{\mathrm{s}}_{ds}(B)$ and $G^{\mathrm{ps}}_{ds}(B)$, which involve quarks of different flavors but with same electric charge. From Eqs. (3.19), (3.29) and (3.32) we have

$$\begin{aligned}
G^{\substack{\mathrm{s}\\ \mathrm{ps}}}_{ds}(B) = g &+ \frac{g^2N_c}{4\pi^2}\left[\Lambda^2 - M_d^2\ln\left(\frac{\Lambda^2+M_d^2}{M_d^2}\right)\right] + \frac{g^2N_c}{4\pi^2}\left[\Lambda^2 - M_s^2\ln\left(\frac{\Lambda^2+M_s^2}{M_s^2}\right)\right] \\
&- \frac{g^2N_c}{4\pi^2}(M_d \pm M_s)^2\int_0^1 dx\left\{\ln\left[\frac{\Lambda^2+D^2_{ds}(0)}{D^2_{ds}(0)}\right] + \frac{D^2_{ds}(0)}{\Lambda^2+D^2_{ds}(0)} - 1\right\} \\
&+ \frac{g^2N_c|q_dB|}{2\pi^2}\int_0^\infty du\int_{-1}^1 dv\,\frac{e^{-\frac{u}{2}(1+v)M_d^2-\frac{u}{2}(1-v)M_s^2}}{2} \\
&\times\left[\frac{1\mp uM_dM_s}{u\tanh(|q_dB|u)} + \frac{|q_dB|}{\sinh^2(|q_dB|u)} - \frac{2\mp uM_dM_s}{|q_dB|u^2}\right],
\end{aligned} \tag{3.35}$$

where $D^2_{ds}(0) = xM_d^2 + (1-x)M_s^2$. Unfortunately, the integrals in Eq. (3.35) cannot be solved in closed form and we need to rely on numerical methods. Choosing to apply the trapezoidal rule, we find the results shown in Figures 3.5 and 3.6 for the correction to the effective couplings as functions of the magnetic field.

Like in the case where $f = g$, Figures 3.5 and 3.6 show a decreasing effective scalar coupling and an increasing effective pseudoscalar coupling. We also note that the corrections to the pseudoscalar coupling assume high values even for small field strengths.

The high values for the corrections to the effective couplings in Figures 3.3, 3.4 and 3.6 in the weak field regime could be a consequence of our parameter choice, which was made by fitting meson properties back in Section 2.4. These values can be compared with the results from Ref. [22], where another set of parameters was chosen and smaller values for the corrections to the NJL coupling constant were obtained. We could change our parameter choice in an attempt to obtain more reliable results for those corrections. However, since the scope of our work is to analyze the consequences of a B-dependent

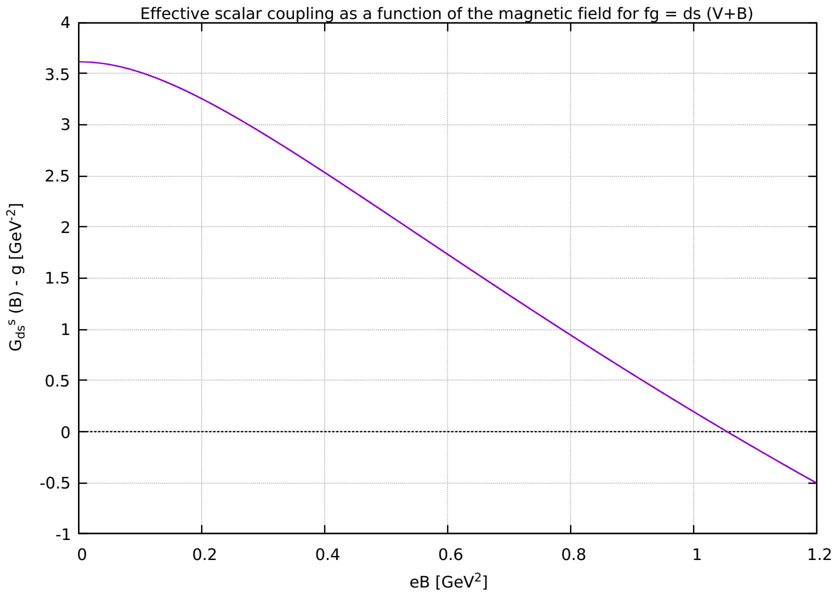

Figure 3.5: Correction to the effective scalar coupling as a function of the magnetic field, $G^{\mathrm{s}}_{ds}(B) - g$, given by Eq. (3.35), which considers both vacuum and pure magnetic contributions. Here we used $\Lambda = 914.6\,\mathrm{MeV}$ and $g = 9.76\,\mathrm{GeV}^{-2}$ as well as the values of the effective quark masses displayed in Figure 3.1.

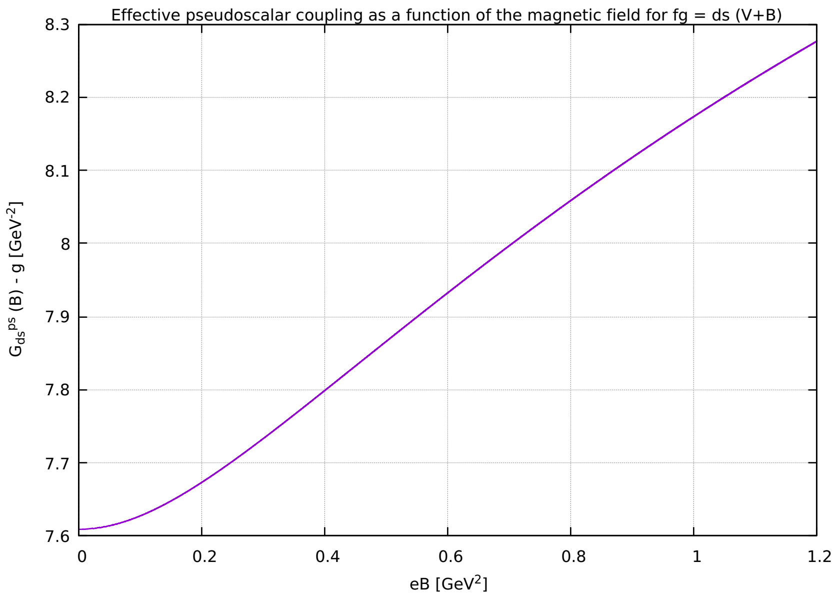

Figure 3.6: Correction to the effective pseudoscalar coupling as a function of the magnetic field, $G^{\mathrm{ps}}_{ds}(B) - g$, given by Eq. (3.35), which considers both vacuum and pure magnetic contributions. Here we used $\Lambda = 914.6\,\mathrm{MeV}$ and $g = 9.76\,\mathrm{GeV}^{-2}$ as well as the values of the effective quark masses displayed in Figure 3.1.

coupling, we might as well simply discard the vacuum terms and consider only the pure magnetic contributions to the effective coupling.

3.4.4.2 Only B

By dropping the vacuum regularization-dependent contribution, Eqs. (3.34) become

$$\begin{aligned}G^{\mathrm{s}}_{ff}(B) = g + \frac{g^2 N_c M_f^2}{2\pi^2}\Bigg[1 + \frac{|q_f B|}{M_f^2}\ln\left(\frac{M_f^2}{4\pi|q_f B|}\right) + \frac{2|q_f B|}{M_f^2}\ln\Gamma\left(\frac{M_f^2}{2|q_f B|}\right) \\ +2\psi\left(\frac{M_f^2}{2|q_f B|}\right) - 3\ln\left(\frac{M_f^2}{2|q_f B|}\right) + 2\frac{|q_f B|}{M_f^2}\Bigg],\end{aligned} \tag{3.36a}$$

$$G^{\mathrm{ps}}_{ff}(B) = g + \frac{g^2 N_c M_f^2}{2\pi^2}\left[1 + \frac{|q_f B|}{M_f^2}\ln\left(\frac{M_f^2}{4\pi|q_f B|}\right) + \frac{2|q_f B|}{M_f^2}\ln\Gamma\left(\frac{M_f^2}{2|q_f B|}\right) - \ln\left(\frac{M_f^2}{2|q_f B|}\right)\right], \tag{3.36b}$$

while, for Eq. (3.35), we have

$$\begin{aligned}G^{\substack{\mathrm{s}\\\mathrm{ps}}}_{ds}(B) = g + \frac{g^2 N_c |q_d B|}{2\pi^2}\int_0^\infty du \int_{-1}^{1} dv\, \frac{e^{-\frac{u}{2}(1+v)M_d^2 - \frac{u}{2}(1-v)M_s^2}}{2} \\ \times\left[\frac{1 \mp u M_d M_s}{u\tanh\left(|q_d B| u\right)} + \frac{|q_d B|}{\sinh^2\left(|q_d B| u\right)} - \frac{2 \mp u M_d M_s}{|q_d B| u^2}\right].\end{aligned} \tag{3.37}$$

Note that none of the equations above contain explicit dependence of the cutoff parameter.

For the couplings containing only the pure magnetic contribution to their corrections, we have the results shown in Figures 3.7, 3.8, 3.9 and 3.10.

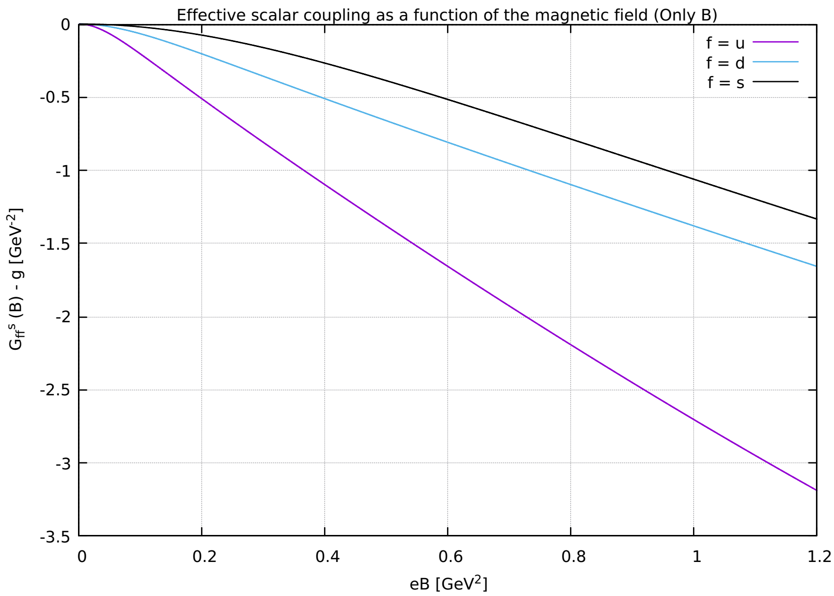

Figure 3.7: Correction to the effective scalar coupling as a function of the magnetic field, $G^{\mathrm{s}}_{ff}(B) - g$, given by Eq. (3.36a), which considers only the pure magnetic contribution. Here we used $g = 9.76\,\mathrm{GeV}^{-2}$ as well as the values of the effective quark masses displayed in Figure 3.1.

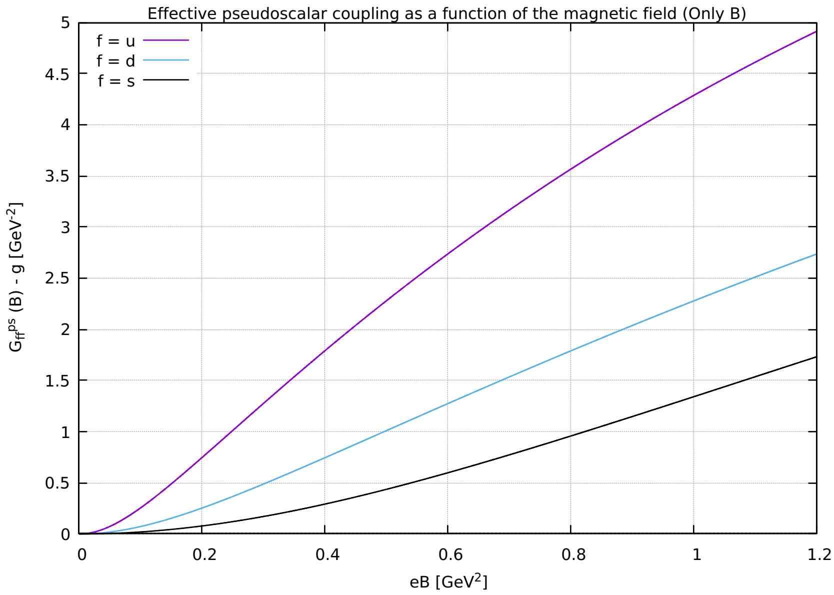

Figure 3.8: Correction to the effective pseudoscalar coupling as a function of the magnetic field, $G_{ff}^{\text{ps}}(B) - g$, given by Eq. (3.36b), which considers only the pure magnetic contribution. Here we used $g = 9.76\,\text{GeV}^{-2}$ as well as the values of the effective quark masses displayed in Figure 3.1.

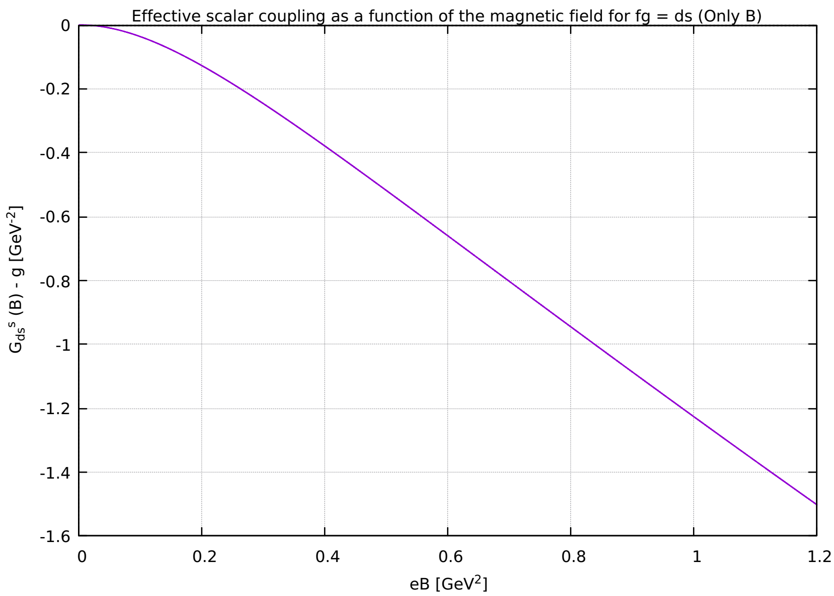

Figure 3.9: Correction to the effective scalar coupling as a function of the magnetic field, $G_{ds}^{\text{s}}(B) - g$, given by Eq. (3.37), which considers only the pure magnetic contribution. Here we used $g = 9.76\,\text{GeV}^{-2}$ as well as the values of the effective quark masses displayed in Figure 3.1.

Just like in the previous case, Figures 3.7 and 3.9 show decreasing effective scalar couplings, while Figures 3.8 and 3.10 show increasing effective pseudoscalar couplings. Both in the scalar and pseudoscalar cases, for $f = g$, the quark up exhibits a greater response to the applied field, since now there is no vacuum contribution to compete with.

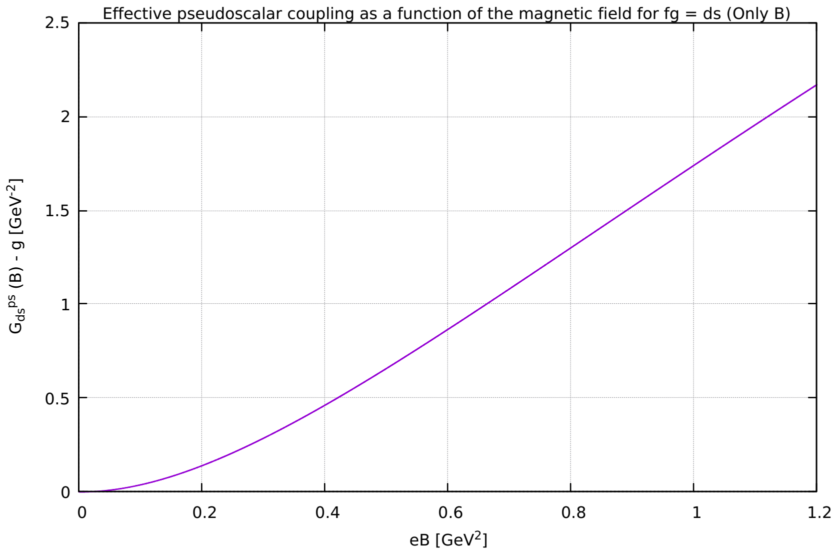

Figure 3.10: Correction to the effective pseudoscalar coupling as a function of the magnetic field, $G^{\rm ps}_{ds}(B) - g$, given by Eq. (3.37), which considers only the pure magnetic contribution. Here we used $g = 9.76\,{\rm GeV}^{-2}$ as well as the values of the effective quark masses displayed in Figure 3.1.

We have now shown how vacuum polarization naturally leads to a flavor- and B-dependent coupling in the NJL model in the presence of a uniform magnetic field, thus serving as a mechanism for obtaining such dependence. Since we are interested in the effects of the external field in the NJL coupling, from now on we shall consider only the pure magnetic contributions to the coupling constant corrections. Hence, in the remainder of this work, when we talk about B-dependent couplings we are considering them to be given by the ones displayed in Figures 3.7, 3.8, 3.9 and 3.10.

CHAPTER 4

Quark and meson masses with B-dependent couplings

In Chapter 3 we showed how a flavor- and B-dependent coupling arises in the NJL model from vacuum polarization. Having proposed a mechanism for the dependence of the coupling constants on the external field, let us now consider a NJL Lagrangian with such couplings and see the effects on the constituent quark masses as well as on meson masses.

4.1 Quark masses with B-dependent couplings

We start with the following Lagrangian,

$$\mathcal{L}'_{\rm NJL} = \bar{\psi}\Big(i\not{D} - m\Big)\psi + \frac{1}{2}G^{\rm s}_{ab}\Big(\bar{\psi}\lambda^a\psi\Big)\Big(\bar{\psi}\lambda^b\psi\Big) + \frac{1}{2}G^{\rm ps}_{ab}\Big(\bar{\psi}i\gamma_5\lambda^a\psi\Big)\Big(\bar{\psi}i\gamma_5\lambda^b\psi\Big), \qquad (4.1)$$

where, like before, ψ is the quark field, m is the current quark mass matrix and $D_\mu = \partial_\mu - iQA_\mu$ is the covariant derivative. The B-dependence of the couplings is left implicit for now.

We may proceed exactly as we did in Section 2.3 to obtain an effective action in terms of interacting quark fields by separating the quark field bilinears into two components like in Eq. (2.9). Then, the Lagrangian becomes

$$\begin{aligned}
\mathcal{L}'_{\rm NJL} = {}& \bar{\psi}_c\Big(i\not{D} - m\Big)\psi_c + \bar{\psi}\Big(i\not{D} - m\Big)\psi \\
&+ \frac{1}{2}G^{\rm s}_{ab}\Big(\bar{\psi}\lambda^a\psi\Big)_c\Big(\bar{\psi}\lambda^b\psi\Big)_c + \frac{1}{2}G^{\rm ps}_{ab}\Big(\bar{\psi}i\gamma_5\lambda^a\psi\Big)_c\Big(\bar{\psi}i\gamma_5\lambda^b\psi\Big)_c \\
&+ \frac{1}{2}G^{\rm s}_{ab}\Big(\bar{\psi}\lambda^a\psi\Big)\Big(\bar{\psi}\lambda^b\psi\Big) + \frac{1}{2}G^{\rm ps}_{ab}\Big(\bar{\psi}i\gamma_5\lambda^a\psi\Big)\Big(\bar{\psi}i\gamma_5\lambda^b\psi\Big) \\
&+ \frac{1}{2}G^{\rm s}_{ab}\Big[\Big(\bar{\psi}\lambda^a\psi\Big)_c\Big(\bar{\psi}\lambda^b\psi\Big) + \Big(\bar{\psi}\lambda^a\psi\Big)\Big(\bar{\psi}\lambda^b\psi\Big)_c\Big] \\
&+ \frac{1}{2}G^{\rm ps}_{ab}\Big[\Big(\bar{\psi}i\gamma_5\lambda^a\psi\Big)_c\Big(\bar{\psi}i\gamma_5\lambda^b\psi\Big) + \Big(\bar{\psi}i\gamma_5\lambda^a\psi\Big)\Big(\bar{\psi}i\gamma_5\lambda^b\psi\Big)_c\Big].
\end{aligned}$$

In order to integrate out the quark component $(\bar{\psi}\psi)_c$ with the help of the usual SU(3) auxiliary fields, we introduce the identities

$$1 = N\int \mathcal{D}S\ \exp\left\{-\frac{i}{2}\left[S^a - g\left(\bar{\psi}\lambda^a\psi\right)_c\right]\frac{G^{\rm s}_{ab}}{g^2}\left[S^b - g\left(\bar{\psi}\lambda^b\psi\right)_c\right]\right\},$$

$$1 = N\int \mathcal{D}P\ \exp\left\{-\frac{i}{2}\left[P^a - g\left(\bar{\psi}i\gamma_5\lambda^a\psi\right)_c\right]\frac{G^{\rm ps}_{ab}}{g^2}\left[P^b - g\left(\bar{\psi}i\gamma_5\lambda^b\psi\right)_c\right]\right\},$$

in the generating functional

$$Z'[\bar{\eta},\eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\ \exp\left\{i\left[S'_{\rm NJL}[\bar{\psi},\psi] + \int d^4x\,(\bar{\psi}\eta + \bar{\eta}\psi)\right]\right\},$$

where $S'_{\rm NJL} = \int d^4x\,\mathcal{L}'_{\rm NJL}$. Then, we repeat the procedure of Section 2.3, which now leads us to obtain the effective action given by

$$\begin{aligned} S'_{\rm eff} = &-i{\rm Tr}\ln\left(-i\mathcal{S}^{-1}\right) - \int d^4x\left(\frac{G^{\rm s}_{ab}}{2g^2}S^aS^b + \frac{G^{\rm ps}_{ab}}{2g^2}P^aP^b\right) \\ &+ \int d^4x\left[\bar{\psi}\left(i\not{\partial} - m\right)\psi + \frac{1}{2}G^{\rm s}_{ab}\left(\bar{\psi}\lambda^a\psi\right)\left(\bar{\psi}\lambda^b\psi\right) + \frac{1}{2}G^{\rm ps}_{ab}\left(\bar{\psi}i\gamma_5\lambda^a\psi\right)\left(\bar{\psi}i\gamma_5\lambda^b\psi\right)\right], \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \mathcal{S}^{-1}(x-y) = &\left\{i\not{\partial} - M' + \frac{1}{2}G^{\rm s}_{ab}\left[\lambda^a\left(\bar{\psi}\lambda^b\psi\right) + \lambda^b\left(\bar{\psi}\lambda^a\psi\right)\right]\right. \\ &\left.+\frac{1}{2}G^{\rm ps}_{ab}\left[\lambda^a\left(\bar{\psi}i\gamma_5\lambda^b\psi\right) + \lambda^b\left(\bar{\psi}i\gamma_5\lambda^a\psi\right)\right]\right\}\delta^4(x-y), \end{aligned} \tag{4.3}$$

and the effective mass matrix is

$$M' = m + \frac{G^{\rm s}_{ab}}{2g}\left(S^a\lambda^b + S^b\lambda^a\right) + i\gamma_5\frac{G^{\rm ps}_{ab}}{2g}\left(P^a\lambda^b + P^b\lambda^a\right). \tag{4.4}$$

To obtain the new gap equation, we can proceed as in Section 2.3 by imposing the stationary condition. However, a simpler strategy may be adopted. Recall from Section 3.3 that the first order term in the quark determinant expansion resulted in a correction to mass term given by the gap equation. Since this mechanism did not depend on the coupling constant it naturally applies to our present case. The only difference we can expect to obtain is on which coupling enters the gap equation. Thus, in order to find the new constituent quark masses we expand the quark determinant up to first order in the quark currents, leading us to the mass corrections $M'_f - m_f$ which now take into account the effects of the flavor- and B-dependent couplings.

Hence, we write

$$S'_{\rm det} = -i{\rm Tr}\ln\left\{1 + S^{(0)}\left\{\frac{1}{2}G^{\rm s}_{ab}\left[\lambda^a\left(\bar{\psi}\lambda^b\psi\right) + \lambda^b\left(\bar{\psi}\lambda^a\psi\right)\right]\right.\right.$$

$$
\begin{aligned}
&\qquad +\frac{1}{2}G^{\text{ps}}_{ab}\Big[\lambda^a\Big(\bar\psi i\gamma_5\lambda^b\psi\Big)+\lambda^b\Big(\bar\psi i\gamma_5\lambda^a\psi\Big)\Big]\Big\}\Big\}\\
&\simeq -i\text{Tr}\left\{S^{(0)}\left\{\frac{1}{2}G^{\text{s}}_{ab}\Big[\lambda^a\Big(\bar\psi\lambda^b\psi\Big)+\lambda^b\Big(\bar\psi\lambda^a\psi\Big)\Big]\right.\right.\\
&\qquad \left.\left.+\frac{1}{2}G^{\text{ps}}_{ab}\Big[\lambda^a\Big(\bar\psi i\gamma_5\lambda^b\psi\Big)+\lambda^b\Big(\bar\psi i\gamma_5\lambda^a\psi\Big)\Big]\right\}\right\}\\
&= -\text{tr}\Big\{iS^{(0)}(0)\int d^4x\,\frac{1}{2}G^{\text{s}}_{ab}\Big[\lambda^a\Big(\bar\psi\lambda^b\psi\Big)+\lambda^b\Big(\bar\psi\lambda^a\psi\Big)\Big]\Big\},
\end{aligned}
$$

in analogy with Eq. (3.9). Additionally, we used the fact that the term with the pseudoscalar currents has vanishing Dirac trace.

In order to compute the flavor trace in the expression above, we use the relations (3.10). Then, it follows from Eqs. (3.11) and (3.22), and after some long and tedious algebra, that

$$
\frac{1}{2}G^{\text{s}}_{ab}\Big\{\text{tr}_F\Big[iS^{(0)}(0)\lambda^a\Big]\Big(\bar\psi\lambda^b\psi\Big)+\text{tr}_F\Big[iS^{(0)}(0)\lambda^b\Big]\Big(\bar\psi\lambda^a\psi\Big)\Big\}=2\sum_{f=u,d,s}G^{\text{s}}_{ff}iS^{(0)}_f(0)\bar\psi_f\psi_f.
$$

Therefore, the contribution of the quark determinant is simply

$$
\begin{aligned}
S'_{\text{det}} &= -\sum_{f=u,d,s}\int d^4x\,2G^{\text{s}}_{ff}\text{tr}_{DC}\Big[iS^{(0)}_f(0)\Big]\bar\psi_f\psi_f\\
&= \sum_{f=u,d,s}\int d^4x\,2G^{\text{s}}_{ff}\left\langle\bar\psi_f\psi_f\right\rangle_B\bar\psi_f\psi_f,
\end{aligned}
$$

which simply yields an effective mass given by

$$
M'_f = m_f - 2G^{\text{s}}_{ff}(B)\left\langle\bar\psi_f\psi_f\right\rangle_B. \tag{4.5}
$$

This has the same form as the gap equation (2.20) with g replaced by $G^{\text{s}}_{ff}(B)$ given by Eq. (3.36a). Taking the chiral condensate to be given by Eq. (3.5) with M_f replaced by M'_f, the new constituent quark masses become the solutions of

$$
\begin{aligned}
M'_f = m_f + \frac{G^{\text{s}}_{ff}(B)N_cM'_f}{2\pi^2}&\left[\Lambda^2 - M'^2_f\ln\left(\frac{\Lambda^2+M'^2_f}{M'^2_f}\right)\right.\\
&\left.+M'^2_f\left(1-\ln\frac{M'^2_f}{2|q_fB|}\right)+|q_fB|\ln\frac{M'^2_f}{4\pi|q_fB|}+2|q_fB|\ln\Gamma\left(\frac{M'^2_f}{2|q_fB|}\right)\right].
\end{aligned} \tag{4.6}
$$

The solutions of Eq. (4.6) are shown in Figure 4.1, where we included the solutions of the gap equation with B-independent coupling, Eq. (3.6), for comparison. As we can see, the effect of the B-dependent coupling, which decreases with the magnetic field, is to slow down the increase of the effective masses with B. This effect becomes more evident for higher field strengths and is greater for the quark up. In general, the phenomenon of magnetic catalysis is still observed when considering the new couplings, but it is reduced.

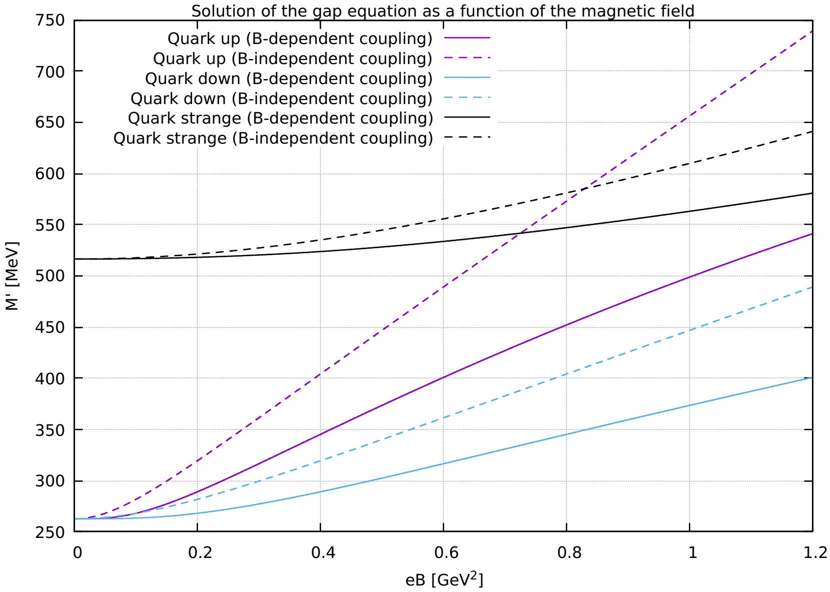

Figure 4.1: Solution of the gap equation as a function of the magnetic field with B-dependent couplings, Eq. (4.6). The dashed lines refer to the effective masses obtained from the gap equation with B-independent coupling. Here we used $\Lambda = 914.6\,\mathrm{MeV}$, $g = 9.76\,\mathrm{GeV}^{-2}$, $m_{ud} = 6\,\mathrm{MeV}$ and $m_s = 165.7\,\mathrm{MeV}$.

In order to investigate if, and to what extent, the inclusion of such flavor- and B-dependence in the NJL coupling leads to an improvement, we now proceed to compare our results with lattice QCD results.

4.2 Comparison with LQCD

In this section, we shall compare our results from the NJL model with B-dependent couplings with the lattice QCD results from Ref. [49]. To do so, we define the quantity

$$\Sigma_f(B) = \frac{2m_{ud}}{m_\pi^2 f_\pi^2}\left|\left\langle\bar{\psi}_f\psi_f\right\rangle_B - \left\langle\bar{\psi}_f\psi_f\right\rangle_{B=0}\right| + 1, \qquad f = u, d, \tag{4.7}$$

where m_π and f_π are the zero-field pion mass and decay constant, respectively. Here, we take $m_\pi = 135\,\mathrm{MeV}$ and $f_\pi = 86\,\mathrm{MeV}$ [49].

In the NJL model with an applied external magnetic field, the renormalized quark condensate was found to be given by Eq. (3.4) in terms of the constituent quark masses and as a function of B. Then, we can compute the quantity $\Sigma_f(B)$ by using the effective masses calculated either with or without B-dependent couplings and compare the results.

In Figures 4.2 and 4.3 we plot the quantities $(\Sigma_u + \Sigma_d)/2$ and $\Sigma_u - \Sigma_d$, respectively, as functions of the magnetic field. The points refer to LQCD results taken from Ref. [49], while the curves refer to the NJL model predictions, being that the yellow curve

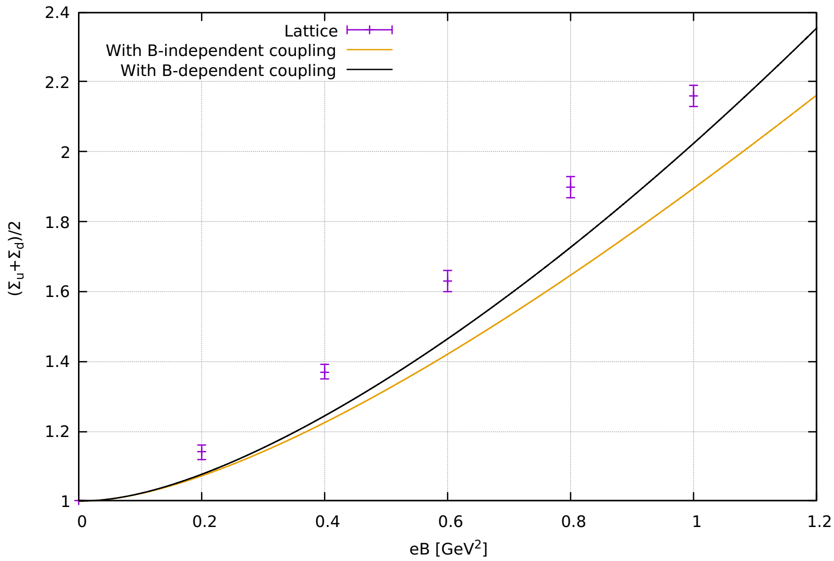

Figure 4.2: Comparison with LQCD results for $(\Sigma_u + \Sigma_d)/2$, with $\Sigma_f(B)$ given by Eq. (4.7). The points stand for lattice results while the curves stand for the NJL predictions. Here we took $m_{ud} = 6\,\text{MeV}$, $m_\pi = 135\,\text{MeV}$ and $f_\pi = 86\,\text{MeV}$. The quark effective masses were obtained with the parameters $\Lambda = 914.6\,\text{MeV}$ and $g = 9.76\,\text{GeV}^{-2}$.

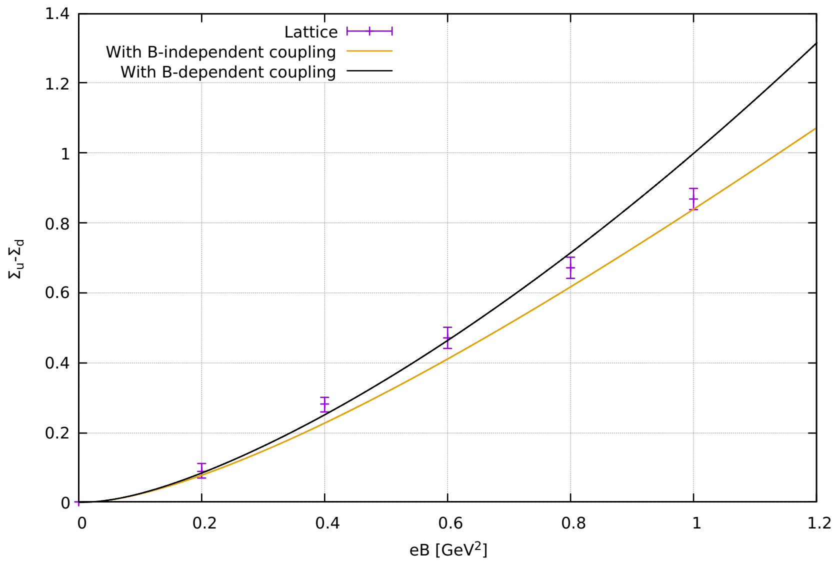

Figure 4.3: Comparison with LQCD results for $\Sigma_u - \Sigma_d$, with $\Sigma_f(B)$ given by Eq. (4.7). The points stand for lattice results while the curves stand for the NJL predictions. Here we took $m_{ud} = 6\,\text{MeV}$, $m_\pi = 135\,\text{MeV}$ and $f_\pi = 86\,\text{MeV}$. The quark effective masses were obtained with the parameters $\Lambda = 914.6\,\text{MeV}$ and $g = 9.76\,\text{GeV}^{-2}$.

considers the effective quark masses to be given by the gap equation with B-independent coupling, Eq. (3.6), while the black curve considers the effective masses to be given by the gap equation with B-dependent couplings, Eq. (4.6). While the effect of the B-dependent coupling is to slow down the increase of the effective quark masses with B, the opposite behavior is obtained for the condensates. This can be understood in a naive manner by recalling that the chiral condensate is given by the trace of the quark propagator, which is proportional to the inverse of the effective masses.

From Figure 4.2 we see that our inclusion of the flavor- and B-dependence in the NJL coupling was not enough to reproduce the lattice results with precision, but it leads to a closer behavior than the curve with fixed coupling, especially for strong fields. In contrast, in Figure 4.3 we have a better agreement between NJL and LQCD results coming from the curve with B-dependent couplings for intermediate field strengths, $\sim 0.4 - 0.8\,\mathrm{GeV}^2$. However, this curve begins to deviate from the lattice results for strong fields.

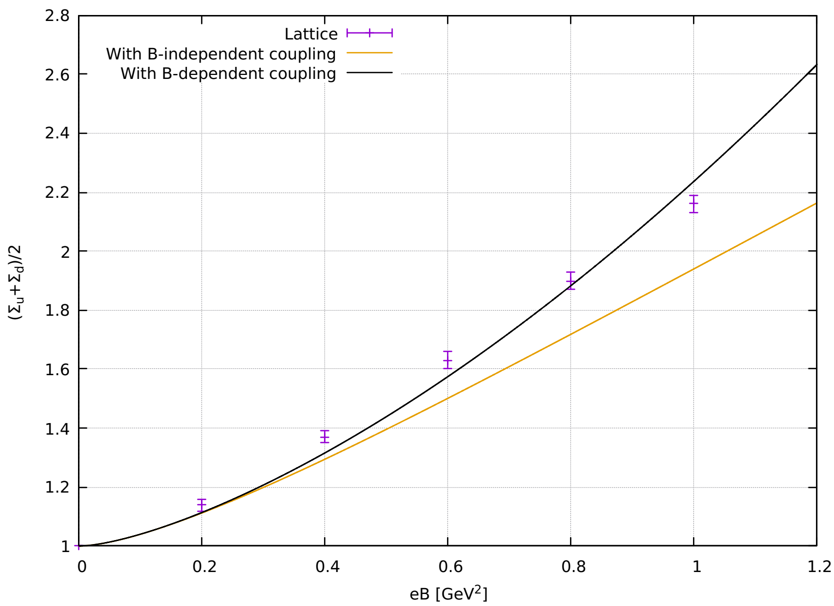

Figure 4.4: Comparison with LQCD results for $(\Sigma_u + \Sigma_d)/2$, with $\Sigma_f(B)$ given by Eq. (4.7). The points stand for lattice results while the curves stand for the NJL predictions. Here we took $m_{ud} = 6.5\,\mathrm{MeV}$, $m_\pi = 135\,\mathrm{MeV}$ and $f_\pi = 86\,\mathrm{MeV}$. The quark effective masses were obtained with the parameters $\Lambda = 400.0\,\mathrm{MeV}$ and $g = 25.0\,\mathrm{GeV}^{-2}$.

We emphasize that the model predictions depend on our parameter choice. Since we chose the NJL free parameters so that meson properties are set to their physical values, one may wonder if there is another set of values that yields better comparisons with LQCD. If, for example, we use the extremely unusual (and maybe even problematic) set of parameters given by $\Lambda = 400.0\,\mathrm{MeV}$, $g = 25.0\,\mathrm{GeV}^{-2}$ and $m_{ud} = 6.5\,\mathrm{MeV}$, we find the results shown in Figures 4.4 and 4.5. These new set improves the comparison for the average of the condensates in the intermediate field regime (Figure 4.4) and worsens the comparison for the difference of the condensates (Figure 4.5). In general, we are then led to conclude that considering couplings that depend on the external magnetic field, as well as on the quark flavors, which arise from vacuum polarization seems to lead in the right direction to conciliate NJL predictions and LQCD results.

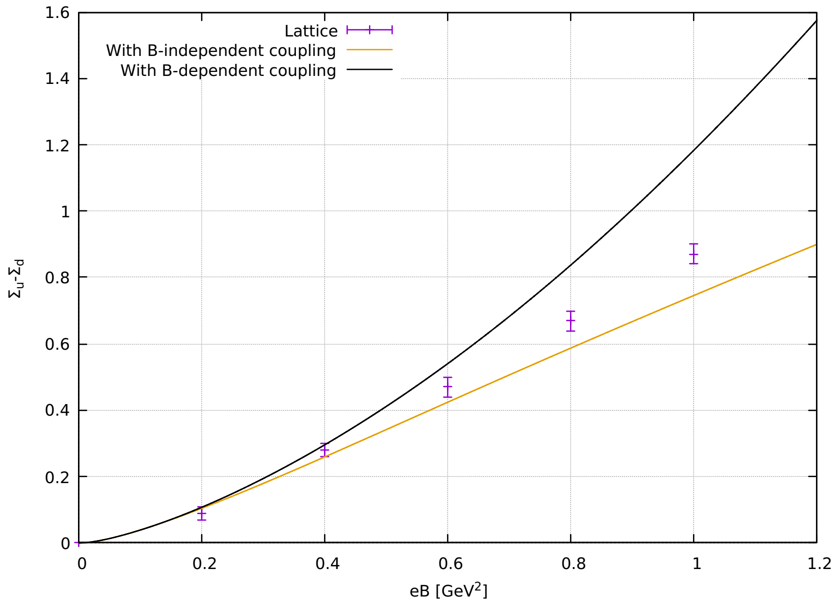

Figure 4.5: Comparison with LQCD results for $\Sigma_u - \Sigma_d$, with $\Sigma_f(B)$ given by Eq. (4.7). The points stand for lattice results while the curves stand for the NJL predictions. Here we took $m_{ud} = 6.5\,\mathrm{MeV}$, $m_\pi = 135\,\mathrm{MeV}$ and $f_\pi = 86\,\mathrm{MeV}$. The quark effective masses were obtained with the parameters $\Lambda = 400.0\,\mathrm{MeV}$ and $g = 25.0\,\mathrm{GeV}^{-2}$.

4.3 Meson masses in the presence of a uniform magnetic field

In order to see the effects of the B-dependent couplings in observables, we now proceed to compute pseudoscalar neutral meson masses in the presence of the applied magnetic field. In Section 2.4 we calculated the π^0 and K^0 masses in the absence of external fields in order the fix the NJL model parameters. Now, all we need to do is modify that procedure in order to obtain the meson masses in the presence of a uniform magnetic field.

4.3.1 The meson polarization tensors

The first modification we need to do concerns the polarization tensors. Recall that

$$\Pi^{\mathrm{ps}}_{\pi^0}(k^2) = \frac{1}{2}\Big[\Pi^{\mathrm{ps}}_{uu}(k^2) + \Pi^{\mathrm{ps}}_{dd}(k^2)\Big],$$

$$\Pi^{\mathrm{ps}}_{K^0}(k^2) = \Pi^{\mathrm{ps}}_{sd}(k^2),$$

where

$$\Pi^{\mathrm{ps}}_{fg}(k^2) = 2iN_c \int \frac{d^4p}{(2\pi)^4} \mathrm{tr}_D\Big[S^{(0)}_f(p) i\gamma_5 S^{(0)}_g(p-k) i\gamma_5\Big],$$

with $k_\mu = (k_0, \mathbf{0})$ being the meson four-momentum and $S_f^{(0)}(p)$ being the quark propagator in momentum space. The only difference with respect to the procedure from Section 2.4 is that now $S_f^{(0)}(p)$ is the propagator in the presence of the magnetic field, namely the one given by Eq. (3.23).

Let us denote

$$\Pi_{fg}^{\rm ps}(B;k^2) = 2iN_c \int \frac{d^4p}{(2\pi)^4}\mathrm{tr}_D\Big[S_f^{(0)}(p)i\gamma_5 S_g^{(0)}(\ell)i\gamma_5\Big], \tag{4.8}$$

where $\ell = p - k$. The procedure here is quite similar to the one from Section 3.4 where we had to compute the quark polarization functions, with the addition that now we have extra terms coming from the meson four-momentum k_μ. Hence we shall list only some main steps. Using the properties of the Dirac trace listed in Appendix B as well as $\ell^\mu = (p_0 - k_0, \mathbf{p})$, one finds

$$\begin{aligned}
&\mathrm{tr}_D\Big[S_f^{(0)}(p)i\gamma_5 S_g^{(0)}(\ell)i\gamma_5\Big] \\
&= -4\int_0^\infty\int_0^\infty dsdr\, e^{-isM_f^2 - ir\left(M_g^2 - k_\parallel^2\right)} \\
&\quad\times \exp\left\{i(s+r)p_\parallel^2 - 2ir(p\cdot k)_\parallel - i\left[\frac{\tan(|q_fB|s)}{|q_fB|} + \frac{\tan(|q_gB|s)}{|q_gB|}\right]p_\perp^2\right\} \\
&\quad\times\Big\{\Big[-M_fM_g + p_\parallel^2 - (p\cdot k)_\parallel\Big][1 - \mathrm{sign}(q_fq_g)\tan(|q_fB|s)\tan(|q_gB|r)] \\
&\qquad - p_\perp^2\sec^2(|q_fB|s)\sec^2(|q_gB|r)\Big\}.
\end{aligned}$$

We are denoting the constituent quark masses simply by M_f for now, as a general notation; later we can consider the cases where these masses are obtained with and without B-dependent couplings, separately.

After rotating the momenta to Euclidean space and introducing the imaginary proper time variables as usual, the quark polarization functions become

$$\begin{aligned}
\Pi_{fg}^{\rm ps}(B;k^2) =& -8N_c\int_0^\infty\int_0^\infty dsdr\, e^{-sM_f^2 - r\left(M_g^2 + k_{\parallel E}^2\right)}\int\frac{d^2p_{\parallel E}}{(2\pi)^2}\frac{d^2p_{\perp E}}{(2\pi)^2} \\
&\times\exp\left\{-(s+r)p_{\parallel E}^2 + 2r(p\cdot k)_{\parallel E} - \left[\frac{\tanh(|q_fB|s)}{|q_fB|} + \frac{\tanh(|q_gB|r)}{|q_gB|}\right]p_{\perp E}^2\right\} \\
&\times\Big\{\Big[-M_fM_g - p_{\parallel E}^2 + (p\cdot k)_{\parallel E}\Big][1 + \mathrm{sign}(q_fq_g)\tanh(|q_fB|s)\tanh(|q_gB|r)] \\
&\qquad - p_{\perp E}^2\,\mathrm{sech}^2(|q_fB|s)\,\mathrm{sech}^2(|q_gB|r)\Big\}.
\end{aligned}$$

The parallel momentum integrals are given by

$$\int\frac{d^2p_{\parallel E}}{(2\pi)^2}\,e^{-(s+r)p_{\parallel E}^2 + 2r(p\cdot k)_{\parallel E}} = \frac{e^{r^2k_{\parallel E}^2/(s+r)}}{4\pi(s+r)},$$

$$\int \frac{d^2 p_{\parallel E}}{(2\pi)^2} \left[p_{\parallel E}^2 - (p \cdot k)_{\parallel E}\right] e^{-(s+r)p_{\parallel E}^2 + 2r(p\cdot k)_{\parallel E}} = \frac{e^{r^2 k_{\parallel E}^2/(s+r)}}{4\pi(s+r)^2}\left(1 - \frac{sr}{s+r}k_{\parallel E}^2\right),$$

while the perpendicular momentum integrals are the same ones we computed in Section 3.4. Then, setting $q_f = q_g$, we find

$$\begin{aligned}\Pi_{fg}^{\rm ps}(B;k^2) = \frac{N_c|q_f B|}{2\pi^2}\int_0^\infty\int_0^\infty dsdr\,\frac{e^{-sM_f^2 - rM_g^2 - \frac{sr}{s+r}k_E^2}}{s+r}\\ \times\left[\frac{1 + M_f M_g(s+r) - \frac{sr}{s+r}k_E^2}{(s+r)\tanh\left(|q_f B|(s+r)\right)} + \frac{|q_f B|}{\sinh^2\left(|q_f B|(s+r)\right)}\right].\end{aligned} \tag{4.9}$$

Note that Eq. (4.9) reduces to Eq. (3.25) when we set $k_E^2 = 0$, as expected. In the limit $B \to 0$, Eq. (4.9) becomes

$$\Pi_{fg}^{\rm ps}(B=0;k^2) = \frac{N_c}{2\pi^2}\int_0^\infty\int_0^\infty dsdr\, e^{-sM_f^2 - rM_g^2 - \frac{sr}{s+r}k_E^2}\left[\frac{2 + M_f M_g(s+r) - \frac{sr}{s+r}k_E^2}{(s+r)^3}\right],$$

and we may write

$$\begin{aligned}\Pi_{fg}^{\rm ps}(B;k^2) = \Pi_{fg}^{\rm ps}(B=0;k^2) + \frac{N_c|q_f B|}{2\pi^2}\int_0^\infty\int_0^\infty dsdr\,\frac{e^{-sM_f^2 - rM_g^2 + \frac{sr}{s+r}k^2}}{s+r}\\ \times\left[\frac{1 + M_f M_g(s+r) + \frac{sr}{s+r}k^2}{(s+r)\tanh\left(|q_f B|(s+r)\right)} + \frac{|q_f B|}{\sinh^2\left(|q_f B|(s+r)\right)} - \frac{2 + M_f M_g(s+r) + \frac{sr}{s+r}k^2}{(s+r)^3}\right],\end{aligned} \tag{4.10}$$

after we rotated the momentum k back to Minkowski space. With $\Pi_{fg}^{\rm ps}(B=0;k^2)$ given by Eq. (2.39), we finally write

$$\begin{aligned}\Pi_{fg}^{\rm ps}(B;k^2) = \frac{N_c}{4\pi^2}\left[\Lambda^2 - M_f^2\ln\left(\frac{\Lambda^2 + M_f^2}{M_f^2}\right)\right] + \frac{N_c}{4\pi^2}\left[\Lambda^2 - M_g^2\ln\left(\frac{\Lambda^2 + M_g^2}{M_g^2}\right)\right]\\ + \frac{N_c}{4\pi^2}\left[k^2 - (M_f - M_g)^2\right]\int_0^1 dx\left\{\ln\left[\frac{\Lambda^2 + D_{fg}^2(k^2)}{D_{fg}^2(k^2)}\right] + \frac{D_{fg}^2(k^2)}{\Lambda^2 + D_{fg}^2(k^2)} - 1\right\}\\ + \frac{N_c|q_f B|}{2\pi^2}\int_0^\infty\int_0^\infty dsdr\,\frac{e^{-sM_f^2 - rM_g^2 + \frac{sr}{s+r}k^2}}{s+r}\\ \times\left[\frac{1 + M_f M_g(s+r) + \frac{sr}{s+r}k^2}{(s+r)\tanh\left(|q_f B|(s+r)\right)} + \frac{|q_f B|}{\sinh^2\left(|q_f B|(s+r)\right)} - \frac{2 + M_f M_g(s+r) + \frac{sr}{s+r}k^2}{(s+r)^3}\right],\end{aligned} \tag{4.11}$$

where $D_{fg}^2(k^2) = -x(1-x)k^2 + xM_f^2 + (1-x)M_g^2$.

4.3.2 Neutral pion and kaon masses

Let us now return to our usual notation. Let M_f denote the constituent quark masses that are obtained from the gap equation with fixed coupling, Eq. (3.6), and M_f'

denote the constituent quark masses that are obtained from the gap equation with B-dependent coupling, Eq. (4.6). Then we denote by $\Pi^{\rm ps}_{\pi^0}$ the neutral pion polarization tensor computed with effective masses M_f and $\Pi'^{\rm ps}_{\pi^0}$ the neutral pion polarization tensor computed with effective masses M'_f; similar notations will be used for the kaon polarization tensors.

In the absence of flavor-dependent couplings, the bound state equations required to be solved in order to obtain the neutral pion and kaon masses have the same form as we found in Section 2.4, namely

$$\begin{aligned} 1 - g\Pi^{\rm ps}_{\pi^0}(B; m_\pi^2) &= 0 \\ 1 - g\Pi^{\rm ps}_{K^0}(B; m_K^2) &= 0 \end{aligned} \tag{4.12}$$

where $\Pi^{\rm ps}_{\pi^0} = (\Pi^{\rm ps}_{uu} + \Pi^{\rm ps}_{dd})/2$ and $\Pi^{\rm ps}_{K^0} = \Pi^{\rm ps}_{ds}$, with $\Pi^{\rm ps}_{fg}(B; k^2)$ given by Eq. (4.11).

The bound state equations must be modified to include the flavor-dependent couplings. Here we will not consider the couplings G_{ab} with $a \neq b$ for simplicity. In fact, as it can be see from Eqs. (3.22), we can expect to have $G_{ab} << G_{aa}$ for $a \neq b$. Then, repeating the procedure of Section 2.4 for the effective action (4.2) with only diagonal coupling terms, we find the new bound state equations to be given by

$$\begin{aligned} 1 - \frac{g^2}{G^{\rm ps}_{33}(B)}\Pi'^{\rm ps}_{\pi^0}(B; m_\pi^2) &= 0 \\ 1 - \frac{g^2}{G^{\rm ps}_{66}(B)}\Pi'^{\rm ps}_{K^0}(B; m_K^2) &= 0 \end{aligned} \tag{4.13}$$

where $G^{\rm ps}_{33} = (G^{\rm ps}_{uu} + G^{\rm ps}_{dd})/2$ and $G^{\rm ps}_{66} = G^{\rm ps}_{77} = G^{\rm ps}_{ds}$.

Eqs. (4.12) and (4.13) need to be solved numerically, as well as the integrals in Eq. (4.11). Once again, we choose to apply the trapezoidal rule. The results for the pseudoscalar neutral meson masses as a function of the applied magnetic field are shown in Figure 4.6.

For the π^0 meson mass when the B-dependent coupling is used, Figure 4.6 shows an increase of m_π with the magnetic field for weak and intermediate field strengths, followed by a slight decrease for strong fields. This behavior is in contrast with LQCD results [50] and also with NJL predictions that employ a magnetic field dependent coupling which decreases with B [19]. For the K^0 meson mass, the B-dependent coupling calculation still results in a decreasing behavior with the magnetic field. However, this decrease is weaker than the one for the B-independent coupling calculation, which again is in contrast with LQCD results [51] and NJL with decreasing coupling $G(B)$ predictions [21]. This is because the vacuum polarization corrections to the NJL coupling found in Chapter 3 leads to pseudoscalar effective couplings that increase with the magnetic field, in contrast with the scalar ones. Since the reproduction of LQCD results with the NJL model requires a decreasing coupling with B [52], the disagreement between the results for

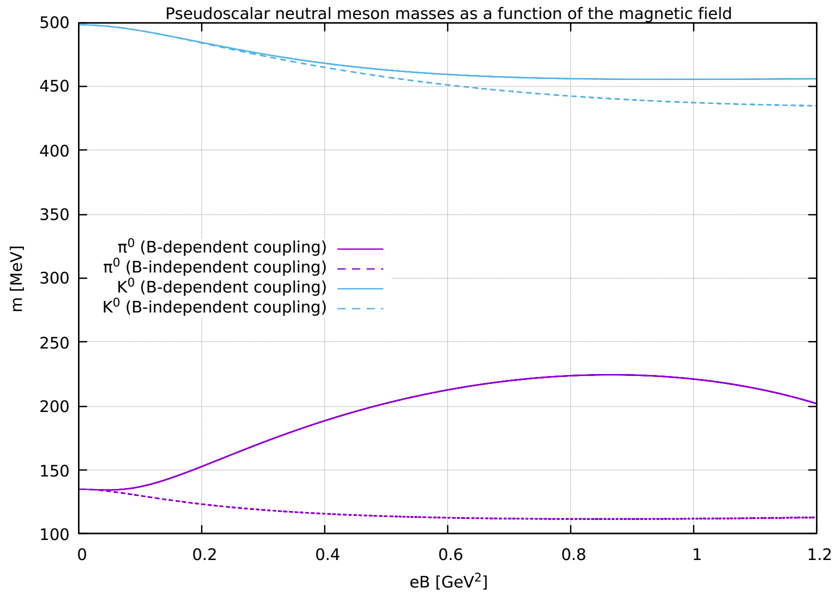

Figure 4.6: Pseudoscalar neutral meson masses as a function of the applied magnetic field. Here we used $\Lambda = 914.6\,\text{MeV}$ and $g = 9.76\,\text{GeV}^{-2}$.

using an increasing pseudoscalar coupling and the results from the literature is somewhat expected.

CHAPTER 5

Concluding remarks

Motivated by recent works which concluded that the NJL model is able to reproduce lattice QCD results provided a magnetic field dependent coupling is employed, we obtained a mechanism for which this dependence is acquired by considering vacuum polarization effects.

Starting with the U(3) NJL Lagrangian minimally coupled with a uniform magnetic field, we separated the quark field bilinears into two components, one corresponding to the quarks that condense in the ground state into scalar quark-antiquark condensates, and other corresponding to the interacting quasiparticle quarks. We then integrated out the first component with the introduction of the usual SU(3) auxiliary fields and an effective action in terms of the interacting quarks was obtained.

The gap equation for the constituent quark masses was derived by extremizing the effective action with respect to the auxiliary variables at zero quark field. This equation was solved by using the four-momentum cutoff regularization scheme and its solution showed the increase of mass with the magnetic field strength, exhibiting the phenomenon of magnetic catalysis.

The next step was the expansion of the quark determinant in powers of the quark currents. While the first-order term provided a correction to the quark masses given by the gap equation, the second-order term yielded corrections to an effective coupling dependent on the magnetic field strength and also on the flavors of the interaction term involved. Discarding the vacuum terms in order to exclusively analyze the magnetic explicit regularization-independent contribution to the effective coupling, we have found scalar couplings which decrease with increasing magnetic field strength, and pseudoscalar couplings with the opposite behavior.

Considering a NJL model with these flavor- and B-dependent couplings, we have found new expressions for the constituent quark masses, which now depended on the scalar couplings, and for the bound state equation whose solutions yields the pseudoscalar meson masses, which now depended on the pseudoscalar couplings. For the new constituent quark masses, the scalar B-dependent couplings served to make the increase of the effective masses with B weaker than the case with fixed coupling. Comparisons with lattice QCD results show that the inclusion of these scalar B-dependent couplings from vacuum polarization is a step in the right direction to conciliate NJL predictions and LQCD simulations.

As for the meson masses, the inclusion of the pseudoscalar coupling, which we found to increase with the magnetic field, led to a π^0 meson mass that increases with B for weak and intermediate field strengths, and to a K^0 meson mass that decreases with B weaker than it does when the B-dependent coupling is not considered. Both these results seem to be in contrast with the literature. However, this was somewhat expected since it has been established that the reproduction of LQCD results with the NJL model requires a decreasing coupling with B.

Thus, we have concluded that the vacuum polarization corrections to the NJL coupling in the presence of a uniform magnetic field may improve the results that involve the couplings of the scalar currents, but not the ones that involve the couplings of the pseudoscalar ones. Further investigations may include the computation of the third-order term in the quark determinant expansion, which shall result in corrections to the coupling of the U(3) 't Hooft determinant term and are expected to play an important role in computing the meson masses. Additionally, one may also compute the charged meson masses and the meson decay constants in the presence of the applied magnetic field, which requires to carefully deal with the Schwinger phases that do not cancel out in those cases. Finally, our entire analyses can be also done at finite temperature in order to obtain effective couplings that depend on B and T, for example.

APPENDIX A

Useful relations, integrals and expansions

In this Appendix we list some useful results, such as expansions and integrals that were needed at some point in the text, as well as some definitions of special functions. Most of these equations were taken from Ref. [53] while others are well-known relations listed for reference purposes. A few results that were not found in the literature are demonstrated in Section A.4.

A.1 Definitions of special functions and constants

Euler constant

$$\gamma_E = \lim_{n\to\infty}\left[\sum_{k=1}^{n-1}\frac{1}{k} - \ln n\right] = 0.57721566490\ldots \tag{A.1}$$

Exponential integral function

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty}\frac{e^{-t}}{t}\,dt \tag{A.2}$$

Gamma function

$$\Gamma(x) = \int_0^{\infty} e^{-t}t^{x-1}\,dt \tag{A.3}$$

Generalized zeta function

$$\zeta(x,a) = \sum_{n=1}^{\infty}\frac{1}{(a+n)^x} \tag{A.4}$$

Euler psi function

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x}\ln\Gamma(x) \tag{A.5}$$

Beta function

$$\mathrm{B}(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{A.6}$$

A.2 Useful relations and expansions

$$\tanh(a+b) = \frac{\tanh a + \tanh b}{1 + \tanh a \, \tanh b} \tag{A.7}$$

$$\sinh(a+b) = \frac{\tanh a + \tanh b}{\operatorname{sech} a \, \operatorname{sech} b} \tag{A.8}$$

$$\coth x = \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \dots \tag{A.9}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} \tag{A.10}$$

$$\frac{1}{\sinh^2 x} = \frac{1}{x^2} - \frac{1}{3} + \frac{x^2}{15} - \frac{2x^4}{189} + \dots \tag{A.11}$$

$$\mathrm{Ei}(x) = \gamma_E + \ln(-x) + \sum_{k=1}^{\infty} \frac{x^k}{k \cdot k!}, \quad x < 0 \tag{A.12}$$

$$x^{-\epsilon} = 1 - \epsilon \ln x + O(\epsilon^2) \tag{A.13}$$

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + O(\epsilon^2) \tag{A.14}$$

$$\Gamma(\epsilon - 1) = -\frac{1}{\epsilon} + \gamma_E - 1 + O(\epsilon) \tag{A.15}$$

$$\zeta(\epsilon, x) = \frac{1}{2} - x + \epsilon \, \frac{\mathrm{d}}{\mathrm{d}\epsilon} \zeta(\epsilon, x)\bigg|_{\epsilon=0} + O(\epsilon^2) \tag{A.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \zeta(x, a)\bigg|_{x=0} = \ln \Gamma(a) - \frac{1}{2} \ln 2\pi \tag{A.17}$$

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln 2\pi + O\left(\frac{1}{x}\right), \quad x \to \infty \tag{A.18}$$

$$\psi(x+1) + \psi(x) = \frac{1}{x} + 2\psi(x) \tag{A.19}$$

$$\mathrm{B}\left(\frac{a^2}{2}-\frac{\epsilon}{2}+1,\epsilon-1\right)=-\frac{a^2}{2}\left(\frac{1}{\epsilon}+1-\gamma_E\right)+\frac{a^2}{4}\left[\psi\left(\frac{a^2}{2}\right)+\psi\left(\frac{a^2}{2}+1\right)\right]+O(\epsilon) \quad \text{(A.20)}$$

A.3 Useful integrals

$$\int_{-\infty}^{\infty} e^{ia^2x^2}dx=\sqrt{\frac{i\pi}{a^2}} \quad \text{(A.21)}$$

$$\int_u^\infty dx\frac{e^{-px}}{x^{n+1}}=(-1)^{n+1}\frac{p^n\mathrm{Ei}(-pu)}{n!}+\frac{e^{-pu}}{u^n}\sum_{k=0}^{n-1}\frac{(-1)^k p^k u^k}{n(n-1)\dots(n-k)} \quad \text{(A.22)}$$

$$\int_0^\infty x^{\mu-1}e^{-\beta x}\coth x\,dx=\Gamma(\mu)\left[2^{1-\mu}\zeta\left(\mu,\frac{\beta}{2}\right)-\beta^{-\mu}\right] \quad \text{(A.23)}$$

$$\int_0^\infty \frac{dx}{x^2}e^{-a^2x}\left(x\coth x-1\right)=\ln\frac{1}{\pi a^2}+a^2+\left(2-a^2\right)\ln\frac{a^2}{2}+2\ln\Gamma\left(\frac{a^2}{2}\right)^{1} \quad \text{(A.24)}$$

$$\int_0^\infty \frac{dx}{x}\,e^{-\beta x}(1-x\coth x)=\psi\left(\frac{\beta}{2}\right)-\ln\frac{\beta}{2}+\frac{1}{\beta} \quad \text{(A.25)}$$

$$\int_0^\infty dx\,e^{-\mu x}\sinh^\nu\beta x=\frac{1}{2^{\nu+1}\beta}\mathrm{B}\left(\frac{\mu}{2\beta}-\frac{\nu}{2},\nu+1\right) \quad \text{(A.26)}$$

$$\int_0^\infty dx\,e^{-a^2x}\left(\frac{1}{\sinh^2 x}-\frac{1}{x^2}\right)=1+a^2\left[\psi\left(\frac{a^2}{2}\right)-\ln\left(\frac{a^2}{2}\right)\right]^{2} \quad \text{(A.27)}$$

1. See proof in Section A.4.
2. See proof in Section A.4.

A.4 Some demonstrations

Eq. (A.24) Let us denote

$$I_a = \int_0^\infty \frac{dx}{x^2} e^{-a^2 x} \left(x \coth x - 1 \right).$$

Although this integral has a finite value, we will solve it by separating it into two divergent integrals and showing that the divergences cancel each other [54, 55]. We begin by writing

$$\begin{aligned} I_a &= \int_0^\infty \frac{dx}{x} e^{-a^2 x} \coth x - \int_0^\infty \frac{dx}{x^2} e^{-a^2 x} \\ &= \lim_{\eta \to 0} \left(\int_0^\infty dx\, x^{-1+\eta} e^{-a^2 x} \coth x - \int_0^\infty dx\, x^{-2+\eta} e^{-a^2 x} \right) \\ &= \lim_{\eta \to 0} \left[\int_0^\infty dx\, x^{\eta-1} e^{-a^2 x} \coth x - \frac{1}{(a^2)^{\eta-1}} \int_0^\infty dx\, x^{\eta-2} e^{-x} \right] \\ &= \lim_{\eta \to 0} \left\{ \Gamma(\eta) \left[2^{1-\eta} \zeta \left(\eta, \frac{a^2}{2} \right) - (a^2)^{-\eta} \right] - \frac{1}{(a^2)^{\eta-1}} \Gamma(\eta - 1) \right\}, \end{aligned}$$

where we used the definition of the Gamma function and the result in Eq. (A.23). Using Eqs. (A.13) to (A.17) in order to properly take the limit $\eta \to 0$ in the expression above, we write

$$\begin{aligned} I_a &= \lim_{\eta \to 0} \left(\frac{1}{\eta} - \gamma_E \right) \left[(2 - 2\eta \ln 2) \left(\frac{1}{2} - \frac{a^2}{2} + \eta \left[\ln \Gamma \left(\frac{a^2}{2} \right) - \frac{1}{2} \ln 2\pi \right] \right) - 1 + \eta \ln a^2 \right] \\ &\quad - \lim_{\eta \to 0} \left(a^2 - \eta a^2 \ln a^2 \right) \left(-\frac{1}{\eta} + \gamma_E - 1 \right) \\ &= \lim_{\eta \to 0} \left[-\frac{a^2}{\eta} + 2 \ln \Gamma \left(\frac{a^2}{2} \right) - \ln 2\pi + \ln \frac{a^2}{2} - a^2 \ln \frac{a^2}{2} + a^2 \gamma_E + \frac{a^2}{\eta} - a^2 \gamma_E + a^2 \right] \\ &= \ln \frac{1}{\pi a^2} + \left(2 - a^2 \right) \ln \frac{a^2}{2} + 2 \ln \Gamma \left(\frac{a^2}{2} \right) + a^2. \end{aligned}$$

Eq. (A.27) Let us denote

$$Y_a = \int_0^\infty dx\, e^{-a^2 x} \left(\frac{1}{\sinh^2 x} - \frac{1}{x^2} \right).$$

We proceed as we did for Eq. (A.24) by separating the finite integral into two divergent ones,

$$\begin{aligned} Y_a &= \int_0^\infty dx\, \frac{e^{-a^2 x}}{\sinh^2 x} - \int_0^\infty \frac{dx}{x^2} e^{-a^2 x} \\ &= \lim_{\eta \to 0} \left[\int_0^\infty dx\, e^{-a^2 x} \sinh^{\eta-2} x - \frac{1}{(a^2)^{\eta-1}} \int_0^\infty dx\, x^{\eta-2} e^{-x} \right] \end{aligned}$$

$$= \lim_{\eta \to 0} \left[\frac{1}{2^{\eta-1}} \mathrm{B}\left(\frac{a^2}{2} - \frac{\eta}{2} + 1, \eta - 1 \right) - \frac{1}{(a^2)^{\eta-1}} \Gamma(\eta - 1) \right],$$

where we used the definition of the Gamma function and Eq. (A.26). The limit $\eta \to 0$ can be taken by using expansions (A.13), (A.15) and (A.20) with relation (A.19). The result is

$$\begin{aligned} Y_a = \lim_{\eta \to 0} \Bigg\{ &(2 - 2\eta \ln 2) \frac{a^2}{2} \left[-\frac{1}{\eta} - 1 + \gamma_E + \frac{1}{a^2} + \psi\left(\frac{a^2}{2} \right) \right] \\ &- \left(a^2 - a^2 \eta \ln a^2 \right) \left(-\frac{1}{\eta} + \gamma_E - 1 \right) \Bigg\} \\ = 1 + a^2 &\left[\psi\left(\frac{a^2}{2} \right) - \ln \frac{a^2}{2} \right]. \end{aligned}$$

APPENDIX B

Pauli, Dirac and Gell-Mann matrices

In this Appendix, we list some properties of the Pauli, Dirac and Gell-Mann matrices that may have been used in the text. Here the traces will be simply be denoted by tr as there is no potential to confusion regarding on which space the trace acts on. Additionally, we denote the $n \times n$ identity matrix as $\mathbb{1}_{n\times n}$.

Pauli matrices The Pauli sigma matrices σ_i, $i = 1, 2, 3$, satisfy the following relations [56]:

1. $\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = \mathbb{1}_{2\times 2}$;
2. $\det(\sigma_i) = -1$;
3. $\mathrm{tr}(\sigma_i) = 0$;
4. $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$;
5. $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}_{2\times 2}$;
6. $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$.

Explicitly, they are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For an arbitrary vector $\mathbf{a} = a\hat{\mathbf{e}}_n$ we have

$$e^{i\mathbf{a}\cdot\boldsymbol{\sigma}} = \mathbb{1}_{2\times 2}\cos a + i(\hat{\mathbf{e}}_n \cdot \boldsymbol{\sigma})\sin a.$$

Dirac matrices The Dirac matrices γ^μ are a set of four 4×4 matrices satisfying the anticommutation relations [33]:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{1}_{4\times 4}.$$

The Weyl or chiral representation of the Dirac matrices, in 2×2 block form, reads

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}.$$

We also introduce an additional gamma matrix,

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3,$$

which satisfies

1. $(\gamma^5)^2 = \mathbb{1}_{4\times 4}$;
2. $\{\gamma^5, \gamma^\mu\} = 0$.

Useful trace relations involving the gamma matrices are listed below:

1. $\text{tr}(\text{any odd number of } \gamma' s) = 0$;
2. $\text{tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$;
3. $\text{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$;
4. $\text{tr}(\gamma^5) = 0$;
5. $\text{tr}(\gamma^\mu\gamma^\nu\gamma^5) = 0$;
6. $\text{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5) = -4i\epsilon^{\mu\nu\rho\sigma}$.

Gell-Mann matrices The eight 3×3 independent traceless Hermitian Gell-Mann matrices λ_a, $a = 1, \ldots, 8$, satisfy [57]

$$[\lambda_a, \lambda_b] = 2if_{abc}\lambda_c,$$
$$\{\lambda_a, \lambda_b\} = 2d_{abc}\lambda_c + \frac{4}{3}\delta_{ab}\mathbb{1}_{3\times 3},$$

where the completely antisymmetric coefficients f_{abc} (the SU(3) structure constants) are given by

$$f_{abc} = -\frac{i}{4}\text{tr}([\lambda_a, \lambda_b]\lambda_c),$$

and the completely symmetric coefficients d_{abc} are given by

$$d_{abc} = \frac{1}{4}\mathrm{tr}(\{\lambda_a, \lambda_b\}\lambda_c).$$

Explicitly, the Gell-Mann matrices read

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},$$

and we define the additional λ_0 matrix as

$$\lambda_0 = \sqrt{\frac{2}{3}}\mathbb{1}_{3\times 3} = \sqrt{\frac{2}{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The trace of the pairwise product of Gell-Mann matrices satisfies

$$\mathrm{tr}(\lambda_a\lambda_b) = 2\delta_{ab}.$$

APPENDIX C

The fermion propagator in a uniform magnetic field

In this Appendix we derive an expression for the fermion propagator in a uniform magnetic field following Schwinger's proper-time method [26].

C.1 Fermion propagator in a constant electromagnetic field

The propagator of the Dirac field in the presence of the gauge field A_μ, $G(x, y)$, is formally defined by

$$\Big(i\not{D} - m\Big)G(x, y) = \delta^4(x - y), \tag{C.1}$$

where $D_\mu = \partial_\mu - iqA_\mu$ is the covariant derivative, with q being the fermion electric charge and m standing for its mass. Schwinger's technique begins by regarding $G(x, y)$ as the matrix element of an operator G [26, 27], namely $G(x, y) = \langle y|G|x\rangle$. Then, in terms of this operator, Eq. (C.1) is written as

$$\Big(\not{\Pi} - m\Big)G = 1, \tag{C.2}$$

where $\Pi_\mu = P_\mu + qA_\mu$ denotes the conjugated momentum operator. Eq. (C.2) can be formally solved by writing the following integral representation for G:

$$G = \frac{1}{\not{\Pi} - m} = \frac{\not{\Pi} + m}{\not{\Pi}^2 - m^2 + i\epsilon} = -i\int_0^\infty ds\,\Big(\not{\Pi} + m\Big)\exp\Big[-i\Big(m^2 - \not{\Pi}^2 - i\epsilon\Big)s\Big].$$

Here we used the Feynman prescription and introduced the factor $i\epsilon$ in order to compute the integral. It is understood that the limit $\epsilon \to 0$ is to be taken and we will simply omit this factor from now on.

Then, for $G(x,y) = \langle y|G|x\rangle$ we have

$$G(x,y) = -i\int_0^\infty ds\, e^{-im^2 s}\,\langle y|\Big(\not{\Pi} + m\Big)U(s)|x\rangle\,,$$

where

$$U(s) = e^{-iHs} \quad \text{with} \quad H = -\not{\Pi}^2.$$

The operator $H = -\not{\Pi}^2$ can be regarded as an effective Hamiltonian while s can be interpreted as a time variable, which is known as the Schwinger proper-time. Therefore the unitary operator $U(s)$ can be viewed as the time-evolution operator whose action on the state $|x\rangle$ may be written as

$$|x(s)\rangle = U(s)\,|x(0)\rangle\,.$$

We now rewrite $G(x,y)$ as

$$\begin{aligned} G(x,y) &= -i\int_0^\infty ds\, e^{-im^2 s}\,\langle y(0)|\Big(\not{\Pi} + m\Big)|x(s)\rangle \\ &= -i\int_0^\infty ds\, e^{-im^2 s}[\gamma^\mu\,\langle y(0)|\Pi_\mu(0)|x(s)\rangle + m\,\langle y(0)|x(s)\rangle]. \end{aligned} \tag{C.3}$$

The transformation function $\langle y(0)|x(s)\rangle$, fundamental to the evaluation of $G(x,y)$ in Eq. (C.3), is characterized by the following equations:

$$i\frac{\partial}{\partial s}\,\langle y(0)|x(s)\rangle = \langle y(0)|H|x(s)\rangle\,, \tag{C.4a}$$

$$[i\partial_\mu + qA_\mu(x)]\,\langle y(0)|x(s)\rangle = \langle y(0)|\Pi_\mu(s)|x(s)\rangle\,, \tag{C.4b}$$

$$\Big[-i\partial_{\mu_y} + qA_\mu(y)\Big]\,\langle y(0)|x(s)\rangle = \langle y(0)|\Pi_\mu(0)|x(s)\rangle\,, \tag{C.4c}$$

with the boundary condition

$$\lim_{s\to 0}\,\langle y(0)|x(s)\rangle = \delta^4(x-y). \tag{C.5}$$

The first equation in (C.4) is simply the Schrödinger equation for the time-evolution of the state $|x(s)\rangle$ while the others follow from the definition of the conjugated momentum operator Π_μ. Here we have assumed that $\Pi_\mu(s)$ operates on $|x(s)\rangle$ and $\Pi_\mu(0)$ operates on $|x(0)\rangle$ [27].

In order to evaluate Eqs. (C.4), we first use the commutation relations

$$[\Pi_\mu, x_\nu] = ig_{\mu\nu},$$

$$[\Pi_\mu, \Pi_\nu] = iqF_{\mu\nu},$$

to write the Hamiltonian H as

$$H = -\not{\Pi}^2 = -\Pi^2 - \frac{q}{2}\sigma^{\mu\nu}F_{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field-strength tensor of the gauge field and

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu].$$

The operators x_μ and Π_μ satisfy the equations of motion

$$\frac{\mathrm{d}x_\mu}{\mathrm{d}s} = i[H, x_\mu] = i\left[x_\mu, \Pi^2\right] = i[x_\mu, \Pi_\nu]2\Pi^\nu = 2\Pi_\mu, \tag{C.6a}$$

$$\begin{aligned}\frac{\mathrm{d}\Pi_\mu}{\mathrm{d}s} &= i[H, \Pi_\mu] = i\left[\Pi_\mu, \Pi^2\right] + \frac{i}{2}q\sigma^{\lambda\nu}[\Pi_\mu, F_{\lambda\nu}] \\ &= i([\Pi_\mu, \Pi_\nu]\Pi^\nu + \Pi^\nu[\Pi_\mu, \Pi_\nu]) + \frac{i}{2}q\sigma^{\lambda\nu}[\Pi_\mu, x_\rho]\partial^\rho F_{\lambda\nu} \\ &= -2qF_{\mu\nu}\Pi^\nu, \end{aligned}\tag{C.6b}$$

where, in the last step, we assumed a constant field strength $F_{\mu\nu}$, in which case $\partial^\rho F_{\lambda\nu} = 0$ and $[\Pi_\mu, F_{\lambda\nu}] = 0$. In matrix notation, Eqs. (C.6) read

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s} = 2\mathbf{\Pi}, \qquad \frac{\mathrm{d}\mathbf{\Pi}}{\mathrm{d}s} = -2q\mathbf{F}\mathbf{\Pi},$$

for which the solutions are

$$\mathbf{\Pi}(s) = e^{-2q\mathbf{F}s}\mathbf{\Pi}(0),$$

$$\mathbf{x}(s) - \mathbf{x}(0) = \left(1 - e^{-2q\mathbf{F}s}\right)(q\mathbf{F})^{-1}\mathbf{\Pi}(0) = 2e^{-q\mathbf{F}s}\sinh(q\mathbf{F}s)(q\mathbf{F})^{-1}\mathbf{\Pi}(0).$$

If we write $\mathbf{\Pi}(0)$ in terms of $\mathbf{x}(s) - \mathbf{x}(0)$ in the second equation and substitute the result into the first one, we find

$$\mathbf{\Pi}^T(s) = [\mathbf{x}(s) - \mathbf{x}(0)]^T \frac{1}{2} q\mathbf{F}\sinh^{-1}(q\mathbf{F}s)\, e^{q\mathbf{F}s},$$

after using the antisymmetry of $\mathbf{F}$. With this expression, we can write the Hamiltonian as

$$H = -\Pi^2 - \frac{q}{2}\sigma^{\mu\nu}F_{\mu\nu} = [\mathbf{x}(s) - \mathbf{x}(0)]^T\,\mathbf{K}\,[\mathbf{x}(s) - \mathbf{x}(0)] + \frac{q}{2}\mathrm{tr}\left(\boldsymbol{\sigma}\mathbf{F}\right),$$

where

$$\mathbf{K} = \frac{1}{4}(q\mathbf{F})^2\sinh^{-2}(q\mathbf{F}s),$$

and here the trace tr is over the indices $\mu, \nu, \ldots$.

Now, let us note that $\mathbf{x}(s)$ does not commute with $\mathbf{x}(0)$ due to the dependence of $\mathbf{x}(s)$ on $\mathbf{\Pi}(0)$. Then, since the Hamiltonian is to be evaluated in position eigenstates like in Eq. (C.4a), it will be useful to write it so that $\mathbf{x}(s)$ is on the left and $\mathbf{x}(0)$ is on the right [34]. To do that, we use

$$[\mathbf{x}(s), \mathbf{x}(0)] = -2ie^{-q\mathbf{F}s} \sinh(q\mathbf{F}s)(q\mathbf{F})^{-1},$$

to write

$$\begin{aligned}
\mathbf{\Pi}^2(s) &= \mathbf{x}^T(s)\mathbf{K}\mathbf{x}(s) - 2\mathbf{x}^T(s)\mathbf{K}\mathbf{x}(0) + \mathbf{x}^T(0)\mathbf{K}\mathbf{x}(0) + K^{\mu\nu}[x_\mu(s), x_\nu(0)] \\
&= \mathbf{x}^T(s)\mathbf{K}\mathbf{x}(s) - 2\mathbf{x}^T(s)\mathbf{K}\mathbf{x}(0) + \mathbf{x}^T(0)\mathbf{K}\mathbf{x}(0) + \mathrm{tr}(\mathbf{K}[\mathbf{x}(s), \mathbf{x}(0)]) \\
&= \mathbf{x}^T(s)\mathbf{K}\mathbf{x}(s) - 2\mathbf{x}^T(s)\mathbf{K}\mathbf{x}(0) + \mathbf{x}^T(0)\mathbf{K}\mathbf{x}(0) - \frac{i}{2}\mathrm{tr}\left[Q\mathbf{F}\frac{e^{-Q\mathbf{F}s}}{\sinh(Q\mathbf{F}s)}\right].
\end{aligned}$$

Then, since $\frac{e^{-x}}{\sinh x} = \coth x - 1$ and $\mathrm{tr}(\mathbf{F}) = 0$, we find the matrix element in Eq. (C.4a) to be

$$\langle y(0)|H|x(s)\rangle = \left\{\frac{q}{2}\mathrm{tr}(\boldsymbol{\sigma}\mathbf{F}) - (\mathbf{x}-\mathbf{y})^T\mathbf{K}(\mathbf{x}-\mathbf{y}) - \frac{i}{2}\mathrm{tr}[q\mathbf{F}\coth(q\mathbf{F}s)]\right\}\langle y(0)|x(s)\rangle.$$

Substituting this result back into Eq. (C.4a) and solving the differential equation leads to

$$\begin{aligned}
\ln(\langle y(0)|x(s)\rangle) = &-\frac{i}{2}q\mathrm{tr}(\boldsymbol{\sigma}\mathbf{F})\, s - \frac{i}{4}(\mathbf{x}-\mathbf{y})^T q\mathbf{F}\coth(q\mathbf{F}s)(\mathbf{x}-\mathbf{y}) \\
&-\frac{1}{2}\mathrm{tr}\ln\left[(q\mathbf{F})^{-1}\sinh(q\mathbf{F}s)\right] + \text{constant},
\end{aligned}$$

where we add $\ln s^2$ on both sides to obtain [27]

$$\begin{aligned}
\langle y(0)|x(s)\rangle = C(\mathbf{x}, \mathbf{y})\, s^{-2} \exp\left\{-\frac{1}{2}\mathrm{tr}\ln\left[(q\mathbf{F}s)^{-1}\sinh(q\mathbf{F}s)\right]\right\} \\
\times \exp\left[-\frac{i}{4}(\mathbf{x}-\mathbf{y})^T q\mathbf{F}\coth(q\mathbf{F}s)(\mathbf{x}-\mathbf{y}) + \frac{i}{2}q\sigma_{\mu\nu}F^{\mu\nu}\, s\right].
\end{aligned} \tag{C.7}$$

To determine the factor $C(\mathbf{x}, \mathbf{y})$, we can use Eqs. (C.4b) and (C.4c). The right-hand side of these two equations are given by

$$\begin{aligned}
\langle y(0)|\mathbf{\Pi}(s)|x(s)\rangle &= \langle y(0)|\left[\frac{1}{2}q\mathbf{F}\sinh^{-1}(q\mathbf{F}s)\, e^{-q\mathbf{F}s}(\mathbf{x}(s) - \mathbf{x}(0))\right]|x(s)\rangle \\
&= \frac{1}{2}[q\mathbf{F}\coth(q\mathbf{F}s) - q\mathbf{F}](\mathbf{x}-\mathbf{y})\langle y(0)|x(s)\rangle,
\end{aligned} \tag{C.8a}$$

$$\begin{aligned}
\langle y(0)|\mathbf{\Pi}(0)|x(s)\rangle &= \langle y(0)|\left[\frac{1}{2}q\mathbf{F}\sinh^{-1}(q\mathbf{F}s)\, e^{q\mathbf{F}s}(\mathbf{x}(s) - \mathbf{x}(0))\right]|x(s)\rangle \\
&= \frac{1}{2}[q\mathbf{F}\coth(q\mathbf{F}s) + q\mathbf{F}](\mathbf{x}-\mathbf{y})\langle y(0)|x(s)\rangle.
\end{aligned} \tag{C.8b}$$

Therefore, substituting Eq. (C.7) into Eqs. (C.4b) and (C.4c), and using Eqs. (C.8), one arrives at

$$\left[i\partial_\mu + qA_\mu(x) - \frac{1}{2}qF_{\mu\nu}(y-x)^\nu\right]C(x,y) = 0,$$

$$\left[-i\partial_{\mu_y} + qA_\mu(y) + \frac{1}{2}qF_{\mu\nu}(y-x)^\nu\right]C(x,y) = 0,$$

from which we find $C(x,y)$ to be

$$C(x,y) = C\exp\left\{iq\int_y^x d\xi^\mu\,\left[A_\mu(\xi) + \frac{1}{2}F_{\mu\nu}\left(\xi-y\right)^\nu\right]\right\}. \tag{C.9}$$

Lastly, we apply the boundary condition (C.5) to determine the constant C. We are interested in the behavior of the solution (C.7) as $s\to 0$. Since $\sinh x \to x$ for $x << 1$, we have $\ln[x^{-1}\sinh x] \to \ln(1) = 0$. Also, $\coth x \to \frac{1}{x}$ as $x\to 0$, so that in order to find C we write

$$C\frac{1}{s^2}\,\exp\left[-\frac{i}{4}(x-y)^\mu\frac{1}{s}(x-y)_\mu\right] \to \delta^4(x-y).$$

This condition is equivalent to

$$\int d^4x\,Cs^{-2}\exp\left(-\frac{i}{4}\frac{x^2}{s}\right) = 1,$$

from where we find C to be

$$C = -i\left(4\pi\right)^{-2}, \tag{C.10}$$

after using the result (A.21).

At last, we are ready to put the pieces together. From Eqs. (C.3), (C.7), (C.8), (C.9) and (C.10), we find

$$G(x,y) = \Phi(x,y)\mathcal{G}(x-y), \tag{C.11a}$$

where

$$\Phi(x,y) \equiv \exp\left\{iq\int_y^x d\xi^\mu\,\left[A_\mu(\xi) + \frac{1}{2}F_{\mu\nu}\left(\xi-y\right)^\nu\right]\right\}, \tag{C.11b}$$

$$\begin{aligned}\mathcal{G}(x-y) \equiv -\left(4\pi\right)^{-2}\int_0^\infty \frac{ds}{s^2}&\left[m + \frac{1}{2}\gamma\cdot\left[q\mathbf{F}\coth\left(q\mathbf{F}s\right) + q\mathbf{F}\right](\mathbf{x}-\mathbf{y})\right]\\ &\times\exp\left\{-im^2s - \frac{1}{2}\mathrm{tr}\ln\left[\left(q\mathbf{F}s\right)^{-1}\sinh\left(q\mathbf{F}s\right)\right]\right\}\\ \times\exp&\left[-\frac{i}{4}(\mathbf{x}-\mathbf{y})^T q\mathbf{F}\coth\left(q\mathbf{F}s\right)(\mathbf{x}-\mathbf{y}) + \frac{i}{2}q\sigma_{\mu\nu}F^{\mu\nu}\,s\right].\end{aligned} \tag{C.11c}$$

Note that the phase factor $\Phi(x,y)$ is explicitly gauge dependent and breaks the translation invariance of the propagator.

C.2 Propagator in the presence of a uniform magnetic field

Now, let us specialize to the case where the background field is a homogeneous magnetic field $\mathbf{B} = B\hat{\mathbf{e}}_z$ such that $F_{12} = -F_{21} = B$. Then we can write the propagator in terms of B. To do so, we begin by analyzing

$$\sigma_{\mu\nu}F^{\mu\nu} = \frac{i}{2}\left[\gamma_1,\gamma_2\right]F^{12} + \frac{i}{2}\left[\gamma_2,\gamma_1\right]F^{21} = i\left[\gamma_1,\gamma_2\right]F^{12} = i\left[\sigma_2,\sigma_1\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}F^{12},$$

$$\sigma_{\mu\nu}F^{\mu\nu} = 2B\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{C.12}$$

Next we evaluate $\exp\left[-\frac{1}{2}\mathrm{tr}\ln\left(\mathbf{F}^{-1}\sinh\mathbf{F}\right)\right]$. First, we write

$$\sinh\mathbf{F} = \sum_{k=0}^{\infty}\frac{\mathbf{F}^{2k+1}}{(2k+1)!} = \mathbf{F} + \frac{\mathbf{F}^3}{3!} + \frac{\mathbf{F}^5}{5!} + \frac{\mathbf{F}^7}{7!} + \frac{\mathbf{F}^9}{9!} + \ldots .$$

Now, $\lim_{x\to 0}\frac{\sinh x}{x} = 1$ and $\ln 1 = 0$. Thus, the only contributions to the exponent will come from the non-zero terms in $F^{\mu\nu}$, namely $\mu,\nu = 1,2$. So, working with two dimensions, we define the matrix

$$\mathbf{T} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that

$$\mathbf{F} = B\mathbf{T}, \quad \mathbf{F}^3 = -B^3\mathbf{T}, \quad \mathbf{F}^5 = B^5\mathbf{T}, \ldots .$$

Thus,

$$\sinh\mathbf{F} = \mathbf{T}\left(B - \frac{B^3}{3!} + \frac{B^5}{5!} - \frac{B^7}{7!} + \ldots\right) = \mathbf{T}\sin B,$$

and

$$\mathbf{F}^{-1}\sinh\mathbf{F} = \mathbf{T}^{-1}\mathbf{T}\frac{\sin B}{B} = \frac{\sin B}{B}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

With this result, we find

$$\exp\left[-\frac{1}{2}\mathrm{tr}\ln\left(\mathbf{F}^{-1}\sinh\mathbf{F}\right)\right] = \frac{B}{\sin B}. \tag{C.13}$$

Now consider the expression $\gamma\cdot\left(\mathbf{F}\coth\mathbf{F} + \mathbf{F}\right)\mathbf{x} = \gamma_\mu\left(F^{\mu\nu}\coth F_{\nu\rho} + F^{\mu\rho}\right)x_\rho$. Once again the elements of $\mathbf{F}$ vanish for $\mu = \rho = 0$ or $\mu = \rho = 3$, and since $\mathbf{F}\coth\mathbf{F} \to 1$ as $\mathbf{F} \to 0$, we can simply separate these contributions to the expression and write

$$\gamma\cdot\left(\mathbf{F}\coth\mathbf{F} + \mathbf{F}\right)\mathbf{x} = \left(\gamma\cdot x\right)_{\parallel} - \gamma\cdot\left[B\mathbf{T}\coth(B\mathbf{T})\right]x_\perp + \gamma_1F^{12}x_2 + \gamma_2F^{21}x_1,$$

where, for two arbitrary 4−vectors a^μ and b^μ, we denote

$$(a \cdot b)_\parallel = a^0 b^0 - a^3 b^3,$$
$$(a \cdot b)_\perp = a^1 b^1 + a^2 b^2.$$

Explicitly, we have

$$\begin{aligned} B\mathbf{T}\coth(B\mathbf{T}) &= \frac{1}{(B\mathbf{T})^{-1}\sinh(B\mathbf{T})}\cosh(B\mathbf{T}) = \frac{B}{\sin B}\sum_{k=0}^{\infty}(-1)^k \frac{B^{2k}}{(2k)!}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \\ &= \frac{B}{\sin B}\cos B\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \end{aligned}$$

and

$$\begin{aligned} \gamma_1 F^{12} x_2 + \gamma_2 F^{21} x_1 = B\left(\gamma_1 x_2 - \gamma_2 x_1\right) &= -B\left(\gamma_1\gamma_2\gamma_2 x_2 - \gamma_2\gamma_1\gamma_1 x_1\right) \\ = -B\gamma_1\gamma_2\left(\gamma_2 x_2 + \gamma_1 x_1\right) &= -B\gamma_1\gamma_2\left(\gamma\cdot x\right)_\perp = iB\boldsymbol{\sigma}_3\left(\gamma\cdot x\right)_\perp, \end{aligned}$$

since

$$\boldsymbol{\sigma}_3 \equiv \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = i\gamma_1\gamma_2. \tag{C.14}$$

We then arrive at

$$\gamma\cdot\left(\mathbf{F}\coth\mathbf{F} + \mathbf{F}\right)\mathbf{x} = \left(\gamma\cdot x\right)_\parallel - \frac{B}{\sin B}\left(\gamma\cdot x\right)_\perp e^{-iB\boldsymbol{\sigma}_3}. \tag{C.15}$$

Lastly, proceeding as we did for the last expression, we obtain

$$\mathbf{x}^T\left(\mathbf{F}\coth\mathbf{F}\right)\mathbf{x} = x_\parallel^2 - B\cot B\, x_\perp^2. \tag{C.16}$$

Using the relations (C.12), (C.13), (C.15) and (C.16), we can write

$$\begin{aligned} \mathcal{G}(x-y) = -\left(4\pi\right)^{-2}\int_0^\infty \frac{ds}{s^2}\frac{qBs}{\sin\left(qBs\right)}\exp\left(-im^2 s + iqBs\boldsymbol{\sigma}_3\right) \\ \times\exp\left\{-\frac{i}{4s}\left[(x-y)_\parallel^2 - qBs\cot\left(qBs\right)(x-y)_\perp^2\right]\right\} \\ \times\left\{m + \frac{1}{2s}\left[\gamma\cdot(x-y)_\parallel - \frac{qBs}{\sin\left(qBs\right)}\gamma\cdot(x-y)_\perp e^{-iqBs\boldsymbol{\sigma}_3}\right]\right\}. \end{aligned} \tag{C.17}$$

This is the expression for the translation invariant propagator function in the presence of a homogeneous background magnetic field. Since quarks can have either positive or negative electric charges it will be useful to rewrite this expression in terms only of the magnitude of that charge, namely,

$$\begin{aligned}\mathcal{G}(x-y) = &-(4\pi)^{-2}\int_0^\infty \frac{ds}{s^2}\frac{|qB|s}{\sin(|qB|s)}\exp\left(-im^2s + i\,\text{sign}(qB)|qB|s\boldsymbol{\sigma}_3\right)\\ &\times\exp\left\{-\frac{i}{4s}\left[(x-y)_\parallel^2 - |qB|s\cot(|qB|s)\,(x-y)_\perp^2\right]\right\}\\ &\times\left\{m + \frac{1}{2s}\left[\gamma\cdot(x-y)_\parallel - \frac{|qB|s}{\sin(|qB|s)}\gamma\cdot(x-y)_\perp e^{-i\,\text{sign}(qB)|qB|s\boldsymbol{\sigma}_3}\right]\right\},\end{aligned} \tag{C.18}$$

where

$$\text{sign}(x) = \begin{cases} +1 & \text{if } x>0\\ -1 & \text{if } x<0\end{cases}.$$

Eq. (C.18) is the final result we need for the propagator. However, we eventually may find more convenient to write an expression for the propagator in momentum space, which will be done in the next section.

C.3 Propagator in momentum space

The expression for the translation invariant propagator function, Eq. (C.18), can be cast in the form

$$\mathcal{G}(x-y) = \int\frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}\mathcal{G}(p),$$

with

$$\begin{aligned}\mathcal{G}(p) &= \int d^4x\, e^{ipx}\mathcal{G}(x)\\ &= -(4\pi)^{-2}\int_0^\infty\frac{ds}{s^2}\frac{|qB|s}{\sin(|qB|s)}\exp\left(-im^2s + i\,\text{sign}(qB)|qB|s\boldsymbol{\sigma}_3\right)(\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3),\end{aligned}$$

where

$$\begin{aligned}\mathcal{I}_1 &= m\int d^2x_\parallel\int d^2x_\perp e^{ipx_\parallel}e^{-ipx_\perp}\exp\left[-\frac{1}{4s}\left(x_\parallel^2 - |qB|s\cot(|qB|s)\,x_\perp^2\right)\right]\\ &= \frac{i(4\pi)^2s^2}{|qB|s\cot(|qB|s)}m\exp\left[is\left(p_\parallel^2 - \frac{p_\perp^2}{|qB|s\cot(|qB|s)}\right)\right],\end{aligned}$$

$$\begin{aligned}\mathcal{I}_2 &= \frac{1}{2s}\int d^2x_\parallel\int d^2x_\perp e^{ipx_\parallel}e^{-ipx_\perp}\gamma\cdot x_\parallel\exp\left[-\frac{1}{4s}\left(x_\parallel^2 - |qB|s\cot(|qB|s)\,x_\perp^2\right)\right]\\ &= \frac{i(4\pi)^2s^2}{|qB|s\cot(|qB|s)}\gamma\cdot p_\parallel\exp\left[is\left(p_\parallel^2 - \frac{p_\perp^2}{|qB|s\cot(|qB|s)}\right)\right],\end{aligned}$$

$$\mathcal{I}_3 = -\frac{1}{2s}\frac{|qB|s}{\sin(|qB|s)}e^{-i\,\text{sign}(qB)|qB|s\boldsymbol{\sigma}_3}\int d^2x_\parallel\exp\left[-i\left(\frac{1}{4s}x_\parallel^2 - p\cdot x_\parallel\right)\right]$$

$$\times \int d^2x_\perp \gamma \cdot x_\perp \exp\left[-i\left(-\frac{1}{4s}|qB|s\cot\left(|qB|s\right)x_\perp^2 + p\cdot x_\perp\right)\right]$$
$$= \frac{-i(4\pi)^2 s^2}{|qB|s\cos^2(|qB|s)}\sin(|qB|s)e^{-i\,\mathrm{sign}(qB)|qB|s\boldsymbol{\sigma}_3}\gamma\cdot p_\perp \exp\left[is\left(p_\parallel^2 - \frac{p_\perp^2}{|qB|s\cot(|qB|s)}\right)\right]$$

Putting the pieces together leaves us with

$$\mathcal{G}(p) = -i\int_0^\infty \frac{ds}{\cos(|qB|s)}\exp\left\{-is\left[m^2 - p_\parallel^2 + \frac{\tan(|qB|s)}{|qB|s}p_\perp^2\right]\right\}$$
$$\times\left[e^{i\,\mathrm{sign}(qB)|qB|s\sigma_3}\left(m+\gamma\cdot p_\parallel\right) - \frac{\gamma\cdot p_\perp}{\cos(|qB|s)}\right].$$

By using

$$e^{i\theta\sigma_3} = \cos\theta + i\sigma_3\sin\theta,$$

and Eq. (C.14), we can also write

$$\mathcal{G}(p) = -i\int_0^\infty ds\,\exp\left\{-is\left[m^2 - p_\parallel^2 + \frac{\tan(|qB|s)}{|qB|s}p_\perp^2\right]\right\}$$
$$\times\Big\{[1-\mathrm{sign}(qB)\gamma_1\gamma_2\tan(|qB|s)]\Big(m+\gamma\cdot p_\parallel\Big) - \gamma\cdot p_\perp\Big[1+\tan^2\left(|qB|s\right)\Big]\Big\}. \tag{C.19}$$

APPENDIX D

Flavor trace of the second order term in the quark determinant expansion

In this Appendix we give a few details on the flavor trace of the second order term in the quark determinant expansion. Both terms in Eq. (3.15) have the general form $K\lambda^a K\lambda^b$, where K is a combination of the quark propagator matrix and some flavor identity operator and λ^a, $= 0, \ldots, 8$, represents the Gell-Mann matrices. The important feature of the operator K is its flavor structure,

$$K = \begin{pmatrix} K_u & 0 & 0 \\ 0 & K_d & 0 \\ 0 & 0 & K_s \end{pmatrix}.$$

Since the indices a, b can each take on the values $0, \ldots, 8$, we have 81 possible combinations of the product $K\lambda^a K\lambda^b$. Fortunately, a lot of them have vanishing trace and we list below only the nonzero results,

$$\begin{aligned}
\mathrm{tr}_F\Big(K\lambda^0 K\lambda^0\Big) &= \frac{2}{3}\Big(K_u^2 + K_d^2 + K_s^2\Big), \\
\mathrm{tr}_F\Big(K\lambda^0 K\lambda^3\Big) = \mathrm{tr}_F\Big(K\lambda^3 K\lambda^0\Big) &= \sqrt{\frac{2}{3}}\Big(K_u^2 - K_d^2\Big), \\
\mathrm{tr}_F\Big(K\lambda^0 K\lambda^8\Big) = \mathrm{tr}_F\Big(K\lambda^8 K\lambda^0\Big) &= \frac{\sqrt{2}}{3}\Big(K_u^2 + K_d^2 - 2K_s^2\Big), \\
\mathrm{tr}_F\Big(K\lambda^1 K\lambda^1\Big) = \mathrm{tr}_F\Big(K\lambda^2 K\lambda^2\Big) &= 2K_u K_d, \\
\mathrm{tr}_F\Big(K\lambda^3 K\lambda^3\Big) &= K_u^2 + K_d^2, \\
\mathrm{tr}_F\Big(K\lambda^3 K\lambda^8\Big) = \mathrm{tr}_F\Big(K\lambda^8 K\lambda^3\Big) &= \frac{1}{\sqrt{3}}\Big(K_u^2 - K_d^2\Big), \\
\mathrm{tr}_F\Big(K\lambda^4 K\lambda^4\Big) = \mathrm{tr}_F\Big(K\lambda^5 K\lambda^5\Big) &= 2K_u K_s,
\end{aligned}$$

$$\mathrm{tr}_F\Big(K\lambda^6K\lambda^6\Big) = \mathrm{tr}_F\Big(K\lambda^7K\lambda^7\Big) = 2K_dK_s,$$
$$\mathrm{tr}_F\Big(K\lambda^8K\lambda^8\Big) = \frac{1}{3}\Big(K_u^2 + K_d^2 + 4K_s^2\Big).$$

Now consider the term with the scalar currents, $\mathrm{tr}_F\Big(K\lambda^aK\lambda^b\Big)j_s^aj_s^b$, where $j_s^a = \bar{\psi}\lambda^a\psi$, $\psi = (u,d,s)^T$ and a sum over repeated indices is implicit. We can work only with this term since the one with the pseudoscalar currents is completely analogous. Explicitly, we have

$$\begin{aligned}\mathrm{tr}_F\Big(K\lambda^aK\lambda^bj_s^aj_s^b\Big) = {}& \frac{2}{3}\Big(K_u^2+K_d^2+K_s^2\Big)(j_s^0)^2 + 2K_uK_d\Big[(j_s^1)^2+(j_s^2)^2\Big] + \Big(K_u^2+K_d^2\Big)(j_s^3)^2\\ &+2K_uK_s\Big[(j_s^4)^2+(j_s^5)^2\Big] + 2K_dK_s\Big[(j_s^6)^2+(j_s^7)^2\Big] + \frac{1}{3}\Big(K_u^2+K_d^2+4K_s^2\Big)(j_s^8)^2\\ &+2\sqrt{\frac{2}{3}}\Big(K_u^2-K_d^2\Big)j_s^0j_s^3 + \frac{2\sqrt{2}}{3}\Big(K_u^2+K_d^2-2K_s^2\Big)j_s^0j_s^8 + \frac{2}{\sqrt{3}}\Big(K_u^2-K_d^2\Big)j_s^3j_s^8.\end{aligned}$$

Using the results below,

$$\begin{aligned}
\Big(j_s^0\Big)^2 &= \frac{2}{3}\Big[(\bar{u}u)^2 + \Big(\bar{d}d\Big)^2 + (\bar{s}s)^2 + 2(\bar{u}u)\Big(\bar{d}d\Big) + 2(\bar{u}u)(\bar{s}s) + 2\Big(\bar{d}d\Big)(\bar{s}s)\Big],\\
\Big(j_s^1\Big)^2 &= (\bar{u}d)^2 + \Big(\bar{d}u\Big)^2 + 2(\bar{u}d)\Big(\bar{d}u\Big),\\
\Big(j_s^2\Big)^2 &= -(\bar{u}d)^2 - \Big(\bar{d}u\Big)^2 + 2(\bar{u}d)\Big(\bar{d}u\Big),\\
\Big(j_s^3\Big)^2 &= (\bar{u}u)^2 + \Big(\bar{d}d\Big)^2 - 2(\bar{u}u)\Big(\bar{d}d\Big),\\
\Big(j_s^4\Big)^2 &= (\bar{u}s)^2 + (\bar{s}u)^2 + 2(\bar{u}s)(\bar{s}u),\\
\Big(j_s^5\Big)^2 &= -(\bar{u}s)^2 - (\bar{s}u)^2 + 2(\bar{u}s)(\bar{s}u),\\
\Big(j_s^6\Big)^2 &= \Big(\bar{d}s\Big)^2 + (\bar{s}d)^2 + 2\Big(\bar{d}s\Big)(\bar{s}d),\\
\Big(j_s^7\Big)^2 &= -\Big(\bar{d}s\Big)^2 - (\bar{s}d)^2 + 2\Big(\bar{d}s\Big)(\bar{s}d),\\
\Big(j_s^8\Big)^2 &= \frac{1}{3}\Big[(\bar{u}u)^2 + \Big(\bar{d}d\Big)^2 + 4(\bar{s}s)^2 + 2(\bar{u}u)\Big(\bar{d}d\Big) - 4(\bar{u}u)(\bar{s}s) - 4\Big(\bar{d}d\Big)(\bar{s}s)\Big],\\
j_s^0j_s^3 &= \sqrt{\frac{2}{3}}\Big[(\bar{u}u)^2 - \Big(\bar{d}d\Big)^2 + (\bar{u}u)(\bar{s}s) - \Big(\bar{d}d\Big)(\bar{s}s)\Big],\\
j_s^0j_s^8 &= \frac{\sqrt{2}}{3}\Big[(\bar{u}u)^2 + \Big(\bar{d}d\Big)^2 - 2(\bar{s}s)^2 + 2(\bar{u}u)\Big(\bar{d}d\Big) - (\bar{u}u)(\bar{s}s) - \Big(\bar{d}d\Big)(\bar{s}s)\Big],\\
j_s^3j_s^8 &= \frac{1}{\sqrt{3}}\Big[(\bar{u}u)^2 - \Big(\bar{d}d\Big)^2 - 2(\bar{u}u)(\bar{s}s) + 2\Big(\bar{d}d\Big)(\bar{s}s)\Big],
\end{aligned}$$

we obtain, after some long and tedious but straightforward algebra,

$$\begin{aligned}\mathrm{tr}_F\Big(K\lambda^aK\lambda^bj_s^aj_s^b\Big) = 4\,\Big[&K_u^2(\bar{u}u)^2 + K_d^2(\bar{d}d)^2 + K_s^2(\bar{s}s)^2 + 2K_uK_d(\bar{u}d)(\bar{d}u)\\ &+2K_uK_s(\bar{u}s)(\bar{s}u) + 2K_dK_s(\bar{d}s)(\bar{s}d)\Big],\end{aligned}$$

or simply,

$$\mathrm{tr}_F\Big(K\lambda^a K\lambda^b j_s^a j_s^b\Big) = 4\sum_{f,g=u,d,s} K_f K_g(\bar{\psi}_f\psi_g)(\bar{\psi}_g\psi_f).$$

Taking $K = S^{(0)}(p)$ for the term with the scalar currents, we have

$$\mathrm{tr}_F\Big[S^{(0)}(p)\lambda^a S^{(0)}(p)\lambda^b j_s^a j_s^b\Big] = 4\sum_{f,g=u,d,s} S_f^{(0)}(p)S_g^{(0)}(p)(\bar{\psi}_f\psi_g)(\bar{\psi}_g\psi_f).$$

Similarly, for the term with the pseudoscalar currents we find

$$\mathrm{tr}_F\Big[S^{(0)}(p)i\gamma_5\lambda^a S^{(0)}(p)i\gamma_5\lambda^b j_s^a j_s^b\Big] = 4\sum_{f,g=u,d,s} S_f^{(0)}(p)i\gamma_5 S_g^{(0)}(p)i\gamma_5(\bar{\psi}_f i\gamma_5\psi_g)(\bar{\psi}_g i\gamma_5\psi_f).$$

Bibliography

[1] ZHANG, R.; FU, W. jie; LIU, Y. xin. Properties of mesons in a strong magnetic field. **The European Physical Journal C**, Springer Science and Business Media LLC, v. 76, n. 6, jun 2016.

[2] MIRANSKY, V. A.; SHOVKOVY, I. A. Magnetic catalysis and anisotropic confinement in QCD. **Physical Review D**, American Physical Society (APS), v. 66, n. 4, aug 2002.

[3] ANDERSEN, J. O.; NAYLOR, W. R.; TRANBERG, A. Phase diagram of QCD in a magnetic field. **Reviews of Modern Physics**, American Physical Society (APS), v. 88, n. 2, apr 2016.

[4] SKOKOV, V.; ILLARIONOV, A.; TONEEV, V. Estimate of the magnetic field strength in heavy-ion collisions. **International Journal of Modern Physics A**, World Scientific Pub Co Pte Lt, v. 24, n. 31, p. 5925–5932, dec 2009.

[5] HATSUDA, T.; KUNIHIRO, T. QCD phenomenology based on a chiral effective lagrangian. **Physics Reports**, Elsevier BV, v. 247, n. 5-6, p. 221–367, oct 1994.

[6] SHOVKOVY, I. A. Magnetic catalysis: A review. In: **Strongly Interacting Matter in Magnetic Fields**. [S.l.]: Springer Berlin Heidelberg, 2013. p. 13–49.

[7] GUSYNIN, V.; MIRANSKY, V.; SHOVKOVY, I. Dimensional reduction and catalysis of dynamical symmetry breaking by a magnetic field. **Nuclear Physics B**, Elsevier BV, v. 462, n. 2-3, p. 249–290, mar 1996.

[8] MIRANSKY, V. A.; SHOVKOVY, I. A. Quantum field theory in a magnetic field: From quantum chromodynamics to graphene and dirac semimetals. **Physics Reports**, Elsevier BV, v. 576, p. 1–209, apr 2015.

[9] NAMBU, Y.; JONA-LASINIO, G. Dynamical model of elementary particles based on an analogy with superconductivity. I. **Physical Review**, American Physical Society (APS), v. 122, n. 1, p. 345–358, apr 1961.

[10] NAMBU, Y.; JONA-LASINIO, G. Dynamical model of elementary particles based on an analogy with superconductivity. II. **Physical Review**, American Physical Society (APS), v. 124, n. 1, p. 246–254, oct 1961.

[11] KLEVANSKY, S. P. The Nambu—Jona-Lasinio model of quantum chromodynamics. **Reviews of Modern Physics**, American Physical Society (APS), v. 64, n. 3, p. 649–708, jul 1992.

[12] VOGL, U.; WEISE, W. The Nambu and Jona-Lasinio model: Its implications for hadrons and nuclei. **Progress in Particle and Nuclear Physics**, Elsevier BV, v. 27, p. 195–272, jan 1991.

[13] FUKUSHIMA, K.; HIDAKA, Y. Magnetic catalysis versus magnetic inhibition. **Physical Review Letters**, American Physical Society (APS), v. 110, n. 3, jan 2013.

[14] BALI, G. S. et al. The QCD phase diagram for external magnetic fields. **Journal of High Energy Physics**, Springer Science and Business Media LLC, v. 2012, n. 2, feb 2012.

[15] FERRER, E.; INCERA, V. de la; WEN, X. Quark antiscreening at strong magnetic field and inverse magnetic catalysis. **Physical Review D**, American Physical Society (APS), v. 91, n. 5, mar 2015.

[16] AYALA, A. et al. Thermomagnetic evolution of the QCD strong coupling. **Physical Review D**, American Physical Society (APS), v. 98, n. 3, aug 2018.

[17] FARIAS, R. L. S. et al. Thermo-magnetic effects in quark matter: Nambu-Jona-Lasinio model constrained by lattice QCD. **The European Physical Journal A**, Springer Science and Business Media LLC, v. 53, n. 5, may 2017.

[18] FERREIRA, M. et al. Inverse magnetic catalysis in the (2+1)-flavor Nambu–Jona-Lasinio and Polyakov–Nambu–Jona-Lasinio models. **Physical Review D**, American Physical Society (APS), v. 89, n. 11, jun 2014.

[19] AVANCINI, S. S. et al. π_0 pole mass calculation in a strong magnetic field and lattice constraints. **Physics Letters B**, Elsevier BV, v. 767, p. 247–252, apr 2017.

[20] ENDRŐDI, G.; MARKÓ, G. Magnetized baryons and the QCD phase diagram: NJL model meets the lattice. **Journal of High Energy Physics**, Springer Science and Business Media LLC, v. 2019, n. 8, aug 2019.

[21] AVANCINI, S. S. et al. Light pseudoscalar meson masses under strong magnetic fields within the SU(3) Nambu–Jona-Lasinio model. **Physical Review D**, American Physical Society (APS), v. 104, n. 9, nov 2021.

[22] PAULO, A.; BRAGHIN, F. L. Vacuum polarization corrections to low energy quark effective couplings. **Physical Review D**, American Physical Society (APS), v. 90, n. 1, jul 2014.

[23] BRAGHIN, F. L. Low energy constituent quark and pion effective couplings in a weak external magnetic field. **The European Physical Journal A**, Springer Science and Business Media LLC, v. 54, n. 3, mar 2018.

[24] BRAGHIN, F. L. SU(2) low energy quark effective couplings in weak external magnetic field. **Physical Review D**, American Physical Society (APS), v. 94, n. 7, oct 2016.

[25] BRAGHIN, F. L. Flavor-dependent U(3) Nambu–Jona-Lasinio coupling constant. **Physical Review D**, American Physical Society (APS), v. 103, n. 9, may 2021.

[26] SCHWINGER, J. On gauge invariance and vacuum polarization. **Physical Review**, American Physical Society (APS), v. 82, n. 5, p. 664–679, jun 1951.

[27] CHYI, T.-K. et al. Weak-field expansion for processes in a homogeneous background magnetic field. **Physical review D: Particles and fields**, v. 62, 12 1999.

[28] GRIFFITHS, D. **Introduction to Elementary Particles**. [S.l.]: Wiley-VCH GmbH, 2008. ISBN 3527406018.

[29] NAGASHIMA, Y. **Elementary particle physics**. Weinheim Chichester: Wiley-VCH John Wiley distributor, 2010. v. 1. ISBN 9783527409624.

[30] NAGASHIMA, Y. **Elementary particle physics**. Weinheim Chichester: Wiley-VCH John Wiley distributor, 2012. v. 2. ISBN 3527409661.

[31] CHENG, T.-P.; LI, L.-F. **Gauge theory of elementary particle physics**. Oxford Oxfordshire New York: Clarendon Press Oxford University Press, 1984. ISBN 0198519613.

[32] AITCHISON, I. J. R.; HEY, A. J. **Gauge theories in particle physics: a practical introduction**. Boca Raton, FL: CRC Press, 2012. v. 2. ISBN 1466513071.

[33] PESKIN, M. E.; SCHROEDER, D. V. **An introduction to quantum field theory**. Boca Raton: CRC Press, 2019. ISBN 0367320568.

[34] SCHWARTZ, M. D. **Quantum Field Theory and the Standard Model**. [S.l.]: Cambridge University Pr., 2014. ISBN 1107034736.

[35] SKANDS, P. Introduction to QCD. In: **Searching for New Physics at Small and Large Scales**. [S.l.]: WORLD SCIENTIFIC, 2013.

[36] ZYLA and P. A. et al. Review of particle physics. **Progress of Theoretical and Experimental Physics**, Oxford University Press (OUP), v. 2020, n. 8, aug 2020.

[37] PAIS, H.; MENEZES, D. P.; PROVIDÊNCIA, C. Neutron stars: From the inner crust to the core with the (extended) Nambu–Jona-Lasinio model. **Physical Review C**, American Physical Society (APS), v. 93, n. 6, jun 2016.

[38] SREDNICKI, M. **Quantum Field Theory**. [S.l.]: Cambridge University Press, 2019. ISBN 0521864496.

[39] ’tHOOFT, G. Symmetry breaking through bell-jackiw anomalies. **Physical Review Letters**, American Physical Society (APS), v. 37, n. 1, p. 8–11, jul 1976.

[40] MOSEL, U. **Path Integrals in Field Theory**. [S.l.]: Springer Berlin Heidelberg, 2003. ISBN 3540403825.

[41] REHBERG, P.; KLEVANSKY, S. P.; HÜFNER, J. Hadronization in the SU(3) Nambu–Jona-Lasinio model. **Physical Review C**, American Physical Society (APS), v. 53, n. 1, p. 410–429, jan 1996.

[42] KLIMT, S. et al. Generalized SU(3) Nambu-Jona-Lasinio model. **Nuclear Physics A**, Elsevier BV, v. 516, n. 3-4, p. 429–468, oct 1990.

[43] KLEVANSKY, S. P.; LEMMER, R. H. Chiral-symmetry restoration in the Nambu–Jona-Lasinio model with a constant electromagnetic field. **Physical Review D**, American Physical Society (APS), v. 39, n. 11, p. 3478–3489, jun 1989.

[44] MENEZES, D. P. et al. Quark matter under strong magnetic fields in the Nambu–Jona-Lasinio model. **Physical Review C**, American Physical Society (APS), v. 79, n. 3, mar 2009.

[45] AVANCINI, S. S.; TAVARES, W. R.; PINTO, M. B. Properties of magnetized neutral mesons within a full RPA evaluation. **Physical Review D**, American Physical Society (APS), v. 93, n. 1, jan 2016.

[46] AVANCINI, S. S. et al. Regularizing thermo and magnetic contributions within nonrenormalizable theories. **Physical Review D**, American Physical Society (APS), v. 103, n. 5, mar 2021.

[47] COPPOLA, M.; DUMM, D. G.; SCOCCOLA, N. Charged pion masses under strong magnetic fields in the NJL model. **Physics Letters B**, Elsevier BV, v. 782, p. 155–161, jul 2018.

[48] LI, J.; CAO, G.; HE, L. Gauge independence of pion masses in a magnetic field within the Nambu–Jona-Lasinio model. **Physical Review D**, American Physical Society (APS), v. 104, n. 7, oct 2021.

[49] BALI, G. S. et al. QCD quark condensate in external magnetic fields. **Physical Review D**, American Physical Society (APS), v. 86, n. 7, oct 2012.

[50] BALI, G. et al. Meson masses in electromagnetic fields with wilson fermions. **Physical Review D**, American Physical Society (APS), v. 97, n. 3, feb 2018.

[51] DING, H.-T. et al. Chiral properties of (2+1)-flavor QCD in strong magnetic fields at zero temperature. **Physical Review D**, American Physical Society (APS), v. 104, n. 1, jul 2021.

[52] FARIAS, R. L. S. et al. Importance of asymptotic freedom for the pseudocritical temperature in magnetized quark matter. **Physical Review C**, American Physical Society (APS), v. 90, n. 2, aug 2014.

[53] GRADSHTEYN, I. S.; RYZHIK, I. M. **Table of integrals, series and products**. Oxford: Academic, 2007. ISBN 9780123736376.

[54] DITTRICH, H. G. W. **Probing the Quantum Vacuum**. [S.l.]: Springer Berlin Heidelberg, 2000. ISBN 3540674284.

[55] AVANCINI, S. S. et al. NJL-type models in the presence of intense magnetic fields: The role of the regularization prescription. **Physical Review D**, American Physical Society (APS), v. 99, n. 11, jun 2019.

[56] COHEN-TANNOUDJI, C.; DIU, B.; LALOE, F. **Quantum Mechanics 01**. [S.l.]: Wiley-VCH GmbH, 2019. ISBN 3527345531.

[57] COSTA, G.; FOGLI, G. **Symmetries and Group Theory in Particle Physics**. [S.l.]: Springer Berlin Heidelberg, 2012. ISBN 3642154816.

[58] D'ELIA, M.; NEGRO, F. Chiral properties of strong interactions in a magnetic background. **Physical Review D**, American Physical Society (APS), v. 83, n. 11, jun 2011.

[59] KLEVANSKY, S. P.; JAENICKE, J.; LEMMER, R. H. Collective modes of the Nambu–Jona-Lasinio model with an external U(1) gauge field. **Physical Review D**, American Physical Society (APS), v. 43, n. 9, p. 3040–3048, may 1991.

[60] AHMAD, A. et al. Flavor, temperature and magnetic field dependence of the QCD phase diagram: magnetic catalysis and its inverse. **Journal of Physics G: Nuclear and Particle Physics**, IOP Publishing, v. 48, n. 7, p. 075002, may 2021.

[61] GUSYNIN, V.; MIRANSKY, V.; SHOVKOVY, I. Dimensional reduction and dynamical chiral symmetry breaking by a magnetic field in 3 + 1 dimensions. **Physics Letters B**, Elsevier BV, v. 349, n. 4, p. 477–483, apr 1995.

[62] CAO, G. Effects of a parallel electromagnetic field in the three-flavor Nambu–Jona-Lasinio model. **Physical Review D**, American Physical Society (APS), v. 101, n. 9, may 2020.

[63] SHI, S. et al. Fermion self-energy and pseudovector condensate in NJL model with external magnetic field. 2018.

[64] GHOSH, S. et al. Impact of magnetic field on shear viscosity of quark matter in Nambu–Jona-Lasinio model. **Physical Review D**, American Physical Society (APS), v. 100, n. 3, aug 2019.

[65] GALILO, B. V.; NEDELKO, S. N. Impact of the strong electromagnetic field on the QCD effective potential for homogeneous abelian gluon field configurations. **Physical Review D**, American Physical Society (APS), v. 84, n. 9, nov 2011.

[66] TAVARES, W. R. et al. Nambu–Jona-Lasinio SU(3) model constrained by lattice QCD: thermomagnetic effects in the magnetization. **The European Physical Journal A**, Springer Science and Business Media LLC, v. 57, n. 9, sep 2021.

[67] DAS, A.; HAQUE, N. Neutral pion mass in the linear sigma model coupled to quarks at arbitrary magnetic field. **Physical Review D**, American Physical Society (APS), v. 101, n. 7, apr 2020.

[68] GORBAR, E. V.; MIRANSKY, V. A.; SHOVKOVY, I. A. Normal ground state of dense relativistic matter in a magnetic field. **Physical Review D**, American Physical Society (APS), v. 83, n. 8, apr 2011.

[69] OZAKI, S. QCD effective potential with strong U(1) magnetic fields. **Physical Review D**, American Physical Society (APS), v. 89, n. 5, mar 2014.

[70] CAO, G. Recent progresses on QCD phases in a strong magnetic field: views from Nambu–Jona-Lasinio model. **The European Physical Journal A**, Springer Science and Business Media LLC, v. 57, n. 9, sep 2021.